On the existence of slice theorems for moduli spaces on fiber bundles

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Abstract

After a short introduction to the finite dimensional case of orbit spaces and a summary of the most important results on Hilbert manifolds and smooth infinite dimensional manifolds, we consider three orbit spaces related to the space $\mathcal{C}\text{onn}(E)$ of connections on a fiber bundle $E \to M$ and its gauge group $\mathcal{G}\text{au}(E)$.

This investigation is motivated by a decomposition of the space of metrics $\mathcal{M}\text{et}(E)$ on the total space of the bundle into three parts on which $\mathcal{G}\text{au}(E)$ acts, one of them being $\mathcal{C}\text{onn}(E)$. For the orbit spaces related to $\mathcal{M}\text{et}(E)$ and to $\mathcal{C}\text{onn}(E) \times \mathcal{M}\text{et}(VE)$ the direct sum of the space of connections and the space of fiber metrics on the vertical bundle, respectively, slice theorems will be proven, which lead to stratifications of the orbit spaces.

For the orbit space related to the space of connections on E counterexamples will show that, except for the trivial cases of zero dimensional fiber or zero dimensional base, no such slice theorem can exist.

Kurzfassung

Nach einer kurzen Einführung in Orbiträume im endlichdimensionalen Fall und einer Zusammenfassung der wichtigsten Ergebnisse über Hilbert Mannigfaltigkeiten und glatte unendlichdimensionale Mannigfaltigkeiten betrachten wir drei Orbiträume, die mit den Konnexionen $\mathcal{C}\text{onn}(E)$ eines Faserbündels $E \to M$ und dessen Eichgruppe $\mathcal{G}\text{au}(E)$ zusammenhängen.

Dies ist motiviert durch die Zerlegung des Raumes der Metriken $\mathcal{M}\text{et}(E)$ auf dem Totalraum des Bündels in drei Teile, auf denen jeweils die Eichgruppe wirkt. Einer dieser Teile ist der Raum der Konnexionen $\mathcal{C}\text{onn}(E)$. Für den Orbitraum zu $\mathcal{M}\text{et}(E)$ und dem Orbitraum zu $\mathcal{C}\text{onn}(E) \times \mathcal{M}\text{et}(VE)$, der direkten Summe der Räume der Konnexionen und der Fasermetriken auf dem vertikalen Bündel, wird ein Scheibensatz bewiesen, der zu einer Stratifizierung der Orbiträume führt.

Für den Orbitraum, der zum Raum der Konnexionen gehört, wird mit Hilfe von Gegenbeispielen gezeigt, daß kein Scheibensatz existieren kann, ausgenommen in den trivialen Fällen von nulldimensionaler Basis oder nulldimensionaler Faser.

Preface

At first I would like to thank my supervisor Prof. Peter W. Michor from the Institute of Mathematics of the University of Vienna for the possibility to write this thesis, for the problem, the necessary papers, his support, and his patience.

Furthermore, I want to send special thanks to my former teacher Prof. Wolfgang Herfort from the University for Technology of Vienna for introducing me to the theory of Hilbert spaces, especially Sobolev spaces, which was an important background for this work.

Thanks, also, to my colleague Andreas Cap for much help and many fruitful discussions, whenever questions arose while trying to solve the problem, and to Prof. Dan Burghelea of Ohio State University for a special discussion and the reference to the paper of J. Cerf.

Finally, I have to thank the Institute of Mathematics of the University of Vienna, especially its head Prof. Harald Rindler, and the Erwin Schrödinger Institute of Mathematical Physics, for a position and for providing me with all the means to finish this thesis.

Vienna, November 17th 1996 — Hermann Schichl

Vorwort

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Zuletzt gebühren noch dem Institut für Mathematik der Universität Wien, dabei ganz besonders dem Institutsvorstand Prof. Harald Rindler, und dem Erwin Schrödinger Institut für Mathematische Physik Dank für eine Stelle bzw. die zur Verfügung Stellung von Mitteln, ohne die die Fertigstellung dieser Dissertation nicht möglich gewesen wäre.

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Contents

1. Introduction

In modern mathematics and physics actions of Lie groups on manifolds and the resulting orbit spaces (moduli spaces) are of great interest. For example, the moduli space of principal connections on a principal fiber bundle modulo the group of principal bundle automorphisms is the proper configuration space for Yang–Mills field theory (as e.g. outlined in [Gribov 1977], [Singer 1978], and [Narasimhan, Ramadas 1979]).

Usually, when symmetries and invariance groups are considered, a problem reduces to the corresponding orbit space, and therefore the structure of these spaces has to be investigated. This structure theory is quite complicated in general, since these spaces usually are singular spaces and not again manifolds. In fact, only if the action of the Lie group is free (i.e. all isotropy subgroups of single points are trivial), the resulting orbit space bears a manifold structure and forms together with the manifold and the quotient map a principal fiber bundle, whose structure is well known. More often, the orbit space admits a stratification into smooth manifolds with an open and dense largest stratum, the set of principal orbits (see section 2). This stratified space can then be treated almost like a manifold when taking special care. The existence of such a stratification is usually shown by proving the existence of slices at every point for the group action.

All these problems arise already, if both the Lie group and the manifold are finite dimensional. As shown in section 4, both notions can be very widely generalized to infinite dimensions, and again the structure of the, now often infinite dimensional, moduli spaces is interesting. For example the above mentioned configuration space of Yang–Mills theory is constructed from infinite dimensional spaces. Again, a slice theorem for the action is the way to prove existence of a (generalized) stratification of the moduli space. This slice theorem turns out to be more difficult than in the finite dimensional case, because of the lack of an inverse function theorem. In order to reduce this problem to a very special inverse function theorem with very strong prerequisites, an approach using Hilbert manifolds and Sobolev completions has to be taken in order to construct the slice via inverse limits of Hilbert manifolds.

In spite of these technical difficulties, these infinite dimensional generalizations are very interesting, since many problems in modern theoretical physics lead to moduli spaces in the infinite dimensional setting. Not only the above mentioned Yang–Mills theory is considered, but also the space of Riemannian metrics modulo the group of diffeomorphisms, which appears in General Relativity, (principal) fiber bundle connections modulo the gauge group arise from gauge theories, and also one approach to quantization, the geometric quantization method, is taken via orbit spaces, as described in [Kirillov 1982], [Kirillov 1990], and [Vizman 1994].

Also, very recent research in theoretical physics is connected to moduli spaces: e.g. invariance of Euler numbers of moduli spaces of instantons on 4–manifolds [Vafa, Witten 1990], moduli spaces of parabolic Higgs bundles, which are connected to Higgs fields [Maruyama, Yokogawa 1992], [Nakajima 1996].

In algebraic topology moduli spaces play an important role, either, [Maruyama 1996], [Simpson 1994], [Simpson 1995], and also the definition of the famous Donaldson Polynomials involves moduli spaces ([Donaldson 1990]).

In this thesis, I will try to analyze the structure of some moduli spaces, which are connected with the space of connections on a general fiber bundle with compact fiber and compact base space. In section 2 some known results about actions of finite dimensional groups on finite dimensional manifolds will be recalled, such that the notion of slices will become clear. Then in sections 3 and 4 the basic facts about Hilbert manifolds and about infinite dimensional smooth manifolds and Lie groups will be introduced. Slice theorems for two moduli spaces will be shown in section 5. In section 6, finally, counterexamples will show that there cannot exist a slice theorem for the moduli space $\mathcal{C}\mathrm{onn}(E)/\,\mathcal{G}\mathrm{au}(E)$ of connections on a general fiber bundle modulo the gauge group.

The result is connected with a slice theorem for $\mathcal{M}\mathrm{et}(M)/\,\mathcal{D}\mathrm{iff}(M)$, the orbit space of metrics on a manifold with respect to the action of the group of diffeomorphisms proved by [Ebin 1968], another slice theorem proved by [Kondracki, Rogulski 1986] for $\mathcal{C}\mathrm{onn}(P)/\,\mathcal{G}\mathrm{au}(P)$, the space of connections on a *principal* fiber bundle modulo the gauge group, [Cerf] for $\mathfrak{F}/\,\mathcal{D}\mathrm{iff}(\mathbb{R})$ the space of functions of finite codimension at critical points on $\mathbb{R}$ modulo the diffeomorphism group. For a general survey on slice theorems and slices see [Isenberg, Marsden 1982], where a slice theorem for the space of solutions of Einstein's equations modulo the diffeomorphism group is proven.

Finally, the non-existence of the slice theorem in the case connections on a fiber bundle modulo the gauge group is connected to the fact, that $C^\infty(S^1,\mathbb{R})/\,\mathcal{D}\mathrm{iff}(S^1)$, where the diffeomorphisms act by composition, admits in general no slices, except when restricted to the space of functions of finite codimension at critical points.

2. G–manifolds and Lie group actions

In this section, I want to introduce the basic facts about finite dimensional Lie groups acting on finite dimensional manifolds (some of them without proofs). These facts will be taken for motivation of the terms in infinite dimensions. Moreover, a basic example will be discussed, which helps to understand the properties of slices and sections.

Throughout this chapter all manifolds and groups are supposed to be finite dimensional, except if stated otherwise explicitly.

2.1. Definition. Let G be a Lie group, M a smooth manifold. A *smooth action* of G on M is a C^∞ mapping $l : G \times M \to M$ ($l(g,x) = l_g(x) = l^x(g) = gx$), such that

$$ex = x \quad \forall x \in M$$
$$(g_1 g_2)x = g_1(g_2 x) \quad \forall g_1, g_2 \in G, x \in M.$$

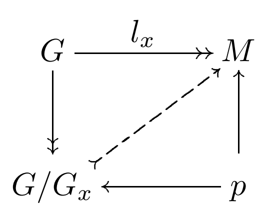
We say that G acts on M, or M is a G–manifold. Furthermore, a G–action on M is called

(1) *linear*, if M is a vector space, and the action is a representation.
(2) *affine*, if M is an affine space, and all the l_g are affine transformations.
(3) *orthogonal*, if (M,γ) is a Euclidean space, and the action is a subgroup of $O(M,\gamma)$.
(4) *isometric*, if (M,γ) is a Riemannian manifold, and every l_g is an isometry.
(5) *symplectic*, if (M,ω) is a symplectic manifold, and every l_g is a symplectomorphism.

2.2. Definition. Let M be a G–manifold, $x \in M$

The set $G \cdot x := \{gx \mid g \in G\}$ is called the *(G–)orbit* of x.
The closed subgroup $G_x := \{g \in G \mid gx = x\}$ of G is called the *isotropy subgroup* of x.

Then $T_e G =: \mathfrak{g}$, $T_0(G/G_x) \simeq \mathfrak{g}/\operatorname{Lie}(G_x)$, and



where the mapping $G/G_x \to M$ is an initial immersion with image $G \cdot x$.

2.3. Lemma.

(1) $G_{gx} = g G_x g^{-1}$
(2) $G \cdot x \cap G \cdot y \neq \emptyset \implies G \cdot x = G \cdot y$
(3) $T_x(G \cdot x) = T_e(l_x).\mathfrak{g}$

Proof. Is clear.

2.4. Definition. For a G–manifold M, let M/G be the space of all G–orbits equipped with the quotient topology; $\pi : M \to M/G$. Then M/G is called the *orbit space (moduli space)* of M with respect to G.

The set of closed subgroups of G bears an equivalence relation $H \sim H' :\iff H = gH'g^{-1}$ for some $g \in G$. The equivalence classes are called *conjugacy classes.*

The set of conjugacy classes admits a partial order $(H) \le (H')$ if $H \subseteq gH'g^{-1}$ for some $g \in G$.

For an orbit $G \cdot x$, by Lemma 2.3(1), the isotropy subgroups G_{gx} form a conjugacy class (G_x), which is called the *isotropy type* of the orbit $G \cdot x$. Two orbits are said to be of the same type if their isotropy types coincide.

2.5. Definition. For two G–manifolds M, N and a smooth mapping $f : M \to N$ we say f is *equivariant* iff $f(gx) = g\,f(x)$.

2.6. Definition. Let M be a G–manifold. An orbit $G \cdot x$ is called a *principal orbit*, if there exists an open neighborhood U of x in M with $\forall y \in U : \exists f : G \cdot x \to G \cdot y$ G–equivariant, which is equivalent to $\forall y \in U : \exists a \in G : G_x \subseteq aG_y a^{-1}$.

$x \in M$ is called a *regular point* if $G \cdot x$ is a principal orbit, and is called *singular point* otherwise.

$M_{\text{reg}} := \{\text{regular points}\}$, $M_{\text{sing}} := \{\text{singular points}\}$.

2.7. Definition. Let $x \in M$, M a G–manifold. A subset S of M is called a *slice* at x, if there exists a G–invariant open neighborhood U of the orbit $G \cdot x$ and a smooth equivariant retraction $r : U \to G \cdot x$, such that $S = r^{-1}(x)$.

2.8. Proposition. *Let M be a G–manifold and S a slice at x. Then*

(1) *$x \in S$ and $G_x \cdot S \subseteq S$*
(2) *$g \cdot S \cap S \neq \emptyset \implies g \in G_x$*
(3) *$G \cdot S = \{gy | g \in G, y \in S\} = U$ with U as in definition 2.7.*

Proof. Let r be the (smooth, equivariant) retraction. Then we know that r is a submersion and $S = r^{-1}(x)$. Thus $G_y \subseteq G_x$ $\forall y \in S$. Therefore, $r|_{G \cdot y} : G \cdot y \to G \cdot x$ is also a submersion $\forall y$, which implies that x is a regular value for r. That suffices to show that $S = r^{-1}(x)$ is a sub-manifold of U and also of M.
Thus we have $y \in S, gy \in S \implies r(gy) = x = gr(x) = gx \implies g \in G_x$, which in return implies (2).
Then $g \in G_x, s \in S \implies r(gs) = gr(s) = gx = x \implies G_x \cdot S \subseteq S$, which shows (1).
(3) can be shown as follows: $y \in U \implies \exists g \in G : gr(y) = x \implies r(gy) = gr(y) = x \implies gy \in S$. □

2.9. Corollary. *Let S be a slice at x for the G–manifold M. Then*

(1) *S is a G_x–manifold*
(2) *For $y \in S$ is $G_y \subseteq G_x$.*
(3) *If $G \cdot x$ is a principal orbit and G_x compact, then $G_x = G_y$ $\forall y \in S$, i.e. all orbits near $G \cdot x$ are principal, too.*
(4) *Two G_x–orbits $G_x \cdot s_1$ and $G_x \cdot s_2$ are of the same type, iff the two G–orbits $G \cdot s_1$ and $G \cdot s_2$ in M are of the same type.*
(5) *S/G_x is isomorphic to $G \cdot S/G$, an open neighborhood of $G \cdot x$ in the orbit space M/G.*

Proof.

(1) is clear
(2) is clear
(3) $y \in S \implies G_y \subseteq G_x(\text{compact}) \implies G_y$ is compact. $G \cdot x$ is principal $\implies$ for y near x is G_x conjugate to a subgroup of G_y. Therefore $G_x = G_y$.
(4) $K = G_x$, $s \in S$, K acts on S. $K_s = G_s$ (see Proposition 2.8(2)). $G_x s_1 = G_x s_2 \implies K_{s_1}$ is conjugate to K_{s_2} in G_x. Hence G_{s_1} is conjugate to G_{s_2}.
(5) follows from 2.8(2) and (3).

□

2.10. Example. An elementary example will be useful to illustrate the properties of slices a bit.

Take $M = \mathbb{R}^n$ and $G = SO(n)$ acting by rotations about $0 \in \mathbb{R}^n$. The orbit $G \cdot 0 = \{0\}$, $G_0 = G$, and the slice S at zero may be chosen to be any open ball centered at 0. For any other point $x \in \mathbb{R}^n$ $G \cdot x$ is the sphere of radius $\|x\|_2$ centered at 0, $G_x = \{e\}$, and any sufficiently short line segment transversal to $G \cdot x$ through x can be chosen as S.

2.11. Proposition. *If S is a slice at x in a G–manifold M. Then there exists a G–equivariant diffeomorphism $G[S] = G \times_{G_x} S \to G \cdot S$, which maps the "zero-section" $G \times_{G_x} \{x\}$ onto $G \cdot x$.*

Proof. The map $f : G[S] \to G \cdot S$ given by $f : [(g, s)] \mapsto g \cdot s$ is smooth and has the required properties. □

2.12. Definition. An action $l : G \times M \to M$ is called *proper*, if one, hence all, of the following equivalent conditions is satisfied.

(1) $(l, \mathrm{Id}) : G \times M \to M \times M$ $(g, x) \mapsto (gx, x)$ is proper
(2) $g_n x_n \to y$ and $x_n \to x$ imply g_n has a convergent subsequence.
(3) $K, L \subset M$ compact $\implies \{g \in G | gK \cap L \neq \emptyset\}$ is compact

M is then called a *proper G–manifold.*

Proof. (1) $\implies$ (2) is clear. Suppose (2) $\implies$ (3) does not hold. Then $\exists (g_n)$ without cluster point, $g_n K \cap L \neq \emptyset$. Choose $x_n \in K$ with $g_n x_n \in L$. Without loss of generality $x_n \to x$, $g_n x_n \to y$, which is a contradiction.
(3) $\implies$ (1): $(l, \mathrm{Id})^{-1}(L \times K) = \{(g, x) : x \in K, gx \in L\} \underset{\text{closed}}{\subseteq} \{g \in G : gK \cap L \neq \emptyset\} \times K$ which is compact. □

2.13. Lemma. *(M, γ) a Riemannian manifold, $l : G \times M \to M$ an effective, isometric action, such that $\check{l}(G)$ is closed in $\mathrm{Isom}(M, \gamma)$. Then l is a proper action.*

Now we have collected most of the important results and terms for the first important theorem, which was proven by R. Palais in 1961.

2.14. Slice Theorem. *Let M be a G–manifold, $x \in M$ such that G_x is compact and for all open neighborhoods U of G_x in G there is an open neighborhood V of x in M with $\{g \in G | gV \cap V \neq \emptyset\} \subseteq U$. Then there exists a slice at x.*

Idea of proof. Take a Riemannian metric on M, and construct a G_x–invariant metric $\tilde{\gamma}$. Now construct a so called "*almost slice*" $\tilde{S}$: $\tilde{S} := \exp_x^{\tilde{\gamma}}(T_x(G \cdot x)^{\perp} \cap B(\varepsilon))$. If

you take an open neighborhood U of 0 in G/G_x, such that there exists a section $\chi : U \to G$ with $\chi(0) = e$. Then
$f : U \times \tilde{S} \to M$ $(u, s) \mapsto \chi(u) \cdot s$ is a diffeomorphism (maybe for somewhat smaller U, $\tilde{S}$) onto an open neighborhood of x in M. Using the assumption of the theorem, you get a neighborhood V of x in M, such that $S = \tilde{S} \cap V$ is the required slice.

The complete proof can be found in [Palais 1961].

In the proof of the infinite dimensional slice theorem most of the "easy-to-construct" things like the manifold G/G_x, χ, and $T_x(G \cdot x)^\perp$ will make much more difficulty, and constructing these will be the first crucial step in obtaining the slice theorem.

2.15. Theorem. *In every proper G–manifold M, there exists a slice at every point $x \in M$.*

Proof. G_x is compact, since the action is proper. Now let U be an open neighborhood of G_x in G. Then there exists an open neighborhood V of G_x with $G_x \cdot V = V$. ($G_x \cdot G_x = G_x$; Thus $\forall (a, b) \in G_x \times G_x$ exist open neighborhoods $A_{a,b}$ of a in G and $B_{a,b}$ of b in G such that $A_{a,b} \cdot B_{a,b} \subseteq U$. Since G_x is compact $G_x \underset{\text{open}}{\subseteq} \bigcup_{i=1}^n B_{a,b_i} =: B_a$. Let $A_a := \bigcap_{i=1}^n A_{a,b_i}$. Then $A_a \cdot B_a \subseteq U$. Furthermore $G_x \underset{\text{open}}{\subseteq} \bigcup_{j=1}^n A_{a_j}$ and $W := \bigcap_{j=1}^n B_{b_i}$ is open in G, $G_x \subseteq W$, $G_x \cdot W \subseteq U$, $V = G_x \cdot W$.)

Now let $U = G_x U$ be a neighborhood of cosets. $l^x : G \to G \cdot x$ is a closed mapping. Thus $(G \setminus U) \cdot x$ is closed in $G \cdot x$, therefore in M. This implies the existence of a neighborhood W of x such that $W \cap (G \setminus U) \cdot x = \emptyset$. Without loss of generality let W be compact. Then $\{g \in G \mid gW \cap W \neq \emptyset\}$ is compact (2.12(3)). $K := \{g \in G \setminus U \mid gW \cap W \neq \emptyset\}$ is a compact subset of $G \setminus U$. $k \in K \implies k\,x \in (G \setminus U) \cdot x \implies k\,x \in M \setminus W$ (which is open). Therefore, there exist neighborhoods Q_k of k, V_k of x such that $Q_k \cdot V_k \subseteq M \setminus W$. Some $Q_{k_1} \dots Q_{k_n}$ cover K, $V := \bigcap_{i=1}^n V_{k_i}$ is neighborhood of x in M, without loss of generality $V \subseteq W$.

Let $gV \cap V \neq \emptyset$. Then $gW \cap W \neq \emptyset$, and therefore $g \in U \cup K$. If $g \in K$, then there exists i such that $g \in Q_{k_i}$. But then $gV \subseteq Q_{k_i} V \subseteq M \setminus W \subseteq M \setminus V$ and this implies $gV \cap V = \emptyset$ which is a contradiction. Therefore $g \in U$. □

2.16. Theorem (Palais 1961)**.** *For a proper G–manifold M, $x \in M$ the following are equivalent*

1. *G_x is compact and there exists a slice at x.*
2. *There exists a neighborhood U of x in M, s.t. $\{g \in G | gV \cap V \neq \emptyset\}$ has compact closure in G.*

Proof. in [Palais 1961].

2.17. Corollary. *M proper G–manifold $\implies$ M/G is completely regular and locally compact (hence* T_2*).*

2.18. Lemma. *If M is a proper G–manifold, then there exists a principal orbit type.*

2.19. Theorem. *M a proper G–manifold. Then the set of all regular points in M is open and dense.*

2.20. Theorem. *M a proper G–manifold, $x \in M$. Then x has a G–invariant open neighborhood U, such that U contains only finitely many orbit types.*

2.21. Theorem. *M a proper G–manifold, then the space M_{sing}/G of all singular G–orbits does not locally dissect M/G.*

2.22. Corollary. *M connected proper G–manifold. Then*

(1) *M/G is connected*
(2) *M has exactly one principal orbit type.*
(3) *The set of all principal orbits is open and dense.*

Most of these facts essentially follow from the existence of slices. In infinite dimensions analogous results can be likewise proven, if the existence of such slices is ensured.

However, first we will have to define most of the involved terms like manifold, Lie group, tangent space, ... in infinite dimensions. The next two chapters are devoted to this.

3. Banach and Hilbert manifolds

In this chapter I will give a brief excursion on Banach and Hilbert manifolds which is in part excerpted from [Lang 1995]. More detail can (e.g.) be found there.

3.1. Definition. A *topological vector space* is a vector space E over $\mathbb{R}$ equipped with a topology such that the operations $+ : E \times E \to E$ (addition of vectors) and $\cdot : \mathbb{R} \times E \to E$ (multiplication with scalars) are continuous.

Throughout our presentation all topological vector spaces will be Hausdorff and *locally convex* (i.e. every neighborhood U of $0 \in E$ contains an open convex neighborhood V of 0.)

The set of continuous linear maps $\varphi : E \to F$ (E and F being two topological vector spaces) will be denoted by $L(E, F)$, and we set $L(E) := L(E, \mathbb{R})$.

The vector space of n–linear maps $\psi : E \times \cdots \times E \to F$ will be denoted $L^n(E, F)$, and as above $L^n(E) := L^n(E, \mathbb{R})$.

3.2. Definition. A locally convex topological space E will be called a *Fréchet space* if its topology is metrizable (i.e. there exists a metric $d : E \times E \to \mathbb{R}_0^+$, such that every neighborhood U of $0 \in E$ contains a ball $B_\varepsilon := \{v \in E | d(0, e) < \varepsilon\}$.), and it is complete (i.e. all Cauchy sequences converge).

A Fréchet space space E will be called a *Banach space* if its metric is defined by a norm $\|.\| : E \to \mathbb{R}_0^+$ (i.e. $d(v, w) = \|v - w\|$). It is well known, that for Banach spaces E, F the norm $\|A\| := sup_{\|x\|_E = 1} \|Ax\|_F$ for $A \in L(E, F)$ makes $L(E, F)$ into a Banach space.

A Banach space is called a *Hilbert space* if the norm is defined by an inner product $\langle ., . \rangle : E \times E \to \mathbb{R}$ (i.e. $\|v\| = \sqrt{\langle v, v \rangle}$).

3.3. Proposition. *Let E, F be Banach spaces, $f \in L(E, F)$. Assume $G \subset F$ is an (algebraic) linear complement to* $\operatorname{im} f$ *and G is closed in F. Then* $\operatorname{im} f$ *is closed in F and $F = \operatorname{im} f \oplus G$.*

Proof. See [Palais 1965, proof of Theorem 1] □

3.4. Definition. Let E, F be two Banach spaces, and $f : U \underset{\text{open}}{\subset} E \to F$ be a continuous map. We say that f is *differentiable at a point* $x_0 \in U$ if there exists a $\lambda \in L(E, F)$ such that

$$\lim_{y \to 0} \frac{\|f(x_0 + y) - f(x_0) - \lambda(y)\|}{\|y\|} = 0.$$

λ is then uniquely determined, and we set $Df(x_0) := \lambda$ and call it the *derivative* of f at x_0. If f is differentiable at every point $x \in U$ then we get a map $Df : U \to L(E, F)$, and we say f is differentiable.

If Df is again continuous we say that f is of class C^1. Maps of class C^p for $p \geq 1$ are then defined inductively. The p–th derivative of f will be $D^p f := D(D^{p-1} f)$, a map of U to $L(E, L(E, \ldots, L(E, F) \ldots))$, which can be identified with $L^p(E, F)$. A map f is said to be of class C^p if $D^k f$ exists for $1 \leq k \leq p$, and is continuous.

The usual results like chain rule, and linearity hold as in the finite dimensional case.

3.5. Definition. Let E_1, E_2, F be Banach spaces, $U_i \underset{\text{open}}{\subset} E_i$ and $f : U_1 \times U_2 \to F$ be a continuous map. If $(u, v) \in U \times V$ and we keep v fixed, then $f(\quad, v) : U_2 \to F$, and we define the *partial derivative* as

$$\partial_1 f(u, v) := D(f(\quad, v))(u),$$

which is a map $U \times V \to L(E_1, F)$. Similarly, we may define $\partial_2 f$. The total derivative and the partials are related as follows.

Proposition. *Let E_i, F be Banach spaces, $U_i \underset{\text{open}}{\subset} E_i$, and $f : U_1 \times \cdots \times U_n \to F$ continuous. Then f is of class C^p if and only if each partial derivative $\partial_i f : U_1 \times \cdots \times U_n \to L(E_i, F)$ exists and is of class C^{p-1}. In that case for $v = (v_1, \ldots, v_n) \in U_1 \times \cdots \times U_n$, and $(w_1, \ldots, w_n) \in E_1 \times \cdots \times E_n$ we have*

$$Df(v) \cdot (w_1, \ldots, w_n) = \sum_{i=1}^{n} \partial_i f(v) \cdot w_i.$$

3.6. The inverse mapping theorem. The inverse mapping theorem is one of the main reasons for considering Banach and Hilbert spaces (and the yet to be defined Banach and Hilbert manifolds), since in most of the smooth infinite dimensional spaces of chapter 4 this theorem does not hold. This arises as the main difficulty in proving theorems for these spaces, and long ways have to be taken to circumvent this difficulty. Most of these ways lead to Banach and Hilbert completions of the spaces.

Both the inverse function theorem and the existence theorem for differential equations (which is extremely important also) are based on the following

Shrinking Lemma. *Let E be a complete metric space, with distance function d, and let $f : E \to E$. Assume the existence of a constant $0 < C < 1$, such that, for any $v, w \in E$, we have*

$$d(f(v), f(w)) \leq C d(v, w).$$

Then f has a unique fixed point x ($x = f(x)$). Given any point $x_0 \in E$, then $x = \lim_{n\to\infty} f^n(x_0)$ with $f^n(v) = f(f^{n-1}(v))$.

Inverse mapping theorem. *Let E, F be Banach spaces, $U \underset{\text{open}}{\subset} E$, and let $f : U \to F$ be a C^p–map with $p \geq 1$. Assume that for some point $x_0 \in U$, the derivative $Df(x_0) : E \to F$ is a top-linear isomorphism (the inverse is continuous also). Then f is a local C^p–isomorphism at x_0. (i.e. there exists an open neighborhood V of x_0 such that $f|_V$ is a C^p–isomorphism onto an open subset of F.)*

Proof. Since a top-linear isomorphism is C^∞, we may assume without loss of generality that $E = F$, and $Df(x_0)$ is the identity (Consider $Df(x_0)^{-1} \circ f$ instead of f). Furthermore, we may assume that $x_0 = 0$ and $f(x_0) = 0$.

Set $g(x) = x - f(x)$. Then $Dg(x_0) = 0$, and by continuity there exists $r > 0$ such that, if $\|x\| < 2r$, we have $\|Dg(x)\| < \frac{1}{2}$. Then $\|g(x)\| \leq \frac{1}{2}$ by the mean value theorem. Thus $g(\overline{B}_r(0)) \subset \overline{B}_{\frac{r}{2}}(0)$.
Claim: For $y \in \overline{B}_{\frac{1}{2}}(0)$ there exists $x \in \overline{B}_r(0)$ such that $f(x) = y$.
Proof: Consider the map

$$g_y = y + x - f(x).$$

If $\|y\| \leq \frac{r}{2}$ and $\|x\| \leq r$, then $\|g_y(x)\| \leq r$, and hence g_y may be viewed as a mapping of the complete metric space $\overline{B}_r(0)$ into itself. g_y is a contracting map, since

$$\|g_y(v) - g_y(w)\| = \|g(v) - g(w)\| \leq \tfrac{1}{2}\|v - w\|$$

by the mean value theorem for $v, w \in \overline{B}_r(0)$. By the Shrinking Lemma we find that g_y has a unique fixed point x, which is the solution we were looking for.

By this claim we obtain a local inverse f^{-1}. It is continuous, since

$$\|v - w\| \leq \|f(v) - f(w)\| + \|g(v) - g(w)\| \leq 2\|f(v) - f(w)\|.$$

Furthermore, f^{-1} is differentiable in $B_{\frac{r}{2}}(0)$ by the following argument: Let $v_1, v_2 \in \overline{B}_r(0)$, $w_1, w_2 \in \overline{B}_{\frac{r}{2}}(0)$, with $f(v_i) = w_i$. Then

$$\begin{aligned}
\|f^{-1}(w_1) - f^{-1}(w_2) - Df(v_2)^{-1}(w_1 - w_2)\| = \\
&= \|v_1 - v_2 - Df(v_2)^{-1}(f(v_1) - f(v_2))\| \leq \\
&\leq \|Df(v_2)^{-1}\|\, \|Df(v_2)(v_1 - v_2) - f(v_1) + f(v_2)\| = \\
&= o(|v_1 - v_2|) = o(|w_1 - w_2|),
\end{aligned}$$

where the first o–term is correct, since f is differentiable, and the last equality follows from the already proved continuity of f^{-1}. So the differentiability of f^{-1} is proved, and its derivative is $D(f^{-1})(y) = Df(f^{-1}(y))^{-1}$ for $y \in B_{\frac{r}{2}}(0)$. Since f^{-1}, Df are continuous and taking inverses is C^∞, $D(f^{-1})$ is continuous, so f^{-1} is C^1. By induction, it follows that f^{-1} is C^p if f is. $\square$

3.7. The Implicit Mapping Theorem. *Let E, F, G be Banach spaces, $U \underset{\text{open}}{\subseteq} E$, $V \underset{\text{open}}{\subseteq} F$, and let $f : U \times V \to G$ be a C^p mapping. Let $(u_0, v_0) \in U \times V$, and assume that*

$$\partial_2 f(u_0, v_0) : F \to G$$

is a top-linear isomorphism. Let $f(u_0, v_0) = C$. Then there exists a continuous map $g : U_0 \to V$ defined on an open neighborhood U_0 of u_0 such that $g(u_0) = v_0$, and such that

$$f(x, g(x)) = C$$

for all $x \in U_0$. If U_0 is taken sufficiently small then g is uniquely determined, and is also of class C^p.

Proof. Without loss of generality we may assume that $\partial_2 f(u_0, v_0)$ is the identity (simply replace f by $\partial_2 f(u_0, v_0)^{-1} \circ f$). Consider the map $\varphi : U \times V \to E \times F$ $\varphi(x, y) = (x, f(x, y))$. Then we compute

$$D\varphi(u_0, v_0) = \begin{pmatrix} \mathrm{Id}_E & 0 \\ \partial_1 f(u_0, v_0) & \partial_2 f(u_0, v_0) \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_E & 0 \\ \partial_1 f(u_0, v_0) & \mathrm{Id}_F \end{pmatrix},$$

which obviously is invertible. Hence, φ is locally invertible by 3.6. Since we have $\varphi^{-1}(x, z) = (x, h(x, z))$ for some mapping h of class C^p. We set $g(x) = h(x, C)$. Then g is also of class C^p and

$$(x, f(x, g(x))) = \varphi(x, g(x)) = \varphi(x, h(x, C)) = \varphi(\varphi^{-1}(x, C)) = (x, C).$$

This proves the existence of g. For the uniqueness we suppose that g' is a continuous map defined near u_0 such that $g'(u_0) = v_0$ and $f(x, g'(x)) = C$ for all x near u_0. Then $g'(x)$ is near v_0 for such x, and hence $\varphi(x, g'(x)) = (x, C)$. Since φ is invertible near (u_0, v_0), it follows that there is a unique point (x, y) near (u_0, v_0) such that $\varphi(x, y) = (x, C)$. Let U_0 be a small ball on which g is defined. If g' is also defined on U_0, then the argument above shows that g and g' coincide on some smaller neighborhood of u_0. Let $x \in U_0$ and let $u' = x - u_0$. Consider the set $\{t \in [0,1] | g(u_0 + tu') = g'(u_0 + tu')\}$. This set is nonempty, so let T be its least upper bound. By continuity we get $g(u_0 + Tu') = g'(u_0 + Tu')$. If $T < 1$ we can apply the existence and uniqueness part we have already proved to show that g and g' are equal in a neighborhood of $u_0 + Tu'$. Therefore, $T = 1$, and the uniqueness is proved, as well as the theorem. □

3.8. Banach Manifolds. Let M be a topological Hausdorff space. A *chart* on M is a triple (U, u, E), where $U \underset{\text{open}}{\subseteq} M$ and $u : U \to V \underset{\text{open}}{\subseteq} E$ is a homeomorphism onto an open set in a Banach space E.

An *atlas of class* C^p $(0 \le p \le \infty)$ is a set $A = \{(U_i, u_i, E_i) | i \in I\}$ of charts satisfying the following conditions:

(1) $\{U_i | i \in I\}$ is a covering of M
(2) For each pair of indices $(i, j) \in I \times I$ the map

$$u_j \circ u_i^{-1} : u_i(U_i \cap U_j) \to u_j(U_i \cap U_j)$$

is a C^p diffeomorphism.

If $p \ge 1$ we see by (2) that if $U_i \cap U_j$ is nonempty $E_i \simeq E_j$. Therefore, on each connected component of M we can assume that all E_i are equal, say E. In the following we will assume that all E_i are isomorphic, which is not a big assumption, since we could prove all theorems for each connected component separately.

We say that two atlases A_1 and A_2 are C^p equivalent if the atlas $A_1 \cup A_2$ is a C^p atlas. An equivalence class of C^p atlases is said to define the structure of a C^p *Banach manifold* on M, and if all E_i are isomorphic to the space E we say M is modeled on E. If the modeling space E is a Hilbert space, we call M a *Hilbert manifold.*

Let M and N be two Banach manifolds, and let $f : M \to N$ be a continuous map. We shall say $f \in C^p(M, N)$ if for all $x \in M$, there exist charts (U, u) of M with $x \in U$ and (V, v) of N with $f(x) \in N$ such that $v \circ f \circ u^{-1} : u(U) \to v(V)$ is C^p as a mapping of Banach spaces. A bijective mapping is said to be a C^p diffeomorphism if f and f^{-1} are both C^p.

Manifolds, mappings, ... of class C^∞ will also be called *smooth* in the sequel.

3.9. Submanifolds, Immersions, Submersions. Let M be a C^p Banach manifold. A subset $N \subset M$ is called *submanifold* of M if at each point $x \in N$ there exists a chart (U, u) of M such that there are two Banach spaces E_1, E_2 with $E_1 \times E_2 \simeq E$, and $u(U) = V_1 \times V_2$ with $V_i \underset{\text{open}}{\subseteq} E_i$ and $u(N \cap U) = V_1 \times \{0\}$.

Then the collection of pairs $(U \cap N, \mathrm{pr}_1 \circ (u|_{U \cap N}))$ is an atlas of class C^p for N.

This structure satisfies a universal mapping property, which characterizes it: Given any map $f : Z \to M$ from a manifold Z into M such that $f(Z)$ is contained in N. Let $f_N : Z \to N$ be the induced map. Then f is C^p if and only if f_N is C^p.

A submanifold N is always locally closed in M (i.e. every point $x \in N$ has an open neighborhood U in M such that $N \cap U$ is closed in U). We say that N is a closed submanifold of M if N is a closed topological subspace of M.

Let $f : Z \to M$ be a C^p map, and let $z \in Z$. We say that f is an *immersion at* z if there exists an open neighborhood V of z such that $f|_U$ induces an isomorphism of V onto a submanifold of M. We call f an *immersion* if it is an immersion at every point $z \in Z$. An immersion f is called a (closed) *embedding* if it gives an isomorphism onto a (closed) submanifold of M.

A mapping $f : M \to Z$ is called a *submersion at a point* $x \in M$ if there exists a chart (U, u) at x and a chart (V, v) at $f(x)$ such that u gives an isomorphism of U on a product $U_1 \times U_2$ (U_1, U_2 open in some Banach spaces), and such that the map $vfu^{-1} : U_1 \times U_2 \to V$ is a projection. We say that f is a *submersion* if it is a submersion at every point. Submersions are open mappings.

A criterion for immersions and submersions will be given in 3.10 using the tangent space:
On every point $x \in M$ we consider triples (U, u, X) where (U, u) is a chart at x and $X \in E$. We call two such triples (U, u, X), (V, v, Y) equivalent if and only if $D(vu^{-1})(u(x)).X = Y$. An equivalence class of such triples is called a tangent vector of M at x. The set T_xM of such tangent vectors is called the *tangent space* of M at x. Each chart (U, u) determines a bijection of T_xM to the modeling space E of M by which T_xM gets the structure of a Banach space. Using the tangent spaces, we can interpret the derivative of a C^p mapping $f : M \to N$ by means of charts as a continuous linear mapping $T_xf : T_xM \to T_{f(x)}N$ (essentially as in the finite dimensional case).

Proposition. *Let M and N be Banach manifolds of class C^p ($p \geq 1$), and let $f : M \to N$ be a C^p mapping. Take $x \in M$. Then*

(1) *f is an immersion at x if and only if the map T_xf is injective and splits (i.e. it exists a top-linear isomorphism $\alpha : T_{f(x)}N \to F_1 \times F_2$ such that $\alpha \circ T_xf$ induces a top-linear isomorphism of T_xM onto $F_1 \times \{0\}$.)*
(2) *f is a submersion at x if and only if the map T_xf is surjective and its kernel splits (i.e. is closed and has closed complement).*

Proof. This is an immediate consequence of the inverse mapping theorem. □

3.10. Partitions of unity. Unlike for finite dimensional manifolds the existence of partitions of unity is even in the Banach case not always satisfied. The problem for constructing differentiable partitions of unity is the existence of a differentiable norm. However, if we put strong restrictions on the topology of M, we will get the existence:

Theorem. *Let M be a Banach manifold modeled on E which is locally compact, and whose topology has a countable base. Then M admits partitions of unity: i.e. for every open covering $\{V_i\}$ there exists a subordinate open covering $\{U_i\}$ of M and a family of functions $f_i : M \to \mathbb{R}$ satisfying the following conditions*

(1) *For all $x \in M$ we have $f_i(x) \geq 0$.*
(2) *The support of f_i is contained in U_i.*
(3) *The covering is locally finite (i.e. every $x \in M$ has a neighborhood which intersects only finitely many U_i).*

(4) *For each $x \in M$ we have $\sum_i f_i(x) = 1$.*

Proof. The proof will be left out, but can be found in [Lang 1995, II §3]. □

However, since the only locally compact Banach spaces are finite dimensional we have not won too much. The proof can be carried over to the situation where M is paracompact, and there is a differentiable norm on E which is equivalent to the original one, but it is difficult to check the existence of such a norm. However, in one case we can always construct differentiable partitions of unity:

3.11. Theorem. *Let M be a paracompact manifold of class C^p modeled on a separable Hilbert space H. Then M admits partitions of unity of class C^p.*

Proof. For this proof we will need a few definitions and some lemmas.

Definition. A subset V of a metric space (X, d) is called *scalloped*, if there exist open balls $B_{r_i}(x_i)$ in X such that

$$V = B_{r_0}(x_0) \cap C(\overline{B}_{r_1}) \cap \cdots \cap C(\overline{B}_{r_n}).$$

where $C(A)$ shall denote the set theoretical complement of A.

Lemma 1. *Let (X, d) be a metric space and $\{B_{r_i}(x_i)\}$ ($i \in \mathbb{N}$) a countable covering of a subset W by open balls. Then there exists a locally finite open covering $\{V_i\}$ ($i \in \mathbb{N}$) of W such that $V_i \subset B_{r_i}(x_i)$ for all i, and such that V_i is scalloped for all i.*

Proof. Define V_i inductively by the following construction. Let $V_1 := B_{r_1}(x_1)$. Then set $r'_{ji} := r_j - \frac{1}{i}$, and let

$$V_i := B_{r_i}(x_i) \cap \bigcap_{j=1}^{i-1} C(\overline{B}_{r'_{ji}}(x_j)),$$

replacing all balls of negative radius by the empty set. By construction, each V_i is scalloped and is contained in $B_{r_i}(x_i)$. Take $x \in W$, and let k be the smallest index such that $x \in B_{r_k}(x_k)$. Then $x \in V_k$, because otherwise, x would be in $C(V_k)$. But

$$C(V_k) = C(B_{r_k}) \cup \bigcup_{j=1}^{k-1} \overline{B}_{r'_{kj}}(x_j),$$

and thus x lies in some $B_{r_j}(x_j)$ with $j \leq k-1$ which is a contradiction.

For proving the locally finiteness, take again $x \in W$. Then $x \in B_{r_k}(x_k)$ for some k. Let $\varepsilon > 0$ be so small that $B_\varepsilon(x) \subset B_{r_k}(x_k)$. For all sufficiently large i the ball $B_{\frac{\varepsilon}{2}}(x) \subset \overline{B}_{r'_{ki}}(x_k)$, and therefore by construction $B_{\frac{\varepsilon}{2}}(x) \cap V_i = \emptyset$. Thus $B_{\frac{\varepsilon}{2}}(x)$ meets only finitely many V_i. □

Lemma 2. *Let U be an open ball in the Hilbert space H, and let V be a scalloped open subset. Then there exists a C^∞-function $\psi : H \to \mathbb{R}$ such that $\psi|_{C(V)} \equiv 0$ and $\psi|_V > 0$.*

Proof. Since V is scalloped, we have $V = B_{r_0}(x_0) \cap \bigcap_{i=1}^n C(\overline{B}_{r_i}(x_i))$. For $i = 1, \dots, n$ choose a function $\varphi_i : H \to \mathbb{R}$ such that

$$\begin{aligned} 0 < \varphi_i(x) \leq 1 \quad & \text{if } x \in C(\overline{B}_{r_i}(x_i)) \\ \varphi_i(x) = 0 \quad & \text{if } x \in \overline{B}_{r_i}(x_i). \end{aligned}$$

Let $\varphi_0 : H \to \mathbb{R}$ be a function such that $\varphi(x) > 0$ on U and $\varphi(x) = 0$ outside U. Set

$$\psi(x) := \prod_{i=0}^{n} \varphi_i(x).$$

Then ψ satisfies the requirement. □

Proposition. *Let A_1, A_2 be two nonempty, closed, disjoint subsets of a separable Hilbert space H. Then there exists a smooth function $\psi : H \to \mathbb{R}$ such that $\psi|_{A_1} \equiv 0$ and $\psi|_{A_2} \equiv 1$, and $0 \leq \psi(x) \leq 1$ for all x (i.e. H is smoothly normal).*

Proof. By the theorem of Lindelöf, we can find countably many open balls $B_{r_i}(x_i)$ ($i \in \mathbb{N}$) covering A_2 such that each $B_{r_i}(x_i)$ is contained in $C(A_1)$. Let $W = \bigcup_{i \in \mathbb{N}} B_{r_i}(x_i)$. By Lemma 1 we can find a locally finite refinement $\{V_i\}$ of scalloped open sets. By Lemma 2, we find functions φ_i positive on V_i and zero outside of V_i. Let $\varphi = \sum_{i \in \mathbb{N}} \varphi_i$ (the sum is finite at each point of W, since the V_i are a locally finite covering). Then φ is positive on A_2, and $\varphi|_{A_1} \equiv 0$.

Let U be the open neighborhood of A_2 on which $\varphi > 0$. Then A_2 and $C(U)$ are disjoint closed sets. We then apply the construction above to get another function $\sigma : H \to \mathbb{R}$ which is positive on $C(U)$ and is identically zero on A_2. By setting

$$\psi := \frac{\varphi}{\varphi + \sigma}$$

we get the required function. □

Proof of Theorem 3.11. Let $B_r(x)$ be an open ball in H. By

$$y \mapsto \frac{y}{\sqrt{r^2 - \langle y, y \rangle}}$$

$B_r(x)$ is diffeomorphic to H. Take any point $x \in M$, and a neighborhood V of x. We can find a chart (U, u) of M at x such that $u(U) = H$, and $U \in V$. Given an open covering of M we, therefore, can find an atlas $\{(U_\alpha, u_\alpha)\}$ such that $u_\alpha(U_\alpha) = H$ for all α, and the U_α are subordinate to the given covering. By paracompactness, we can find a refinement $\{\tilde{U}_i\}$ of the covering $\{U_\alpha\}$ which is locally finite. Each $\tilde{U}_i$ is contained in some $U_{\alpha(i)}$. Let $\tilde{u}_i = u_i|_{\tilde{U}_i}$. Again by paracompactness we find open refinements $\{V_i\}$ and $\{W_i\}$ such that

$$\overline{W}_i \subset V_i \subset \overline{V}_i \subset \tilde{U}_i.$$

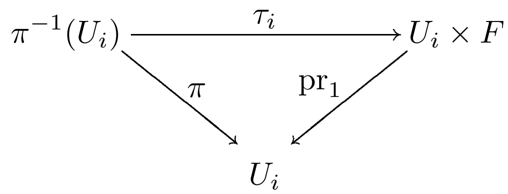
By construction $\tilde{u}_i(\overline{W}_i)$ and $\tilde{u}_i(\overline{V}_i)$ are closed in H. By the proposition, we can find functions φ_i on H with $\varphi_i|_{\tilde{u}_i(\overline{W}_i)} = 1$ and $\varphi_i|_{H-\tilde{u}_i(V_i)} = 0$, being between 0 and 1, otherwise. Set $\psi_i = \varphi_i \circ u_i$. Then ψ_i is 0 on $M - V_i$ and 1 on $\overline{W}_i$. Set $\psi = \sum_i \psi_i$, and $f_i = \frac{\psi_i}{\psi}$. Then the $\{f_i\}$ are the desired partition of unity. □

Since partitions of unity are the only known means of gluing together local mappings, this theorem gives a hint on the importance of Hilbert manifolds (i.e. manifolds modeled on Hilbert spaces). A very important class of Hilbert spaces will be considered in paragraph 3.21.

3.12. Vector bundles. The partitions of unity discussed above are an essential tool when considering vector bundles.

Let M be a C^p Banach manifold modeled on a Banach space B, let E be another Banach manifold, and $\pi : E \to M$ be a C^p map. Let F be a Banach space. Let $\{U_i\}$ be an open covering of M, and for each i suppose that we have a mapping $\tau_i : \pi^{-1}(U_i) \to U_i \times F$ satisfying the following conditions:

(1) The map τ_i is a C^p diffeomorphism such that the following diagram commutes:



In particular, we obtain an isomorphism on each fiber

$$\tau_{ix} : \pi^{-1}(x) \to F.$$

(2) For each pair of open sets U_i, U_j the map

$$\tau_{jx}\tau_{ix}^{-1} : F \to F$$

is a top-linear isomorphism.

(3) If U_i and U_j are two members of the covering, then the map of $U_i \cap U_j$ into $L(F, F)$ given by

$$x \mapsto (\tau_j \tau_i^{-1})_x$$

is a C^p mapping.

Then we call (U_i, τ_i) a *trivializing covering* for π or E, and that $\{\tau_i\}$ are its trivializing maps. If $x \in U_i$, we say that (U_i, τ_i) trivializes at x. Two trivializing coverings are called to be equivalent if their union is a trivializing covering. An equivalence class of such trivializing coverings is said to give the quadruple (E, π, M, F) the structure of a *vector bundle*. M is called the *base (space)*, E the *total space*, π the *bundle (or footpoint) projection*. The Banach space F is called the *standard fiber*. The space $\pi^{-1}(x)$ is called the *fiber over* x.

Note the difference to the finite dimensional case: (3) is implied by (2) there. In the infinite dimensional case it has to be stated explicitly.

The maps $\tau_{ijx} = \tau_{ix} \circ \tau_{jx}^{-1}$, are called the *transition functions* associated with the covering. They satisfy the so called *cocycle condition*

$$\tau_{kjx} \circ \tau_{jix} = \tau_{kix}, \quad \text{(i.p. } \tau_{ijx} = \tau_{jix}^{-1}).$$

As in the finite dimensional case, the cocycle of transition functions characterizes the vector bundle.

A vector bundle (E, π, M, F) is called *trivializable* if it is isomorphic to $(M \times F, \mathrm{pr}_1, M, F)$.

Let (E, π, M, F) and (E', π', M', F') be two vector bundles. A *C^p vector bundle morphism* f between these bundles consists of a pair of C^p mappings $f_0 : M \to M'$ and $f : E \to E'$ satisfying the following conditions:

(1) The diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow[f_0]{} & M' \end{array}$$

is commutative for each $x \in M$ $f_x : E_x \to E'_{f(x)}$, and the induced map is a continuous linear map.

(2) For each $x_0 \in M$ there exist trivializing maps

$$\tau : \pi^{-1}(U) \to U \times F$$
$$\tau' : \pi'^{-1}(U') \to U' \times F'$$

at x_0 and $f(x_0)$ respectively, such that $f_0(U)$ is contained in U', and such that the map

$$U \to L(F, F')$$
$$x \mapsto \tau'_{f_0(x)} \circ f_x \circ \tau_x^{-1}$$

is C^p.

We will usually write $f : E \to E'$ to denote a vector bundle morphism.

Let (E, π, M, F) be a vector bundle, and $f : N \to M$ a C^p map. Then $(f^*(E), f^*(\pi), N, F)$ is a vector bundle called the *pull back of E along f*, and the pair $(f, \pi^*(f))$ is a vector bundle morphism.

$$\begin{array}{ccc} f^*(E) & \xrightarrow{\pi^*(f)} & E \\ {\scriptstyle f^*(\pi)}\downarrow & & \downarrow{\scriptstyle\pi} \\ N & \xrightarrow[f]{} & M \end{array}$$

An important vector bundle is the *tangent bundle* of a manifold. Let M be a manifold of class C^p with $p \geq 1$. Let TM be the disjoint union of the vector spaces T_xM from 3.9. Let $\pi : TM \to M$ map T_xM to x, and set $F = B$ the modeling space of M. Take an atlas (U_i, u_i) of M. From the definition of tangent vectors as triples (U_i, u_i, X_i) we immediately get a bijection

$$\tau_i : \pi^{-1}(U_i) = TU_i \to U_i \times F$$

which commutes with the projection on U_i, that is such that

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\tau_i} & U_i \times F \\ & \searrow^{\pi} \quad {}^{\mathrm{pr}_1}\swarrow & \\ & U_i & \end{array}$$

is commutative. Furthermore, if we set for any two charts (U_i, u_i) and (U_j, u_j) $u_{ij} = u_j u_i^{-1}$, then we obtain transition mappings

$$\tau_{ji} = \tau_j \tau_i^{-1} : u_i(U_i \cap U_j) \times F \to u_j(U_i \cap U_j) \times F$$

by the formula

$$\tau_{ji}(x, X) = (u_{ji}(x), Du_{ji}(x) \cdot X)$$

for $x \in U_i \cap U_j$ and $X \in F$. Since the derivative Du_{ji} is of class C^{p-1} and is an isomorphism at x, we find all conditions for a vector bundle satisfied. Therefore, TM is a vector bundle of class C^{p-1}.

Given a C^p map $f : M \to N$, we can define $Tf : TM \to TN$ to be simply $T_x f$ on each fiber $T_x M$. It is easy to check that Tf is a vector bundle morphism $TM \to TN$ of class C^{p-1} called the *tangent map* of f. Locally, the map is given as $Tf(x, X) = (f(x), Df(x) \cdot X)$.

Another useful definition follows: A mapping $f : E \to E'$ between vector bundles (E, π, M, F) and (E', π', M, F') is called *fiber preserving*, if $f(\pi^{-1}(x)) \subset \pi'^{-1}(x)$ for all $x \in M$.

3.13. Sections of bundles, vector fields. Let M be a C^p manifold, and take a C^q vector bundle (M, E, π, F) over M $(q \leq p)$. A C^r *section of* E $(r \leq q)$ is a C^r map $\xi : M \to E$ with $\pi \circ \xi = \mathrm{Id}_M$. The set of all such sections will be denoted by $C^r(E)$.

If $E = TM$ such a section ξ of class C^{p-1} will be called a *(time-independent) vector field* on M. The set of all vector fields on M will be denoted by $\mathfrak{X}(M)$.

Like in the finite dimensional case, some constructions can be applied to vector bundles, but more care has to be taken with the topology: $V \oplus W$, $V \otimes W$, $V^* = L(V, \mathbb{R})$, and $\Lambda^n V$ can e.g. be constructed. Sections of tensor products of TE and TE^* are also called *tensor fields.*

3.14. The existence theorem for differential equations. There is an existence theorem for the flow of vector fields similar to the finite dimensional case. Since we will only need the existence of local flows, only that result will be mentioned.

Let $f : J \times U \to E$ be a C^p mapping $(p \geq 0)$, $0 \in J$ an open interval in $\mathbb{R}$ and $U \subset E$ open (i.e. the local representation of a time-dependent vector field).

For a point $x_0 \in U$, an *integral curve* for f with *initial condition* x_0 is a mapping of class C^r $(r \geq 1)$

$$\begin{aligned} \alpha &: J_0 \to U \\ \alpha(0) &= x_0 \\ \alpha'(t) &= f(t, \alpha(t)), \end{aligned}$$

where $0 \in J_0$ is an open subinterval of J.

A *local flow* for f at x_0 is a mapping

$$\alpha : J_0 \times U_0 \to U$$
$$\alpha_x(t) = \alpha(t, x),$$

where $0 \in J_0$ is an open subinterval of J, and $x_0 \in U_0 \subset U$ is an open subset, and α_x is an integral curve for f with initial condition x.

Having these definitions in mind, we find the following results, similar to the finite dimensional case.

Proposition. *In the situation above, let $0 < a < 1$ be a real number such that $\overline{B}_{3a}(x_0)$ lies in U. Assume that f is a continuous map bounded by a constant $L \geq 1$ on $J \times U$ and satisfies a Lipschitz condition on U, uniformly with respect to J, with constant $K \geq 1$. If $b < \frac{a}{LK}$, then for each $x \in \overline{B}_a(x_0)$ there exists a unique local flow*

$$\alpha : (-b, b) \times B_a(x_0) \to U.$$

If f is of class C^p, so is each integral curve α_x.

The local flow α is continuous, and the map $x \mapsto \alpha_x$ of $\overline{B}_a(x_0)$ into the space of curves satisfies a Lipschitz condition.

If we take f C^p with $p \geq 1$ then we get stronger results.

Theorem. *Let f be a (local) vector field on U of class C^p $(p \geq 1)$, and let $x_0 \in U$.*

Then there exists a unique local flow for f at x_0. We can select a (maximal) open subinterval J_0 of J containing 0 and an open subset U_0 of U containing x_0, such that the unique local flow

$$\alpha : J_0 \times U_0 \to U$$

is of class C^p, and such that $\partial_2\alpha$ satisfies the differential equation

$$\partial_1\partial_2\alpha(t, x) = \partial_2 f(t, \alpha(t, x))\partial_2\alpha(t, x)$$

on $J_0 \times U_0$ with initial condition $\partial_2\alpha(0, x) = \mathrm{Id}$. Usually, α will then be denoted by Fl^f.

Proof. The proof can e.g. be found in [Lang 1995, IV.§1]. It depends heavily on the Shrinking Lemma, and on the Banach space norm.

3.15. Corollary. *Let U, V be open sets in Banach spaces E, F respectively. Let J be an open interval of $\mathbb{R}$ containing 0, and let $g : J \times U \times V \to F$ be a C^r map $(r \geq 1)$. Let (u_0, v_0) be a point in $U \times V$. Then there exist open balls J_0, U_0, V_0 centered at 0, u_0, v_0 respectively, and a unique map of class C^r $h : J_0 \times U_0 \times V_0 \to V$ such that $h(0, u, v) = v$ and*

$$\partial_1 h(t, u, v) = g(t, u, h(t, u, v))$$

for all $(t, u, v) \in J_0 \times U_0 \times V_0$.

Proof. This follows from the existence and uniqueness of the local flow of the vector field on $U \times V$ $G : J \times U \times V \to E \times F$ given by $G(t, u, v) = (0, g(t, u, v))$. Then $h(t, u, v) = \mathrm{pr}_2 \circ \mathrm{Fl}^G(t, u, v)$. $\square$

3.16. Corollary. *The function h from Corollary 3.15 satisfies the equation*

$$\partial_1\partial_2 h(t,u,v)\cdot x = \partial_2 g(t,u,h(t,u,v))\cdot x + \partial_3 g(t,u,h(t,u,v))\cdot \partial_2 h(t,u,v)\cdot x$$

for all $x \in E$.

Proof. This is just calculation, using the result above. □

3.17. Corollary. *Let* J *be an open interval of* $\mathbb{R}$ *containing* 0 *and take* $U \underset{\text{open}}{\subseteq} E$. *Let* $f : J \times U \to E$ *be a continuous map, which is Lipschitz on* U *uniformly for every compact subinterval of* J. *Let* $t_0 \in J$ *and let* φ_1, φ_2 *be two* C^1 *maps such that* $\varphi_1(t_0) = \varphi_2(t_0)$ *and satisfying the relation*

$$\varphi_i'(t) = f(t, \varphi_i(t))$$

for all $t \in J$. *Then* $\varphi_1(t) = \varphi_2(t)$.

Proof. This follows directly from the existence and uniqueness result for differential equations. □

3.18. Integrable subbundles. Let V be a tangent subbundle over M. We say V is *integrable at a point* x_0 if there exists a submanifold N of M containing x_0 such that the tangent map of the inclusion $i : N \to M$ induces a vector bundle isomorphism of TN with the subbundle V restricted to N. Equivalent is, for each $y \in N$ the tangent map $T_y j : T_y N \to T_y M$ induces a top-linear isomorphism of $T_y N$ onto V_y.

We say that V is *integrable* if it is integrable at every point.

3.19. Frobenius' Theorem. *Let* M *be a Banach manifold of class* C^p *for* $p \geq 2$ *and let* S *be a subbundle of* TM. *Then* S *is integrable if and only if* S *is involutive (i.e. for each point* $z \in M$ *and vector fields* X, Y *defined on an open neighborhood of* z *which lie in* V, *the bracket* $[X,Y]$ *also lies in* S*).*

Proof. The part integrable $\Longrightarrow$ involutive follows just by the functoriality of vector fields and their relations under tangent maps. The converse is the difficult direction.

The proof will be carried out locally. We first try to find a suitable description of the bundle S in local terms.

Take $z \in M$. We can then find a product decomposition of an open neighborhood W of z, say $U \times V$, open subsets of Banach spaces E and F, respectively, such that the point has coordinates (u_0, v_0) and such that $S|_W$ can be written as the image of an injective vector bundle map $f : U \times V \times E \hookrightarrow U \times V \times E \times F$ with $f(u_0, v_0) : E \hookrightarrow E \times F$ is the canonical embedding $E \hookrightarrow E \times \{0\}$. Without loss of generality we may assume that $\mathrm{pr}_1 \circ f(u,v) = \mathrm{Id}_E$ for all $(u,v) \in U \times V$. Thus we may describe f by a C^{p-1} mapping (also called) $f : U \times V \to L(E,F)$.

Further note that a subbundle S of TM is integrable at a point $z \in M$ if and only if there exists an open neighborhood W of z and a diffeomorphism $\varphi : U \times V \to W$ of a product of open subsets of Banach spaces onto W such that the composition $T_1(U \times V) \hookrightarrow T(U \times V) \stackrel{T\varphi}{\to} TW$ is a bundle isomorphism onto $S|_W$, where $T_1(U \times V)$ is the subbundle of $T(U \times V)$ whose fibers are $T_x U \times 0 \subseteq T_x U \times T_y V = T_{(x,y)}(U \times V)$, $x \in U$, $y \in V$.

Now take the local representations of a vector field X over $W = U \times V$. Then $X \in S|_W$ if and only if $X_2(u,v) = f(u,v) \cdot X_1(u,v)$, where X_1 and X_2 are the projections of X to E and F respectively. In other words, iff X is of the form $X(u,v) = (X_1(u,v), f(u,v).X_2(u,v))$, for some C^{p-1} map $X_1 : U \times V \to E$. If X, Y are vector fields of this type, then $[X,Y] \in S|_W$ if and only if

$$Df \cdot X \cdot Y_1 = Df \cdot Y \cdot X_1,$$

which can be calculated from the local representation of [,]. Having expressed all data locally, the following result remains to be shown.

Proposition. *Let U, V be open subsets of Banach spaces E, F respectively. Let $f : U \times V \to L(E,F)$ be a C^r map ($r \geq 1$). Assume that if $X_1, Y_1 : U \times V \to E$ are two C^r maps and that*

$$Df \cdot (X_1, f \cdot X_1) \cdot Y_1 = Df \cdot (Y_1, f \cdot Y_1) \cdot X_1.$$

Let $(u_0, v_0) \in U \times V$. Then there exist open neighborhoods $U_0 \subset U$, $V_0 \subset V$ of u_0, v_0 respectively, and a unique C^r map $\alpha : U_0 \times V_0 \to V$ such that

$$\partial_1 \alpha(u,v) = f(u, \alpha(u,v)),$$

and $\alpha(u_0, v) = v$ for all $(u,v) \in U_0 \times V_0$.

Proof. By acting by a translation we can without loss of generality assume that $(u_0, v_0) = (0,0) \in E \times F$. Now set $g(t,u,v) = f(tu,v) \cdot u$. $u \in B_\varepsilon(0) \subset U$ a small ball in E. Then by Corollary 3.15 we obtain $h : J_0 \times E_0 \times V_0 \to V$ with initial condition $h(0,u,v) = v$ for all $u \in E_0$, satisfying the differential equation $\partial_1 h(t,u,v) = f(tu, h(t,u,v)) \cdot u$. Changing variables by $t = at'$ and $u = a^{-1}u'$ for a small $a > 0$, we can assume that $1 \in J_0$, provided E_0 is small enough.

Set $\alpha(u,v) = h(1,u,v)$. Then we have to calculate $\partial_2 h(t,u,v)$. From Corollary 3.16 we obtain for any vector $x \in E$,

$$\begin{aligned} \partial_1 \partial_2 h(t,u,v) \cdot x = {} & t \partial_1 f(tu, h(t,u,v)) \cdot x \cdot u + \\ & + \partial_2 f(tu, h(t,u,v)) \cdot \partial_2 h(t,u,v) \cdot x \cdot u + f(tu, h(t,u,v)) \cdot x. \end{aligned}$$

Now let $k(t) = \partial_2 h(t,u,v) \cdot x - t f(tu, h(t,u,v)) \cdot x$. Then $k(0) = 0$, and using the local version of the integrability for the fields u and x, we get

$$Dk(t) = \partial_2 f(tu, h(t,u,v)) \cdot k(t) \cdot u.$$

By Corollary 3.17 we know that $k(t) \equiv 0$ is the unique solution. Thus

$$\partial_1 h(t,u,v) = t f(tu, h(t,u,v)),$$

and hence

$$\partial_1 \alpha(u,v) = f(u, \alpha(u,v)).$$

□

Having this result, we can set $\varphi : U_0 \times V_0 \to U \times V$ as $\varphi(u,v) = (u, \alpha(u,v))$. Then

$$D\varphi(u_0,v_0) = \begin{pmatrix} \mathrm{Id} & 0 \\ f(u_0,v_0) & \mathrm{Id} \end{pmatrix},$$

which, obviously, is a top-linear isomorphism. Thus by the inverse mapping theorem 3.6 φ is a local diffeomorphism at (u_0,v_0). Furthermore, for $(x,y) \in E \times F$ we have

$$\partial_1\varphi(u,v)\cdot(x,y) = (x, \partial_1\alpha(u,v)\cdot x) = (x, f(u,\alpha(u,v))\cdot x),$$

which shows that the bundle is integrable. □

3.20. Corollary. *Let M be a Banach manifold, S an involutive subbundle of TX. Then for any $x \in M$, there is a neighborhood W of x and a diffeomorphism $\varphi : U \times V \to W$ (U, V open neighborhood in Banach spaces) such that $\varphi(0,0) = x$, and the composition $T_1(U\times V) \hookrightarrow T(U\times V) \xrightarrow{T\varphi} TW$ is a bundle isomorphism onto $S|_W$, where $T_1(U\times V)$ is the subbundle of $T(U\times V)$ whose fibers are $T_xU \times 0 \subseteq T_xU \times T_yV = T_{(x,y)}(U\times V)$, $x \in U$, $y \in V$.*

Proof. This is just a reformulation of the local version of V being integrable.

3.21. Sobolev spaces. These important spaces have been developed in [Sobolev 1936]. Lets start with the space $L^2(\mathbb{R}^m,\mathbb{C}^n)$ of all Lebesgue square integrable functions. This space is, as is well known, a Hilbert space with the inner product

$$\langle f,g\rangle_0 := \int \langle f(x), g(x)\rangle\, dx.$$

Definition. The space of rapidly decreasing functions $S(\mathbb{R}^m,\mathbb{C}^n)$ is the vector space of all C^∞–functions $f : \mathbb{R}^m \to \mathbb{C}^n$, which satisfy that for every multiindex α and every $p \in \mathbb{N}_0$ exists a $c_{\alpha p} \geq 0$ such that for all $x \in \mathbb{R}^n$

$$\|x\|^p \|D^\alpha f(x)\| \leq c_{\alpha p}.$$

Obviously, $S(\mathbb{R}^m,\mathbb{C}^n) \subset L^2(\mathbb{R}^m,\mathbb{R}^n)$.

For all $f \in S(\mathbb{R}^m,\mathbb{C}^n)$ we define the *Fourier transformation* F_0 as

$$(F_0 f)(x) := (2\pi)^{-\frac{m}{2}} \int e^{-i\langle x,y\rangle} f(y)\, dy.$$

Having this, we get the well known result

Proposition. *$F_0 S(\mathbb{R}^m,\mathbb{C}^n) \subset S(\mathbb{R}^m,\mathbb{C}^n)$, and for every $f \in S(\mathbb{R}^m,\mathbb{C}^n)$ and every multiindex α*

$$D^\alpha F_0 f = (-1)^{|\alpha|} F_0 M_\alpha f, \qquad M_\alpha F_0 f = F_0 D^\alpha f,$$

where $(M_\alpha f)(x) = x^\alpha f(x)$ componentwise.

F_0 is a bijective linear mapping of $S(\mathbb{R}^m,\mathbb{C}^n)$ onto itself and

$$(F_0^{-1} g)(x) = (2\pi)^{-\frac{m}{2}} \int e^{i\langle x,y\rangle} g(y)\, dy.$$

Furthermore, $(F_0 f)(x) = (F_0^{-1} f)(-x)$ for all $f \in S(\mathbb{R}^m,\mathbb{C}^n)$ and $F_0^4 = \mathrm{Id}$.

The following describes the extension of F_0 to $L^2(\mathbb{R}^m,\mathbb{C}^n)$, which is well known, also.

Theorem. *F_0 and F_0^{-1} preserve the L^2–norm, and there exist unique extensions F, $\tilde{F}$ of F_0 and F_0^{-1}, respectively, as bounded unitary operators on $L^2(\mathbb{R}^m, \mathbb{C}^n)$, and $\tilde{F} = F^* = F^{-1}$, $F^4 = \mathrm{Id}$. The operator F is called Fourier transformation on $L^2(\mathbb{R}^m, \mathbb{C}^n)$.*

Furthermore, the following are equivalent

(1) $Ff \cdot Fg \in L^2(\mathbb{R}^m, \mathbb{C}^n)$,
(2) $F^{-1}f \cdot F^{-1}g \in L^2(\mathbb{R}^m, \mathbb{C}^n)$,
(3) $f * g \in L^2(\mathbb{R}^m, \mathbb{C}^n)$,

and in that case $f * g = F^{-1}(Ff \cdot Fg)$.

Definition. In the following set for $x \in \mathbb{R}^m$

$$k_s(x) := (1 + \|x\|^2)^{\frac{s}{2}}$$

and

$$L^2_s(\mathbb{R}^m, \mathbb{C}^n) := \{f \in L^2(\mathbb{R}^m, \mathbb{C}^n) | k_s f \in L^2(\mathbb{R}^m, \mathbb{C}^n)\}.$$

$L^2_s(\mathbb{R}^m, \mathbb{C}^n)$ is a dense subspace of $L^2(\mathbb{R}^m, \mathbb{C}^n)$.

$$\langle f, g\rangle_{(s)} := \int \langle f(x), g(x)\rangle k_s(x)^2 \, dx$$

for $f, g \in L^2_s(\mathbb{R}^m, \mathbb{C}^n)$ defines an inner product on $L^2_s(\mathbb{R}^m, \mathbb{C}^n)$. This inner product makes $L^2_s(\mathbb{R}^m, \mathbb{C}^n)$ to a separable Hilbert space, isomorphic to $L^2(\mathbb{R}^m, \mathbb{C}^n)$ by $U_s : f \mapsto k_s f : L^2_s(\mathbb{R}^m, \mathbb{C}^m) \to L^2(\mathbb{R}^m, \mathbb{C}^n)$.

The *Sobolev space* of order s is defined by

$$H^s(\mathbb{R}^m, \mathbb{C}^n) := \{f \in L^2(\mathbb{R}^m, \mathbb{C}^n) | Ff \in L^2_s(\mathbb{R}^m, \mathbb{C}^n)\} = F^{-1} L^2_s(\mathbb{R}^m, \mathbb{C}^n).$$

$H^s(\mathbb{R}^m, \mathbb{C}^n)$ is a dense subspace of $L^2(\mathbb{R}^m, \mathbb{C}^n)$. It is a separable Hilbert space by defining for $f, g \in H^s(\mathbb{R}^m, \mathbb{C}^n)$

$$\langle f, g\rangle_s := \langle Ff, Fg\rangle_{(s)}$$
$$\|f\|_s := \sqrt{\langle f, f\rangle_s}.$$

Obviously, $H^0(\mathbb{R}^m, \mathbb{C}^n) = L^2(\mathbb{R}^m, \mathbb{C}^n)$.

The functions in $H^s(\mathbb{R}^m, \mathbb{C}^n)$ are in a weak sense differentiable:

Theorem.

(1) *Let $s \geq 1$, $w_j = (\delta_{j1}, \ldots, \delta_{jm})$, $(M_j g)(x) = x_j g(x)$ and $f_{j,\varepsilon}(x) = f(x + \varepsilon w_j)$ for $j = 1, 2, \ldots, m$, and δ_{ij} denotes the Kronecker δ symbol. Then for all $f \in H^s(\mathbb{R}^m, \mathbb{C}^n)$ and $j = 1, 2, \ldots, m$*

$$\lim_{\varepsilon \to 0} \frac{1}{i\varepsilon}(f_{j,\varepsilon} - f) = F^{-1} M_j F f.$$

in the L^2–sense. Is $f \in S(\mathbb{R}^m, \mathbb{C}^n)$, then the limit coincides with $D^\alpha f$ with $\alpha = (\delta_{j1}, \ldots, \delta_{jm})$. We write $D^\alpha f$ for this limit in any case, even if $f \notin S(\mathbb{R}^m, \mathbb{C}^n)$.

(2) *If α is a multiindex with $|\alpha| \leq s$, then the derivative $D^\alpha f$ can be calculated iteratively. The order of differentiation can be exchanged.*
(3) *If $s \in \mathbb{N}_0$, then $\|f\|_s$ is equivalent to the norms*

$$\|f\|_{s,0} = \sqrt{\sum_{|\alpha| \leq s} \|D^\alpha f\|^2}$$

$$\|f\|_{s,1} = \sqrt{\|f\|_2^2 + \sum_{|\alpha| = s} \|D^\alpha f\|^2}.$$

(4) *For all $g \in H^{|\alpha|}(\mathbb{R}^m, \mathbb{C}^n)$ is $\langle D^\alpha f, g\rangle_0 = \langle f, D^\alpha g\rangle_0$.*
(5) *The function $D^\alpha f \in L^2(\mathbb{R}^m, \mathbb{C}^n)$ is uniquely determined by*

$$\langle D^\alpha f, g\rangle_0 = \langle f, D^\alpha g\rangle_0$$

for all $g \in C_0^\infty(\mathbb{R}^m, \mathbb{C}^n)$, the space of all functions $\mathbb{R}^m \to \mathbb{C}^n$ of compact support.
(6) *For $s \geq 0$ are $C_0^\infty(\mathbb{R}^m, \mathbb{C}^n) \subset S(\mathbb{R}^m, \mathbb{C}^n)$ and $C_0^\infty(\mathbb{R}^m, \mathbb{R}^n) \subset H^s(\mathbb{R}^m, \mathbb{C}^n)$ dense subspaces with respect to the norm $\| \quad \|_s$.*

Definition. The spaces $L^2(\mathbb{R}^m, \mathbb{R}^n)$, $L_s^2(\mathbb{R}^m, \mathbb{R}^n)$, and $H^s(\mathbb{R}^m, \mathbb{R}^n)$ shall be just the subspaces of almost everywhere real valued functions of the spaces $L^2(\mathbb{R}^m, \mathbb{C}^n)$, $L_s^2(\mathbb{R}^m, \mathbb{C}^n)$, and $H^s(\mathbb{R}^m, \mathbb{C}^n)$, respectively. The results above are true for the spaces of real valued functions, also.

The result, which is one of the reasons for the importance of Sobolev spaces is the

3.22. Sobolev Lemma. *Let $s > k + \frac{m}{2}$, then the inclusions $H^s(\mathbb{R}^m, \mathbb{C}^n) \subset C^k(\mathbb{R}^m, \mathbb{C}^n)$ and $H^s(\mathbb{R}^m, \mathbb{R}^n) \subset C^k(\mathbb{R}^m, \mathbb{R}^n)$ are continuous linear maps.*

Proof. in [Sobolev 1938].

Analogously, we define the spaces $H^s(U, \mathbb{R}^n)$ for open subsets U of $\mathbb{R}^m$.
Next we will prove some useful results, which we will need in the next sections.

3.23. Lemma. *Let D^n denote the open unit ball in $\mathbb{R}^n$, and let $f : D_n \to D_n$ and $g : D_n \to \mathbb{R}^k$ be H^s–maps ($s > \frac{n}{2} + 1$) such that Df has everywhere maximal rank. Then $g \circ f \in H^s(D_n, \mathbb{R}^k)$, and the map $(f, g) \to g \circ f$ is jointly continuous near (f, g), as a map $\circ : H^s(D_n, D_n) \times H^s(D_n, \mathbb{R}^k) \to H^s(D_n, \mathbb{R}^k)$.*

Proof. Recall that by the Sobolev Lemma 3.22 $H^s(D_n, D_n) \subset C^1(D_n, D_n)$ is continuous. By induction, we will prove that $\circ : H^s(D_n, D_n) \times H^r(D_n, \mathbb{R}^k) \to H^r(D_n, \mathbb{R}^k)$ is continuous.

For $k = 0$ we check $\int_{D_n} \|g \circ f\|^2$. But $\int_{D_n} \|g \circ f\|^2 = \int_{f(D_n)} \|g\|^2 1/|\det(J(f))|$ with $J(f)$ the Jacobian of f. Since D_n is compact, $J(f)$ is bounded away from zero. Therefore, $\circ(H^s(D_n, D_n) \times H^0(D_n, \mathbb{R}^k)) \subset H^0(D_n, \mathbb{R}^k)$. Take $\varepsilon > 0$, f' such that $\int_{D_n} \|f - f'\| < \varepsilon/(4 \max_{x \in D_n} \|D_x g\|^2)$ (this is possible since the L^1 norm is weaker than the H^s norm) and $(\int_{D_n}\|g-g'\|)^2<\varepsilon/(4\max_{x\in D_n}\{1/|\det(J(f'))(x)|\})$. Further choose δ such that $\max_{x\in D_n}\{1/|\det(J(f))(x)|,1/|\det(J(f'))(x)|\}<\varepsilon/4\delta$, and pick $g^\infty\in C^\infty(D_n,\mathbb{R}^k)$ so that $\int_{D_n}\|g^\infty-g\|<\delta$. Then we compute

$$\begin{aligned}\int_{D_n}\|g\circ f-g'\circ f'\|^2 &\le \int_{D_n}\|g\circ f-g^\infty\circ f\|^2+\int_{D_n}\|g^\infty\circ f-g^\infty\circ f'\|^2+\\ &\quad+\int_{D_n}\|g^\infty\circ f'-g\circ f'\|^2+\int_{D_n}\|g\circ f'-g'\circ f'\|^2\\ &\le \int_{D_n}\|g^\infty-g\|^2\frac{1}{|\det(J(f))|}+\max_{x\in D_n}\|D_xg\|^2\int_{D_n}\|f-f'\|+\\ &\quad+\int_{D_n}\|g^\infty-g\|^2\frac{1}{|\det(J(f'))|}+\int_{D_n}\|g-g'\|\frac{1}{|\det(J(f'))|}\\ &<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}<\varepsilon,\end{aligned}$$

which proves the continuity of $\circ: H^s(D_n,D_n)\times H^0(D_n,\mathbb{R}^k)\to H^0(D_n,\mathbb{R}^k)$.

For the inductive step we will need the following

3.24. Lemma. *Let $l>\frac{n}{2}$, $k\le l$. Let B be any bilinear map $B:\mathbb{R}^p\times\mathbb{R}^q\to\mathbb{R}^r$. Then $\tilde{B}: H^l(B_n,\mathbb{R}^p)\times H^k(B_n,\mathbb{R}^q)\to H^k(D_n,\mathbb{R}^r)$ defined by $\tilde{B}(f,g)(x)=B(f(x),g(x))$ is a continuous bilinear map.*

Proof. In [Palais 1968, 9.13] □

Next we assume the lemma for $H^s\times H^r\to H^r$ for $r<s$. We then prove it for $H^s\times H^{r+1}\to H^{r+1}$. Since $g\circ f\in H^{r+1}(D_n,\mathbb{R}^k)$ if $D(g\circ f)\in H^r(D_n,L(\mathbb{R}^n,\mathbb{R}^k))$ we compute $D(g\circ f)=(Dg\circ f)\cdot Df$. But $Dg\in H^r(D_n,L(\mathbb{R}^n,\mathbb{R}^k))$, thus $Dg\circ f\in H^r(D_n,L(\mathbb{R}^n,\mathbb{R}^k))$ by the induction assumption. Furthermore, $Df\in H^{s-1}(D_n,L(\mathbb{R}^n,\mathbb{R}^n))$, and $s-1\ge r$, so we find that by Lemma 3.24, $(Dg\circ f)\cdot Df\in H^r(D_n,L(\mathbb{R}^n,\mathbb{R}^k))$.

Take (f',g') near (f,g) in $H^s(D_n,D_n)\times H^{r+1}(D_n,\mathbb{R}^k)$. Then (f',Dg') is near (f,Dg) in $H^s(D_n,D_n)\times H^r(D_n,L(\mathbb{R}^n,\mathbb{R}^k))$, so $Dg\circ f$ is near $Dg'\circ f'$ by the induction assumption. Also Df' is near Df in $H^{s-1}(D_n,L(\mathbb{R}^n,\mathbb{R}^n))$. Since we have $r\le s-1$, $s-1>\frac{n}{2}$, $(Dg'\circ f')\cdot Df'$ is near $(Dg\circ f)\cdot Df$ in $H^r(D_n,L(\mathbb{R}^n,\mathbb{R}^k))$ by Lemma 3.24. □

3.25. Sobolev completions of spaces of vector bundle sections. To overcome certain difficulties which arise while working with manifolds modeled on Fréchet spaces (or on general convenient spaces (see section 4)), we construct Hilbert manifold completions of these spaces in the following way.

For any vector bundle V over M construct the s-th jet bundle $J^s(V)$, and endow $J^s(V)$ with an inner product $\langle\ ,\ \rangle_s$. Taking any volume form $d\,\mathrm{vol}$ on M, we get an inner product $(\ ,\ )_s$ on $C^\infty(J^s(V))$, the space of smooth $J^s(V)$–sections by

$$(a,b)_s:=\int_M\langle a,b\rangle_s d\,\mathrm{vol}\,.$$

Since there exists the natural map $j^s: C^\infty(V)\to C^\infty(J^s(V))$, $(\ ,\ )_s$ defines an inner product on $C^\infty(V)$, also.

Define $H^s(V)$ as the Hilbert space completion of $C^\infty(V)$ with respect to $(\ \ ,\ \ )_s$. Taking other choices for $d\,\mathrm{vol}$ and $\langle\ \ ,\ \ \rangle_s$ changes the inner product, but any two such constructed inner products are equivalent.

3.26. Lemma (Sobolev Lemma). *Let $s > k + \frac{1}{2}\dim(M)$, then the inclusion $H^s(V) \subset C^k(V)$ is a continuous linear map.*

Proof. See [Palais 1965], but follows essentially from the Sobolev Lemma 3.22.

3.27. Theorem. *If V and W are vector bundles over M and $f : V \to W$ is a smooth fiber preserving map, then for $s > \frac{1}{2}\dim(M)$, the map $\tilde{f} : H^s(M,V) \to H^s(M,W)$ defined by $\tilde{f}(\alpha) = f \circ \alpha$ is smooth, and its derivatives satisfy the formula $D^k_\alpha \tilde{f}(x_1,\dots,x_k)(p) = D^k_{\alpha(p)} f(x_1(p),\dots,x_k(p))$, with $p \in M$.*

Proof. In [Palais 1968, 11.3]. $\square$

3.28. Differential operators. Let V and W be vector bundles over M, and let $D : C^\infty(V) \to C^\infty(W)$ be a k-th order differential operator. Let V and W have smooth inner products $\langle\ \ ,\ \ \rangle_V$ and $\langle\ \ ,\ \ \rangle_W$, respectively, and let vol be a smooth volume form on M so that $H^0(V)$ and $H^0(W)$ have explicit inner products.

A k-th order differential operator $D^* : C^\infty(W) \to C^\infty(V)$ is called an *adjoint* of D if for all $v \in C^\infty(V)$, $w \in C^\infty(W)$, $\int_M \langle Dv, w\rangle_W d\,\mathrm{vol} = \int_M \langle v, D^*w\rangle_V d\,\mathrm{vol}$. Every operator D has a unique adjoint.

For any $x \in M$, and $\xi \in T^*_x$, the *symbol* of D at ξ, $\sigma_\xi(D)$ is a linear map $V_x \to W_x$. It can be shown that

$$\sigma_\xi(D_1 \circ D_2) = \sigma_\xi(D_1) \circ \sigma_\xi(D_2), \quad \text{and} \quad \sigma_\xi(D^*) = \sigma_\xi(D)^*,$$

where $\sigma_\xi(D)^* : W_x \to V_x$ is the adjoint of $\sigma_\xi(D)$ (with respect to the given inner products on V_x and W_x).

We say that D has *injective symbol* if $\sigma_\xi(D)$ is an injective map for all $\xi \neq 0$, and we call D *elliptic*, if $\sigma_\xi(D)$ is an isomorphism for $\xi \neq 0$.

The equations above show that $D^* \circ D$ is elliptic if and only if D has injective symbol.

Any k-th order operator D extends uniquely from a map $C^\infty(V) \to C^\infty(W)$ to a continuous linear map $D_s : H^s(V) \to H^{s-k}(W)$.

3.29. Proposition. *If D is a k-th order elliptic operator from V to W. Then*

(1) $\ker D = \ker D_s$ *is a finite dimensional subspace of* $C^\infty(V)$, *and similarly* $\ker D^* = \ker D^*_s$ *is a finite dimensional subspace of* $C^\infty(W)$.
(2) $H^{s-k}(W) = \mathrm{im}\, D_s \oplus \ker D^*$, *in particular* $\mathrm{im}\, D^s$ *is closed in* $H^{s-k}(W)$.

Proof. The proof can be found in [Palais 1965, pp.178–179]. $\square$

3.30. Proposition. *If D has injective symbol then*

(1) $H^{s-k}(V) = \mathrm{im}(D^* \circ D)_{s+k} + \ker D^* \circ D$.
(2) $\ker D^* \circ D = \ker D$ *and* $\mathrm{im}(D^* \circ D)_{s+k} = \mathrm{im}\, D^*_s$.
(3) $\mathrm{im}\, D_{s+k}$ *is closed in* $H^s(W)$, *and* $H^s(W) = \mathrm{im}\, D_{s+k} \oplus \ker D^*_s$.

Proof. (1) follows from 3.29(2) and the fact that $D^* \circ D$ is selfadjoint.
(2): Let $\langle \quad , \quad \rangle_{V,0}$ be the inner product on $H^0(V)$ and $\langle \quad , \quad \rangle_{W,0}$ be the one on $H^0(W)$. Of course, $\ker(D^* \circ D) \supset \ker D$. Also if $D^* Dv = 0$, $0 = \langle D^* Dv, v \rangle_{V,0} = \langle Dv, Dv \rangle_{W,0}$, so $Dv = 0$. Therefore $\ker D^* \circ D = \ker D$. Furthermore, $\operatorname{im}(D^* \circ D)_{s+k} \subset \operatorname{im} D^*_s$, since $(D^* \circ D)_{s+k} = D^*_s \circ D_{s+k}$. By (1) it has only to be shown that $\operatorname{im} D^*_s \cap \ker D^* \circ D = \{0\}$ or that $\operatorname{im} D^*_s \cap \ker D = \{0\}$. If $Dv = 0$ and $v = D^* w$ then $D \circ D^* w = 0$, so $0 = \langle D \circ D^* w, w \rangle_{W,0} = \langle D^* w, D^* w \rangle_{V,0}$, and so $v = D^* w = 0$, which in turn implies (2).
(3): Since $\ker D^*_s$ is closed by Lemma 3.3 we need only show that $H^s(W) = \operatorname{im} D_{s+k} \oplus \ker D^*_s$ is true in the algebraic sense.

Let $w \in H^s(W)$ such that $D^*_s w = 0$ and $w = D_{s+k} v$. Then $0 = \langle D^*_s \circ D_{s+k} v, v \rangle_{E,0} = \langle D_{s+k} v, D_{s+k} \rangle_{F,0}$, so $w = 0$. Therefore, $\operatorname{im} D_{s+k} \cap \ker D^*_s = \{0\}$.

$D^*_s(H^s(W)) = D^*_s \circ D_{s+k}(H^{s+k}(V))$ by (2), $H^s(W) = D^{*\,-1}_s(D^*_s(H_s(W)))$, and thus

$$H^s(W) = D^{*\,-1}_s(D^*_s \circ D_{s+k}(H^{s+k}(V))) = \ker D^*_s + \operatorname{im} D_{s+k}.$$

This implies (3). $\square$

4. Smooth infinite dimensional manifolds

Throughout this and the following chapters, I will use the term *smooth manifold* in the sense of [Kriegl, Michor 1997]. I will use the notion of Frölicher–Kriegl calculus and of convenient vector spaces.

I will not define more than the most basic facts of the Frölicher–Kriegl calculus, since that would exceed the goal of this thesis. However, the most important results can be found in [Frölicher, Kriegl 1988] or [Kriegl, Michor 1997]. Most of the results, which are presented in this chapter are taken from [Kolář et al. 1993], [Michor 1988], [Michor 1991], and [Kriegl, Michor 1997].

First we will define a suitable generalization of Fréchet spaces which will fit extraodinary well in a category theoretical way to infinite dimensional calculus. These spaces will later serve for the local models of manifolds.

4.1. Definition. Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ will be called *differentiable* if the derivative $c'(t) = \lim_{h\to 0} \frac{1}{h}(c(t+h) - c(t))$ at t exists for all t. A curve c is called *smooth* (or C^∞) if all iterated derivatives exist. The set of all smooth curves will be denoted by $C^\infty(\mathbb{R}, E)$. It will be equipped with the bornologification of the topology of uniform convergence on compact sets, in all derivatives separately. (Note: Smoothness is not primarily a topological concept, but rather a concept of bounded sets, hence a bornological concept. Therefore, the bornologification.)

A sequence $\{x_n\}$ in E is called *Mackey–convergent* to x if there exists a positive sequence $\{\mu_n\}$ in $\mathbb{R}$ with $\mu_n \to 0$ and $\frac{1}{\mu_n}(x_n - x)$ is bounded.

The c^∞*–topology* on a locally convex vector space is the final topology with respect to all smooth curves $\mathbb{R} \to E$. (Note: The c^∞–topology is in general not a vector space topology on E. However, the finest locally convex topology coarser than the c^∞–topology is the bornologification of the original locally convex topology.)

A locally convex vector space E is called *convenient* (or c^∞–complete) if one of the following equivalent conditions is satisfied. (There are more equivalent conditions than these, which can be found together with the proof of equivalence in [Kriegl, Michor 1997, Theorem 1.22].)

(1) Any Mackey-Cauchy sequence converges (i.e. E is Mackey–complete).
(2) E is c^∞–closed in any locally convex space.
(3) For any smooth curve $c_1 \in C^\infty(\mathbb{R}, E)$ there exists a curve $c_2 \in C^\infty(\mathbb{R}, E)$ with $c_2' = c_1$ (i.e. the existence of an *antiderivative*).
(4) If $c : \mathbb{R} \to E$ is a curve such that $\ell \circ c : \mathbb{R} \to \mathbb{R}$ is smooth for all $\ell \in E^*$, then c is smooth.

4.2. Theorem. *The following constructions preserve c^∞–completeness:*

(1) *limits,*
(2) *direct sums,*
(3) *strict inductive limits of sequences of closed embeddings,*
(4) *formation of $\ell^\infty(X, \quad)$, where X is a set together with a family $\mathfrak{B}$ of subsets of X containing the finite ones, which are called bounded, and $\ell^\infty(X, F)$*

denotes the space of all functions $f : X \to F$ *bounded on all* $B \in \mathfrak{B}$, *supplied with the topology of uniform convergence on the sets in* $\mathfrak{B}$.

Proof. See [Kriegl, Michor 1997, Theorem 1.23].

4.3. Definition. A mapping $f : E \supseteq U \to F$ between convenient spaces, defined on a c^∞–open subset U of E is called *smooth* (C^∞) if it maps smooth curves in U to smooth curves in F.

By $C^\infty(U, F)$ we will denote the space of all smooth maps $U \to F$. This space is locally convex, with pointwise linear structure and the bornologification of the initial topology with respect to all mappings $c^* : C^\infty(U, F) \to C^\infty(\mathbb{R}, F)$ for $c \in C^\infty(\mathbb{R}, U)$. Then $C^\infty(U, F)$ is a convenient vector space.

4.4. Proposition. *A (multi)linear map* $f : E_1 \times \cdots \times E_n \to F$, *where* E_i, F *are convenient spaces, is smooth if and only if* f *is bounded.*

We equip the space $L(E_1, \dots, E_n; F)$ *of all such maps with the topology of uniform convergence on bounded sets. Then* $L(E_1, \dots, E_n; F)$ *is a closed linear subspace of* $C^\infty(E_1 \times \cdots \times E_n, F)$, *hence convenient.*

There are natural bornological isomorphisms

$$L(E_1, \dots, E_n; F) \simeq L(E_1, \dots, E_l; L(E_{l+1}, \dots, E_n; F)).$$

This is called the exponential law for the linear maps.

Proof. See [Kriegl, Michor 1997, Propostion 3.2]

4.5. Theorem (Cartesian closedness). *The category of convenient vector spaces and smooth mappings is cartesian closed. So there is natural bijection*

$$C^\infty(E \times F, G) \simeq C^\infty(E, C^\infty(F, G)).$$

Furthermore, the following canonical mappings are smooth.

$$\begin{aligned}
\operatorname{ev} &: C^\infty(E, F) \times E \to F, \quad \operatorname{ev}(f, x) = f(x) \\
\operatorname{ins} &: E \to C^\infty(F, E \times F), \quad \operatorname{ins}(x)(y) = (x, y) \\
(\quad)^\wedge &: C^\infty(E, C^\infty(F, G)) \to C^\infty(E \times F, G) \\
(\quad)^\vee &: C^\infty(E \times F, G) \to C^\infty(E, C^\infty(F, G)) \\
\operatorname{comp} &: C^\infty(F, G) \times C^\infty(E, F) \to C^\infty(E, G), \quad \operatorname{comp}(f, g)(x) = f(g(x)) \\
C^\infty(\quad,\quad) &: C^\infty(F, F') \times C^\infty(E', E) \to C^\infty(C^\infty(E, F), C^\infty(E', F')) \\
&\quad (f, g) \mapsto (h \mapsto f \circ h \circ g) \\
\prod &: \prod C^\infty(E_i, F_i) \to C^\infty(\prod E_i, \prod F_i)
\end{aligned}$$

Proof. See [Kriegl, Michor 1997, 1.36]

The following lemma provides strong means for proving many results in the Frölicher–Kriegl calculus.

4.6. Lemma. Uniform boundedness principle. *Let E be a locally convex vector space and let S be a point seperating set of bounded linear mappings with common domain E. Then the following conditions are equivalent.*

(1) *If F is a c^∞–complete locally convex vector space and $f : F \to E$ is linear and $\lambda \circ f$ is bounded for all $\lambda \in S$, then f is bounded.*

(2) *If $\{b_n\}$ is an unbounded sequence in E with $\lambda(b_n)$ bounded for all $\lambda \in S$, then there is some $\{t_n\} \in \ell^1$ such that $\sum t_n b_n$ does not converge in E for the initial locally convex topology induced by S.*

We then say that E satisfies the uniform S-boundedness principle if these conditions are satisfied.

A convenient vector space E satisfies the uniform S-boundedness principle for each point separating set S of bounded linear mappings on E if and only if there exists no strictly weaker ultrabornological topology than the bornological topology of E.

The space $C^\infty(U, E)$ satisfies the uniform boundedness principle for the set $S := \{\mathrm{ev}_x : x \in U\}$.

Proof. See [Kriegl, Michor 1997, 3.21–3.25]

The importance of the Frölicher–Kriegl calculus is also due to the fact that the function spaces in finite dimensions are all convenient vector spaces.

4.7. Proposition. *Let M be a smooth finite-dimensional paracompact manifold. Then the space $C^\infty(M, \mathbb{R})$ of all smooth functions on M is a convenient vector space and satisfies the uniform boundedness principle for the point evaluations. The structure is e.g. given by the following description: The initial structure with respect to the cone*

$$C^\infty(M, \mathbb{R}) \xrightarrow{c^*} C^\infty(\mathbb{R}, \mathbb{R})$$

for all $c^* \in C^\infty(\mathbb{R}, M)$.

Proof. Other equivalent descriptions and the proof can be found in [Kriegl, Michor 1997, 3.31].

By considering *smooth spaces*, which are the category theoretical basis of the smooth calculus, many results can be proved by using the Cartesian closedness. However, one cannot differentiate in smooth spaces, so I will not show the development of the theory here, and rather give a cite: [Kriegl, Michor 1997, section 4].

A very important notion, as always in the theory of manifolds, is the existence of smooth partitions of unity, since they are the only known means of gluing local results together to yield global results.

4.8. Definition. A convenient vector space is said to be *smoothly normal* if for any two closed disjoint subsets $A_1, A_2 \subset X$ there is a smooth function f with $f|_{A_1} \equiv 0$ and $f|_{A_2} \equiv 1$.

It is called *smoothly paracompact* if it is paracompact and smoothly normal.

Then the vector space admits *smooth bump functions* (i.e. for any neighborhood U of x there exists a smooth function f such that $f(x) = 1$ and the carrier $\mathrm{carr}(f) \subset U$).

Note: A nuclear convenient space admits smooth bump functions.

4.9. Definition. A chart (U, u) on a set M is a bijection $u : U \to u(U) \subset E_U$ from a subset $U \subset M$ onto a c^∞–open subset of a convenient vector space E_U. For two charts (U_α, u_α) and (U_β, u_β) on M the mapping $u_{\alpha\beta} := u_\alpha \circ u_\beta : u_\beta(U_\alpha \cap U_\beta) \to u_\alpha(U_\alpha \cap U_\beta)$ is called the *chart changing.*

A family $(U_\alpha, u_\alpha)_{\alpha\in A}$ of charts is called an *atlas* of M, if $\{U_\alpha\}$ is a covering of M and all chart changings are defined on c^∞–open subsets. Such an atlas is called C^∞ if all chart changings are smooth. Two C^∞–atlases are called C^∞–equivalent if their union is again a C^∞–atlas. An equivalence class of C^∞–atlases is called a C^∞*–structure* on M. A set M together with a C^∞–structure is called C^∞–manifold.

A mapping $f : M \to N$ between manifolds is called *smooth* if for each $x \in M$ and each chart (V, v) on N with $f(x) \in V$ there is a chart (U, u) in M with $x \in U$, $f(U) \subset V$, such that $v \circ f \circ u^{-1}$ is smooth. This is the case if and only if $f \circ c$ is smooth for each smooth curve $c : \mathbb{R} \to M$.

We will denote by $C^\infty(M, N)$ the space of all smooth mappings $f : M \to N$. A smooth bijective mapping $f : M \to N$ is called a *diffeomorphism* if f^{-1} is smooth also.

The natural topology on a C^∞–manifold M is the identification topology with respect to some C^∞–atlas $(U_\alpha, u_\alpha)_{\alpha\in A}$, where $W \subset M$ is open if and only if $u_\alpha(U_\alpha \cap W)$ is c^∞–open in E_{U_α} for all α.

M is called *smoothly Hausdorff* if the smooth functions in $C^\infty(M, \mathbb{R})$ separate points in M.

A C^∞–manifold M will be called a *smooth manifold* if it is smoothly Hausdorff and pure (i.e. the isomorphism type of the modeling spaces E_α, which is constant on each connected component of M, is constant on M). If a smooth manifold, which is smoothly paracompact, is modeled on a convenient vector space which is smoothly normal then M admits smooth partitions of unity.

$N \subset M$ is called a *submanifold*, if for each $x \in N$ there is a chart (U, u) of M such that $u(U \cap N) = u(U) \cap F_U$, where F_U is a closed linear subspace of the convenient model space E_U. Of course, N is itself a manifold with $(U \cap N, u|_{U\cap N})$ as charts for all (U, u), which are as above.

A submanifold N is called *splitting submanifold* of M if there is a cover of N by submanifold charts (U, u) as above such that the $F_U \subset E_U$ are complemented (i.e. splitting) linear subspaces.

4.10. Tangent bundles. In finite dimensions, and up to Banach manifolds, the two descriptions of the tangent spaces as spaces of tangent vectors, and on the other hand, as spaces of derivations, coincide. This is in general not the case for manifolds modeled on convenient vector spaces. Therefore, we will spend some time on the definition of the tangent bundles.

Let E be a convenient vector space, U an open subset of E, $a \in U$. A *kinematic tangent vector* with footpoint a is a pair (a, X) with $X \in E$. Let the kinematic tangent space $T_aU \simeq E$ be the space of all kinematic tangent vectors with footpoint a. It consists of all derivatives $c'(0)$ at 0 of smooth curves $c : \mathbb{R} \to E$ with $c(0) = a$, thus the name kinematic. Similar to the finite dimensional case, a kinematic tangent vector induces a *continuous derivation over* ev_a.

An *operational tangent vector* with footpoint a is a bounded derivation $\partial : C_a^\infty(U, \mathbb{R}) \to \mathbb{R}$ over ev_a. Let the operational tangent space D_aU be the space of all such derivations. It can be equipped with a convenient vector space structure, and for all $a \in U$ these spaces are isomorphic. $DU := \bigcup_{a\in U} D_aU$.

However, usually T_aE and D_aE are not isomorphic! For more information see [Kriegl, Michor 1997, 19.6]

We can then define the kinematic and operational tangent bundles as quotient sets with respect to an equivalence relation. Let us start with an atlas for the manifold M $(U_\alpha, u_\alpha : U_\alpha \to E_\alpha)_{\alpha\in A}$. Then we define manifolds

$$TM := \left(\bigcup_{\alpha\in A} U_\alpha \times E_\alpha \times \{\alpha\} \right) \Big/ \sim,$$
$$(x, v, \alpha) \sim (y, w, \beta) \iff x = y \wedge d(u_{\alpha\beta})(u_\beta(x))w = v$$

with charts (TU_α, Tu_α), where $TU_\alpha = \pi_M^{-1}(U_\alpha)$ (π_M being the footpoint projection) and $Tu_\alpha([x, v, \alpha]) = (u_\alpha(x), v)$, and

$$DM := \left(\bigcup_{\alpha\in A} D(u_\alpha(U_\alpha)) \times \{\alpha\} \right) \Big/ \sim,$$
$$(\partial, \alpha) \sim (\partial', \beta) \iff D(u_{\alpha\beta})\partial' = \partial$$

with charts (DU_α, Du_α), where $DU_\alpha = \pi_M^{-1}(U_\alpha)$ and $Du_\alpha([\partial, \alpha]) = \partial$. There is an embedding $TM \hookrightarrow DM$.

As expected, a smooth mapping $f : M \to N$ induces smooth mappings $Tf : TM \to TN$ and $Df : DM \to DN$, which are fiber linear, and Tf is the restriction of Df to TM.

4.11. Vector bundles. Let $p : E \to M$ be a smooth mapping between manifolds. A *vector bundle chart* on (E, p, M) is a pair (U, ψ), where U is an open subset of M, and ψ is a fiber respecting diffeomorphism as in the following diagram:

$$\begin{array}{ccc} E|_U := p^{-1}(U) & \xrightarrow{\quad\psi\quad} & U \times V \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U & \end{array}$$

V is a convenient vector space, called the *standard fiber.* Two vector bundle charts (U_α, ψ_α) and (U_β, ψ_β) are called *compatible*, if $\psi_\alpha \circ \psi_\beta^{-1}$ is a fiber linear isomorphism, i.e. $(\psi_\alpha \circ \psi_\beta^{-1})(x, v) = (x, \psi_{\alpha\beta}(x)v)$ for some mapping $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)$. This mapping is then unique and smooth into $L(V, V)$ and is called *transition function.*

A *vector bundle atlas* is a collection of pairwise compatible vector bundle charts $(U_\alpha, u_\alpha)_{\alpha\in A}$ such that the U_α are an open cover of M. Two vector bundle atlasses are called equivalent if their union is again a vector bundle atlas.

A *smooth vector bundle* (E, p, M) consists of smooth manifolds E (the *total space*), M (the *base space*), a smooth mapping $p : E \to M$ (the *projection*, which turns out to be a surjective submersion), and an equivalence class of vector bundle atlasses (the *vector bundle structure*). Similar to the finite dimensional case, we can describe vector bundles over M by the cocycle of transition functions, and the isomorphism classes by cohomological means.

TM and DM are two important examples of vector bundles. Like in the finite dimensional case, and in the Banach case, one can perform certain operations on vector bundles. However, one has to take much more care in bornological and topological questions. For two vector bundles (E, p, M) and (F, q, M) the following constructions are, e.g., possible, yielding again vector bundles over M: $\Lambda^k E$, $E \oplus F$, E^*, $\Lambda E := \bigoplus_{k \in \mathbb{N}_0} \Lambda^k E$, $E \hat{\otimes} F$, $L(E, F)$.

4.12. Sections of vector bundles. There is a unique structure of a convenient vector space on each fiber E_x of a vector bundle (E, p, M), induced by the vector bundle charts. So $0_x \in E_x$ is a special element, and $0 : M \to E$, $0(x) = 0_x$ is a smooth mapping, the *zero section.*

A *section* s of (E, p, M) is a smooth mapping $s : M \to E$ with $p \circ s = \mathrm{Id}_M$. The space of all smooth sections of the bundle (E, p, M) is denoted by $C^\infty(E)$. It is a vector space with fiberwise addition and scalar multiplication. It is equipped with the structure of a convenient vector space given by the closed embedding

$$C^\infty(E) \to \prod_\alpha C^\infty(U_\alpha, V)$$
$$s \mapsto \mathrm{pr}_2 \circ \psi_\alpha \circ (s|_{U_\alpha})$$

where (U_α, ψ_α) is a vector bundle atlas. This structure is independent of the choice of the atlas.

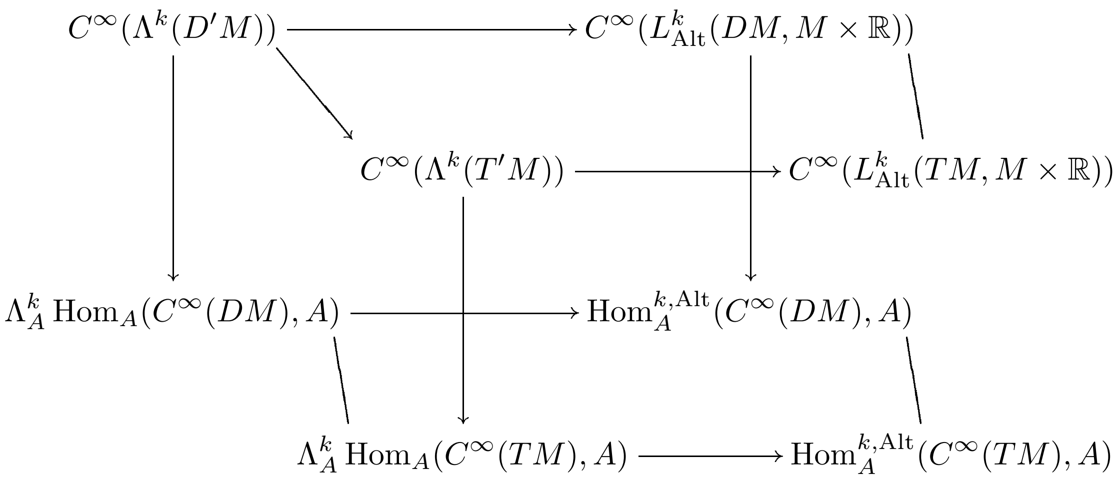
The space $C^\infty(E)$ satisfies the uniform boundedness principle with respect to the point evaluations $\mathrm{ev}_x : C^\infty(E) \to E_x$ for all $x \in M$.

As usual the sections of the tangent bundles are called (here kinematic, and operational respectively) *vector fields* on M.

4.13. Differential Forms. The definition of the tangent bundles has provided us with a severe technical difficulty, but if we want to define differential forms, these difficulties even increase.

The first problem is the definition of the cotangent spaces, since there are of course also two of them: Take the covariant smooth functor, which takes the convenient vector space E to its dual E'. Applying this functor to the tangent bundles TM and DM, we get the *kinematic* $(T'M)$ and *operational $(D'M)$ cotangent bundles*, respectively. By taking spaces of sections, we can define the *kinematic* and *operational* 1*-forms.*

However, for the definition of higher order forms, we have to take deeper investigations. There are at least eight possible choices for the space of k-forms, which coincide in the finite dimensional case.



where $A := C^\infty(M, \mathbb{R})$ and Λ^k denotes the bornological exterior product and Λ^k_A the convenient module exterior product.

A careful examination in [Kriegl, Michor 1997] shows that the space with the most fruitful features leads to the definition

$$\Omega^k(M) := C^\infty(L^k_{\text{Alt}}(TM, M \times \mathbb{R})).$$

This space is isomorphic as convenient vector space to the closed linear subspace of $C^\infty(TM \times_M \cdots \times_M TM, \mathbb{R})$ consisting of all fiberwise k-linear alternating smooth functions. Using this definition, all the important mappings

$$\begin{aligned}
&d : \Omega^k(M) \to \Omega^{k+1}(M) \\
&i : C^\infty(TM) \times \Omega^k(M) \to \Omega^{k-1}(M) \\
&\mathcal{L} : C^\infty(TM) \times \Omega^k(M) \to \Omega^k(M) \\
&f^* : \Omega^k(M) \to \Omega^k(N)
\end{aligned}$$

are smooth, and there is a working notion of De Rham cohomology. However, in this thesis we will not be concerned with this.

4.14. The Frölicher–Nijenhuis Bracket. Now consider the graded commutative algebra

$$\Omega(M) = \bigoplus_{k \geq 0} \Omega^k(M) = \bigoplus_{k=-\infty}^{\infty} \Omega^k(M)$$

of differential forms on M, where $\Omega^k(M) = 0$ for $k < 0$. A *graded derivation* of degree k on $\Omega(M)$ is a bounded linear map $D : \Omega(M) \to \Omega(M)$ with $D(\Omega^l(M)) \subseteq \Omega^{k+l}(M)$ and

$$D(\phi \wedge \psi) = D(\phi) \wedge \psi + (-1)^{kl} \phi \wedge D(\psi) \quad \text{for } \phi \in \Omega^l(M).$$

The space of all such derivations is called $\operatorname{Der}_k \Omega(M)$. The space

$$\operatorname{Der} \Omega(M) = \bigoplus_k \operatorname{Der}_k \Omega(M)$$

with the graded commutator $[D_1, D_2] = D_1 \circ D_2 - (-1)^{k_1 k_2} D_2 \circ D_1$ for $D_i \in \operatorname{Der}_{k_i} \Omega(M)$ is a graded Lie algebra. The bracket is graded anticommutative and satisfies the graded Jacobi identity

$$[D_1, [D_2, D_3]] = [[D_1, D_2], D_3] + (-1)^{k_1 k_2} [D_2, [D_1, D_3]].$$

A derivation $D \in \operatorname{Der} \Omega(M)$ is called *algebraic* if $D|_{\Omega^0(M)} = 0$. Then $D(f.\phi) = f.D(\phi)$ for $f \in C^\infty(M, \mathbb{R})$ and $\phi \in \Omega(M)$.

For the definition of the Frölicher–Nijenhuis bracket we will have to consider the *space of vector valued kinematic differential forms*

$$\Omega(M; TM) = \bigoplus_{k \geq 0} \Omega^k(M; TM) = \bigoplus_{k \geq 0} C^\infty(L^k_{\text{Alt}}(TM; TM)).$$

For $K \in \Omega^k(M; TM)$ and $L \in \Omega^l(M; TM)$ the formula

$$\begin{aligned}
&[K, L](X_1, \dots, X_{k+l}) = \\
&= \tfrac{1}{k!l!} \operatorname{sgn} \sigma \sum_\sigma [K(X_{\sigma 1}, \dots, X_{\sigma k}), L(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)})] + \\
&+ (-1)^k \Bigg(\tfrac{1}{k!(l-1)!} \sum_\sigma \operatorname{sgn} \sigma L([X_{\sigma 1}, K(X_{\sigma 2}, \dots, X_{\sigma(k+1)})], X_{\sigma(k+2)}, \dots) - \\
&\quad - \tfrac{1}{2!(k-1)!(l-1)!} \sum_\sigma \operatorname{sgn} \sigma L(K([X_{\sigma 1}, X_{\sigma 2}], X_{\sigma 3}, \dots), X_{\sigma(k+2)}, \dots) \Bigg) + \\
&+ (-1)^{l(k+1)} \Bigg(\tfrac{1}{l!(k-1)!} \sum_\sigma \operatorname{sgn} \sigma K([X_{\sigma 1}, L(X_{\sigma 2}, \dots, X_{\sigma(l+1)})], X_{\sigma(l+2)}, \dots) - \\
&\quad - \tfrac{1}{2!(k-1)!(l-1)!} \sum_\sigma \operatorname{sgn} \sigma K(L([X_{\sigma 1}, X_{\sigma 2}], X_{\sigma 3}, \dots), X_{\sigma(l+2)}, \dots) \Bigg)
\end{aligned}$$

defines a graded anticommutative bracket, which satisfies the graded Jacobi identity, the *Frölicher–Nijenhuis bracket*. This makes $\Omega(M; TM)$ to a graded Lie algebra. In the above situation $[K, L] \in \Omega^{k+l}(M; TM)$ and

$$[\mathcal{L}(K), \mathcal{L}(L)] = \mathcal{L}([K, L]) \in \operatorname{Der} \Omega(M),$$

Id_{TM} is in the center, and the mapping $\mathcal{L} : \Omega^*(M; TM) \to \operatorname{Der}_* \Omega(M)$ is an injective homomorphism of graded Lie algebras.

4.15. f–relatedness of the Frölicher–Nijenhuis bracket. Let $f : M \to N$ be a smooth mapping between manifolds. Two vector valued forms $K \in \Omega^k(M; TM)$ and $L \in \Omega^l(M; TM)$ are called *f–related* (*f–dependent*), if for all $X_i \in T_x M$

$$T_x f^{-1} L_{f(x)}(T_x f \cdot X_1, \dots, T_x f \cdot X_k) = K_x(X_1, \cdots, X_k).$$

Theorem. *If K_j and L_j are f–related for $j = 1, 2$ then their Frölicher–Nijenhuis brackets $[K_1, K_2]$ and $[L_1, L_2]$ are also f–related.*

4.16. Lie groups. A smooth manifold G is called a *Lie group* if there are smooth mappings $\mu : G \times G \to G$ and $\nu : G \to G$, such that (G, μ, ν) is a group with *multiplication* μ, *inversion* ν, and unit element e. $\mu_a(b) = \mu^b(a) = \mu(a,b) = a \cdot b$

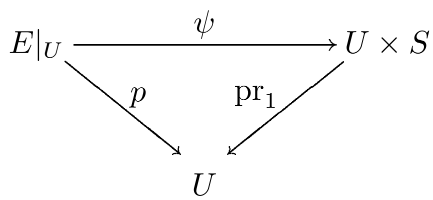
A kinematic vector field ξ on G is called *left invariant*, if $\mu_a^* \xi = \xi$ for all $a \in G$, where $\mu_a^* \xi = T\mu_{a^{-1}} \circ \xi \circ \mu_a$. The vector space $\mathfrak{X}_L(G)$ of all left invariant vector fields on G is closed under the Lie bracket, so it is a sub Lie algebra of $\mathfrak{X}(G)$. Since every left invariant vector field ξ is uniquely determined by $\xi(e) \in T_eG$ ($\xi(a) = T_e\mu_a.\xi(e)$), the Lie algebra of left invariant vector fields is linearly isomorphic to T_eG. This isomorphism induces on T_eG a Lie algebra structure. This Lie algebra is called the *Lie algebra of* G, and is denoted by $\mathrm{Lie}(G)$ or by $\mathfrak{g}$.

An important example of an infinite dimensional Lie group is the group $\mathcal{D}\mathrm{iff}(M)$ of diffeomorphisms of a compact finite dimensional manifold. Its Lie algebra is the algebra of vector fields $\mathfrak{X}(M)$.

$\mathcal{D}\mathrm{iff}(M)$ is a *regular Lie group*, which is an extremely important subset of all Lie groups. For more information on regular Lie groups see [Kriegl, Michor 1997, section 29].

Now, we have mentioned the most important definitions and results of the Frölicher–Kriegl calculus applied to differential geometry. In the following we will concentrate on fiber bundles and the basis for the slice theorems we will consider in the next section.

4.17. Definition. A *fiber bundle* (E, p, M, S) consists of smooth (here finite dimensional) manifolds E, M, S and a smooth mapping $p : E \to S$. Moreover, each $x \in M$ possesses an open neighborhood U, such that $E|_U := p^{-1}(U)$ is diffeomorphic to $U \times S$ via a fiber respecting diffeomorphism.



E is then called the *total space*, M the *basis* (or *base space*), and S the *standard fiber*. (U, ψ), as above, is called a *fiber chart*; (U_α, ψ_α), such that (U_α) cover M is a *(fiber) bundle atlas*. As for vector bundles there is a cocycle of transition functions constructed as follows. For two charts (U_α, ψ_α) and (U_β, u_β), we consider the mapping

$$\psi_\alpha \circ \psi_\beta^{-1}(x, s) = (x, \psi_{\alpha\beta}(x, s)),$$

where $\psi_{\alpha\beta} : (U_{\alpha\beta} := U_\alpha \cap U_\beta) \times S \to S$ is smooth, and $\psi_{\alpha\beta}(x, \ )$ is a diffeomorphism of S for each $x \in U_{\alpha\beta}$. Thus $\psi_{\alpha\beta} : U_{\alpha\beta} \to \mathcal{D}\mathrm{iff}(S)$. They again satisfy the cocycle conditions for $x \in U_\alpha \cap U_\beta \cap U_\gamma$

$$\psi_{\alpha\beta}(x) \circ \psi_{\beta\gamma}(x) = \psi_{\alpha\gamma}(x)$$
$$\psi_{\alpha\alpha}(x) = \mathrm{Id}_S\,.$$

As usual, a cocycle of transition functions reproduces the fiber bundle. For the notions presented here see [Michor 1988], [Michor 1991], and [Kolář et al. 1993].

4.18. Lemma. *Let M, N be finite dimensional manifolds, let $p : N \to M$ be a proper surjective submersion (fibered manifold), and let M be connected. Then (N, p, M, S) is a fiber bundle, where S is diffeomorphic to $p^{-1}(x)$ for an $x \in M$.*

Proof. We have to construct a fiber chart at each $x_0 \in M$. Take (U, u) a chart centered at x_0 on M such that $u(U) \cong \mathbb{R}^m$. For each $x \in U$ let $\xi_x(y) := (T_y u)^{-1}.u(x)$, then $\xi_x \in \mathfrak{X}(U)$, depending smoothly on $x \in U$, such that $u(\mathrm{Fl}^{\xi_x}_t u^{-1}(z)) = z + t.u(x)$. Thus, each ξ_x is a complete vector field on U. Since p is a proper submersion, with the help of a partition of unity on $p^{-1}(U)$ we may construct vector fields $\eta_x \in \mathfrak{X}(p^{-1}(U))$ which depend smoothly on $x \in U$ and are p–related to $\xi_x : Tp.\eta_x = \xi_x \circ p$. Therefore, $p \circ \mathrm{Fl}^{\eta_x}_t = \mathrm{Fl}^{\xi_x}_t \circ p$ and $\mathrm{Fl}^{\eta_x}_t$ is fiber respecting, and since each fiber is compact and ξ_x is complete, η_x has a global flow, too. If we define $S := p^{-1}(x_0)$ then $\phi : U \times S \to p^{-1}(U)$, defined by $\phi(x, y) = \mathrm{Fl}^{\eta_x}_1(y)$, is a fiber respecting diffeomorphism, and thus (U, ϕ) is a fiber chart. Since M is connected, the fibers $p^{-1}(x)$ are all diffeomorphic. □

4.19. Definition. Let (E, p, M, S) be a fiber bundle. We consider the fiber linear mapping $Tp : TE \to TM$ and its kernel $VE := \ker Tp$, the *vertical bundle* of E.

A vector valued 1–form $\Phi \in \Omega^1(E; VE)$ which satisfies $\Phi \circ \Phi = \Phi$ and $\operatorname{im} \Phi = VE$ (a projection $TE \twoheadrightarrow VE$) is called a connection on (E, p, M, S).

Since $\ker \Phi$ is of constant rank, $\ker \Phi$ is a subbundle of TE, called the space of *horizontal vectors* or *horizontal bundle* HE. (Of course, $TE = VE \oplus HE$.)

Consider $(Tp, \pi_E) : TE \to TM \times_M E$. Then $(Tp, \pi_E)^{-1}(0_{p(u)}, u) = V_u E$. Therefore, $(Tp, \pi_E)|_{HE} : HE \to TM \times_M E$ is an injective, fiber linear mapping, and thus a fiber linear isomorphism. $C := [(Tp, \pi_E)|_{HE}]^{-1} : TM \times_M E \to HE \hookrightarrow TE$ is right inverse to (Tp, π_E), called the *horizontal lift with respect to* Φ. The connection between Φ and C is as follows,

$$\Phi(\xi_u) = \xi_u - C(Tp.\xi_u, u) \qquad \xi_u \in T_u E.$$

$\chi := \mathrm{Id}_{TE} - \Phi = C \circ (Tp, \pi_E)$ is called the *horizontal projection.*

4.20. Pullback bundle. Let (E, p, M, S) be a fiber bundle and $f : N \to M$ a smooth mapping. Since p is a submersion, f and p are transversal. Thus the *pullback* f^*E of the fiber bundle E along f exists.

$$\begin{array}{ccc} f^*E & \xrightarrow{p^*f} & E \\ \big\downarrow{\scriptstyle f^*p} & & \big\downarrow{\scriptstyle p} \\ N & \xrightarrow[f]{} & M \end{array}$$

Proposition.

(1) *(f^*E, f^*p, N, S) is again a fiber bundle and p^*f is a fiberwise diffeomorphism.*

(2) *If $\Phi \in \Omega^1(E; TE)$ is a connection on E, then $f^*\Phi$ defined by*

$$(f^*\Phi)_u(x) := T_u(p^*f)^{-1}.\Phi.T_u(p^*f).X$$

for $X \in T_u E)$ is a connection on f^*E. $f^*\Phi$ and Φ are p^*f–dependend.

Proof. (1): If (U_α, ψ_α) is a fiber bundle atlas of (E, p, M, S), then the collection $(f^{-1}(U_\alpha), (f^*p, pr_2 \circ \psi_\alpha \circ p^* f))$ is obviously a fiber bundle atlas for (f^*E, f^*p, N, S), by the formal universal properties of a pullback.
(2) is obvious. □

4.21. Remark. *Parallel transport.*
Let Φ be a connection on the fiber bundle (E, p, M, S) and let $c : (a, b) \to M$ be a smooth curve with $0 \in (a, b)$, $c(0) = x$. Then there exists a neighborhood U of $E_x \times \{0\}$ in $E_x \times (a, b)$ and a smooth mapping $\operatorname{Pt}_c : U \to E$, such that

1. *$p(\operatorname{Pt}(c, t, u_x)) = c(t)$, if defined and $\operatorname{Pt}(c, 0, u_x) = u_x$.*
2. *$\Phi(\frac{d}{dt} \operatorname{Pt}(c, t, u_x)) = 0_{c(t)}$, if defined.*
3. *Reparametrization invariance: If $f : (a', b') \to (a, b)$ is a smooth mapping with $0 \in (a', b')$, then $\operatorname{Pt}(c, f(t), u_x) = \operatorname{Pt}(c \circ f, t, \operatorname{Pt}(c, f(0), u_x))$, where defined.*
4. *U is maximal for (1) and (2).*
5. *parallel transport is smooth as a mapping $C^\infty(\mathbb{R}, M) \times_{(\mathrm{ev}_0, M, p)} E \times \mathbb{R} \supset U \xrightarrow{\operatorname{Pt}} E$, where U is its domain of definition.*

4.22. Definition. Let (E, p, M, S) be a fiber bundle and (U_α, ψ_α) a fixed bundle atlas. Then Frölicher–Kriegl calculus implies that $C^\infty(U_{\alpha\beta}, C^\infty(S, S)) \subseteq C^\infty(U_{\alpha\beta} \times S, S)$ and equality if and only if S is compact. Therefore, we will restrict ourselves to compact S from now on.

We define the *non–linear frame bundle* of (E, p, M, S) as

$$\mathcal{D}\mathrm{iff}\{S, E\} := \bigcup_{x \in M} \mathcal{D}\mathrm{iff}(S, E_x)$$

together with the differentiable structure which is obtained if the functor $\mathcal{D}\mathrm{iff}(S, \quad)$ is applied to the cocycle of transition functions $(\psi_{\alpha\beta})$. The cocycle for $\mathcal{D}\mathrm{iff}\{S, E\}$, which is constructed in this procedure describes the structure of a smooth principal fiber bundle (see [Michor 1991]) with structure group $\mathcal{D}\mathrm{iff}(S)$. (The right action is given by composition from the right.)

Since $\mathrm{ev} : \mathcal{D}\mathrm{iff}(S) \times S \to S$ is smooth, we may consider the associated bundle

$$\mathcal{D}\mathrm{iff}\{S, E\}[S, \mathrm{ev}] = \frac{\mathcal{D}\mathrm{iff}\{S, E\} \times S}{\mathcal{D}\mathrm{iff}(S)}.$$

$\mathrm{ev} : \mathcal{D}\mathrm{iff}\{S, E\} \times S \to E$ is invariant under the $\mathcal{D}\mathrm{iff}(S)$ action and thus factorizes to a smooth mapping $\mathcal{D}\mathrm{iff}\{S, E\}[S, \mathrm{ev}] \to E$ as it is shown in the following diagram.

$$\begin{array}{ccc}
\mathcal{D}\mathrm{iff}\{S, E\} \times S & \xrightarrow[\mathrm{pr}]{} & \frac{\mathcal{D}\mathrm{iff}\{S, E\} \times S}{\mathcal{D}\mathrm{iff}(S)} \\
\Big\downarrow \mathrm{ev} & & \Big\| \\
E & \xleftarrow[\text{is Diffeomorphism}]{} & \mathcal{D}\mathrm{iff}\{S, E\}[S, \mathrm{ev}]
\end{array}$$

Therefore, the name non–linear frame bundle is justified.

4.23. Lemma. *$\mathcal{D}\mathrm{iff}\{S,E\}$ is a smooth splitting submanifold of $\mathcal{E}\mathrm{mb}(S,E)$, with the obvious embedding.*

4.24. Connections on $\mathcal{D}\mathrm{iff}\{S,E\}$ and E. Let $\Phi \in \Omega^1(E,TE)$ be a connection on E. We want to lift it to a principal connection on $\mathcal{D}\mathrm{iff}\{S,E\}$. If we use a result of [Michor 1980], we get

$$T\,\mathcal{D}\mathrm{iff}\{S,E\} = \bigcup_{x\in M}\Big\{ f \in C^\infty(S, TE|_{E_x}) \;\Big|\; Tp\circ f = \text{one point in } T_xM \text{ and } \pi_E\circ f \in \mathcal{D}\mathrm{iff}(S,E_x)\Big\}$$

If we consider $\omega(f) := T(\pi_E\circ f)^{-1}\circ\Phi\circ f : S \to TE \to VE \to TS$ for $f \in T\,\mathcal{D}\mathrm{iff}\{S,E\}$, then $\omega(f)$ is a vector field and we get

Lemma. *$\omega \in \Omega^1(\mathcal{D}\mathrm{iff}\{S,E\};\mathfrak{X}(S))$ is a principal connection, and the induced connection on $E = \mathcal{D}\mathrm{iff}\{S,E\}[S,\mathrm{ev}]$ coincides with Φ.*

and

Theorem. *Let (E,p,M,S) be a fiber bundle with compact standard fiber S. Then there is a bijective correspondence between connections on E and principal connections on $\mathcal{D}\mathrm{iff}\{S,E\}$.*

Proofs. See [Michor 1991, 13.3]

4.25. Definition. $\mathcal{D}\mathrm{iff}\{E,E\} := \bigcup_{x\in M}\mathcal{D}\mathrm{iff}(E_x,E_x)$ with the smooth structure, which is described by the cocycle $\mathcal{D}\mathrm{iff}(\psi^{-1}_{\alpha\beta},\psi_{\alpha\beta}) = (\psi_{\alpha\beta})_*(\psi_{\alpha\beta})^*$, when again $(\psi_{\alpha\beta})$ is the cocycle for (E,p,M,S).

Lemma. *The associated bundle $\mathcal{D}\mathrm{iff}\{S,E\}[\mathcal{D}\mathrm{iff}(S),\mathrm{conj}]$ is isomorphic to the fiber bundle $\mathcal{D}\mathrm{iff}\{E,E\}$.*

Proof. The mapping $A : \mathcal{D}\mathrm{iff}\{S,E\}\times\mathcal{D}\mathrm{iff}(S) \to \mathcal{D}\mathrm{iff}\{E,E\}$, given by $A(f,g) := f\circ g\circ f^{-1} : E_x \to S \to S \to E_x$ for $f \in \mathcal{D}\mathrm{iff}(S,E_x)$ is $\mathcal{D}\mathrm{iff}(S)$ invariant. Thus, it factors to a smooth mapping $\mathcal{D}\mathrm{iff}\{S,E\}[\mathcal{D}\mathrm{iff}(S)] \to \mathcal{D}\mathrm{iff}\{E,E\}$. It is bijective and admits smooth inverses locally over M, so it is a fiber respecting diffeomorphism. □

4.26. Definition. The *gauge group* $\mathcal{G}\mathrm{au}(E)$ of the bundle (E,p,M,S) is the group of all principal fiber bundle automorphisms of the $\mathcal{D}\mathrm{iff}(S)$–bundle $\mathcal{D}\mathrm{iff}\{S,E\}$, which cover the identity on M. Lemma 4.25 implies that $\mathcal{G}\mathrm{au}(S)$ is equal to the space of sections of the bundle $\mathcal{D}\mathrm{iff}\{E,E\} = \mathcal{D}\mathrm{iff}\{S,E\}[\mathcal{D}\mathrm{iff}(S),\mathrm{conj}]$.

4.27. Results. Bearing in mind, that we have restricted ourselves to compact S, we get the following results.

Theorem. *The gauge group $\mathcal{G}\mathrm{au}(E) = C^\infty(\mathcal{D}\mathrm{iff}\{E,E\})$ is a splitting closed subgroup of $\mathcal{D}\mathrm{iff}(E)$. It admits an exponential mapping, which is not surjective on any neighborhood of the identity. The Lie algebra consists of all vertical vector fields with compact support on E with the negative of the usual Lie bracket.*

Proof. Since S is compact, we see from a local application of the exponential law (see [Frölicher, Kriegl 1988]) that $C^\infty(\mathcal{D}\mathrm{iff}\{E,E\}\to M) \hookrightarrow \mathcal{D}\mathrm{iff}(E)$ is an embedding of a splitting submanifold, which proves the first assertion. A curve of principal bundle automorphisms of $\mathcal{D}\mathrm{iff}\{S,E\} \to M$ through the identity is a smooth curve through the identity in $\mathcal{D}\mathrm{iff}(E)$ consisting of fiber representing maps. The derivative of such a curve is thus an arbitrary vertical vector field with compact support. The space of all these derivatives is therefore the Lie algebra of the gauge group, with the negative of the usual Lie bracket. The exponential mapping is given by the flow of such vector fields. Since on each fiber it is just isomorphic to the exponential mapping of $\mathcal{D}\mathrm{iff}(S)$, it has all properties of the latter. □

4.28. Definition. Let $J^1(E) \to E$ the affine 1–jet bundle of sections of $E \to M$. We have $J^1(E) = \{l \in L(T_xM, T_uE) : Tp \circ l = \mathrm{Id}_{T_xM}, u \in E, p(u) = x\}$. Then a section of $J^1(E) \to E$ is just a horizontal lift mapping $TM \times_M E \to TE$, which is fiber linear over E. Therefore, it describes a connection like in 4.19. Thus, we may view $C^\infty(J^1(E) \to E) \cong \mathcal{C}\mathrm{onn}(E)$ as the *space of connections* on E.

4.29. Curvature. If $\Phi \in \mathcal{C}\mathrm{onn}(E)$ then $\Phi \in \Omega^1(E; VE)$ and $\Phi \circ \Phi = \Phi$. Using the Frölicher–Nijenhuis bracket (cf. 4.14), we may define

$$R = \tfrac{1}{2}[\Phi, \Phi] = \tfrac{1}{2}[\mathrm{Id} - \Phi, \mathrm{Id} - \Phi] \in \Omega^2(E; VE).$$

the *curvature* of Φ. R satisfies the *Bianchi identity* $[\Phi, R] = 0$.

4.30. The action of $\mathcal{G}\mathrm{au}(E)$ on $\mathcal{C}\mathrm{onn}(E)$. Let again $\mathcal{C}\mathrm{onn}(E) = \{\Phi \in \Omega^1(E; TE) : \Phi \circ \Phi = \Phi, \Phi(TE) = VE\}$. Since S is compact, $\mathcal{C}\mathrm{onn}(E)$ is diffeomorphic to $C^\infty(J^1(E))$. The action of $\gamma \in \mathcal{G}\mathrm{au}(E) \subseteq \mathcal{D}\mathrm{iff}(E)$ on $\Phi \in \mathcal{C}\mathrm{onn}(E)$ is given by

$$\gamma_* \Phi = (\gamma^{-1})^* \Phi = T\gamma \circ \Phi \circ T\gamma^{-1}$$

Then from [Michor 1991, Theorem 6.6] we get

Theorem. *The action of the gauge group $\mathcal{G}\mathrm{au}(E)$ on the space of connections $\mathcal{C}\mathrm{onn}(E)$ is smooth.*

4.31. Remark. *The infinitesimal action can be found as follows. Let X be a vertical vector field with compact support on E and Fl_t^X the global flow. Then we find*

$$\left.\frac{d}{dt}\right|_0 (\mathrm{Fl}_t^X)^* \Phi = \mathcal{L}_X \Phi = [X, \Phi]$$

$$T_\Phi \mathcal{C}\mathrm{onn}(E) = \{\Psi \in \Omega^1(E; TE) : \Psi|_{VE} = 0\}$$

The infinitesimal orbit at Φ in $T_\Phi \mathcal{C}\mathrm{onn}(E)$ is $\{[X, \Phi] : X \in C_c^\infty(VE)\}$.

The isotropy subgroup I_Φ of a connection Φ is $\{\gamma \in \mathcal{G}\mathrm{au}(E) : \gamma^* \Phi = \Phi\}$, clearly just the group of those f, which respect the horizontal bundle $HE = \ker(\Phi)$. This group is often infinite dimensional. The non-compactness of this group gives rise to some difficulty, and is the essential reason why there exists no slice theorem for $\mathcal{C}\mathrm{onn}(E)/\,\mathcal{G}\mathrm{au}(E)$.

4.32. Definition. The *orbit space* (moduli space) $\mathcal{C}\mathrm{onn}(E)/\,\mathcal{G}\mathrm{au}(E)$ is the space of $\mathcal{G}\mathrm{au}(E)$–orbits in $\mathcal{C}\mathrm{onn}(E)$.

This space is the main interest of our studies. In the next chapters we will formulate the problem, mention similar results, and consider slice theorems for moduli spaces related to this moduli space.

5. Slice theorems for $\mathcal{M}\mathrm{et}(E)/\mathcal{G}\mathrm{au}(E)$ and $(\mathcal{C}\mathrm{onn}(E) \times \mathcal{M}\mathrm{et}(VE))/\mathcal{G}\mathrm{au}(E)$

First, we will discuss the slice theorem for $\mathcal{M}\mathrm{et}(E)/\mathcal{G}\mathrm{au}(E)$, which will be proved closely following the proof of the slice theorem for $\mathcal{M}\mathrm{et}(E)/\mathcal{D}\mathrm{iff}(E)$ by [Ebin 1968]. Later we will use that result and a "decomposition theorem" from [Gil–Medrano et al. 1992] to carry the result from $\mathcal{M}\mathrm{et}(E)/\mathcal{G}\mathrm{au}(E)$ to $(\mathcal{C}\mathrm{onn}(E) \times \mathcal{M}\mathrm{et}(VE))/\mathcal{G}\mathrm{au}(E)$. However, if we want to use the same result to remove the $\mathcal{M}\mathrm{et}(VE)$ term in $(\mathcal{C}\mathrm{onn}(E) \times \mathcal{M}\mathrm{et}(VE))/\mathcal{G}\mathrm{au}(E)$ difficulties arise, but that will be described in the following chapter.

5.1. Definition. Let (E, p, M, S) be a fiber bundle, M and S both compact, and let $\mathcal{M}\mathrm{et}(E)$ be the space of Riemannian metrics on E. Since the group $\mathcal{G}\mathrm{au}(E)$ is a subgroup of $\mathcal{D}\mathrm{iff}(E)$, it acts on $\mathcal{M}\mathrm{et}(E)$, also. Let $\mathcal{M}\mathrm{et}(E)/\mathcal{G}\mathrm{au}(E)$ be the corresponding moduli space.

5.2. Remark. The method of proving a slice theorem for this space follows essentially the proof for the action of a compact finite dimensional Lie group on a finite dimensional manifold as e.g. in [Borel 1960] or [Palais 1961].

However, if one tries to copy that proof difficulties arise. First, $\mathcal{G}\mathrm{au}(E)$ is not compact, not even locally compact. Therefore, integration with respect to the Haar measure yielding a $\mathcal{G}\mathrm{au}(E)$–invariant metric on E is not possible. Also, we cannot easily conclude that an orbit $\mathcal{G}\mathrm{au}(E) \cdot x$ is closed in E, and that $\mathcal{G}\mathrm{au}(E)/\mathcal{G}\mathrm{au}(E)_x$ is a manifold and in fact homeomorphic to $\mathcal{G}\mathrm{au}(E) \cdot x$.

Furthermore, the problem which forces us to take the detour via Hilbert manifolds is a topological one. $\mathcal{M}\mathrm{et}(E)$ is a manifold modeled on neighborhoods of a Fréchet space, and $\mathcal{G}\mathrm{au}(E)$ is also modeled on Fréchet spaces. The lack of an implicit function theorem in Fréchet spaces has the very disturbing consequence, that the exponential mapping, Exp, for $\mathcal{G}\mathrm{au}(E)$, although it can be proved to exist, is not a diffeomorphism onto any neighborhood of the identity, which is used very heavily in the usual proof of a slice theorem. However, in spite of the detour, we will need a very restricted inverse function theorem on Fréchet spaces, but with strong prerequisites.

5.3. Hilbert manifold completions. To overcome the topological difficulties mentioned above, we consider the Sobolev completions of the manifolds $\mathcal{M}\mathrm{et}(E)$ and $\mathcal{G}\mathrm{au}(E)$.

Recall the constructions from 3.25, and now construct analogously for an arbitrary fiber bundle (F, π, N, S') the space of H^s–sections $H^s(F)$, which consists of all sections, whose partial derivatives in local coordinates up to order s are square integrable. $H^s(F)$ is a Hilbert manifold modeled on the Sobolev space $H^s(\mathbb{R}^n, \mathbb{R}^m)$, where $n = \dim(N)$ and $m = \dim(S')$.

Again, the Sobolev lemma holds: For $s > k + \frac{n}{2}$ the inclusion map $H^s(F) \subset C^k(F)$ into the Banach manifold $C^k(F)$ is continuous.

This result is proven in [Palais 1968].

5.4. The space of metrics $\mathcal{M}\mathrm{et}^s(E)$. Now consider the vector bundle S^2T^*E over E. $\mathcal{M}\mathrm{et}(E)$ is the space of smooth sections of this bundle, which are positive definite everywhere. Now consider $C^0(S^2T^*E)$, the space of continuous sections of S^2T^*E. It follows that $C^0\,\mathcal{M}\mathrm{et}(E) \subset C^0(S^2T^*E)$, the subset of everywhere positive definite sections is an open subspace.

Now define the Sobolev extension $\mathcal{M}\mathrm{et}^s(E)$ of $\mathcal{M}\mathrm{et}(E)$ for $s > \frac{n}{2}$ as $H^s(S^2T^*E) \cap C^0\,\mathcal{M}\mathrm{et}(E)$. $\mathcal{M}\mathrm{et}^s(E)$ is an open subset of $H^s(S^2T^*E)$, a convex positive cone, and therefore a Hilbert manifold.

The tangent space $T_g\,\mathcal{M}\mathrm{et}^s(E)$ can be canonically identified at each point g with $H^s(S^2T^*E)$.

We can construct a Riemannian structure on $\mathcal{M}\mathrm{et}^s(E)$ as follows: Since every $g \in \mathcal{M}\mathrm{et}^s(E)$ defines a Riemannian structure on TE, it induces a (C^0) Riemannian structure $\tilde{g}$ on S^2T^*E and a (C^0) volume form $d\,\mathrm{vol}^g$ on E. Now define for $\phi, \psi \in H^s(S^2T^*E) = T_g\,\mathcal{M}\mathrm{et}^s(E)$ an inner product

$$(\phi, \psi)^g_s := \int_E \tilde{g}(\phi, \psi)\, d\,\mathrm{vol}^g,$$

which induces the H^0–topology on $H^s(S^2T^*E)$. Therefore, $H^s(S^2T^*E)$ is unfortunately not complete under $(\quad,\quad)^g_s$. Thus, we cannot define a real Riemannian structure on $\mathcal{M}\mathrm{et}^s(E)$. However, $(\quad,\quad)^g_s$ is a positive definite inner product everywhere, therefore we call $G_s(\phi, \psi) := (g \mapsto (\phi_g, \psi_g)^g_s)$ a weak Riemannian structure on $\mathcal{M}\mathrm{et}^s(E)$.

Lemma. *The weak Riemannian structure G_s on $\mathcal{M}\mathrm{et}^s(E)$ is smooth.*

Proof. It suffices to show that the function $G_s(\phi, \psi)_. : \mathcal{M}\mathrm{et}^s(E) \to \mathbb{R}$ $g \mapsto (\phi, \psi)^g_s$ is smooth for $\phi, \psi \in H^s(S^2T^*E)$, uniformly in H^s–norms of ϕ and ψ.

Take $g_0 \in \mathcal{M}\mathrm{et}^s(E)$, and let $d\,\mathrm{vol}^{g_0}$ be the corresponding volume form on E. Consider $f : \mathcal{M}\mathrm{et}^s(E) \to H^s(E, \mathbb{R})$ defined by the equation $f(g)d\,\mathrm{vol}^{g_0} = d\,\mathrm{vol}^g$ and $h : \mathcal{M}\mathrm{et}^s(E) \to H^s(E, \mathbb{R})$, $h(g) = \tilde{g}(\phi, \psi)$.

Now let F be the subbundle in S^2T^*E of positive definite forms on each T_pE. Obviously, there exist functions $\bar{f}, \bar{h} : F \to E \times \mathbb{R}$ with $f(g) = \bar{f} \circ g$, $h(g) = \bar{h} \circ g$. Thus, by Theorem 3.27, f and h, and hence fh are smooth functions from $H^s(F) = \mathcal{M}\mathrm{et}^s(E) \to H^s(E, \mathbb{R})$. $\square$

5.5. Connection and Exponential mapping on $\mathcal{M}\mathrm{et}^s(E)$.

Proposition.

(1) *The weak Riemannian structure defined above admits a unique connection ∇ which respects G_s.*
(2) *This connection defines an exponential mapping, which is a local diffeomorphism $T\,\mathcal{M}\mathrm{et}^s(E) \to \mathcal{M}\mathrm{et}^s(E)$ at the zero section.*

Proof. (1): By [Lang 1995, Theorem VIII.4.1] there exists for any Riemannian metric g on a Hilbert manifold a unique connection ∇ with

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{1}$$

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0. \tag{2}$$

However, since we have only a weak Riemannian metric, a problem arises. Assume, that we have already found such a connection. In our case, we may compute

$$\begin{aligned} XG^s(Y,Z) &= G^s(\nabla_X Y, Z) + G^s(Y, \nabla_X Z) \\ YG^s(Z,X) &= G^s(\nabla_Y Z, X) + G^s(Z, \nabla_Y X) \\ ZG^s(X,Y) &= G^s(\nabla_Z X, Y) + G^s(X, \nabla_Z Y) \end{aligned} \quad \left| \begin{aligned} &+ \\ &+ \\ &- \end{aligned} \right.$$

giving

$$\begin{aligned} &XG^s(Y,Z) + YG^s(Z,X) - ZG^s(X,Y) = \\ &\quad = G^s(\nabla_X Y + \nabla_Y X, Z) + G^s(\nabla_X Z - \nabla_Z X, Y) + G^s(\nabla_Y Z - \nabla_Z Y, X) = \\ &\quad = G^s([X,Z],Y) + G^s([Y,Z],X) + G^s(2\nabla_X Y - [X,Y], Z), \end{aligned} \tag{3}$$

and thus

$$\begin{aligned} 2G^s(\nabla_X Y, Z) = XG^s(Y,Z) + YG^s(Z,X) - ZG^s(X,Y) + \\ + G^s([X,Y],Z) - G^s([Y,Z],X) + G^s([Z,X],Y). \end{aligned}$$

Now we would like to conclude that $\nabla_X Y$ is uniquely determined by this equation. But, since we G^s is only a weak Riemannian metric, we cannot conclude that a solution $\nabla_X Y$ exists, although if it exist it is uniquely determined.

Since G^s defines the H^0–topology on $H^s(S^2T^*E)$, we may conclude, that a solution exists in $H^0(S^2T^*E)$. Everything that remains to show now is that this solution, in fact, lies in $H^s(S^2T^*E)$.

Equip $\mathcal{M}\mathrm{et}^s(E)$ with the canonical coordinate system it inherits as a subset of $H^s(S^2T^*E)$, and fix $g \in \mathcal{M}\mathrm{et}^s(E)$. If we consider g as a section of $\mathrm{Hom}(TE, T^*E)$, we can write any element of $H^s(S^2T^*E)$ as $g \cdot a$ where $a \in H^s(\mathrm{Hom}(TE,TE))$. Thus, vector fields X, Y, and Z can be written as $g \cdot a$, $g \cdot b$, and $g \cdot c$ in the canonical coordinate system. Then for $h \in \mathcal{M}\mathrm{et}^s(E)$, $\tilde{h}(Y,Z) = \mathrm{tr}(h^{-1}gbh^{-1}gc))$, where h^{-1} is regarded as a section of $\mathrm{Hom}(T^*E, TE)$. Furthermore, $(Y,Z)_s^h = \int_E \mathrm{tr}(h^{-1}gbh^{-1}gc)\sqrt{\det(hg^{-1})}\, d\,\mathrm{vol}^g$. Let $h_t := h + tga$. Then

$$\begin{aligned} XG_s(Y,Z)_g &= \frac{d}{dt}\bigg|_{t=0} \Big(\int_E \mathrm{tr}(h_t^{-1}gbh_t^{-1}gc\sqrt{\det(h_t g^{-1})}\, d\,\mathrm{vol}^g)\Big) \\ &= \int_E \big(-\mathrm{tr}(h^{-1}gah^{-1}gbh^{-1}gc + h^{-1}gbh^{-1}gah^{-1}gc) + \\ &\quad + \tfrac{1}{2}\,\mathrm{tr}(h^{-1}gbh^{-1}gc)\,\mathrm{tr}(h^{-1}ga)\big)\sqrt{\det(hg^{-1})}\, d\,\mathrm{vol}^g \end{aligned}$$

and

$$G_s([X,Y],Z)_g = \int_E \mathrm{tr}(h^{-1}gah^{-1}gbh^{-1}gc - h^{-1}gbh^{-1}gah^{-1}gc)\sqrt{\det(hg^{-1})}\, d\,\mathrm{vol}^g\,.$$

Using equation (3), we end up with

$$\begin{aligned} &(\nabla_X Y)_h = \\ &\quad = g\big(-h^{-1}gah^{-1}gb + \tfrac{1}{4}(\mathrm{tr}(h^{-1}ga)b + \mathrm{tr}(h^{-1}gb)a - \mathrm{tr}(h^{-1}gah^{-1}gb)g^{-1}h)\big). \end{aligned}$$

Since a, b, g, and h are H^s, $(\nabla_X Y)_h$ is H^s as well, so the weak Riemannian structure defines a connection on $\mathcal{M}\mathrm{et}^s(E)$ for any $s > \frac{n}{2}$.
(2): Furthermore, $(\nabla_Y X)_h$ is clearly smooth in h. Therefore, it is a smooth connection and defines a smooth exponential map Exp_s in the usual way, with all standard properties. In particular, Exp_s is a diffeomorphism from a neighborhood of the zero section of $T\,\mathcal{M}\mathrm{et}^s(E) \to \mathcal{M}\mathrm{et}^s(E)$. □

5.6. The group $\mathcal{G}\mathrm{au}^s(E)$. Recalling 5.3 we may restrict our attention to the special fiber bundle $(F = E \times E, \mathrm{pr}_1, E, E)$, whose sections are exactly the mappings $E \to E$. $C^1(F)$ is then the space of all C^1–mappings with the topology of uniform convergence up to the first derivative. Define $C^1\,\mathcal{D}\mathrm{iff}(E) := \{f \in C^1(F) | \exists f^{-1} \in C^1(F)\}$. The Sobolev extension of $\mathcal{D}\mathrm{iff}(E)$ is then defined as $\mathcal{D}\mathrm{iff}^s(E) := C^1\,\mathcal{D}\mathrm{iff}(E) \cap H^s(F)$ for $s > \frac{n}{2}$. Since, by the Sobolev lemma, $H^s(F) \subseteq C^1(F)$ is continuous, $\mathcal{D}\mathrm{iff}^s(F)$ is open in $H^s(F)$, and therefore is also a Hilbert manifold. Locally it looks like $H^s(TE)$.

We know from theorem 4.27 that $\mathcal{G}\mathrm{au}(E)$ is a splitting closed subgroup of $\mathcal{D}\mathrm{iff}(E)$. Thus, we define the Sobolev extension $\mathcal{G}\mathrm{au}^s(E)$ of $\mathcal{G}\mathrm{au}(E)$ as the topological completion of $\mathcal{G}\mathrm{au}(E)$ in $\mathcal{D}\mathrm{iff}^s(E)$.

Remark. *Note, that $\mathcal{G}\mathrm{au}^s(E)$ is exactly the subgroup of $\mathcal{D}\mathrm{iff}^s(E)$ of all fiber respecting diffeomorphisms, which cover the identity on M.*

Proof. Both conditions can be described by continuous equations, and the inverse of a fiber respecting diffeomorphism is also fiber respecting. □

Proposition. *$\mathcal{G}\mathrm{au}^s(E)$ is a topological group. (Note, that it is not a Hilbert Lie group.)*

Proof. Since, by construction, $\mathcal{G}\mathrm{au}^s(E)$ is a group and a closed submanifold of $\mathcal{D}\mathrm{iff}^s(E)$, it remains to show that $\mathcal{D}\mathrm{iff}^s(E)$ is a topological group.

First we determine the tangent space $T\,\mathcal{D}\mathrm{iff}^s(E)$. $T_\phi\,\mathcal{D}\mathrm{iff}^s(E) = H^s(\phi^* TE)$. By the way, $T_\phi\,\mathcal{G}\mathrm{au}^s(E) = H^s_{\mathrm{vert}}(\phi^* TE)$.

In [Palais 1968, §4], it was shown that $H^s(TM)$ is linearly isomorphic to a closed subspace of $\bigoplus_{i=1}^m H^s(B_n^i, \mathbb{R})$, where each B_n^i is the closed n-dimensional disc. From the local structure of $\mathcal{D}\mathrm{iff}^s(E)$, it only remains to show that the composition map $\circ : H^s(D_n, D_n) \times H^s(D_n, \mathbb{R}) \to H^s(D_n, \mathbb{R})$ is continuous. But this is assured by Lemma 3.23. That shows that the composition map is continuous.

The proof, that the inverse–mapping is continuous, is very similar, also. From definition of $\mathcal{D}\mathrm{iff}^s(E)$, we know that for $f \in \mathcal{D}\mathrm{iff}^s(E)$, $f^{-1} \in C^1\,\mathcal{D}\mathrm{iff}(E)$. Therefore, it remains to show that $f^{-1} \in H^s$ or $D(f^{-1}) \in H^{s-1}$. This can be achieved as follows. Consider the map $i : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ the matrix inversion. Then $D(f^{-1}) = i \circ Df \circ f^{-1}$. $Df \in H^{s-1}$ and $s - 1 > \frac{n}{2}$, and therefore by Theorem 3.27 $i \circ Df \in H^{s-1}$. Now we use induction to show that for $0 < k \leq s$, $D^k(f^{-1}) = g_k \circ f^{-1}$ for $g_k \in H^{s-k}$. For $k = 1$ we have shown that already. Assume, the fact is true for k. Then we compute $D^{k+1}(f^{-1}) = Dg_k \circ f^{-1} \cdot D(f^{-1}) = (Dg_k \cdot (i \circ Df)) \circ f^{-1}$. Since $Dg_k \in H^{s-k-1}$ and $i \circ Df \in H^{s-1}$, $g_{k+1} = Dg_k \cdot (i \circ Df) \in H^{s-k-1}$ by lemma 3.24. Now we see that $D^s(f^{-1}) \in H^0$, since $f^{-1} \in C^1\,\mathcal{D}\mathrm{iff}(E)$ already. Thus, $D(f^{-1}) \in H^{s-1}$, so $f^{-1} \in \mathcal{D}\mathrm{iff}^s(E)$. The continuity of the inversion map is proved analogously. Since $C^1\,\mathcal{D}\mathrm{iff}(E)$ is a topological group, f^{-1} is close to f'^{-1}

in the C^1–topology, if f is close to f' in $\mathcal{D}\mathrm{iff}^s(E)$. Using the same induction as above, we show that $g'_1 = i \circ Df'$ is close to g_1 in H^{s-1}, and inductively g'_k is close to g_k in H^{s-k}. Hence, g'_s is close to g_s in H^0. Again, since f^{-1} is close to f'^{-1} in C^1 already, $D^s f'^{-1} = g'_s \circ f'^{-1}$ is close to $D^s f$ in H^0. Thus Df'^{-1} is close to Df^{-1} in H^{s-1}, therefore, f'^{-1} is close to f^{-1} in $\mathcal{D}\mathrm{iff}^s(E)$, which proves continuity of $(\ )^{-1}$. □

5.7. Proposition. *Let $\gamma \in \mathcal{G}\mathrm{au}^s(E)$.*

1. *The right multiplication $r_\gamma : \mathcal{G}\mathrm{au}^s(E) \to \mathcal{G}\mathrm{au}^s(E)$ is smooth.*
2. *The left multiplication $l_\gamma : \mathcal{G}\mathrm{au}^s(E) \to \mathcal{G}\mathrm{au}^s(E)$ is smooth for $\gamma \in \mathcal{G}\mathrm{au}(E)$ (i.e. a smooth gauge transformation).*

Proof. (1): We know already that r_γ is smooth. Locally $\mathcal{G}\mathrm{au}^s(E)$ looks like a closed subset of $H^s(V)$ for a vector bundle V. Locally $r_g a$ is the restriction of $\phi \mapsto \phi \circ \gamma$ $H^s(V) \to H^s(W)$ to that subset for W another vector bundle. Since this map is the restriction of a linear map and continuous it is smooth.
(2): Consider again the trivial fiber bundle $(F = E \times E, \mathrm{pr}_1, E, E)$. Let $\tilde\gamma : E \times E \to E \times E$ be the map $(p, q) \mapsto (p, \gamma(q))$. Since γ is C^∞, so is $\tilde\gamma$. Also $l_\gamma : H^s(F) \to H^s(F)$ is the mapping $\phi \mapsto \tilde\gamma \circ \phi$. This is smooth by theorem 3.27. □

5.8. Proposition. *The action of $\mathcal{G}\mathrm{au}(E)$ on $C^\infty(S^2T^*E)$ can be extended to an action $\ell : \mathcal{G}\mathrm{au}^{s+1}(E) \times H^s(S^2T^*E) \to H^s(S^2T^*E)$, which is continuous and linear (thus smooth) in the second variable. We write $\ell(\gamma, g) = \ell^g(\gamma) = \ell_\gamma(g)$.*

If $g \in C^\infty(S^2T^*E)$ then $\ell^g : \mathcal{G}\mathrm{au}^{s+1}(E) \to H^s(S^2T^*E)$ is smooth.

Proof. We define the extension of the action by the same formula as usual as $(\ell(\gamma, g))_x(X, Y) = g_{\gamma(x)}(T\gamma \cdot TX, T\gamma \cdot TY)$, regarding elements of $\mathcal{G}\mathrm{au}^{s+1}(E)$ as elements in $\mathcal{D}\mathrm{iff}^{s+1}(E)$. First we check that this action is continuous and is well defined (i.e. has range in $H^s(S^2T^*E)$). We check this in local coordinates. There $\mathcal{G}\mathrm{au}^{s+1}(E)$ looks like a closed subset in $H^{s+1}(B_n, \mathbb{R}^n)$ and $H^s(S^2T^*E)$ looks like $H^s(B_n, \mathbb{R}^{\frac{1}{2}n(n+1)})$. The map $\gamma \mapsto T\gamma : H^{s+1}(B_n, \mathbb{R}^n) \to H^s(B_n, \mathbb{R}^{n(n+1)})$ is continuous. The map $(\gamma, g) \mapsto g \circ \gamma : H^{s+1}(B_n, \mathbb{R}^n) \times H^s(B_n, \mathbb{R}^{\frac{1}{2}n(n+1)}) \to H^s(B_n, \mathbb{R}^{\frac{1}{2}n(n+1)})$ is continuous by Lemma 3.23. Considering the map $(T\gamma, g \circ \gamma) \mapsto \ell(\gamma, g) : H^s(B_n, \mathbb{R}^{n(n+1)}) \times H^s(B_n, \mathbb{R}^{\frac{1}{2}n(n+1)}) \to H^s(B_n, \mathbb{R}^{\frac{1}{2}n(n+1)})$, we see that it is bilinear. Therefore, it is smooth by Lemma 3.24. Thus ℓ is continuous and well defined. Obviously, ℓ is linear in the second variable. For $g \in C^\infty(S^2T^*E)$ we see that $(\gamma, g) \mapsto g \circ \gamma$ is smooth by Theorem 3.27. Therefore, ℓ^g is also smooth in that case.

5.9. Proposition.

1. *The weak Riemannian structure G_s on $\mathcal{M}\mathrm{et}^s(E)$ is invariant under the action of $\mathcal{G}\mathrm{au}^{s+1}(E)$.*
2. *There exists a Riemannian structure $\mathcal{G}_s$ on $\mathcal{M}\mathrm{et}^s(E)$ which is invariant under $\mathcal{G}\mathrm{au}^{s+1}(E)$.*

Proof. (1): Take $\gamma \in \mathcal{G}\mathrm{au}^{s+1}(E)$. Then $\ell_\gamma(\mathcal{M}\mathrm{et}^s(E)) = \mathcal{M}\mathrm{et}^s(E)$ and ℓ_γ is linear, so for $g \in \mathcal{M}\mathrm{et}^s(E)$ $T_g\ell_\gamma = \ell_\gamma$. Take $X_g, Y_g \in T_g\,\mathcal{M}\mathrm{et}^s(E) = H^s(S^2T^*E)$. Pick

$p \in E$, then $\tilde{g}(X_g, Y_g)_p = \operatorname{tr}(g^{-1}X_g g^{-1}Y_g)_p$ and

$$\begin{aligned}\widetilde{\ell_\gamma g}(\ell_\gamma.X_g, \ell_\gamma.Y_g)_p &= \\ &= \operatorname{tr}((T\gamma g_{\gamma(p)} T\gamma^\top)^{-1} T\gamma X_{g,\gamma(p)} T\gamma^\top (T\gamma g_{\gamma(p)} T\gamma^\top)^{-1} T\gamma Y_{g,\gamma(p)} T\gamma^\top) = \\ &= \tilde{g}(X_g, Y_g)_{\gamma(p)},\end{aligned}$$

with $^\top$ indicating transposition. γ acts as a diffeomorphism also naturally on volume forms on E, and it is clear that $\gamma^* d\operatorname{vol}^g$ coincides with $d\operatorname{vol}^{\ell_\gamma g}$. ℓ_γ acts as an isometry on $\mathcal{M}\mathrm{et}^s(E)$ since, by the substitution rule, $\int_E \widetilde{\ell_\gamma g}(\ell_\gamma.X_g, \ell_\gamma.Y_g) d\operatorname{vol}^{\ell_\gamma g} = \int_E \tilde{g}(X_g, Y_g) d\operatorname{vol}^g$. Then $\ell_\gamma(\nabla_X Y|_g) = \nabla_{\ell_\gamma(X_g)} \ell_\gamma(Y_g)$ for $X, Y \in \mathfrak{X}(\mathcal{M}\mathrm{et}^s(E))$, and $\operatorname{Exp}_s \circ T\ell_\gamma = \ell_\gamma \circ \operatorname{Exp}_s$ (where one of the sides of the equation is defined, the other is defined and the equation holds).
(2): To construct the $\mathcal{G}\mathrm{au}^{s+1}(E)$–invariant Riemannian structure, we assume that $s > \frac{n}{2} + 1$, for then all elements of $\mathcal{M}\mathrm{et}^s(E)$ are C^1.

It suffices to define $\mathcal{G}_{sg}$ on $H^0(J^s(S^2T^*E))$ to get it on the space $H^s(S^2T^*E) = T_g \mathcal{M}\mathrm{et}^s(E)$. We start by defining $\mathcal{G}_{sg}$ for $g \in \mathcal{M}\mathrm{et}(E)$. To do that we first note, that g induces a smooth isomorphism $J^s(S^2T^*E) \cong \bigoplus_{i=0}^s S^2T^*E \otimes S^iT^*E$, as follows: g defines a covariant derivative for sections of TE, T^*E, and hence on S^2T^*E. Using this covariant derivative, we easily find the required isomorphism. Then we use the inner product induced by g on S^iT^*E and $d\operatorname{vol}^g$ to define the inner product $\mathcal{G}_{sg}$ on $H^0(J^s(S^2T^*E))$.

This, however, is more than we need. We do not need that isomorphism to be smooth. The only thing we have to know is, that it induces a continuous linear map $H^s(S^2T^*E) \to H^0(\bigoplus_{i=0}^s S^2T^*E \otimes S^iT^*E) = \bigoplus_{i=0}^s H^0(S^2T^*E \otimes S^iT^*E)$ for $g \in \mathcal{M}\mathrm{et}^s(E)$. This is what we show. The individual maps $D_g^i : H^s(S^2T^*E) \to H^{s-i}(S^2T^*E \otimes S^iT^*E)$ can be shown to be continuous as follows.

D_g^0 is the identity. By induction, we will assume that D_g^i is linear and continuous, and show that then D_g^{i+1} is continuous. $D_g^{i+1} = \nabla_g^i \circ D_g^i$, where ∇_g^i is the covariant derivative on $S^2T^*E \otimes S^iT^*E$ induced by g. Now for $\phi \in H^{s-i}(S^2T^*E \otimes S^iT^*E)$ in local coordinates we have $\nabla_g^i(\phi) = \frac{\partial \phi}{\partial x} + \Gamma \cdot \phi$, where Γ denotes the Riemann–Christoffel symbols with respect to g. g is H^s, thus Γ is H^{s-1}, and $\frac{\partial \phi}{\partial x}$ is H^{s-i-1}. Therefore, $\nabla_g^i : H^{s-i} \to H^{s-i-1}$ is continuous, so D_g^{i+1} is continuous, as required. Clearly, $\mathcal{G}_s$ gives a weak Riemannian structure for $T\mathcal{M}\mathrm{et}^s(E)$, and $\mathcal{G}_{sg}$ induces the H^s–topology on $T_g \mathcal{M}\mathrm{et}^s(E)$ for $g \in \mathcal{M}\mathrm{et}(E)$. It remains to show, that it also induces the H^s–topology for all $g \in \mathcal{M}\mathrm{et}^s(E)$. By construction, it clearly induces the H^0–topology. Again, by induction, we assume that we have already shown that it induces the H^i–topology $(i < s)$. We now prove, that it then induces the H^{i+1}–topology for $i < s$.

Let $\| \;\; \|^i$ be a fixed norm on $H^s(S^2T^*E)$ giving the H^i topology $(0 \le i \le s)$. That $\mathcal{G}_{sg}$ induces the H^i–topology means that

$$\min_{\|\phi\|^i = 1} \left(\sum_{j=0}^{i} \|D_g^j(\phi)\|_g^0 \right) > 0,$$

where $\| \;\; \|_g^0$ denotes the H^0 norm induced by g. For the inductive step, assume that $\mathcal{G}_{sg}$ does not induce the H^{i+1}–topology. That means, that there exists a sequence

$\{\phi_n\}$ with $\|\phi_n\|^{i+1} = 1$ and $\sum_{j=0}^{i+1} \|D_g^j(\phi_n)\|_g^0 \to 0$. But then, since $\mathcal{G}_{sg}$ induces the H^i–topology, we know $\|\phi_n\|^i \to 0$. But, setting $\psi_n = D_g^i(\phi_n)$, we get $\|\psi_n\|_g^0 \to 0$. We have already seen above, that $D_g^{i+1}(\phi_n) = \frac{\partial \psi_n}{\partial x} + \psi_n \cdot \Gamma$. Thus, $\|\psi_n\|_g^0 \to 0$ implies $\|\psi_n \cdot \Gamma\|_g^0 \to 0$. Hence, since $\|D_g^{i+1}(\phi_n)\|_g^0 \to 0$, $\|\frac{\partial \psi_n}{\partial x}\|_g^0 \to 0$ also. But $\|\phi_n\|^i \to 0$ and $\|\frac{\partial \psi_n}{\partial x}\|_g^0 \to 0$ imply $\|\phi_n\|^{i+1} \to 0$ which is a contradiction. Thus $\mathcal{G}_{sg}$ defines a strong inner product for every $g \in \mathcal{M}\mathrm{et}^s(E)$.

The next thing to show is that $\mathcal{G}_s$ is $\mathcal{G}\mathrm{au}^{s+1}(E)$ invariant. The proof is the same as in (1), but additionally we have to use that $T\gamma(\nabla_X^{\ell_\gamma g} Y) = \nabla_{T\gamma X} T\gamma Y$. It follows that $\gamma^*(D_g^i \phi) = D_{\ell_\gamma g}^i(\gamma^* \phi)$. Thus, the two isomorphisms of $J^s(S^2 T^* E)$ induced by g and $\ell_\gamma g$ commute with ℓ. The remainder is analogous to (1).

What remains is, that $\mathcal{G}_s$ is a continuous Riemannian structure. From the local description, we see that Γ is differentiated no more than $s-1$ times, and it is composed of g and $\frac{\partial g}{\partial x}$. Everything else follows from [Schwartz 1964, pp. 158–159].

It even follows that $\mathcal{G}_s$ is smooth, but we will not use that fact. $\square$

5.10. Construction of the manifold $\mathcal{G}\mathrm{au}^s(E)/\,\mathcal{G}\mathrm{au}(E)_g$. Pick $g \in \mathcal{M}\mathrm{et}(E)$, and let $\mathcal{G}\mathrm{au}^s(E)_g$ be the isotropy subgroup of $\mathcal{G}\mathrm{au}^s(E)$, $s > \frac{n}{2} + 1$. Then $\mathcal{G}\mathrm{au}^s(E)_g$ is the set of H^s diffeomorphisms, which are fiber respecting, cover the identity on M, and are isometries with respect to g. By [Palais 1957] it is well known that any C^1–isometry of a smooth metric is smooth. Therefore, since $C^1\,\mathcal{G}\mathrm{au}(E) \supset \mathcal{G}\mathrm{au}^s(E)$, $\mathcal{G}\mathrm{au}^s(E)_g \subset \mathcal{G}\mathrm{au}(E)$ and $\mathcal{G}\mathrm{au}^s(E)_g = \mathcal{G}\mathrm{au}(E)_g$ for all $s > \frac{n}{2} + 1$. By [Kobayashi, Nomizu 1963] and [Palais 1957] we know the following

Theorem. *Let X be a smooth compact manifold with smooth Riemannian metric g. Let $\phi_n, \phi \in \mathcal{D}\mathrm{iff}(X)_g$. If $\phi_n \to \phi$ and $T\phi_n \to T\phi$ uniformly on X (i.e. C^1–convergence), then $\phi_n \to \phi$ uniformly in all derivatives (i.e. C^∞–convergence). If $\mathcal{D}\mathrm{iff}(X)_g$ is given the topology of uniform C^k–convergence $(1 \le k \le \infty)$, $\mathcal{D}\mathrm{iff}(X)_g$ is a compact Lie group.*

By the first part of this theorem, the topology of $\mathcal{D}\mathrm{iff}(X)_g$ is independent of k. Also $A : \mathcal{D}\mathrm{iff}(X)_g \times X \to X$ defined by $A(\phi, x) = \phi(x)$ induces a natural identification i between the Lie algebra $\mathfrak{G}$ of $\mathcal{D}\mathrm{iff}(X)_g$ and the set $\mathfrak{V}$ of vector fields on X whose one parameter groups of diffeomorphisms lie in $\mathcal{D}\mathrm{iff}(X)_g$. This identification i is defined by $i(Y)_x = T_{(\mathrm{Id},x)} A(Y, 0)$ where Id *is the identity of $\mathcal{D}\mathrm{iff}(X)_g$.*

Since $\mathcal{G}\mathrm{au}(E)_g \subseteq \mathcal{D}\mathrm{iff}(E)_g$ is a closed subgroup, $\mathcal{G}\mathrm{au}(E)_g$ is a compact Lie group. Its topology is also independent of the choice of k. The Lie algebra of $\mathcal{G}\mathrm{au}(E)_g$ can be identified as the Lie subalgebra $\mathfrak{Gau}_g$ of $\mathfrak{G}$ of all vertical vector fields in $\mathfrak{V}$.

5.11. Lemma. *$i : \mathcal{G}\mathrm{au}(E)_g \subseteq \mathcal{G}\mathrm{au}^s(E)$ is a smooth embedding.*

Proof. We prove in two steps and use Corollary 3.15.

Claim 1: $i : \mathcal{G}\mathrm{au}(E)_g \subseteq \mathcal{G}\mathrm{au}^s(E)$ is smooth.

Since, by Proposition 5.7 r_γ is smooth for any $\gamma \in \mathcal{G}\mathrm{au}(E)_g$, and $i = r_\gamma \circ i \circ r_{\gamma^{-1}}$, we only need to show smoothness at the identity $\mathrm{Id} \in \mathcal{G}\mathrm{au}(E)_g$. For doing this, we use a standard chart for $\mathcal{G}\mathrm{au}(E)_g$. Let $\mathrm{Exp}^{\mathfrak{Gau}_g}$ be the exponential mapping of the Lie group $\mathcal{G}\mathrm{au}(E)_g$. It is a diffeomorphism from a neighborhood U of zero in $\mathfrak{Gau}_g$ to a neighborhood of Id.

In Corollary 3.15 set $g : J \times U \times V \to F$ as $g(t,u,v) = u$, and note that $\mathrm{Exp}^{\mathfrak{Gau}_g}(u)$ is the map $v \mapsto h(1,u,v)$. Let $\tilde{h} : U_0 \to \mathcal{G}\mathrm{au}^s(E)$ be defined as $\tilde{h}(u) = (v \mapsto h(1,u,v))$. Then smoothness of i is equivalent to smoothness of $\tilde{h}$. We prove $\frac{d^k}{dv^k}(\tilde{h})(v)(x_1,\dots,x_k) = \partial_2^k h(1,u,v)(x_1,\dots,x_k)$. For $k=1$ this means $\lim_{t\to 0} \frac{h(u+tx_1)-h(u)}{t}$ converges at each point $v \in V$ to $\partial_1 h(1,u,v)(x)$. Since, by the theorem, h is a smooth map, the convergence is uniform in all derivatives. That means, it converges also in the H^s–topology. The same works for $k > 1$, so $\tilde{h}$ is smooth, and so is i.

Claim 2: $i : \mathcal{G}\mathrm{au}(E)_g \subseteq \mathcal{G}\mathrm{au}^s(E)$ is an embedding.

Since $\mathcal{G}\mathrm{au}(E)_g$ is compact, i is a homeomorphism onto its image. $T_{\mathrm{Id}} i : \mathfrak{Gau}_g \to T_{\mathrm{Id}}\,\mathcal{G}\mathrm{au}^s(E)$ is given by $x \mapsto D\tilde{h}(0)(x) = (v \mapsto \partial_2 h(1,0,v)(x))$. If $D\tilde{h}(0)(x) = 0$ then $\partial_2 h(1,0,v)(x) = 0$ for all v. But, by construction, $h(1,x,v) = h(t, \frac{1}{t}x, v)$, so then $\partial_2 h(t,0,v)(x) = 0$, and $0 = \partial_1\partial_2 h(1,0,v)(x) = \partial_2(\partial_1 h(1,0,v))(x) = \partial_2 f(1,0,v)(x) = x$. Thus, $x = 0$, so $T_{\mathrm{Id}} i$ is injective. Its image is finite dimensional, therefore closed in $T_{\mathrm{Id}}\,\mathcal{G}\mathrm{au}^s(E)$. Transporting by right multiplication, we find that Ti is injective on each fiber, and its image is closed in its fiber. □

5.12. Lemma. *Let $S = \bigcup_{\gamma \in \mathcal{G}\mathrm{au}^s(E)} T_{\mathrm{Id}} r_\gamma(\mathfrak{Gau}_g)$. Then S is a smooth involutive subbundle of $T(\mathcal{G}\mathrm{au}^s(E))$.*

The elements of S are the vectors in $T\,\mathcal{G}\mathrm{au}^s(E)$ which are tangent to some coset $\mathcal{G}\mathrm{au}(E)_g\gamma$, i.e. the elements of $T\,\mathcal{G}\mathrm{au}(E)_g\gamma \subset T\,\mathcal{G}\mathrm{au}^s(E)$.

Proof. At first we prove that the composition mapping $\circ : \mathcal{G}\mathrm{au}(E)_g \times \mathcal{G}\mathrm{au}^s(E) \to \mathcal{G}\mathrm{au}^s(E)$ is smooth. We know from Proposition 5.7 that $\circ$ is smooth in either variable individually, that $\partial_i \circ\ .$ $(i = 1,2)$ is continuous in both variables, and that it can locally be described by a map which is linear in the first variable. We only need to show existence and continuity of the higher partial derivatives. But locally

$$\partial_2^k \partial_1^j \circ\ . = \begin{cases} \partial_2^k \circ\ . & j = 1 \\ 0 & j \neq 1. \end{cases}$$

This proves the first step.

Let $\{X_i\}$ be a basis of $\mathfrak{Gau}_g$, and define V_i by $(V_i)_\gamma = Tr_\gamma(X_i)$. V_i is smooth because $\circ$ is, and $\{V_i\}$ is clearly a basis everywhere. Furthermore, $[V_i, V_j]_\gamma = [Tr_\gamma(X_i), Tr_\gamma(X_j)] = Tr_\gamma([X_i, X_j])$ which is in S since $[X_i, X_j] \in \mathfrak{Gau}_g$. Since S has finite dimensional fiber, this suffices to prove the first part of the proposition. The second part is obvious. □

5.13. The manifold structure of $\mathcal{G}\mathrm{au}^s(E)/\mathcal{G}\mathrm{au}(E)_g$. Consider the usual projection $\pi : \mathcal{G}\mathrm{au}^s(E) \twoheadrightarrow \mathcal{G}\mathrm{au}^s(E)/\mathcal{G}\mathrm{au}(E)_g$, and equip $\mathcal{G}\mathrm{au}^s(E)/\mathcal{G}\mathrm{au}(E)_g$ with the quotient topology.

From Frobenius' theorem 3.20 we get for any $\gamma \in \mathcal{G}\mathrm{au}^s(E)$ a neighborhood W of γ and a diffeomorphism $\phi : U \times V \to W$ such that for any fixed $v \in V$, $\phi|_U \times \{v\}$ is a diffeomorphism onto a neighborhood of $\phi(0,v)$ in the coset $\mathcal{G}\mathrm{au}(E)_\gamma \phi(0,v)$. Especially, we can find a neighborhood W of Id small enough so that $\mathcal{G}\mathrm{au}(E)_g \cap W = \phi(U, 0)$.

To assure that $\pi \circ \phi|_{(0\times V_0)} : V_0 \to \mathcal{G}\mathrm{au}^s(E)/\mathcal{G}\mathrm{au}(E)_g$ is a homeomorphism onto a neighborhood of $\mathcal{G}\mathrm{au}(E)_g$ in $\mathcal{G}\mathrm{au}^s(E)/\mathcal{G}\mathrm{au}(E)_g$ for certain smaller V_0, such that we can use that to define charts, we will need the following Lemma.

Lemma. *There exist connected neighborhoods U_0, V_0 of zero in a Banach space, included in U, V, such that if $W_0 = \phi(U_0 \times V_0)$, for any $\gamma \in W_0$, there exists a unique $v \in V_0$ such that $\mathcal{G}\mathrm{au}(E)_g\gamma \cap W_0 = \phi(U_0, v)$.*

Proof. Since $\mathcal{G}\mathrm{au}^s(E)$ is a topological group, we get neighborhoods W_1, W_2 of Id such that $W_1 \cdot W_1 \subset W$, $W_2 \cdot W_2^{-1} \subset W_1$, and $W_1 = \phi(U_1 \times V_1)$, where $U_1 \subset U$, $V_1 \subset V$ are connected balls about 0. Next pick U_0, V_0 so small that $\phi(U_0 \times V_0) = W_0 \subset W_2$. Take $\gamma \in W_0$, and $\eta \in \mathcal{G}\mathrm{au}(E)_g\gamma \cap W_0$. Then $\eta\gamma^{-1} \in \mathcal{G}\mathrm{au}(E)_g\gamma \cap W_0 \cdot W_0^{-1} \subset \mathcal{G}\mathrm{au}(E)_g \cap W_1$. Hence, $\eta\gamma^{-1} \in \phi(U_1, 0)$.

Let $l := \{t \cdot \phi^{-1}(\eta\gamma^{-1}) | t \in [0,1]\} \subset U_1$ be a line. Then $\phi(l)$ is a smooth curve from Id to $\eta\gamma^{-1}$ in $\mathcal{G}\mathrm{au}(E)_g \cap W_1$. Thus, $\phi(l)\gamma$ is a smooth curve c from γ to η in $(\mathcal{G}\mathrm{au}(E)_g \cap W_1)\gamma \subset \mathcal{G}\mathrm{au}(E)_g\gamma \cap W \subset \phi(U \times V)$. From the construction of ϕ, we know that the tangent to c at any point lies in $T\phi(TU)$. Therefore, if $\gamma = \phi(u, v_0)$ and $\eta = \phi(u', v)$, we get $v_0 = v$, and so $\eta \in \phi(U, v)$. Since $\eta \in \mathcal{G}\mathrm{au}(E)_g\gamma \cap W_0$ was arbitrary, we get $\mathcal{G}\mathrm{au}(E)_g\gamma \cap W_0 \subset \phi(U_0, v)$, and $v \in V_0$ is unique, and by the paragraph before the Lemma, $\phi(U_0, v) \subset \mathcal{G}\mathrm{au}(E)_g\gamma \cap W_0$. $\square$

Now define charts for $\mathcal{G}\mathrm{au}^s(E)/\mathcal{G}\mathrm{au}(E)_g$ by right translation

$$\psi_\gamma := \pi \circ r_\gamma \circ \phi : V_0 \to \mathcal{G}\mathrm{au}(E)_g\gamma.$$

ψ_γ clearly is injective and $\psi_\gamma(V_0)$ covers a neighborhood of $\mathcal{G}\mathrm{au}(E)_g\gamma$. We compute

$$\psi_\gamma^{-1} \circ \psi_\eta(v) = \psi_\gamma^{-1}(\mathcal{G}\mathrm{au}(E)_g\phi(0,v) \cdot \eta) = \mathrm{pr}_2\, \phi^{-1}(\mathcal{G}\mathrm{au}(E)_g\phi(0,v) \cdot \eta \cdot \gamma^{-1}),$$

which is a smooth map. Therefore, the maps are smoothly compatible, and hence $\mathcal{G}\mathrm{au}^s(E)/\mathcal{G}\mathrm{au}(E)_g$ is a smooth manifold.

Note, that the usual right action r_γ on $\mathcal{G}\mathrm{au}^s(E)/\mathcal{G}\mathrm{au}(E)_g$ is smooth.

5.14. Proposition.

(1) *The map $\pi : \mathcal{G}\mathrm{au}^s(E) \twoheadrightarrow \mathcal{G}\mathrm{au}^s(E)/\mathcal{G}\mathrm{au}(E)_\gamma$ admits a smooth local cross section at any coset $\mathcal{G}\mathrm{au}(E)_\gamma\eta$.*

(2) *Let N be any manifold. A map $f : \mathcal{G}\mathrm{au}^s(E)/\mathcal{G}\mathrm{au}(E)_\gamma \to N$ is smooth if and only if $f \circ \pi : \mathcal{G}\mathrm{au}^s(E) \to N$ is smooth. In particular, π is smooth.*

Proof. (1): Define $s_\gamma : \mathcal{G}\mathrm{au}^s(E)/\mathcal{G}\mathrm{au}(E)_g \to \mathcal{G}\mathrm{au}^s(E)$ on a neighborhood of γ by

$$s_\gamma := r_\gamma \circ \phi \circ \psi_\gamma^{-1}.$$

$\pi \circ s_\gamma$ is the identity, and $s_\gamma(\mathcal{G}\mathrm{au}(E)_g\gamma) = \gamma$. Furthermore, s_γ is obviously smooth.
(2): $\Longleftarrow$: Suppose $f \circ \pi$ is smooth. Near $\mathcal{G}\mathrm{au}(E)_g\gamma$, $f = f \circ \pi \circ s_\gamma$, thus f is a composition of smooth mappings.
$\Longrightarrow$: π is smooth, since in charts ψ_γ and $r_\gamma \circ \phi$, $\pi \hat{=} \mathrm{pr}_2 : U \times V \to V$. Thus $f \circ \pi$ is smooth, if f is. $\square$

5.15. The map of $\mathcal{G}\mathrm{au}^{s+1}(E)$ onto an orbit through g. Consider again the map ℓ^g. Since $\ell^g(\mathcal{G}\mathrm{au}(E)_g) = g$, ℓ^g induces a map $\tilde{\ell}^g : \mathcal{G}\mathrm{au}^{s+1}(E)/\mathcal{G}\mathrm{au}(E)_g \to \mathcal{M}\mathrm{et}^s(E)$. We will show that $\tilde{\ell}^g$ is a diffeomorphism onto the orbit of $\mathcal{G}\mathrm{au}^{s+1}(E)$ through g.

Lemma. *$\tilde{\ell}^g$ is smooth and injective.*

Proof. ℓ^g is smooth and $\ell^g = \tilde{\ell}^g \circ \pi$. By Proposition 5.14(2) $\tilde{\ell}^g$ is smooth. Since $\tilde{\ell}^g(\mathcal{G}\mathrm{au}(E)_g\gamma) = \tilde{\ell}^g(\mathcal{G}\mathrm{au}(E)_g\eta)$, $\ell^g(\gamma) = \ell^g(\eta)$, hence $\gamma\eta^{-1} \in \mathcal{G}\mathrm{au}(E)_g$, and finally $\mathcal{G}\mathrm{au}(E)_g\gamma = \mathcal{G}\mathrm{au}(E)_g\eta$. □

To show that $\tilde{\ell}^g$ is an immersion, we must show that the tangent mapping $T_p\tilde{\ell}^g : T_p(\mathcal{G}\mathrm{au}^{s+1}(E)/\mathcal{G}\mathrm{au}(E)_g) \to T_{\tilde{\ell}^g(p)}(\mathcal{M}\mathrm{et}^s(E))$ is injective and has closed range. First we will consider the map $T_{\mathrm{Id}}\ell^g : T_{\mathrm{Id}}(\mathcal{G}\mathrm{au}^{s+1}(E)) \to T_g(\mathcal{M}\mathrm{et}^s(E))$. Recalling, that $T_{\mathrm{Id}}(\mathcal{G}\mathrm{au}^{s+1}(E)) \cong H^{s+1}(VE)$ and $T_g\,\mathcal{M}\mathrm{et}^s(E) \cong H^s(S^2T^*E)$, we can formulate the following

5.16. Lemma. *$T_{\mathrm{Id}}\ell^g$, with the identifications defined above, is the first order differential operator $L_e(X) = \mathcal{L}_X(g)$ (the Lie derivative of g with respect to the vertical vector field X).*

Proof. We know that $\mathcal{L}_{\cdot}(g)$ and $T_{\mathrm{Id}}\ell^g$ are both continuous linear maps $H^{s+1}(VE) \to H^s(S^2T^*E)$ because they are of first order. Thus we only have to show that they agree on the dense subset $C^\infty(VE)$. Take any $X \in C^\infty(VE)$. This X generates a one parameter subgroup of diffeomorphisms γ_t. The map $c : \mathbb{R} \to \mathcal{G}\mathrm{au}^{s+1}(E)$ given by $t \mapsto \gamma_t$ is a smooth curve in $\mathcal{G}\mathrm{au}^{s+1}(E)$. $T_sc = (x \mapsto X_{\gamma_s(x)}) \in T_{\gamma_s}(\mathcal{G}\mathrm{au}^{s+1}(E))$.

To compute $T_{\mathrm{Id}}\ell^g(X)$, we compute $\frac{d}{dt}|_0(\ell^g \circ c(t)) = \frac{d}{dt}|_0(\ell_{\gamma_t}(g))$. By definition, at each point $x \in E$, $\frac{d}{dt}|_0(\ell_{\gamma_t}(g_x)) = \mathcal{L}_X(g)_x$. Since ℓ^g is smooth, $\frac{d}{dt}(\ell^g \circ c(t))$ exists in $H^s(S^2T^*E)$, and it is the map $x \mapsto \mathcal{L}_X(g)_x$. □

$T_\gamma\ell^g$ for any $\gamma \in \mathcal{G}\mathrm{au}^{s+1}(E)$ can be computed by right translation to the identity: $T_\gamma\ell^g(T_\gamma(\mathcal{G}\mathrm{au}^{s+1}(E))) = \gamma^* \circ L_e \circ T_\gamma(r_{\gamma^{-1}})(T_\gamma(\mathcal{G}\mathrm{au}^{s+1}(E)))$. Therefore, $T_\gamma\ell^g(T_\gamma\,\mathcal{G}\mathrm{au}^{s+1}(E)) \cong T_{\mathrm{Id}}\ell^g(T_{\mathrm{Id}}\,\mathcal{G}\mathrm{au}^{s+1}(E))$.

Now, if we want to show that $T_\gamma\ell^g(T_\gamma\,\mathcal{G}\mathrm{au}^{s+1}(E))$ is closed in $T_{\ell^g(\gamma)}\,\mathcal{M}\mathrm{et}^s(E) = H^s(S^2T^*E)$, it is sufficient to prove it for $\gamma = \mathrm{Id}$. By 5.16 we, therefore, examine the map L_e.

5.17. Proposition.

(1) *The map $L_e : H^{s+1}(VE) \to H^s(S^2T^*E)$ has closed image, and the complement of* $\mathrm{im}(L_e)$ *is closed.*
(2) $\mathrm{im}(T_{\pi(\gamma)}\tilde{\ell}^g)$ *is closed and has closed complement in $T_{\ell^g(\gamma)}\,\mathcal{M}\mathrm{et}^s(E)$.*

Proof. (1): By Proposition 3.30 we need only show that for $\xi \neq 0 \in T_x^*E$ $\sigma_\xi(L_e)$ is injective, where $\sigma_\xi(L_e)$ denotes the symbol of L_e at $\xi \in T_x^*E$. Let $Y \in T_xE$ and $\eta \in T_x^*E$ be the element associated to Y by the metric g. Then

$$\sigma_\xi(L_e)(Y) = \eta \otimes \xi + \xi \otimes \eta,$$

which is injective.
(2): $\ell^g = \tilde{\ell}^g \circ \pi$ and $T\pi$ is onto $T_{\pi(\gamma)}(\mathcal{G}\mathrm{au}^{s+1}(E)/\mathcal{G}\mathrm{au}(E)_g)$, so the image of $T_\gamma\ell^g$ coincides with the image of $T_{\pi(\gamma)}$. This fact, together with (1) proves (2). □

5.18. Proposition. *$\tilde{\ell}^g : \mathcal{G}\mathrm{au}^{s+1}(E)/\mathcal{G}\mathrm{au}(E)_g \to \mathcal{M}\mathrm{et}^s(E)$ is an injective immersion.*

Proof. For proving this fact, it remains to show that $T\tilde{\ell}^g$ is injective in every tangent space. But we know already that $T_\gamma \ell^g = T_{\pi(\gamma)}\tilde{\ell}^g \circ T_\gamma \pi$, and therefore injectivity can be proven by the following claim:
Claim: If $X \in T_\gamma \mathcal{G}\mathrm{au}^{s+1}(E)$ with $T_\gamma \ell^g(X) = 0$, then $T_\gamma \pi(X) = 0$.
To prove this, we consider as first case $\gamma = \mathrm{Id}$. There $T_{\mathrm{Id}}\ell^g = L_e$. If $L_e(X) = 0$ then X is a smooth vertical vector field and $\mathcal{L}_X(g) = 0$. Thus, the one parameter subgroup $\{\eta_t\}$ generated by X lies in $\mathcal{G}\mathrm{au}(E)_g$. Hence, $\pi(\{\eta_t\}) \equiv \mathcal{G}\mathrm{au}(E)_g$, and therefore $T_{\mathrm{Id}}\pi(X) = 0$. □

The next step will be showing that $\tilde{\ell}^g$ is actually an embedding with closed image in $\mathcal{M}\mathrm{et}^s(E)$.

5.19. Proposition. *If $s > \frac{n}{2} + 2$ then $\tilde{\ell}^g : \mathcal{G}\mathrm{au}^{s+1}(E)/\mathcal{G}\mathrm{au}(E)_g \to \mathcal{M}\mathrm{et}^s(E)$ is a homeomorphism onto a closed subset of $\mathcal{M}\mathrm{et}^s(E)$.*

Proof. Let $\{\gamma_n\}$ be any sequence in $\mathcal{G}\mathrm{au}^{s+1}(E)$ such that $\ell_{\gamma_n}(g) \to g'$. Consider the exponential mappings $\exp, \exp', \exp_n : TE \to E$ of g, g', and $\ell_{\gamma_n}(g)$, respectively.

Since E is compact, there exists a positive real ε such that any ball of radius smaller than ε with respect to g in E is contained in some normal coordinate neighborhood of E. For g' there similarly exists a real number ε'.

Now set $K = \max_{X \in TE}(g'(X,X)/g(X,X))$, and pick a finite set $\{e_i\}$ of points in E such that for each i, U_i is a normal coordinate neighborhood with respect to g centered at e_i, of radius less than $\delta = K^{-\frac{1}{2}} \min(\varepsilon, \varepsilon')$, and the U_i are a covering of E. Further pick $\{X_i^j\}$ such that for each fixed i, $\{X_i^j\}_j$ is an orthonormal basis with respect to g of $T_{e_i}M$.
Claim 1: You can find a finite set $\{f_i\} \subset E$, $\{Y_i^j\} \subset TE$ and a subsequence $\{\gamma_{n_k}\}$ such that $\gamma_{n_k}(e_i) \to f_i$ and $T\gamma_{n_k}(X_i^j) \to Y_i^j$ for all i, j. Furthermore, Y_i^j is a basis of $T_{f_i}E$.
Proof: It is immediate from the compactness of E that there exists a subsequence η_k of γ_n such that for all i there is an f_i and $\eta_k(e_i) \to f_i$. Let $C = \max_{(i,j)}(g'(X_i^j, X_i^j))$. Since $\ell_{\eta_k}(g) \to g'$, for every (i,j) we have $g(T\eta_k X_i^j, T\eta_k X_i^j) \to g'(X_i^j, X_i^j)$. Thus for k large enough $g(T\eta_k X_i^j, T\eta_k X_i^j)^{\frac{1}{2}} \le 2C$. If you define $B_N(E) = \{X \in TE | g(X,X)^{\frac{1}{2}} \le N\}$, then $B_{2L}(E)$ is compact, and $T\eta_k X_i^j \in B_{2L}(E)$ for large k. The existence of a subsequence as asked for in the claim and the existence of the Y_i^j follow again from compactness of E. The Y_i^j obviously are a basis.

From now, we denote γ_{n_k} again by γ_n for convenience.
Claim 2: There exists $\gamma \in C^1\,\mathcal{G}\mathrm{au}(E)$ such that $\gamma_n \to \gamma$.
Proof: Consider a fixed U_i about e_i. If $e \in U_i$, $e = \exp(\sum_j a_i^j X_i^j)$ with $\sum_j (a_i^j)^2 < \delta$. Thus, $\gamma_n(e) = \gamma_n(\sum_j a_i^j X_i^j) = \exp_n \circ T\gamma_n(\sum_j a_i^j X_i^j)$. Since $T\gamma_n(X_i^j) \to Y_i^j$ $\{T\gamma_n(X_i^j)\}$ is a bounded set. Furthermore, on bounded sets $\exp_n$ converges C^1 to $\exp'$ (since $s > \frac{n}{2} + 2$), thus $\gamma_n(e) \to \exp'(\sum_j a_i^j Y_i^j)$. Define $\gamma(e) = \exp'(\sum_j a_i^j Y_i^j)$, then γ_n converges C^1 to γ on U_i. Extend γ to a map on E by gluing together the definitions on all U_i. The two constructions on $e \in U_i \cap U_j$ coincide since the γ_n are maps on E and $\gamma_n(e) \to \gamma(e)$. Therefore, γ is well defined on E, and since $\gamma_n \to \gamma$

on each U_i, $\gamma_n \to \gamma$ on E. On U_i is $\gamma = \exp' \circ \lambda \circ \exp_{e_i}^{-1}$, where $\exp_{e_i} : T_{e_i}E \to E$ is $\exp|_{T_{e_i}E}$, and $\lambda : T_{e_i}E \to T_{f_i}E$ is the linear map given on the basis by $\lambda(X_i^j) = Y_i^j$. Since $g'(Y_i^j, Y_i^j) \leq K$, $\lambda \circ \exp_{e_i}^{-1}(U_i)$ is contained in a neighborhood of zero of g'-radius $\leq \varepsilon'$. This proves that $\exp'|_{\lambda \circ \exp_{e_i}^{-1}(U_i)}$ is a C^1–diffeomorphism. Hence, $\gamma|_{U_i}$ is a C^1–diffeomorphism onto a neighborhood of $\gamma(e_i)$, and thus $\gamma(E)$ is open in E. Since E is compact, $\gamma(E)$ is closed also.
Let d, d_n, d' be the (topological) metrics on E induced by g, $\ell_{\gamma_n}(g)$, and g', respectively. Let l be the Lebesgue number of the covering U_i with respect to d (i.e., if $d(e,f) < l$, there is some U_i such that $e, f \in U_i$.). For we know that $\gamma|_{U_i}$ is injective, we take $e, f \in E$ such that $d(e,f) \geq l$. Then, $d_n(\gamma_n(e), \gamma_n(f)) \geq l$ for k big. But $\ell_{\gamma_n}(g) \to \gamma'$ and $\gamma_n(e) \to \gamma(e)$, $\gamma_n(f) \to \gamma(f)$. Thus, $d'(\gamma(e), \gamma(f)) \geq l$, so $\gamma(e) \neq \gamma(f)$. Therefore, γ is injective, by the above argument it is onto, hence a C^1–diffeomorphism. Since $\gamma_n \to \gamma$ in $C^1\,\mathcal{D}\mathrm{iff}(E)$, $\ell_{\gamma_n}(g) \to \ell_\gamma(g)$ in $C^0\,\mathcal{M}\mathrm{et}(E)$, so $\ell_\gamma(g) = g'$. Next, we have to show that γ actually is in $C^1\,\mathcal{G}\mathrm{au}(E)$.

$\gamma(p^{-1}(x)) \subset p^{-1}(x)$ since all $\gamma_n(p^{-1}(x)) \subset p^{-1}(x)$ and $p^{-1}(x)$ is closed for all $x \in M$, and $p(\gamma(e)) = p(\lim_{n\to\infty} \gamma_n(e)) = \lim_{n\to\infty} p(\gamma_n(e)) = \lim_{n\to\infty} p(e) = p(e)$ since p is continuous and all $\gamma_n \in \mathcal{G}\mathrm{au}^{s+1}(E)$. Hence, $\gamma \in C^1\,\mathcal{G}\mathrm{au}(E)$.

Closely following Palais' idea in Ebin's proof we show what remains to be proved.

We know that given $\ell_{\gamma_n}(g) \to g'$ in $\mathcal{M}\mathrm{et}^s(E)$, there exists a subsequence $\{\gamma_{n_k}\}$ of $\{\gamma_n\}$ such that $\gamma_{n_k} \to \gamma$ in $C^1\,\mathcal{G}\mathrm{au}(E)$ and $\ell_\gamma(g) = g'$. Consider the Riemann-Christoffel-symbols ${}_n\Gamma_{ij}^k$, and ${}'\Gamma_{ij}^k$ of $\ell_{\gamma_n}(g)$, and g', respectively. Set ${}_n\zeta_{ij}^k = {}_n\Gamma_{ij}^k - {}'\Gamma_{ij}^k$. Since $\lambda_{\gamma_n}(g) \to g'$ in $\mathcal{M}\mathrm{et}^s(E)$, the functions ${}_n\zeta_{ij}^k \to 0$ H^{s-1} and hence C^1. Now we denote by γ_n again the subsequence γ_{n_k}. If we represent γ_n in local coordinates by $f_n^i(x^1, \ldots, x^m)$ and γ by $f^i(x^1, \ldots, x^m)$, then

$$ {}_n\Gamma_{ij}^k = \left(\frac{\partial f_n^j}{\partial x^r}\right)^{-1} \left(\frac{\partial f_n^i}{\partial x^s}\right)^{-1} \Gamma_{rs}^t \left(\frac{\partial f_n^k}{\partial x^t}\right) - \left(\frac{\partial^2 f_n^k}{\partial x^s \partial x^r}\right) \left(\frac{\partial f_n^i}{\partial x^s}\right)^{-1} \left(\frac{\partial f_n^j}{\partial x^r}\right)^{-1}. $$

The first term tends C^0 to $(\frac{\partial f^i}{\partial x^r})^{-1}(\frac{\partial f^j}{\partial x^s})^{-1}\Gamma_{rs}^t(\frac{\partial f^k}{\partial x^t})$, so the second term goes to

$$ \left(\frac{\partial f^j}{\partial x^r}\right)^{-1} \left(\frac{\partial f^i}{\partial x^s}\right)^{-1} \Gamma_{rs}^t \left(\frac{\partial f^i}{\partial x^t}\right) - {}'\Gamma_{ij}^k. $$

Therefore,

$$ \left(\frac{\partial^2 f_n^k}{\partial x^r \partial x^s}\right) \to \left(\left(\frac{\partial f^j}{\partial x^r}\right)^{-1} \left(\frac{\partial f^i}{\partial x^s}\right)^{-1} \Gamma_{rs}^t \left(\frac{\partial f^i}{\partial x^t}\right) - {}'\Gamma_{ij}^k\right) \left(\frac{\partial f_n^i}{\partial x^s}\right)^{-1} \left(\frac{\partial f_n^j}{\partial x^r}\right)^{-1} $$

converges C^0. This means that f_n converges C^2, so in particular $\gamma \in C^2\,\mathcal{D}\mathrm{iff}(E)$. As above we show that indeed $\gamma \in C^2\,\mathcal{G}\mathrm{au}(E)$. Then

$$ {}'\Gamma_{ij}^k = \left(\frac{\partial f^j}{\partial x^r}\right)^{-1} \left(\frac{\partial f^i}{\partial x^s}\right)^{-1} \Gamma_{rs}^t \left(\frac{\partial f^k}{\partial x^t}\right) - \left(\frac{\partial^2 f^k}{\partial x^s \partial x^r}\right) \left(\frac{\partial f^i}{\partial x^s}\right)^{-1} \left(\frac{\partial f^j}{\partial x^r}\right)^{-1}. $$

Now assume $\gamma_n \to \gamma$ in $\mathcal{G}\mathrm{au}^t(E)$ for $t \leq s$. Then since ${}_n\zeta_{ij}^k \to 0$ in H^{s-1}, and $\frac{\partial f_n^i}{\partial x^j} \to \frac{\partial f^i}{\partial x^j}$ in H^{t-1} and in C^0, by an application of the Sobolev lemma 3.22,

$\frac{\partial^2 f_n^i}{\partial x^j \partial x^k} \to \frac{\partial^2 f_n^i}{\partial x^j \partial x^k}$ in H^{t-1}. Therefore, we find $f_n \to f$ in H^{t+1}. Thus, by induction we find $\gamma_n \to \gamma$ in $\mathcal{D}\mathrm{iff}^{s+1}(E)$, by the same argument as above we prove $\gamma_n \to \gamma$ in $\mathcal{G}\mathrm{au}^{s+1}(E)$, and $g' = \ell_\gamma(g) \in \tilde{\ell}^g(\mathcal{G}\mathrm{au}^{s+1}(E)/\mathcal{G}\mathrm{au}(E)_g)$, thus the orbits are closed. The last fact to show is that $\mathcal{G}\mathrm{au}(E)_g\gamma_n \to \mathcal{G}\mathrm{au}(E)_g\gamma$ in $\mathcal{G}\mathrm{au}^{s+1}(E)/\mathcal{G}\mathrm{au}(E)_g$. If we assume the contrary, this implies the existence of a neighborhood U of $\mathcal{G}\mathrm{au}(E)_g\gamma$ and of a subsequence $\{\gamma_{n_k}\}$ such that $\gamma_{n_k} \notin U\ \forall k$. But by the above we can find a subsequence $\{\zeta_m\}$ of $\{\gamma_{n_k}\}$ such that $\zeta_m \to \gamma'$ in $\mathcal{G}\mathrm{au}^{s+1}(E)$ and $\ell_{\gamma'}(g) = g'$. Therefore, $\mathcal{G}\mathrm{au}(E)_g\gamma' = \mathcal{G}\mathrm{au}(E)_g\gamma$ and for large m, $\mathcal{G}\mathrm{au}(E)_g\zeta_m \in U$, which contradicts the assumption. $\square$

5.20. Slice theorem. *Let $s > \frac{n}{2} + 2$ and $\ell : \mathcal{G}\mathrm{au}^{s+1}(E) \times \mathcal{M}\mathrm{et}^s(E) \to \mathcal{M}\mathrm{et}^s(E)$ be the usual action. Then there exists for every $g \in \mathcal{M}\mathrm{et}^s(E)$ a submanifold S of $\mathcal{M}\mathrm{et}^s(E)$ containing g, which is diffeomorphic to a ball in Hilbert space, such that:*

(1) *If $\gamma \in \mathcal{G}\mathrm{au}(E)_g$, $\ell(\gamma, S) = S$.*
(2) *If $\gamma \in \mathcal{G}\mathrm{au}^{s+1}(E)$, such that $\ell(\gamma, S) \cap S \neq \emptyset$, then $\gamma \in \mathcal{G}\mathrm{au}(E)_g$.*
(3) *There exists a local cross section $s : \mathcal{G}\mathrm{au}^{s+1}(E)/\mathcal{G}\mathrm{au}(E)_g \to \mathcal{G}\mathrm{au}^{s+1}(E)$ defined on a neighborhood U of the identity coset such that if $F : U \times S \to \mathcal{M}\mathrm{et}^s(E)$ is defined by $F(u,t) := \ell(s(u), t)$, then F is a homeomorphism onto a neighborhood of g.*

Proof. Up to now we have shown that $\tilde{\ell}^g$ is a diffeomorphism of $\mathcal{G}\mathrm{au}^{s+1}(E)/\mathcal{G}\mathrm{au}(E)_g$ onto $O_g^s := \mathcal{G}\mathrm{au}^{s+1}(E) \cdot g$, the orbit through g. First we will construct the normal bundle $\nu(O_g^s)$ of O_g^s in $\mathcal{M}\mathrm{et}^s(E)$.

Obviously, O_g^s is a smooth submanifold of $\mathcal{M}\mathrm{et}^s(E)$, which is by 5.4 equipped with a weak smooth Riemannian structure G_s. We consider $T\mathcal{M}\mathrm{et}^s(E)|_{O_g^s}$, the subset of all tangent vectors in $T\mathcal{M}\mathrm{et}^s(E)$ whose base points are in O_g^s. It is a vector bundle over O_g^s and TO_g^s is a subbundle of it. We define

$$\nu(O_g^s) := \{X \in T\mathcal{M}\mathrm{et}^s(E)|_{O_g^s} | \forall Y \in TO_g^s : G_s(X,Y) = 0\}.$$

Because G_s is only a weak Riemannian metric, it is not automatic that the normal bundle $\nu(O_g^s)$ is a smooth subbundle of $T\mathcal{M}\mathrm{et}^s(E)|_{O_g^s}$. To show this, we will construct a smooth surjective vector bundle map $\Phi : T\mathcal{M}\mathrm{et}^s(E)|_{O_g^s} \to TO_g^s$ whose kernel is $\nu(O_g^s)$.

From 3.30 and 5.17 we know that the fiber of $\nu(O_g^s)$ is $\ker L_e^*$, where $L_e^* : H^s(S^2T^*E) \to H^{s-1}(TE)$. Because the weak Riemannian structure of $\mathcal{M}\mathrm{et}^s(E)$ is preserved by the action of $\mathcal{G}\mathrm{au}^{s+1}(E)$, the fiber of $\nu(O_g^s)$ at any point $\ell_\gamma(g)$ is $\gamma^*(\ker L_e)$.

Set $\Phi := L_e \circ (L_e^* L_e)^{-1} \circ L_e^*$ where we consider $L_e^* : H^s(S^2T^*E) \to H^{s-1}(VE)$, $L_e : H^{s+1}(VE) \to H^s(S^2T^*E)$, and regard $L_e^* \circ L_e$ as a map

$$L_e^* \circ L_e : L_e^* L_e(H^{s+3}(VE)) \subset H^{s+1}(VE) \to L_e^* L_e(H^{s+1}(VE)) \subset H^{s-1}(VE).$$

Since $L_e^* L_e$ is elliptic and selfadjoint, it is an isomorphism on these sets, and since $L_e^*(H^s(S^2T^*E)) = L_e^* L_e(H^{s+1}(VE))$, the composition $L_e \circ (L_e^* L_e)^{-1} \circ L_e^*$ makes sense.

On the fiber at $\ell_\gamma(g)$ define Φ to be $\gamma^* \circ L_e \circ (L_e^* L_e)^{-1} \circ L_e^* \circ \gamma^{*-1}$. Then by 3.30 and the fact that $T_{\ell_\gamma(g)}(O_g^s) = T\ell_\gamma T_g(O_g^s)$, it is obvious that $\nu(O_g^s) = \ker\Phi$ and that Φ is onto. It remains to show that Φ is a smooth bundle map.

Let Φ_γ be Φ restricted to the fiber at $\ell_\gamma(g)$. Assume $s > \frac{n}{2} + 2$.

Next take local trivializations of $T\,\mathcal{M}\mathrm{et}^s(E)|_{O_g^s}$ and TO_g^s over some neighborhood U. There $\gamma \to \Phi_\gamma$ can be viewed as a map $U \to L(H^s(S^2T^*E), L_e(H^{s+1}(VE)))$. Taking U small enough such that there exists by proposition 5.13 a local section $s : U \to \mathcal{G}\mathrm{au}^{s+1}(E)$. In the sequel, we will not distinguish between U and $s(U)$. It suffices to show that $\gamma \to \Phi_\gamma$ is smooth in order to prove that Φ is a smooth bundle map.

Set, therefore, $L_\gamma := \gamma^* \circ L_e \circ (Tr_\gamma)^{-1} = T_\gamma \ell^g$, then $L_\gamma^* = Tr_\gamma \circ L_e^* \circ (\gamma^*)^{-1}$, thus $\Phi_\gamma = L_\gamma \circ (L_\gamma^* \circ L_\gamma)^{-1} \circ L_\gamma^*$.

Since, obviously, $\gamma \to L_\gamma$ is smooth, we only need to show that $\gamma \to L_\gamma^*$ is smooth also. Because then, $L_\gamma^* L_\gamma : T_\gamma(\mathcal{G}\mathrm{au}^{s+1}(E)/\mathcal{G}\mathrm{au}(E)_g) \to T_\gamma(\mathcal{G}\mathrm{au}^{s-1}(E)/\mathcal{G}\mathrm{au}(E)_g)$ is an isomorphism, and $\gamma \to L_\gamma^* L_\gamma$ is smooth. Since we are in the setting of Banach spaces, we conclude that $\gamma \to (L_\gamma^* L_\gamma)^{-1}$ is smooth also, hence $\gamma \to \Phi_\gamma$ is smooth.

Let us concentrate on $\gamma \to L_\gamma^*$. $L_\gamma^* \in L(H^s(S^2T^*E), L_e^*(H^s(V^*E)))$, and we see that $\gamma \to L_\gamma^*$ is smooth if $\gamma \to L_\gamma^*(h)$ is smooth for all $h \in H^s(S^2T^*E)$, and its derivatives are bounded by constants depending only on $\|h\|$. We will prove this fact by computing in local coordinates.

We find from [Palais 1965, chapter 4, §4] that

$$L_e^*(h)^l = \frac{\partial h_{jk} A^{ijkl}}{\partial x^i} + h_{jk} B^{jkl}$$

where A^{ijkl} and B^{jkl} are rational functions of the $\{g_{ij}\}$ and their first derivatives. Let $\{\tau_j^i\}$ be the matrix of first derivatives of γ^{-1}. Then $(\gamma^*)^{-1}(h)_{ij} = \tau_i^k (h_{kl} \circ \gamma^{-1})\tau_j^l$. Also if X^i is a vertical vector field in local coordinates $Tr_\gamma(X^i) = X^i \circ \gamma$, since r_γ looks locally like a linear map (Proposition 5.7). Thus, we compute

$$\begin{aligned} L_\gamma^*(h)^i = \Bigg(&(\tau_j^m \circ \gamma)\frac{\partial h_{mn}}{\partial x^l}(\tau_k^n \circ \gamma) + \frac{\partial \tau_j^m \gamma}{\partial x^l} h(\tau_k^n \circ \gamma) + \\ &+ (\tau_j^m \circ \gamma) h_{mn} \frac{\partial \tau_k^n \circ \gamma}{\partial x^l}\Bigg)(A^{ijkl} \circ \gamma) + (\tau_k^n \circ \gamma) h_{mn} (\tau_l^m \circ \gamma)(B^{ikl} \circ \gamma). \end{aligned}$$

$\tau \circ \gamma$ is the inverse of the matrix of first derivatives of γ, so $L_\gamma^*(h)$ consists of a rational function in the first and second derivatives of γ times the smooth functions A^{ijkl} and B^{ijk} composed with γ. Therefore, using Lemma 3.24 and Theorem 3.27, we see that $\gamma \to L_\gamma^*(h)$ is smooth as a map from $\mathcal{G}\mathrm{au}^{s+1}(E) \to H^{s-1}(VE)$. Each derivative of the map is bounded by a constant involving the functions $\{g_{ij}\}$ and their derivatives, and involving the H^s–norm of h, since h and its first derivatives appear in a linear fashion. Thus $\gamma \to L_\gamma^*$ is smooth.

By the arguments given before, we see that Φ is a smooth bundle projection $T\,\mathcal{M}\mathrm{et}^s(E)|_{O_g^s} \to TO_g^s$ with kernel $\nu(O_g^s)$. Therefore, $\nu(O_g^s)$ is a smooth bundle over O_g^s.

This normal bundle now enables us to construct the requested slice. To accomplish this, we consider the exponential map $\mathrm{Exp}_s : T\,\mathcal{M}\mathrm{et}^s(E) \to \mathcal{M}\mathrm{et}^s(E)$

of the weak Riemannian structure on $\mathcal{M}\mathrm{et}^s(E)$. Then $\mathrm{Exp}_s|_{\nu(O_g^s)}$ is a diffeomorphism from a neighborhood of the zero section of $\nu(O_g^s)$ to a neighborhood of O_g^s in $\mathcal{M}\mathrm{et}^s(E)$. (Here, we essentially need that we are in the Hilbert space setting). There exists a small neighborhood U of g in O_g^s with a local section $s : U \to \mathcal{G}\mathrm{au}^{s+1}(E)$, and a small neighborhood V of zero in $\nu(O_g^s)_g$ such that if $W = \{\gamma^*(v) | v \in V, \gamma \in s(U)\} \subset \nu(O_g^s)$, $\mathrm{Exp}_s|_W$ is a diffeomorphism onto a neighborhood of g in $\mathcal{M}\mathrm{et}^s(E)$.

Consider the strong inner product $\mathcal{G}_{sg}(\quad,\quad)$ on $H^s(S^2T^*E)$ (see Proposition 5.9). As we have seen, $\mathcal{G}\mathrm{au}(E)_g$ acts as a group of orthogonal transformations with respect to $\mathcal{G}_{sg}(\quad,\quad)$. A suitable V for above is then e.g. the set

$$V = \{X \in T\mathcal{M}\mathrm{et}^s(E) | X \in \nu(O_g^s)_g, \mathcal{G}_{sg}(X,X) < \varepsilon^2\}.$$

We choose U and V small enough such that $\mathrm{Exp}_s(W) \cap O_g^s = U$.

By $\mathcal{G}_s(\quad,\quad)$ we denote again the strong Riemannian structure on $\mathcal{M}\mathrm{et}^s(E)$ from 5.9 and by B_g^r the ball of radius r about g with respect to $\mathcal{G}_s$. Then for some $\delta > 0$, $\mathrm{Exp}_s(W) \supset B_g^{2\delta}$. Take $U_1 \subset U$, and pick $\varepsilon_1 < \varepsilon$ giving $V_1 \subset V$ and $W_1 = \{\gamma^*(v) | v \in V_1, \gamma \in s(U_1)\}$ such that $\mathrm{Exp}_s(W_1) \subset B_g^\delta$. Define the slice as $S := \mathrm{Exp}_s(V_1)$. We next prove the three required properties.

(1): Let $\gamma \in \mathcal{G}\mathrm{au}(E)_g$ and $x = \exp(v) \in S$, $v \in V_1$. Then

$$\ell_\gamma(x) = \ell_\gamma(\mathrm{Exp}_s(v)) = \mathrm{Exp}_s(\gamma^*(v))$$

$\gamma^* : \nu(O_g^s)_g \to \nu(O_g^s)_g$, since it preserves $\mathcal{G}_s$ and the weak Riemannian structure G_s. Therefore, $\ell_\gamma(x) = \mathrm{Exp}_s(\gamma^*(v)) \in S$.

(2): Assume that there exist $\gamma \in \mathcal{G}\mathrm{au}^{s+1}(E)$ and $x, y \in S$ such that $\ell_\gamma(x) = y$. Then $\mathcal{G}_s(g,y) < \delta$, and $\delta > \mathcal{G}_s(g,x) = \mathcal{G}_s(\ell_\gamma(g), \ell_\gamma(x))$. Thus $\mathcal{G}_s(g, \ell_\gamma(g)) < 2\delta$ and $\ell_\gamma(g) \in \mathrm{Exp}_s(W)$ and $\ell_\gamma(g) \in U$. If $x = \mathrm{Exp}_s(a)$ and $y = \mathrm{Exp}_s(b)$, then $\mathrm{Exp}_s(b) = \mathrm{Exp}_s(\gamma^*(a))$ and $b, \gamma^*(a) \in W$. Since $\mathrm{Exp}_s|_W$ is injective, $\gamma^*(a) = b$, so since $a, b \in V$, $\ell_\gamma(g) = g$ or $\gamma \in \mathcal{G}\mathrm{au}(E)_g$.

(3): Let s and U be as above. Let $W_2 = \{\gamma^*(v) | v \in V_1, \gamma \in s(U)\}$. Then, since $\mathrm{Exp}_s|_{W_2}$ is a diffeomorphism onto a neighborhood of g, $F : U \times S \to \mathcal{M}\mathrm{et}^s(E)$ defined by $F(u,x) = \ell(s(u),x)$ is a continuous bijection onto $\mathrm{Exp}_s(W_2)$. For $z \in \mathrm{Exp}_s(W_2)$

$$F^{-1}(z) = (\pi\,\mathrm{Exp}_s^{-1}(z), \ell((s \circ \pi \circ \mathrm{Exp}_s^{-1}(z))^{-1}, z))$$

where $\pi : \nu(O_g^s) \to O_g^s$ is the bundle projection map. Therefore, F^{-1} is continuous. $\square$

5.21. Corollary. *Let $s > \frac{n}{2} + 2$ and $\ell : \mathcal{G}\mathrm{au}^{s+1}(E) \times \mathcal{M}\mathrm{et}^s(E) \to \mathcal{M}\mathrm{et}^s(E)$ be the usual action. Then there exists a neighborhood N of O_g^s and a smooth equivariant deformation retract of N onto O_s^g.*

Proof. Following the notation of Theorem 5.20, let $W_3 = \{\gamma^*(v) | v \in V_1, \gamma \in \mathcal{G}\mathrm{au}^{s+1}(E)\}$, and set $N = \mathrm{Exp}_s(W_3)$.

$\mathrm{Exp}_s|_{W_2}$ is a diffeomorphism, and since $\mathrm{Exp}_s(\gamma^*(v)) = \ell_\gamma(\mathrm{Exp}_s(v))$ for any $\gamma^*(v)$, $\mathrm{Exp}_s|_{W_3}$ is a local diffeomorphism. Furthermore,

$$\mathrm{Exp}_s(W_3) = \bigcup_{\gamma \in \mathcal{G}\mathrm{au}^{s+1}(E)} \ell_\gamma(Exp_s(W_2)),$$

so $\operatorname{Exp}_s(W_3)$ is a neighborhood of O_g^s. Next we show that $\operatorname{Exp}_s|_{W_3}$ is injective. Assume $\operatorname{Exp}_s(\gamma_1^*(v_1)) = \operatorname{Exp}_s(\gamma_2^*(v_2))$. Then $\ell_{\gamma_2}^{-1}\ell_{\gamma_1}(\operatorname{Exp}_s(v_1)) = \ell_{\gamma_1\gamma_2^{-1}}\operatorname{Exp}_s(v_2) = \operatorname{Exp}_s(v_2)$. But since $\operatorname{Exp}_s(v_1), \operatorname{Exp}_s(v_2) \in S$ by Theorem 5.20(2) $\gamma_1\gamma_2^{-1} \in \mathcal{G}\mathrm{au}(E)_g$. Thus, $(\gamma_1\gamma_2^{-1})^*(v_1) \in V_1$, so since Exp_s is injective on V_1, $\gamma_1^*(v_1) = \gamma_2^*(v_2)$.

Thus $\operatorname{Exp}_s|_{W_3} : W_3 \to N$ is a diffeomorphism. But then $r : N \times [0,1] \to N$, defined by $r(x,t) = \operatorname{Exp}_s(t\operatorname{Exp}_s^{-1}(x))$ is the required deformation retract. It is equivariant and smooth, since Exp_s is. □

5.22. Smooth Slice theorem. *Let $\ell : \mathcal{G}\mathrm{au}(E) \times \mathcal{M}\mathrm{et}(E) \to \mathcal{M}\mathrm{et}(E)$ be the usual action. Then for every $g \in \mathcal{M}\mathrm{et}(E)$, there exists a contractible subset S of $\mathcal{M}\mathrm{et}(E)$ containing g, such that*

(1) *If $\gamma \in \mathcal{G}\mathrm{au}(E)_g$, $\ell(\gamma, S) = S$.*
(2) *If $\gamma \in \mathcal{G}\mathrm{au}(E)$, such that $\ell(\gamma, S) \cap S \neq \emptyset$, then $\gamma \in \mathcal{G}\mathrm{au}(E)_g$.*
(3) *There exists a local cross section $s : \mathcal{G}\mathrm{au}(E)/\mathcal{G}\mathrm{au}(E)_g \to \mathcal{G}\mathrm{au}(E)$ defined on a neighborhood U of the identity coset such that if $F : U \times S \to \mathcal{M}\mathrm{et}(E)$ is defined by $F(u,t) := \ell(s(u),t)$, then F is a homeomorphism onto a neighborhood of g.*

Proof. If S^s is the slice from Theorem 5.20 define $S := S^s \cap \mathcal{M}\mathrm{et}(E)$.
(1): obvious.
(2): obvious.
(3): Let U^s be the U of Theorem 5.20 and set $U := U^s \cap \mathcal{G}\mathrm{au}(E)/\mathcal{G}\mathrm{au}(E)_g$. Let $u \in U$, so $u = \mathcal{G}\mathrm{au}(E)_g\gamma$, $\gamma \in \mathcal{G}\mathrm{au}(E)$. Let s be the local cross section from 5.20. Then $s(u) = \eta\gamma$, with $\eta \in \mathcal{G}\mathrm{au}(E)_g \subset \mathcal{G}\mathrm{au}(E)$, so $s(u) \in \mathcal{G}\mathrm{au}(E)$. Thus $s(u) \subset \mathcal{G}\mathrm{au}(E)$.

Next assume $x_n \to x$ in U. By the definition of the topology of $\mathcal{G}\mathrm{au}(E)/\mathcal{G}\mathrm{au}(E)_g$, there exist $\{\gamma_n\}, \gamma \in \mathcal{G}\mathrm{au}(E)$ such that $\pi(\gamma_n) = x_n$, $\pi(\gamma) = x$, and $\gamma_n \to \gamma$ in $\mathcal{G}\mathrm{au}(E)$, and there exists $\{\eta_n\}, \eta \in \mathcal{G}\mathrm{au}(E)_g$ with $\eta_n s(x_n) = \gamma_n$ and $\eta s(x) = \gamma$. Because $x_n \to x$ in U^s, $s(x_n) \to s(x)$ in $\mathcal{G}\mathrm{au}^{s+1}(E)$, and since $\mathcal{G}\mathrm{au}^{s+1}(E)$ is a topological group, this means that $\eta_n \to \eta$ in $\mathcal{G}\mathrm{au}^{s+1}(E)$. Therefore, $\eta_n \to \eta$ in $C^1\,\mathcal{G}\mathrm{au}(E)$, and by Theorem 5.10 $\eta_n \to \eta$ in $\mathcal{G}\mathrm{au}(E)$. But since $s(x_n) = \eta_n^{-1}\gamma_n$ and $s(x) = \eta^{-1}\gamma$, we get $s(x_n) \to s(x)$ in $\mathcal{G}\mathrm{au}(E)$, so s is continuous, and s is a local cross section $s : U \to \mathcal{G}\mathrm{au}(E)$.

Define again $F : U \times S \to \mathcal{M}\mathrm{et}(E)$ as $F(u,x) = \ell(s(u),x)$. It remains to show that F is a homeomorphism onto a neighborhood of g in $\mathcal{M}\mathrm{et}(E)$, which reduces to showing that $F(U \times S) = \operatorname{Exp}(W_2) \cap \mathcal{M}\mathrm{et}(E)$ (c.f. Theorem 5.20) which clearly is a neighborhood of g in $\mathcal{M}\mathrm{et}(E)$, and that F^{-1} is continuous. For we know the formula for F^{-1} from Theorem 5.20, it finally suffices to show that $\operatorname{Exp}|_{T\,\mathcal{M}\mathrm{et}(E)\cap W_2} : T\,\mathcal{M}\mathrm{et}(E) \cap W_2 \to W_2 \cap \mathcal{M}\mathrm{et}(E)$ is a homeomorphism in the C^∞ topologies:
Take $\tilde{g} = \operatorname{Exp}(w) \in \mathcal{M}\mathrm{et}(E)$, $w \in W_2$. Then $\operatorname{Exp}^{-1}(\tilde{g}) \in T_h(M)$ for $h \in O_g$, and therefore $\tilde{g} = \ell_{s(h)}(\ell_{\operatorname{Exp}(s(h)^{-1}}w) = F(h, \ell_{s(h)^{-1}}(\tilde{g}))$. So, it remains to show that Exp is a homeomorphism.

This follows from a very restricted implicit function theorem for Fréchet spaces (Theorem 5.23). □

5.23. Theorem. *Let V and W be any vector bundles over E associated to the tangent bundle or the vertical bundle (e.g. TE, VE, T^*E, S^2T^*E, …). Then for* *any $\gamma \in \mathcal{G}\mathrm{au}(E)$ there is a natural linear map $\gamma^* : H^0(V) \to H^0(W)$. Let $A \subset H^s(V)$ and $B \subset H^s(W)$ be submanifolds such that for all $\gamma \in \mathcal{G}\mathrm{au}(E)$, $\gamma^*(A) \subset A$, $\gamma^*(B) \subset B$. Assume $f : A \to B$ is a diffeomorphism such that $f \circ \gamma^* = \gamma^* \circ f$ for all $\gamma \in \mathcal{G}\mathrm{au}(E)$. Then $f(A \cap C^\infty(V)) \subset B \cap C^\infty(W)$, and $f|_{A\cap C^\infty(V)}$ is a homeomorphism onto $B \cap C^\infty(W)$.*

Proof. Since elements of $\mathcal{G}\mathrm{au}(E)$ act naturally on $H^s(V)$ and $H^s(W)$, one can define Lie derivatives for such sections. If $a \in A$, $a \in C^\infty(V)$ iff all its iterated Lie derivatives exist. But $f \circ \gamma^* = \gamma^* \circ f$ implies $\mathcal{L}_X(f(a)) = Tf\,\mathcal{L}(a)$ (or $\mathcal{L}_X(f(a)) = Vf\,\mathcal{L}(a)$ depending which bundles V and W are chosen). This can be extended to higher Lie derivatives. Therefore, if all Lie derivatives of a exist, all Lie derivatives of $f(a)$ exist. Furthermore, $a_n \to a$ is C^∞ if all Lie derivatives of a_n converge to those of a. But then $f(a_n) \to f(a)$ in all Lie derivatives, hence $f(a_n) \to f(a)$ C^∞. Hence f is continuous in the C^∞ topologies. Since the hypothesis of the Theorem is symmetric in f and f^{-1}, we see that f is a homeomorphism. □

Now we have constructed slices for the Hilbert manifold situation and using these for the smooth case. Next we will figure out, how this slice theorem leads to a kind of stratification in the infinite dimensional case, at least to the largest open and dense stratum.

5.24. Theorem. *Let $\mathcal{G}$ be the set of all $g \in \mathcal{M}\mathrm{et}(E)$ with trivial isotropy group.*

(1) *$\mathcal{G}$ is open in $\mathcal{M}\mathrm{et}(E)$*
(2) *$\mathcal{G}$ is dense in $\mathcal{M}\mathrm{et}(E)$ if the base manifold M and the standard fiber S are at least of dimension 1.*

Proof. (1): Take $g \in \mathcal{M}\mathrm{et}(E)$ and V any neighborhood of Id in $\mathcal{G}\mathrm{au}(E)$. Next consider the U, F from Theorem 5.22, set $\tilde U := U \cap \pi(V)$, and let $N = F(\tilde U, S)$. N is a neighborhood of g. For arbitrary $h \in N$, $h = F(\tilde u, x)$, so $k = \ell_{s(\tilde u)^{-1}}(h) \in S$. By Theorem 5.22(2) $\mathcal{G}\mathrm{au}(E)_g = \mathcal{G}\mathrm{au}(E)_k$. Thus if $\gamma = s(\tilde u)$, we get $\gamma^{-1}\,\mathcal{G}\mathrm{au}(E)_k\gamma \subset \mathcal{G}\mathrm{au}(E)_g$. But $\pi(\gamma) = \tilde u \in U$, and therefore $\gamma \in V$. So we have proven that for every nerighborhood V of Id in $\mathcal{G}\mathrm{au}(E)$, there exists a neighborhood N of g such that for every $h \in N$ exists $\gamma \in V$ such that $\mathcal{G}\mathrm{au}(E)_g = \gamma^{-1}\,\mathcal{G}\mathrm{au}(E)_h\gamma$. This implies (1).
(2): Take $g \in \mathcal{M}\mathrm{et}(E)$. We have to show, that in any neighborhood of g we can find h such that $\mathcal{G}\mathrm{au}(E)_h = \{\mathrm{Id}\}$.

Let $S(E)$ be the unit sphere bundle of TE. For $h \in \mathcal{M}\mathrm{et}(E)$ set $f_h : S(E) \to \mathbb{R}$ as

$$f_h(X) = \frac{\mathrm{Ric}(X,X)}{h(X,X)}.$$

where $\mathrm{Ric}(X,X)$ denotes the Ricci tensor of h. In normal coordinates x_n (with respect to g) at $e \in E$.

$$f_g(\partial_n) = \frac{1}{2}\sum_i \left(\frac{\partial^2 g_{ii}}{(\partial x_n)^2} - 2\frac{\partial^2 g_{in}}{\partial x_i \partial x_n} + \frac{\partial^2 g_{nn}}{\partial x_i^2} \right).$$

Next take $\varepsilon \in C^\infty(S^2T^*E)$ such that $\varepsilon_e = 0$ and $\frac{\partial \varepsilon_{ij}}{\partial x_n}|_p = 0$, with ε very near to zero in $C^\infty(S^2T^*E)$ being nonzero only on a given neighborhood U of e, and such that $f_{g+\varepsilon}(\partial_n) > f_g(\partial_n)$. We, furthermore, calculate

$$f_{g+\varepsilon}(\partial_n) = f_g(\partial_n) + \frac{1}{2}\sum_i \left(\frac{\partial^2 \varepsilon_{ii}}{(\partial x_n)^2} - 2\frac{\partial^2 \varepsilon_{in}}{\partial x_i \partial x_n} + \frac{\partial^2 \varepsilon_{nn}}{\partial x_i^2} \right).$$

So we can construct for every metric g, every point $e \in E$, and every neighborhood U of p a metric $\tilde{g}$ which is arbitrarily near g, $g = \tilde{g}$ outside U and $f_{\tilde{g}}(X) = f_g(X)$ for all $X \in S(E)$.

We also see that for any $h \in \mathcal{M}\mathrm{et}(E)$ and $\gamma \in \mathcal{G}\mathrm{au}(E)_h$, $f_h(T\gamma X) = f_h(X)$ for every $X \in S(E)$. Now choose p, $X \in S_p(E)$ such that $f_g(X)$ is maximal. As U take a neighborhood of p which is contained in a normal coordinate neighborhood, and take ε as above. Set $g_1 = g + \varepsilon$, and let $e_1 \in E$ and $X_1 \in S(E)$ be chosen that $f_{g_1}(X_1)$ is again maximal. $e_1 \in U$, since $g = g_1$ outside U. Next select a neighborhood $U_1 \subset U$ of e_1 with radius $r_g(U_1) < \frac{1}{2}r_g(U)$. For the setting e_1, U_1 find a ε_1 as above and iterate the construction to inductively define $h = g+\varepsilon+\sum_i \varepsilon_i$, $\hat{e} = \lim e_i$. We have then got a metric h such that for a $Z \in S_{\hat{e}}(E)$, $f_h(Z) > f_h(Y)$ for all $Y \in S(E) - S_{\hat{e}}(E)$, and $g = h$ outside U. Since for $\gamma \in \mathcal{G}\mathrm{au}(E)_h$ $f_h(X) = f_h(T\gamma X)$ for all $X \in S(E)$, the maximality at $\hat{e}$ implies $\gamma(\hat{e}) = \hat{e}$.

Now set $K^\delta_{\hat{e}} = \{x \in E | d_h(x, \hat{e}) = \delta\}$, where d_h is the metric induced by h. Choose δ in a way that $K^\delta_{\hat{e}} \cap U = \emptyset$ and $K^\delta_{\hat{e}}$ is included in a normal coordinate neighborhood of $\hat{e}$. Using the above procedure, change h to h_1 around $K^\delta_{\hat{e}}$, leaving $h_1 = h$ on U, so that for $\hat{e}_1 \in K^\delta_{\hat{e}}$ and $X_1 \in S_{\hat{e}_1}(E)$ $f_{h_1}(X) > f_{h_1}(X_1) > f_{h_1}(Y)$ for all $Y \in S(K^\delta_{\hat{e}}) - S_{\hat{e}_1}(E)$.

By the same argument as before $\gamma \in \mathcal{G}\mathrm{au}(E)_{h_1}$ implies $\gamma(\hat{e}) = \hat{e}$, $\gamma(\hat{e}_1) = \hat{e}_1$. Take $K^{\delta\delta_1}_{\hat{e}\hat{e}_1} = \{x \in E | d_{h_1}(x, \hat{e}) = \delta, d_{h_1}(x, \hat{e}_1) = \delta_1\}$ and iterate the construction above, giving points $\hat{e}, \hat{e}_1, \ldots, \hat{e}_n$, and metrics $h_1, \ldots, h_n$ with the properties that $\gamma \in \mathcal{G}\mathrm{au}(E)_{h_n}$ implies $\gamma\hat{e} = \hat{e}$ and $\gamma\hat{e}_i = \hat{e}_i$. All $\hat{e}_i$ are in $K^\delta_{\hat{e}}$, so there exist $Y_i \in T_{\hat{e}}(E)$ such that $\hat{e}_i = \exp(Y_i)$ where exp is the exponential map of h_n. The $\{\hat{e}_i\}$ can be chosen in a way that the Y_i form a basis of $T_{\hat{e}}E$. For $\gamma \in \mathcal{G}\mathrm{au}(E)_{h_n}$ we get $\exp(Y_i) = \hat{e}_i = \gamma(\hat{e}_i) = \gamma(\exp(Y_i)) = \exp(T\gamma Y_i)$, hence $Y_i = T\gamma Y_i$, so $T_{\hat{e}}\gamma = \mathrm{Id}$. Since $\mathcal{G}\mathrm{au}(E) \subset \mathcal{D}\mathrm{iff}(E)$, γ is an isometry for h_n, and since $\gamma(\hat{e}) = \hat{e}$ and $T_{\hat{e}}\gamma = \mathrm{Id}$, we get $\gamma = \mathrm{Id}$. Hence, $\mathcal{G}\mathrm{au}(E)_{h_n} = \{\mathrm{Id}\}$ and h_n is arbitrarily close to g. Therefore, $\mathcal{G}$ is dense in $\mathcal{M}\mathrm{et}(E)$. □

5.25.Remark: A stratification for $\mathcal{M}\mathrm{et}(E)/\mathcal{G}\mathrm{au}(E)$. By iterating the constructions above, one can then show that $\mathcal{M}\mathrm{et}(E)/\mathcal{G}\mathrm{au}(E) = M_1 \cup M_2 \cup \ldots$, where $M_1 = \mathcal{G}$ and $M_i \subset \partial M_{i-1}$. All the manifolds are modeled on Fréchet spaces, giving $\mathcal{M}\mathrm{et}(E)/\mathcal{G}\mathrm{au}(E)$ the structure of a *stratified manifold*.

5.26. The slice theorem for $(\mathcal{C}\mathrm{onn}(E) \times \mathcal{M}\mathrm{et}(VE))/\mathcal{G}\mathrm{au}(E)$. Next make use of the following result from [Gil–Medrano et al. 1994]:

Definition. Let M be a smooth (connected) finite dimensional manifold, and let V be a distribution on it. We denote by (M, V, π_V) the vector subbundle determined by V, and by $(M, N = TM/V, \pi_N)$ the normal bundle. $i : V \hookrightarrow TM$ shall denote the embedding, and $p : TM \twoheadrightarrow N$ the epimorphism onto the normal bundle. An *almost product structure* on M is a $P \in C^\infty(L(TM, TM))$ such that $P^2 = \mathrm{Id}$. Every almost product structure on M induces a decomposition of TM of the form

$TM = \ker(P-\mathrm{Id})\oplus\ker(P+\mathrm{Id})$. These subbundles are called vertical and horizontal, and will be denoted V^P and H^P respectively. We are given in a natural way the vertical and horizontal projections $v^P = \frac{1}{2}(P+\mathrm{Id})$ and $h^P = \frac{1}{2}(P-\mathrm{Id})$. The almost product structure P also determines a monomorphism $C_P : N \to TM$, called the horizontal lifting, given by $C_P \circ p = h^P$. It is an isomorphism onto H^P inverse to $p|_{H^P}$.

For a given distribution V in M we will denote by $\mathcal{P}_V(M)$ the space of all almost product structures with $V^P = V$. So choosing an element of $\mathcal{P}_V(M)$ is equivalent to choosing a subbundle of TM complementary to V. This subbundle is given by $\ker(P + \mathrm{Id})$.

Theorem. *Let M be a smooth finite dimensional connected manifold, then there is a real analytic diffeomorphism*

$$\mathcal{M}\mathrm{et}(M) \simeq \mathcal{M}\mathrm{et}(N) \times \mathcal{M}\mathrm{et}(V) \times \mathcal{P}_V(M).$$

where $\mathcal{M}\mathrm{et}(M)$ as usual denotes the Riemannian metrics on M, and $\mathcal{M}\mathrm{et}(N)$ and $\mathcal{M}\mathrm{et}(V)$ denote the bundle metrics on N and V respectively.

Proof. [Gil–Medrano et al. 1994, Proposition 4.4] □

5.27. Application to E and VE. Consider, as before, the vertical bundle of the fiber bundle E. It is, of course, also given by a vertical distribution on E. An almost product structure on E is just a projection onto VE, hence a connection on E. Thus $\mathcal{P}_{VE}(E) = \mathcal{C}\mathrm{onn}(E)$, so the Theorem in 5.26 is applicable to our situation. We, therefore, have

$$\mathcal{M}\mathrm{et}(E) \simeq \mathcal{M}\mathrm{et}(NE) \times \mathcal{M}\mathrm{et}(VE) \times \mathcal{C}\mathrm{onn}(E).$$

Since $\mathcal{G}\mathrm{au}(E)$ acts naturally on $\mathcal{M}\mathrm{et}(E)$, it also acts on the spaces on the right hand side of the above equation. The action on $\mathcal{C}\mathrm{onn}(E)$ is just the usual action.

The action on $g_V \in \mathcal{M}\mathrm{et}(VE)$ is given by

$$(\gamma^* g_V)_p(v^\Phi X, v^\Phi Y) = g_{V\,\gamma(p)}(v^{\gamma^*\Phi} X, v^{\gamma^*\Phi} Y),$$

and the action on $\mathcal{M}\mathrm{et}(NE)$ is trivial.

Using this result, we may use the slice theorem for $\mathcal{M}\mathrm{et}(E)/\mathcal{G}\mathrm{au}(E)$ to gain the following theorem for $(\mathcal{C}\mathrm{onn}(E) \times \mathcal{M}\mathrm{et}(VE))/\mathcal{G}\mathrm{au}(E)$.

5.28. Theorem. *Let $\hat{\ell} : \mathcal{G}\mathrm{au}(E) \times \mathcal{C}\mathrm{onn}(E) \times \mathcal{M}\mathrm{et}(VE) \to \mathcal{C}\mathrm{onn}(E) \times \mathcal{M}\mathrm{et}(VE)$ be the restriction of the action $\ell : \mathcal{G}\mathrm{au}(E) \times \mathcal{M}\mathrm{et}(E) \to \mathcal{M}\mathrm{et}(E)$ to $\mathcal{C}\mathrm{onn}(E) \times \mathcal{M}\mathrm{et}(VE)$. As seen before, $\hat{\ell}$ is an action on $\mathcal{C}\mathrm{onn}(E) \times \mathcal{M}\mathrm{et}(VE)$, and for every $\tilde{g} \in \mathcal{C}\mathrm{onn}(E)\times\mathcal{M}\mathrm{et}(VE)$ there exists a contractible subset S of $\mathcal{C}\mathrm{onn}(E)\times\mathcal{M}\mathrm{et}(VE)$ containing $\tilde{g}$, such that*

1. *If $\gamma \in \mathcal{G}\mathrm{au}(E)_{\tilde{g}}$, $\hat{\ell}(\gamma, S) = S$.*
2. *If $\gamma \in \mathcal{G}\mathrm{au}(E)$, such that $\ell(\gamma, S) \cap S \neq \emptyset$, then $\gamma \in \mathcal{G}\mathrm{au}(E)_{\tilde{g}}$.*
3. *There exists a local cross section $s : \mathcal{G}\mathrm{au}(E)/\mathcal{G}\mathrm{au}(E)_{\tilde{g}} \to \mathcal{G}\mathrm{au}(E)$ defined on a neighborhood U of the identity coset such that if $F : U \times S \to \mathcal{C}\mathrm{onn}(E) \times \mathcal{M}\mathrm{et}(VE)$ is defined by $F(u, t) := \hat{\ell}(s(u), t)$, then F is a homeomorphism onto a neighborhood of $\tilde{g}$.*

Proof. Consider the decomposition $\mathcal{M}\mathrm{et}(E) = \mathcal{C}\mathrm{onn}(E) \times \mathcal{M}\mathrm{et}(VE) \times \mathcal{M}\mathrm{et}(NE)$ which get from 5.27. Choose a $g \in \mathcal{M}\mathrm{et}(E)$ with $(\mathrm{pr}_1 \times \mathrm{pr}_2)(g) = \tilde{g}$. Denote by $S_{\mathcal{M}\mathrm{et}(E)}$ the slice for g of Theorem 5.22, and set $S := \mathrm{pr}_1 \times \mathrm{pr}_2(S_{\mathcal{M}\mathrm{et}(E)})$. Iff $\gamma \in \mathcal{G}\mathrm{au}(E)_{\tilde{g}}$ then $\gamma \in \mathcal{G}\mathrm{au}(E)_g$ since the action on the third factor is trivial. We then have

(1):

$$\begin{aligned}\hat{\ell}_\gamma(S) &= \hat{\ell}_\gamma((\mathrm{pr}_1 \times \mathrm{pr}_2)(S_{\mathcal{M}\mathrm{et}(E)}) = (\mathrm{pr}_1 \times \mathrm{pr}_2)(\ell_\gamma(S_{\mathcal{M}\mathrm{et}(E)})) = \\ &= (\mathrm{pr}_1 \times \mathrm{pr}_2)(S_{\mathcal{M}\mathrm{et}(E)}) = S.\end{aligned}$$

(2): If $\gamma \in \mathcal{G}\mathrm{au}(E)$, such that $\hat{\ell}_\gamma(S) \cap S \neq \emptyset$. Set $\bar{g} := \mathrm{pr}_3(g)$. Let $x, y \in S$ be chosen such that $\hat{\ell}_\gamma(x) = y$. Denote by $i_g : S \to S_{\mathcal{M}\mathrm{et}(E)}$ the map $i_g : x \mapsto (\mathrm{pr}_1 \times \mathrm{pr}_2)^{-1}(x) \times \mathrm{pr}_3^{-1}(\bar{g})$. Then $\ell_\gamma(i_g(x)) = i_g(y)$, and since $S_{\mathcal{M}\mathrm{et}(E)}$ is a slice, we get $\gamma \in \mathcal{G}\mathrm{au}(E)_g$, therefore $\gamma \in \mathcal{G}\mathrm{au}(E)_{\tilde{g}}$.
(3): Take the cross section s from 5.22, the same U as in 5.22, and denote by F' the homeomorphism of 5.22. Then $F = (\mathrm{pr}_1 \times \mathrm{pr}_2) \circ F' \circ (\mathrm{Id}_U \times i_g)$ is a homeomorphism onto the neighborhood $(\mathrm{pr}_1 \times \mathrm{pr}_2)(F'(U \times S_{\mathcal{M}\mathrm{et}(E)}))$ of $\tilde{g}$ as a composition of homeomorphisms. □

5.29. Remark. As in 5.24 one can again show that the set of principal orbits $\mathcal{G}$, where the isotropy group is trivial, is open and dense, and like in 5.25 a stratification into smooth manifolds modeled on Fréchet spaces exists for $(\mathcal{C}\mathrm{onn}(E) \times \mathcal{M}\mathrm{et}(VE))/\mathcal{G}\mathrm{au}(E)$.

In the next chapter we will, however, see that the factor $\mathcal{M}\mathrm{et}(VE)$ is crucial for the existence of a slice theorem. When omitted, it can be shown that there exist elements of $\mathcal{C}\mathrm{onn}(E)$ where no slice can possibly exist.

The non–existence of the slice theorem follows from a counterexample which is inspired by a counterexample for the action of $\mathcal{D}\mathrm{iff}(S^1)$ on $C^\infty(S^1,\mathbb{R})$ and the fact that when slices exist locally the isotropy subgroup of an element cannot increase.

6.1. Proposition. *Let the action of G on M have a slice S at x. Then there is a neighborhood U of x such that*

(1) *if $y \in U \cap S$, then $G_y \subset G_x$,*
(2) *if $y \in U$, then G_y is conjugate to a subgroup of G_x.*

Proof. (1) follows directly from slice property (2) (see e.g. Theorem 5.22) as follows. Let $y \in S$ and some $g \in G_y$ so that $gy = y$. It follows that $gS \cap S \neq \emptyset$, and so by property (2) we must have $g \in G_x$.
(2): Let U be the neighborhood described in slice property (3). Then there must exist some $g \in G$ such that $g^{-1}y \in S$. Applying (1) to $g^{-1}y$, we then find that $g^{-1}G_y k \subset G_x$. $\square$

6.2. The counterexample for $C^\infty(S^1,\mathbb{R})/\mathcal{D}\mathrm{iff}(S^1)$. Let $h(t): S^1 \to \mathbb{R}$ be a smooth bump function with

$$h(t) = \begin{cases} 0, & x \notin [0, \frac{1}{4}] \\ > 0, & x \in]0, \frac{1}{4}[. \end{cases}$$

Then set $h_n(t) = \frac{1}{4^n} h(4^n(x - \frac{1-\frac{1}{4^n}}{3}))$. Then $h_n(t)$ is nonzero in the interval $]\frac{1-\frac{1}{4^n}}{3}, \frac{1-\frac{1}{4^{n+1}}}{3}[$. Defining

$$f(t) = \sum_{n=0}^{\infty} h_n(t) e^{-\frac{1}{(x-\frac{1}{3})^2}}$$

we get a positive smooth function, which has zeros exactly on $t = \frac{1-\frac{1}{4^n}}{3}$, and which is flat at $t = \frac{1}{3}$.

In every neighborhood of f lies a function

$$f_N(t) = \sum_{n=0}^{N} h_n(t) e^{-\frac{1}{(t-\frac{1}{3})^2}}$$

which has only finitely many zeros and is identically zero in the interval $[\frac{1-\frac{1}{4^{N+1}}}{3}, \frac{1}{3}]$.

All diffeomorphisms in the isotropy subgroup of f are also contained in the isotropy subgroup of f_N, but the latter group contains additionally all diffeomorphisms of S^1 which have support only on $[\frac{1-\frac{1}{4^{N+1}}}{3}, \frac{1}{3}]$. This is a contradiction to the fact that locally the isotropy subgroup cannot increase (i.e. to proposition 6.1).

6.3. Remark. The counterexample above also works, when replacing zeros of the function with zeros of the derivative. In the $\mathcal{C}\mathrm{onn}(E)/\mathcal{G}\mathrm{au}(E)$ case neither of these work, since the action is more complicated. So one has to search for an expression in the connection Φ for which the action is simpler. It turns out that the curvature of Φ is such a convenient expression.

6.4. The curvature of a connection. In 4.29 we have defined the curvature R of a connection Φ. For a gauge transformation $\gamma \in \mathcal{G}\mathrm{au}(E)$ we have that Φ and $\gamma^*(\Phi)$ are γ–related. By theorem 4.15 we hence get that R and $\gamma^* R$ are γ–related. But for $\gamma \in \mathcal{G}\mathrm{au}(E)_\Phi$ this means that for all $e \in E$

$$(\gamma^* R)_e = R_e.$$

This is the simple action we were looking for. In order to construct a counterexample to the slice theorem, we have to investigate the local description of the curvature.

6.5. Christoffel forms. Let Φ be a connection on (E, p, M, S), and (U_α, ψ_α) a fiber bundle atlas with transition functions $(\psi_{\alpha\beta})$. Let us consider the connection $(\psi_\alpha^{-1})^*\Phi \in \Omega^1(U_\alpha \times S; U_\alpha \times TS)$. It may be written in the form

$$(\psi_\alpha^{-1})^*(\xi_x, \eta_y) =: -\Gamma^\alpha(\xi_x, y) + \eta_y \quad \text{for } \xi_x \in T_x U_\alpha \text{ and } \eta_y \in T_y S,$$

since it reproduces vertical vectors. The Γ^α are given by

$$(0_x, \Gamma^\alpha(\xi_x, y)) := -T\psi_\alpha.\Phi.T(\psi_\alpha)^{-1}.(\xi_x, 0_y).$$

They can then be considered as elements of the space $\Omega^1(U_\alpha; \mathfrak{X}(S))$. These Γ^α are called the *Christoffel forms* of the connection Φ with respect to the bundle atlas (U_α, ψ_α).

6.6. Lemma. *The curvature R of Φ satisfies the (local) Maurer–Cartan formula.*

$$(\psi_\alpha^{-1})^* R = d\Gamma^\alpha + \tfrac{1}{2}[\Gamma^\alpha, \Gamma^\alpha]_{\mathfrak{X}(S)}.$$

Proof.

$$\begin{aligned}
(\psi_\alpha^{-1})^* & R((\xi_1, \eta_1), (\xi_2, \eta_2)) = \\
&= (\psi_\alpha^{-1})^*\Phi[(\mathrm{Id} - (\psi_\alpha^{-1})^*\Phi)(\xi_1, \eta_1), (\mathrm{Id} - (\psi_\alpha^{-1})^*\Phi)(\xi_2, \eta_2)] = \\
&= (\psi_\alpha^{-1})^*\Phi[(\xi_1, \Gamma^\alpha(\xi_1)), (\xi_2, \Gamma^\alpha(\xi_2))] = \\
&= (\psi_\alpha^{-1})^*\Phi\big([\xi_1, \xi_2], \xi_1\Gamma^\alpha(\xi_2) - \xi_2\Gamma^\alpha(\xi_1) + [\Gamma^\alpha(\xi_1), \Gamma^\alpha(\xi_2)]\big) = \\
&= -\Gamma^\alpha([\xi_1, \xi_2]) + \xi_1\Gamma^\alpha(\xi_2) - \xi_2\Gamma^\alpha(\xi_1) + [\Gamma^\alpha(\xi_1), \Gamma^\alpha(\xi_2)] = \\
&= d\Gamma^\alpha(\xi_1, \xi_2) + [\Gamma^\alpha(\xi_1), \Gamma^\alpha(\xi_2)]_{\mathfrak{X}(S)}. \qquad \square
\end{aligned}$$

6.7. The counterexample for $\dim M \geq 2$. We will construct locally a connection, which satisfies that in any neighborhood there exist connections which have a bigger isotropy subgroup.

Let $n = \dim S$, and let $h : \mathbb{R}^n \to \mathbb{R}$ be a smooth nonnegative bump function, which satisfies $\operatorname{carr} h = \{s \in \mathbb{R}^n | \, \|s - s_0\| < 1\}$. Set

$$h_r(s) := r h(s_0 + \tfrac{1}{r}(s - s_0)),$$

then $\operatorname{carr} h_r = \{s \in \mathbb{R}^n | \ \|s - s_0\| < r\}$. Next define

$$h_r^{s_1}(s) := h(s - (s_1 - s_0)),$$

which implies $\operatorname{carr} h_r^{s_1} = \{s \in \mathbb{R}^n | \ \|s - s_1\| < r\}$. Using these functions, we can define new functions f_k for $k \in \mathbb{N}$ as

$$f_k(s) = h^{s_k}_{\|z\|/2^k}(s),$$

where $z := \frac{s_\infty - s_0}{3}$ for some $s_\infty \in \mathbb{R}^n$ and $s_k := s_0 + z(2\sum_{\ell=0}^k \frac{1}{2^\ell} - 1 - \frac{1}{2^k})$. Further set

$$f^N(s) := e^{-\frac{1}{\|s-s_\infty\|^2}} \sum_{k=0}^N \tfrac{1}{4^k} f_k(s),$$
$$f(s) := \lim_{N\to\infty} f^N(s).$$

The functions f^N and f are smooth, respectively, since all the functions f_k are smooth, and on every point s at most one summand is nonzero. $\operatorname{carr} f^N = \bigcup_{k=0}^N \{s \in \mathbb{R}^n | \ \|s - s_k\| < \frac{1}{2^k}\|z\|\}$, $\operatorname{carr} f = \bigcup_{k=0}^\infty \{s \in \mathbb{R}^n | \ \|s - s_k\| < \frac{1}{2^k}\|z\|\}$, f^N and f vanish in all derivatives in all x_k, and f vanishes in all derivatives in s_∞.

Let $\psi : E|_U \to U \times S$ be a fiber bundle chart of E with a chart $u : U \xrightarrow{\cong} \mathbb{R}^m$ on M, and let $v : V \xrightarrow{\cong} \mathbb{R}^n$ be a chart on S. Choose $g \in C_c^\infty(M, \mathbb{R})$ with $\emptyset \neq \operatorname{supp} g \subset U$ and $dg \wedge du^1 \neq 0$ on an open dense subset of $\operatorname{supp} g$, choose s_0 and s_∞ as inner points in $\operatorname{supp} g$. Then we can define (denoting by u^i and v^j the coordinates in U and V, respectively) a Christoffel form as in 6.5 by

$$\Gamma := g\, du^1 \otimes f(v)\partial_{v^1} \in \Omega^1(U, \mathfrak{X}(S)).$$

This defines a connection Φ on $E|_U$ which can be extended to a connection Φ on E by the following method. Take a smooth functions $k_1, k_2 \geq 0$ on M satisfying $k_1 + k_2 = 1$ and $k_1 = 1$ on $\operatorname{supp} g$ and $\operatorname{supp} k_1 \subset U$ and any connection Φ' on E, and set $\Phi = k_1 \Phi^\Gamma + k_2 \Phi'$, where Φ^Γ denotes the connection which is induced locally by Γ. In any neighborhood of Φ there exists a connection Φ^N defined by

$$\Gamma^N := g\, du^1 \otimes f^N(s)\partial_{v_1} \in \Omega^1(U, \mathfrak{X}(S)),$$

and extended like Φ.

Claim: There is no slice at Φ.

Proof: We have to consider the isotropy subgroups of Φ and Φ^N. Since the connections Φ and Φ^N coincide outside of U, we may investigate them locally on $W = \{u : k_1(u) = 1\} \subset U$. The curvature of Φ is given locally on W by 6.6 as

$$R_U := d\Gamma - \tfrac{1}{2}[\Gamma, \Gamma]^{\mathfrak{X}(S)}_\wedge = dg \wedge du^1 \otimes f(v)\partial_{v^1} - 0.$$

For every element of the gauge group $\mathcal{G}\mathrm{au}(E)$ which is in the isotropy group $\mathcal{G}\mathrm{au}(E)_\Phi$ the local representative over W which looks like $\tilde\gamma : (u,v) \mapsto (u, \gamma(u,v))$ by 6.6 satisfies

$$T_v(\gamma(u,\quad)).\Gamma(\xi_u, v) = \Gamma(\xi_u, \gamma(u,v)) - T_u(\gamma(\quad, v)).\xi_u,$$
$$g(u)du^1 \otimes f(v) \sum_i \frac{\partial\gamma^1}{\partial v^i}\partial_{v^i} = g(u)du^1 \otimes f(\gamma(u,v))\partial_{v^1} - \sum_{i,j}\frac{\partial\gamma^i}{\partial u^j}du^j \otimes \partial_{v^i}.$$

Comparing the coefficients of $du^j \otimes \partial_{v^i}$ we get the following equations for γ over W.

$$\frac{\partial\gamma^i}{\partial u^j} = 0 \quad \text{for } (i,j) \neq (1,1),$$
$$g(u)f(v)\frac{\partial\gamma^1}{\partial v^1} = g(u)f(\gamma(u,v)) - \frac{\partial\gamma^1}{\partial u^1}. \tag{1}$$

Considering next the transformation $\tilde\gamma^* R_U = R_U$ of the curvature 1.2 we get

$$T_v(\gamma(u,\quad)).R_U(\xi_u, \eta_u, v) = R_U(\xi_u, \eta_u, \gamma(u,v)),$$
$$dg \wedge du^1 \otimes f(v) \sum_i \frac{\partial\gamma^1}{\partial v^i}\partial_{v^i} = dg \wedge du^1 \otimes f(\gamma(u,v))\partial_{v^1}.$$

Another comparison of coefficients yields the equations

$$f(v)\frac{\partial\gamma^1}{\partial v^i} = 0 \quad \text{for } i \neq 1,$$
$$f(v)\frac{\partial\gamma^1}{\partial v^1} = f(\gamma(u,v)), \tag{2}$$

whenever $dg \wedge du^1 \neq 0$, but this is true on an open dense subset of $\operatorname{supp} g$. Finally, putting (2) into (1) shows

$$\frac{\partial\gamma^i}{\partial u^j} = 0 \quad \text{for all } i,\ j.$$

Collecting the results on $\operatorname{supp} g$, we see that γ has to be constant in all directions of u. Furthermore, wherever f is nonzero, γ^1 is a function of v^1 only and γ has to map zero sets of f to zero sets of f.

Replacing Γ by Γ^N we get the same results with f replaced by f^N. Since $f = f^N$ wherever f^N is nonzero or f vanishes, γ in the isotropy group of Φ obeys all these equations not only for f but also for f^N on $\operatorname{supp} f^N \cup f \subset (0)$. On $B := \operatorname{carr} f \backslash \operatorname{carr} f^N$ the gauge transformation γ is a function of v^1 only, hence it cannot leave the zero set of f^N by construction of f and f^N. Therefore, γ obeys all equations for f^N whenever it obeys all equations for f, thus every gauge transformation in the isotropy subgroup of Φ is in the isotropy subgroup of Φ^N.

On the other hand, any γ having support in B changing only in v^1 direction not keeping the zero sets of f invariant defines a gauge transformation in the isotropy subgroup of Φ^N which is not in the isotropy subgroup of Φ.

Therefore, there exists in every neighborhood of Φ a connection Φ^N whose isotropy subgroup is bigger than the isotropy subgroup of Φ. Thus, by proposition 6.1 there exists no slice at Φ. □

6.8. The counterexample for $\dim M = 1$. The situation is somewhat different if $\dim M = 1$, i.e. $M = S^1$. In this case, the method of 6.7 is not applicable, since there is no function g satisfying $dg \wedge du^1 \neq 0$ on an open and dense subset of $\operatorname{supp} g$. However, any connection Φ on E is flat. Hence, the horizontal bundle is integrable, the horizontal foliation induced by Φ exists and determines Φ. Any gauge transformation leaving Φ invariant also has to map leaves of the horizontal foliation to other leaves of the horizontal foliation.

We shall construct connections $\Phi^{\lambda'}$ near Φ^λ such that the isotropy groups in $\mathcal{G}\mathrm{au}(E)$ look radically different near the identity, contradicting 6.1.

Let us assume without loss of generality that E is connected, and then, by replacing S^1 by a finite covering if necessary, that the fiber is connected. Then there exists a smooth global section $\chi : S^1 \to E$. By [Michor 1980, Lemma 10.9] there exists a tubular neighborhood $\pi : U \subset E \to \operatorname{im}(\chi)$ such that $\pi = \chi \circ p|U$ (i.e. a tubular neighborhood with vertical fibers). This tubular neighborhood then contains an open thickened sphere bundle with fiber $S^1 \times \mathbb{R}^{n-1}$, and since we are only interested in gauge transformations near Id_E, which e.g. keep a smaller thickened sphere bundle inside the larger one, we may replace E by an S^1-bundle. By replacing the Klein bottle by a 2-fold covering we may finally assume that the bundle is $\mathrm{pr}_1 : S^1 \times S^1 \to S^1$.

Consider now connections where the horizontal foliation is a 1-parameter subgroup with slope λ. We see that the isotropy group equals S^1 if λ is irrational, and equals S^1 times the diffeomorphism group of a closed interval if λ is rational. Since in every neighborhood of a 1-parameter subgroup with irrational slope are some having rational slope, we see that there cannot be a slice theorem for one dimensional M, either. □

The counterexamples constructed in 6.7 and 6.8 imply the following

6.9. Theorem. *Let (E, p, M, S) be a fiber bundle such that M, S are both compact, and $\dim M \geq 1$ and $\dim S \geq 1$. Further denote the usual action by $l : \mathcal{G}\mathrm{au}(E) \times \mathcal{C}\mathrm{onn}(E) \to \mathcal{C}\mathrm{onn}(E)$. Then there exists $\Phi \in \mathcal{C}\mathrm{onn}(E)$ such that there does* **not** *exist a contractible subset S (a slice) of $\mathcal{C}\mathrm{onn}(E)$ containing Φ, such that*

1. *If $\gamma \in \mathcal{G}\mathrm{au}(E)_\Phi$, $\ell(\gamma, S) = S$.*
2. *If $\gamma \in \mathcal{G}\mathrm{au}(E)$, such that $\ell(\gamma, S) \cap S \neq \emptyset$, then $\gamma \in \mathcal{G}\mathrm{au}(E)_\Phi$.*
3. *There exists a local cross section $s : \mathcal{G}\mathrm{au}(E)/\mathcal{G}\mathrm{au}(E)_\Phi \to \mathcal{G}\mathrm{au}(E)$ defined on a neighborhood U of the identity coset such that if $F : U \times S \to \mathcal{C}\mathrm{onn}(E)$ is defined by $F(u,t) := \ell(s(u), t)$, then F is a homeomorphism onto a neighborhood of Φ.*

Proof. See 6.7 and 6.8 □

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