Differential Geometry 1

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Preface

These lecture notes form the basis of an introductory course on differential geometry which I first held in the summer term of 2006. Several boundary conditions made the choice of material to be included quite delicate. On the one hand, in the mathematics curriculum of the Faculty of Mathematics in Vienna, the course 'Differential Geometry 1' is the only compulsory course on the subject for students not specializing in geometry and topology. On the other hand, the course duration is only three hours per week. Therefore, an approach which first focuses on classical differential geometry and then gently moves on to the theory of differentiable manifolds is ruled out by time constraints.
The course therefore puts its main emphasis on a concise introduction to modern differential geometry in order to provide the necessary tools for applications in other branches of mathematics or for a continued study of differential geometry. Nevertheless, an introduction to local curve theory in chapter 1 and applications to the theory of hypersurfaces in chapter 3 are intended to provide a link to more classical aspects of the subject.
Throughout I have tried to motivate all basic concepts thoroughly. As a rule, all proofs are given in full detail, and comprehensibility is given prevalence over elegance whenever the need arises. I have also refrained from including more material than can be covered in one semester in order to make a clear statement on what I consider essential in an introductory course of this kind.
I would like to thank Christoph Marx who typed a first (German) version of these notes and David Langer who supplied the beautiful pictures and diagrams included here. Also, I am grateful for many comments of students participating in the course which, I hope, have led to improvements in the text. Further comments and corrections are always welcome!

Michael Kunzinger, summer term 2008

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Chapter 1

Curves in $\mathbb{R}^n$

1.1 Frenet Curves in $\mathbb{R}^n$

When studying curves as maps c from some interval I in $\mathbb{R}$ to $\mathbb{R}^n$ analytically one needs to make some regularity assumptions on c. Continuity is definitely to weak a requirement as it does not exclude certain pathological examples (think of Peano curves, i.e., continuous curves which completely cover areas in $\mathbb{R}^2$). In particular, if we want to make use of analytical tools we should suppose c to be differentiable. However, even requiring c to be C^∞ does not exclude certain unwanted phenomena like edges (where the derivative of c vanishes). Moreover, geometrically it is natural to require the existence of a nonzero tangent vector in each point of the curve. One is thus led to the following

1.1.1 Definition. *A* regular parametrized curve *is a continuously differentiable map $c : I \to \mathbb{R}^n$ defined on some interval $I \subseteq \mathbb{R}$ such that $\dot{c}(t) \equiv \frac{dc}{dt}(t) \neq 0$ for all $t \in I$.*

When interpreting t as time and c as describing the physical movement of a particle the above definition means that the velocity $\dot{c}$ of c is nowhere zero, i.e., the particle never stops. We call the vector $\dot{c}(t_0)$ the tangent vector of c at t_0 and the line $t \mapsto c(t_0) + (t - t_0)\dot{c}(t_0)$ the tangent of c at $c(t_0)$. By Taylor's theorem, the tangent is a first order approximation to c in t_0:

$$c(t_0 + t) = c(t_0) + t\dot{c}(t_0) + o(t)\,.$$

From the geometric point of view one is not interested in any particular parametrization of a given curve but rather in its shape (which is invariant under re-parametrization):

1.1.2 Definition. *A* regular curve *is an equivalence class of regular parametrized curves with respect to the following equivalence relation: let $c_1 : I_1 \to \mathbb{R}^n$, $c_2 : I_2 \to \mathbb{R}^n$ be regular parametrized curves. Then c_1 is called equivalent with c_2 if there exists a diffeomorphism $\varphi : I_1 \to I_2$ (i.e., φ bijective and φ, φ^{-1} C^1) such that $c_2 \circ \varphi = c_1$ and $\varphi' > 0$ (such φ are called orientation preserving).*

$$\begin{tikzcd}
& \mathbb{R}^n & \\
I_1 \arrow[ur, "c_1"] \arrow[rr, "\varphi"'] & & I_2 \arrow[ul, "c_2"']
\end{tikzcd}$$

Note that we include orientation in our definition of regular curve. One could distinguish between regular oriented curves (with $\varphi' > 0$) and regular curves without this

restriction on φ but we will not do this in the sequel and only consider orientation preserving changes of parameter.
Let $c : [a, b] \to \mathbb{R}^n$ be a regular curve. The length of c is defined as

$$L_a^b(c) := \int_a^b \|\dot{c}(t)\| \, dt$$

Here $\| \, . \, \|$ denotes the euclidean norm in $\mathbb{R}^n$. This notion of length is well defined, i.e., independent of the parametrization. In fact, let $\varphi : [\alpha, \beta] \to [a, b]$ be a parameter transformation as in 1.1.2 above. Then

$$\int_\alpha^\beta \|(c \circ \varphi)^{\cdot}(r)\| \, dr = \int_\alpha^\beta \|\dot{c}(\varphi(r))\| \varphi'(r) \, dr = \int_a^b \|\dot{c}(t)\| \, dt$$

1.1.3 Definition. *A parametrization of a curve c is called* parametrization by arclength *if $\|\dot{c}(t)\| = 1$ for all t.*

Physically speaking, a curve parametrized by arclength has unit speed. Clearly, if $c : [a, b] \to \mathbb{R}^n$ is parametrized by arclength then $L_a^b(c) = b - a$.

1.1.4 Lemma. *Every regular curve possesses a parametrization by arclength. Any two such parametrizations are equivalent via a translation $\varphi : t \mapsto t + a$.*

Proof. Set $s(t) := L_a^t(c)$. Then $s : [a, b] \to [0, l]$ with $l := L_a^b(c)$ and $s'(t) = \|\dot{c}(t)\| > 0$ for all t. Hence s is an orientation-preserving diffeomorphism and we claim that $\bar{c} := c \circ s^{-1}$ is a parametrization of c by arclength. In fact, by the chain rule we have

$$\dot{\bar{c}}(u) = \dot{c}(s^{-1}(u)) \cdot \frac{1}{s'(s^{-1}(u))} = \frac{\dot{c}}{\|\dot{c}\|}(s^{-1}(u)) \, ,$$

so $\|\dot{\bar{c}}(u)\| = 1$ for all $u \in (0, l)$, as claimed.
Suppose, finally, that c and $c \circ \varphi$ are two parametrizations by arclength. Then since $\varphi' > 0$ we have

$$1 = \|(c \circ \varphi)^{\cdot}(t)\| = \|\dot{c}(\varphi(t))\| \cdot \varphi'(t) = \varphi'(t) \, ,$$

so $\varphi = t \mapsto t + a$ for some $a \in \mathbb{R}$. □

In what follows we shall employ the following notational conventions: by $c(t)$ we denote any regular parametrization, whereas we write $c(s)$ for a parametrization by arclength. Accordingly, we set $\dot{c} = \frac{dc}{dt}$ for the tangent vector in an arbitrary parametrization and $c' = \frac{dc}{ds}$ for the tangent vector in the parametrization by arclength. Then we have:

$$\dot{c} = c' \frac{ds}{dt} = \|\dot{c}\| c' \, , \quad \text{and} \quad \|c'\| = 1 \, .$$

1.1.5 Lemma. *If c is parametrized by arclength then $c''(s) \perp c'(s)$ for all s.*

Proof. If we differentiate the equation $1 = \|c'(s)\|^2 = \langle c'(s), c'(s) \rangle$ we obtain

$$0 = \langle c'(s), c''(s) \rangle + \langle c''(s), c'(s) \rangle = 2 \langle c'(s), c''(s) \rangle \, ,$$

hence the claim. □

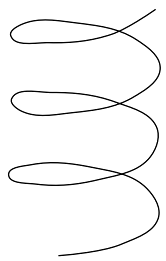
1.1.6 Examples.

(i) For the straight line $c(t) = (at, bt)$ we have $\dot{c}(t) = (a, b)$. Hence c is parametrized by arclength if and only if $a^2+b^2 = 1$. Note also that, e.g., the parametrization $t \mapsto (at^3, bt^3)$, although it describes the same geometric curve, is not regular at $t = 0$.

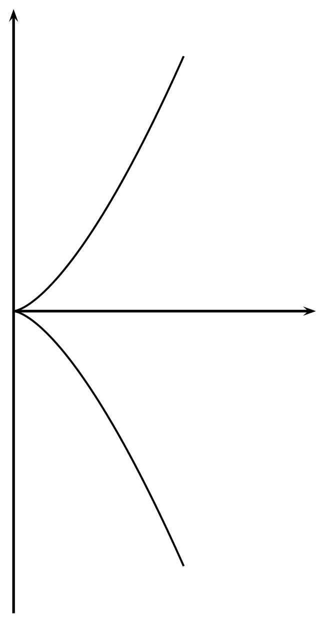
(ii) The assignment $c(s) := \frac{1}{2}(\cos(2s), \sin(2s))$ describes a circle of radius $\frac{1}{2}$. Since $c'(s) = (-\sin(2s),\ \cos(2s))$ we have $\|c'\| = 1$, i.e., c is parametrized by arclength.

(iii) *Circular helix*: Let $c(t) := (a\cos(\alpha t), a\sin(\alpha t), bt)$ with $\alpha,\ a,\ b \in \mathbb{R}$. Then $\dot{c}(t) = (-\alpha a\sin(\alpha t), \alpha a\cos(\alpha t), b)$, so $\|\dot{c}\| = \sqrt{\alpha^2 a^2 + b^2}$. Thus c has constant velocity and $s = t\sqrt{\alpha^2 a^2 + b^2}$ is the parameter of arclength. The circular helix is given geometrically as the image of the point $(a, 0, 0)$ under the following one-parameter group of screw-motions:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \underbrace{\begin{pmatrix} \cos(\alpha t) & -\sin(\alpha t) & 0 \\ \sin(\alpha t) & \cos(\alpha t) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ bt \end{pmatrix}}_{\text{translation}}$$



(iv) *Neil's parabola* (or: *semicubical parabola*) is the curve $c(t) = (t^2, t^3)$. Here, $\dot{c}(t) = (2t, 3t^2)$, so that $\dot{c}(0) = (0, 0)$. Thus c is not a regular parametrization at $t = 0$, although c is of course smooth on all of $\mathbb{R}$. Geometrically we see that at the cusp $c(0, 0)$, c does not have a well-defined tangent vector.



Using Taylor expansion, we may approximate any curve c parametrized by arc length as follows:

$$c(s) = c(0) + sc'(0) + \frac{s^2}{2}c''(0) + \frac{s^3}{6}c'''(0) + o(s^3)\,.$$

Using this expansion up to first order we obtain the tangent $c(0) + sc'(0)$. Up to second order we get the *osculating conic* $c(0) + sc'(0) + \frac{s^2}{2}c''(0)$ which has second order contact with c. Here, two curves are said to have k-th order contact at s if their derivatives up to order k at s coincide.
The above considerations assign a distinguished role to the vectors c', c'', c''', $\ldots$ in describing a given curve c. In particular, if these vectors are linearly independent at each parameter value they fix a natural coordinate system in which to describe c. In what follows we call an orthonormal basis in Euclidean space an n-frame.

1.1.7 Definition. *Let $c : I \to \mathbb{R}^n$ be a regular curve of class C^n parametrized by arclength. c is called a* Frenet *curve if the vectors $c'(s), c''(s), \ldots, c^{(n-1)}(s)$ are linearly independent at each parameter value s. The corresponding Frenet n-frame is then uniquely defined by the following conditions:*

(i) $e_1(s), \ldots, e_n(s)$ are orthonormal and positively oriented for each $s \in I$.

(ii) span$(e_1(s), \ldots, e_k(s))$ = span$(c'(s), \ldots, c^{(k)}(s))$ for all $k \in \{1, \ldots, n-1\}$ and all $s \in I$.

(iii) $\langle c^{(k)}(s), e_k(s)\rangle > 0$ for all $k \in \{1, \ldots, n-1\}$ and all $s \in I$.

To construct $e_1(s), \ldots, e_{n-1}(s)$ from $c'(s), \ldots, c^{(n-1)}(s)$ we use the Gram-Schmidt orthogonalization procedure (where we omit the parameter s for the sake of brevity):

$$\begin{aligned}
e_1 &:= c' \\
e_2 &:= c''/\|c''\| \\
e_3 &:= (c''' - \langle c''', e_1\rangle e_1 - \langle c''', e_2\rangle e_2)/\|\ldots\| \\
&\vdots \\
e_j &:= (c^{(j)} - \sum_{i=1}^{j-1}\langle c^{(j)}, e_i\rangle e_i)/\|\ldots\| \\
&\vdots \\
e_{n-1} &:= (c^{(n-1)} - \sum_{i=1}^{n-2}\langle c^{(n-1)}, e_i\rangle e_i)/\|\ldots\|
\end{aligned}$$

The vector e_n is then uniquely determined by 1.1.7 (i).

1.1.8 Example.

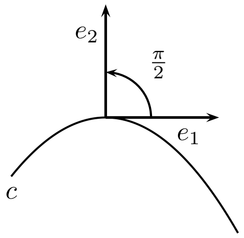
(i) In the plane, every regular curve is Frenet since 1.1.7 does not pose any restriction in case $n = 2$.

(ii) For $n = 3$, i.e., for regular space curves, the only condition remaining is $c'' \neq 0$, i.e., the absence of inflection points.

1.2 Plane and Space Curves, Curvature

1.2.1 Plane Curves

Suppose that c is a regular (hence Frenet, cf. 1.1.8) curve in $\mathbb{R}^2$. Then $e_1 = c'$ is the tangent vector of c and e_2, which is constructed from e_1 by rotating by an angle of $\pi/2$ to the left, is the normal vector of c. We also write $e_2 = e_1^{\perp}$ for short.



Since

$$0 = \langle c', c'\rangle' = 2\langle c', c''\rangle = 2\langle e_1, c''\rangle$$

it follows that c'' and e_2 are in fact parallel, i.e., there exists some function κ with $c''(s) = \kappa(s)e_2(s)$ for all s. κ is called the (oriented) *curvature* of c. The sign of κ indicates the direction in which c (resp. c') is turning: for $\kappa > 0$, the tangent is rotating to the left, for $\kappa < 0$ to the right. For $\kappa = 0$ the tangent is not turning at all. Such points are called *inflection points*. Before turning to a geometric interpretation of the absolute value of κ let us first derive some useful relations.
By definition, $e_1' = c'' = \kappa e_2$. Since e_2 is constructed from e_1 through rotating by $\pi/2$, we conclude by rotating this identity that $e_2' = -\kappa e_1$. Hence we obtain the so-called *Frenet equations* for plane curves:

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}' = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}. \tag{1.2.1}$$

The Frenet equations allow to derive an explicit formula for the curvature of a curve $c(s) = (x(s), y(s))$ in terms of c' and c''. In fact, we have

$$\begin{aligned} \kappa(s) &= \langle \kappa(s)e_2(s), e_2(s)\rangle = \langle c''(s), e_2(s)\rangle = \left\langle \begin{pmatrix} x''(s) \\ y''(s) \end{pmatrix}, \begin{pmatrix} -y'(s) \\ x'(s) \end{pmatrix} \right\rangle \\ &= \det \begin{pmatrix} x'(s) & x''(s) \\ y'(s) & y''(s) \end{pmatrix} = \det(c'(s), c''(s)). \end{aligned}$$

Heuristically, if we consider curves of constant curvature we expect to obtain either straight lines (for $\kappa = 0$) or circles (for $\kappa \neq 0$) since a constant rate of turning of the tangent vector corresponds to driving along a curve with the steering wheel set to a fixed position. This intuitive picture is made precise in the following result.

1.2.1 Theorem. *A regular curve in $\mathbb{R}^2$ has constant curvature κ if and only if it is part of a straight line (for $\kappa = 0$) or of a circle of radius $\frac{1}{|\kappa|}$ (for $\kappa \neq 0$).*

Proof. A straight line obviously has $\kappa = 0$ and conversely $\kappa = 0$ implies $c'' = \kappa e_2 = 0$, i.e., c is a straight line. Suppose now that $k(s) = M + r(\cos(s/r), \sin(s/r))$ is a circle parametrized by arclength. Then $|\kappa(s)| = |k''(s)| = 1/r$ for all s. Note that in this case

$$M = k(s) - r(\cos(s/r), \sin(s/r)) = k(s) + k''(s)r^2 = k(s) + \frac{k''(s)}{|k''(s)|^2} \tag{1.2.2}$$

If, conversely, the curvature κ of a regular curve c is a nonzero constant, guided by (1.2.2) we first note that $M(s) := c(s) + (1/\kappa)e_2(s)$ is constant. In fact, $M'(s) = 0$ for all s by (1.2.1). Moreover, $|M - c(s)| = 1/|\kappa|$ for all s. □

1.2.2 Definition. *Let c be a regular plane curve such that $\kappa(s_0) \neq 0$. Then the circle which has second order contact with c at s_0 is called the* osculating circle *of c at $c(s_0)$.*

Let k denote the osculating circle of c at s_0. Then we have $c(s_0) = k(s_0)$, $c'(s_0) = k'(s_0)$, and $c''(s_0) = k''(s_0)$. By our calculations in the proof of 1.2.1, the osculating circle at $c(s_0)$ therefore has its center at

$$M = k(s_0) + \frac{k''(s_0)}{|k''(s_0)|^2} = c(s_0) + \frac{c''(s_0)}{|c''(s_0)|^2} = c(s_0) + \frac{e_2(s_0)}{\kappa(s_0)}$$

and has radius $1/|\kappa(s_0)|$. In particular, it is uniquely determined. The curve formed by the centers of all osculating circles of c is called the *evolute* of c. It is given by

$$s \mapsto c(s) + \frac{e_2(s)}{\kappa(s)}\,.$$

Note, however, that the evolute of a regular curve in general need not be regular anymore (typically, it will display cusps, similar to Neil's parabola).
So far, we have seen two interpretation of the curvature κ of a regular curve c. Originally, we defined κ as the rate of change of the direction of the tangent vector. Moreover, we established that $1/|\kappa|$ is the radius of the osculating circle of c. A third, physically motivated interpretation of curvature is as follows: for a particle following a trajectory c parametrized by arclength we have $\|c'\| = 1$, i.e., acceleration can only have the effect of changing the direction of the tangent vector, not of changing its norm, i.e., the velocity of the curve. Thus acceleration can only occur orthogonal to c'. Since $c'' = \kappa e_2$ it follows (using Newton's law 'force equals mass times acceleration') that we may interpret curvature as the force to be applied in the normal direction of the trajectory to keep the particle (assumed to have unit mass) on its curved path.

1.2.3 Remark. For a given curvature function κ there is a unique (up to Euclidean motion) Frenet curve c whose curvature is precisely κ. To construct c we will employ the Frenet equations (1.2.1). We first make the ansatz

$$e_1 = (\cos(\alpha(s)), \sin(\alpha(s)))$$

with the function α to be determined. Then

$$e_2(s) = e_1(s)^\perp = (-\sin(\alpha(s)), \cos(\alpha(s)))$$

and by (1.2.1) we need to solve $\kappa e_2 = e_1' = \alpha' e_2$, i.e., $\kappa = \alpha'$. By choosing an adapted coordinate system we may suppose that $c(0) = (0, 0)$ and $e_1(0) = (1, 0)$, so that $\alpha(0) = 0$ and $\alpha(s) = \int_0^s \kappa(t)\,dt$. Then $c(s) = (x(s), y(s))$ with

$$x(s) = \int_0^s \cos\left(\int_0^r \kappa(t)\,dt\right)\,dr\,, \quad y(s) = \int_0^s \sin\left(\int_0^r \kappa(t)\,dt\right)\,dr$$

Note in particular that for κ =const., this precisely reproduces 1.2.1.

1.2.2 Space Curves

As we noted in 1.1.8, a regular curve in $\mathbb{R}^3$ is Frenet if $c''(s) \neq 0$ for all s. Its accompanying 3-frame is given by

$$\begin{aligned}
e_1 &= c', && (\textit{tangent vector}) \\
e_2 &= \frac{c''}{\|c''\|}, && (\textit{principal normal vector}) \\
e_3 &= e_1 \times e_2 && (\textit{binormal vector})
\end{aligned}$$

The *curvature* of c is defined as $\kappa(s) := \|c''(s)\|$. For the derivatives of e_1, e_2, e_3 we calculate:

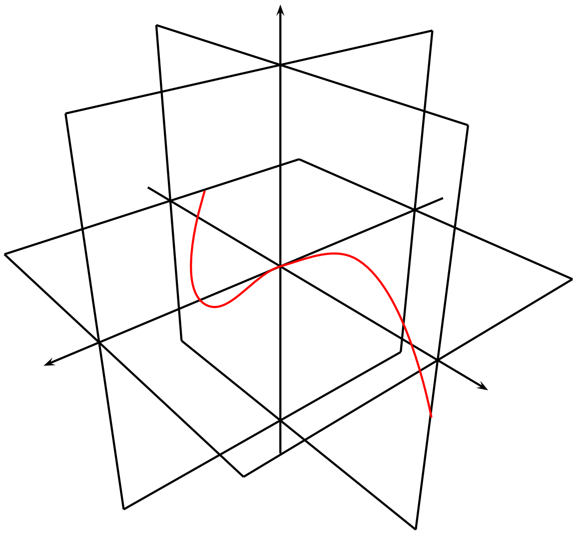
$$\begin{aligned}
e_1' &= c'' = \kappa e_2\,, \\
e_2' &= \langle e_2', e_1\rangle e_1 + \underbrace{\langle e_2', e_2\rangle}_{=0} e_2 + \langle e_2', e_3\rangle e_3 = \langle -e_2, e_1'\rangle e_1 + \underbrace{\langle e_2', e_3\rangle}_{=:\tau} e_3 = -\kappa e_1 + \tau e_3 \\
e_3' &= \langle e_3', e_1\rangle e_1 + \langle e_3', e_2\rangle e_2 + \underbrace{\langle e_3', e_3\rangle}_{=0} e_3 = -\underbrace{\langle e_3, e_1'\rangle}_{=0} e_1 - \underbrace{\langle e_3, e_2'\rangle}_{=\tau} e_2 = -\tau e_2\,.
\end{aligned}$$

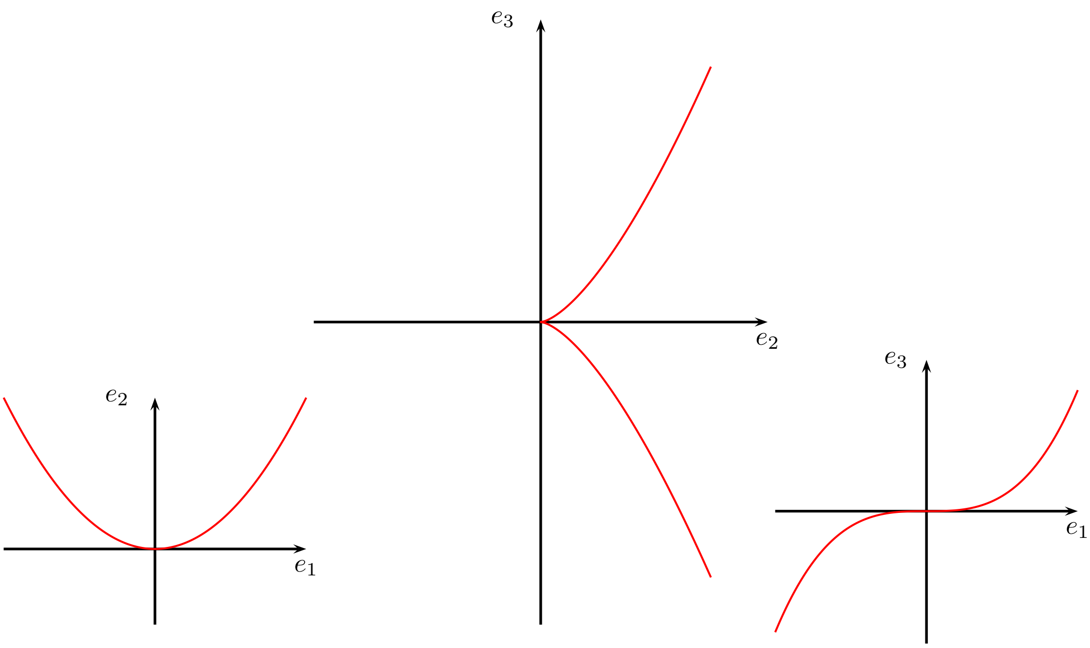
We call $\tau := \langle e_2', e_3\rangle$ the *torsion* of c. Summing up, we obtain the *Frenet equations* for a space curve:

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{1.2.3}$$

To obtain an intuitive understanding of the geometric meaning of torsion, note first that a plane curve $c(s) = (x(s), y(s))$, when viewed as a space curve $(x(s), y(s), 0)$ has vanishing torsion τ. Indeed, since e_3 is constant, this is immediate from (1.2.3). Conversely, if $\tau = 0$ for a space curve c it follows from (1.2.3) that e_3 is constant. Hence c lies in the (e_1, e_2)-plane. Thus τ measures the rate of departure of c from this plane.

As we have just seen, it is instructive to analyze the behavior of a curve by considering its projections onto certain planes spanned by subsets of its accompanying 3-frame, to wit: $\mathrm{span}(e_1, e_2)$, the *osculating plane*, $\mathrm{span}(e_2, e_3)$, the *normal plane*, and $\mathrm{span}(e_1, e_3)$, the *rectifying plane*.





More precisely, we consider the following Taylor expansion of the curve c:

$$c(s) = c(0) + sc'(0) + \frac{s^2}{2}c''(0) + \frac{s^3}{6}c'''(0) + o(s^3)$$

We rewrite this in terms of the accompanying three frame as

$$c(s) = c(0) + \alpha(s)e_1(0) + \beta(s)e_2(0) + \gamma(s)e_3(0) + o(s^3)$$

with $\alpha(s)$, $\beta(s)$, $\gamma(s)$ to be determined. Using (1.2.3), we calculate:

$$\begin{array}{rclclcl} c' &=& e_1 \\ c'' &=& e_1' &=& \kappa e_2 \\ c''' &=& (\kappa e_2)' &=& \kappa' e_2 + \kappa e_2' &=& \kappa' e_2 + \kappa(-\kappa e_1 + \tau e_3)\,. \end{array}$$

Hence, $c(s)$ has the expansion

$$c(0) + \left(s - \frac{s^3\kappa(0)^2}{6}\right) e_1(0) + \left(\frac{s^2\kappa(0)}{2} + \frac{s^3\kappa'(0)}{6}\right) e_2(0) + \frac{s^3\kappa(0)\tau(0)}{6} e_3(0) + o(s^3)\,.$$

The projection in the osculating ((e_1, e_2)-) plane is (up to order two in s) a parabola:

$$c(0) + s e_1(0) + \frac{s^2\kappa(0)}{2} e_2(0) + o(s^2)\,.$$

In the normal ((e_2, e_3)-) plane we obtain a Neil parabola (up to order three):

$$c(0) + \left(\frac{s^2\kappa(0)}{2} + \frac{s^3\kappa'(0)}{6}\right) e_2(0) + \frac{s^3\kappa(0)\tau(0)}{6} e_3(0) + o(s^3)\,.$$

Finally, the projection onto the (e_1, e_3)- (rectifying) plane takes the form of a cubical parabola (up to order three):

$$c(0) + \left(s - \frac{s^3\kappa(0)^2}{6}\right) e_1(0) + \frac{s^3\kappa(0)\tau(0)}{6} e_3(0) + o(s^3)\,.$$

1.3 The Fundamental Theorem of the Local Theory of Curves

In this section we first want to generalize the Frenet equations (1.2.1), (1.2.3) for plane and space curves to the general case of curves in $\mathbb{R}^n$. Thus let c be a Frenet curve in $\mathbb{R}^n$ with accompanying n-frame $e_1, \dots, e_n$. Then we have:

1.3.1 Theorem. *There exist uniquely determined functions $\kappa_1, \dots, \kappa_{n-1}$, the* Frenet-curvatures *of c with $\kappa_1, \dots, \kappa_{n-2} > 0$ and $\kappa_i \in C^{n-1-i}$ (for $1 \le i \le n-1$) such that the Frenet equations hold:*

$$\begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ \vdots \\ \vdots \\ e_{n-1} \\ e_n \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_1 & 0 & 0 & \dots & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 & \ddots & \vdots \\ 0 & -\kappa_2 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & \kappa_{n-1} \\ 0 & \dots & \dots & 0 & -\kappa_{n-1} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ \vdots \\ \vdots \\ e_{n-1} \\ e_n \end{pmatrix} \tag{1.3.1}$$

Proof. We write e_i' in terms of the orthonormal basis $e_1, \dots, e_n$:

$$e_i' = \sum_{j=1}^{n} \langle e_i', e_j \rangle e_j\,. \tag{1.3.2}$$

By construction, $e_i \in \operatorname{span}(c', c'', \dots, c^{(i)})$ for each $i \le n-1$. Differentiating, we obtain that $e_i' \in \operatorname{span}(c', c'', \dots, c^{(i+1)}) = \operatorname{span}(e_1, e_2, \dots, e_{i+1})$. Hence the sum in (1.3.2) only extends up to $i+1$, i.e.,

$$\langle e_i', e_{i+2} \rangle = \langle e_i', e_{i+3} \rangle = \dots = \langle e_i', e_n \rangle = 0\,. \tag{1.3.3}$$

We now set $\kappa_i := \langle e_i', e_{i+1}\rangle$ $(\in C^{n-(i+1)})$. Let $e_i = \sum_{j=1}^{i} a_j c^{(j)}$. Then by 1.1.7,

$$1 = \langle e_i, e_i\rangle = a_i \underbrace{\langle c^{(i)}, e_i\rangle}_{>0},$$

so each a_i is positive. Since, by the product rule, $e_i' = \sum_{j=1}^{i} b_j c^{(j)} + a_i c^{(i+1)}$, we obtain that $\kappa_i = \langle e_i', e_{i+1}\rangle = a_i\langle c^{(i+1)}, e_{i+1}\rangle > 0$ for $i \le n-2$ (by 1.1.7). Since $0 = \langle e_i, e_j\rangle' = \langle e_i', e_j\rangle + \langle e_i, e_j'\rangle$, we conclude from (1.3.3) that in fact $\langle e_i', e_j\rangle = 0$ for $|i-j| \neq 1$. Summing up, we obtain (1.3.1):

$$e_i' = \underbrace{\langle e_i', e_{i-1}\rangle}_{=-\langle e_{i-1}', e_i\rangle} e_{i-1} + \langle e_i', e_{i+1}\rangle e_{i+1} = -\kappa_{i-1}e_{i-1} + \kappa_i e_{i+1}$$

That the κ_i are uniquely determined is immediate from (1.3.1) and the fact that $e_1, \dots, e_n$ forms an orthonormal frame. □

Obviously, (1.3.1) contains (1.2.1) and (1.2.3) as special cases. We may give an interpretation of κ_{n-1} as torsion, similar to the three-dimensional situation. In fact, since $\kappa_i > 0$ for $i \le n-2$, c will lie in a hyperplane (namely the $(e_1, \dots, e_{n-1})$-plane) if and only if the torsion κ_{n-1} vanishes. This in turn is equivalent to e_n being constant, orthogonal to this plane.
Our next result shows that both the Frenet frame and the Frenet curvatures are *geometric* concepts, i.e., they do not depend on any choice of coordinates. This is to say that they do not change under Euclidean motions. A transformation $B : \mathbb{R}^n \to \mathbb{R}^n$ is called a Euclidean motion if it is of the form $B(x) = Ax + b$ for A an orientation preserving rotation (an element of $SL(n, \mathbb{R})$, i.e., $A^{-1} = A^t$ and $\det(A) = 1$) and b a fixed translation vector.

1.3.2 Proposition. *Let c be a Frenet curve in $\mathbb{R}^n$. Then its Frenet frame and its Frenet curvatures are invariant under Euclidean motions.*

Proof. In the notation introduced above, we have to show that if $e_1, \dots, e_n$ is the Frenet frame of c then $Ae_1, \dots Ae_n$ is the Frenet frame of $B \circ c$ and c and $B \circ c$ have the same Frenet curvatures $\kappa_1, \dots, \kappa_{n-1}$. In fact, we have $(B \circ c)^{(i)} = Ac^{(i)}$ for all i, so the claim about the Frenet frame follows immediately from the construction of $e_1, \dots, e_n$ (see 1.1.7). For the curvatures we calculate:

$$(Ae_i)' = Ae_i' = A(-\kappa_{i-1}e_{i-1} + \kappa_i e_{i+1}) = -\kappa_{i-1}(Ae_{i-1}) + \kappa_i(Ae_{i+1})\,.$$

□

As the main result of this section we next show that a Frenet curve in $\mathbb{R}^n$ is entirely determined by its curvatures.

1.3.3 Theorem. *(Fundamental theorem of the local theory of curves)*
Let $\kappa_1, \dots, \kappa_{n-1} : (a, b) \to \mathbb{R}$ be given functions with $\kappa_i \in C^{n-i-1}(a, b)$ for $1 \le i \le n-1$ and $\kappa_1, \dots, \kappa_{n-2} > 0$. Let $s_0 \in (a, b)$, $q_0 \in \mathbb{R}^n$ and a fixed set of positively oriented orthonormal vectors $e_1^{(0)}, \dots, e_n^{(0)}$ be given. Then there exists a unique Frenet curve $c : (a, b) \to \mathbb{R}^n$ such that $c(s_0) = q_0$, $e_1^{(0)}, \dots, e_n^{(0)}$ is the Frenet frame of c at q_0, and $\kappa_1, \dots, \kappa_{n-1}$ are the Frenet curvatures of c.

Proof. We are trying to find a matrix-valued map $F : (a, b) \to \mathbb{R}^{n^2}$, $s \mapsto (e_1(s), \dots, e_n(s))^t$, (i.e., the e_i are the rows of F) where $e_1, \dots, e_n$ is the Frenet frame of our prospective solution curve c. Thus we want $F(s)$ to be an orthogonal matrix with determinant 1 for all parameter values s. Moreover, according to (1.3.1), F should solve the matrix equation

$$F'(s) = K(s)F(s) \tag{1.3.4}$$

with $K(s)$ the skew-symmetric matrix of curvatures (in this case: the given functions $\kappa_1, \ldots, \kappa_{n-1}$) on the right-hand side of (1.3.1).
Now (1.3.4) is a linear system of ordinary differential equations, so there exists a unique solution $F : (a,b) \to \mathbb{R}^{n^2}$ with initial condition $F(s_0) = (e_1^{(0)}, \ldots, e_n^{(0)})^t$. Since F satisfies (1.3.4) we have

$$(FF^t)' = F'F^t + F(F^t)' = F'F^t + F(F')^t = KFF^t + FF^tK^t \, .$$

Thus FF^t solves the system of linear ODEs $X' = KX + XK^t$ with initial condition $F(s_0)F(s_0)^t = I_n$, the $n \times n$ unit matrix. However, since $K + K^T = 0$, I_n itself is the unique solution to this system. It follows that $F(s)F(s)^t = I_n$ for all $s \in (a,b)$, i.e., $F(s)$ is orthogonal for all s. Hence $\det(F(s)) = \pm 1$ for all s. Since $s \mapsto \det(F(s))$ is continuous and $\det(F(s_0)) = \det(I_n) = 1$, it follows that $\det(F(s)) = 1$ for all s. Thus the rows $e_1(s), \ldots, e_n(s)$ of $F(s)$ form a positively oriented orthonormal frame, as desired.
If $e_1(s), \ldots, e_n(s)$ is to be the accompanying n-frame of a Frenet curve c we must have $c' = e_1$. Combined with the prescribed initial condition $c(s_0) = q_0$ this uniquely determines the regular curve c as $c(s) = q_0 + \int_{s_0}^{s} e_1(t)\, dt$. We next show that $e_1(s), \ldots, e_n(s)$ is in fact the Frenet frame of c. To this end, we first show by induction that for $1 \le i \le n$ we have

$$c^{(i)} = \kappa_1 \cdot \kappa_2 \cdot \cdots \cdot \kappa_{i-1} e_i + \sum_{j=1}^{i-1} a_j^i e_j \tag{1.3.5}$$

for certain functions a_j^i. For $i = 1$ this trivially holds since $c' = e_1$. Suppose the result is true for i. Then

$$\begin{aligned} c^{(i+1)} &= (\kappa_1 \ldots \kappa_{i-1}) \cdot e_i' + (\kappa_1 \ldots \kappa_{i-1})' \cdot e_i + \sum_{j=1}^{i-1} ({a_j^i}' e_j + a_j^i e_j') = \\ &= (\kappa_1 \ldots \kappa_{i-1})(-\kappa_{i-1} e_{i-1} + \kappa_i e_{i+1}) + \sum_{j=1}^{i} b_j e_j \\ &= \kappa_1 \ldots \kappa_i e_{i+1} + \sum_{j=1}^{i} c_j e_j \, , \end{aligned}$$

for certain functions b_j, c_j, as claimed. Since $\kappa_1, \ldots \kappa_{n-2} > 0$ it follows that $c', \ldots, c^{(n-1)}$ are linearly independent, i.e., c is a Frenet curve. Moreover, (1.3.5) implies that the e_i constitute the accompanying Frenet frame of c. Finally, since $F' = KF$ it follows that the κ_i are precisely the Frenet curvatures of c (cf. the uniqueness part of 1.3.1). □

Chapter 2

Differentiable Manifolds

The notion of a differentiable manifold is one of the central concepts of modern mathematics. Among others it finds applications in analysis, differential geometry, topology, the theory of Lie groups, ordinary and partial differential equations, as well as in numerous branches of physics, e.g. in mechanics or general relativity.
We start out by studying the special case of submanifolds of $\mathbb{R}^n$, a direct generalization of the notion of surface in $\mathbb{R}^3$ which already displays all the essential characteristics of the concept of abstract manifolds.

2.1 Submanifolds of $\mathbb{R}^n$

To begin with we recall some notions and results from analysis. For simplicity, from now on we will assume all maps to be C^∞.

2.1.1 Theorem. (Inverse Function Theorem) *Let $U \subseteq \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ C^∞, $x_0 \in U$, $y_0 := f(x_0)$ and $Df(x_0)$ invertible ($\det Df(x_0) \neq 0$). Then locally around x_0, f is a diffeomorphism, i.e., there exist $U_1 \subseteq U$ an open neighborhood of x_0, and V_1 an open neighborhood of y_0, such that $f : U_1 \to V_1$ is bijective and $f^{-1} : V_1 \to U_1$ is C^∞.*

2.1.2 Theorem. (Implicit Function Theorem) *Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ open, $f : U \times V \to \mathbb{R}^m$ C^∞, $(x_0, y_0) \in U \times V$, $f(x_0, y_0) = 0$ and let $\frac{\partial f}{\partial y}(x_0, y_0) : \mathbb{R}^m \to \mathbb{R}^m$ be invertible ($\det \frac{\partial f}{\partial y}(x_0, y_0) \neq 0$). Then there exist open neighborhoods $U_1 \subseteq U$ of x_0, $V_1 \subseteq V$ of y_0, such that: $\forall x \in U_1$ $\exists!$ $y = y(x) \in V_1$ with $f(x, y(x)) = 0$. The map $x \mapsto y(x)$ is C^∞.*

2.1.3 Definition. *Let $U \subseteq \mathbb{R}^k$ be open and $\varphi : U \to \mathbb{R}^n$ C^∞. φ is called* regular *if for all $x \in U$ the rank of the Jacobian $D\varphi(x)$ is maximal, hence equal to* $\min(k, n)$. *Then for the rank* $\mathrm{rk}(D\varphi)$ *of $D\varphi$ (also called the rank of φ) we have*

$$\mathrm{rk}(D\varphi(x)) = \dim \mathrm{im}(D\varphi(x)) = \dim(\mathbb{R}^k) - \dim(\ker D\varphi(x)).$$

Thus if $k \leq n$ then $\ker D\varphi(x) = \{0\}$ and $D\varphi(x)$ is injective for all x. In this case φ is called an immersion. *For $k \geq n$, $D\varphi(x)$ is surjective for all x and φ is called a* submersion.

Hence 2.1.1 says that a regular map $f : U \to V$ with U, $V \subseteq \mathbb{R}^n$ open is a local diffeomorphism.

2.1.4 Remark. (Properties of immersions). Let $U \subseteq \mathbb{R}^k$ open and $\varphi : U \to \mathbb{R}^n$ an immersion.

(i) $\mathrm{rk}(D\varphi(x_0)) = k$ means that $\{\frac{\partial\varphi}{\partial x_1}(x), \dots, \frac{\partial\varphi}{\partial x_k}(x)\}$ is linearly independent in $\mathbb{R}^n$.

(ii) Equivalently, there exist indices $1 \le i_1 < i_2 < \dots < i_k \le n$ such that

$$\det \frac{\partial(\varphi_{i_1}, \dots, \varphi_{i_k})}{\partial(x_1, \dots, x_k)}(x_0) \neq 0$$

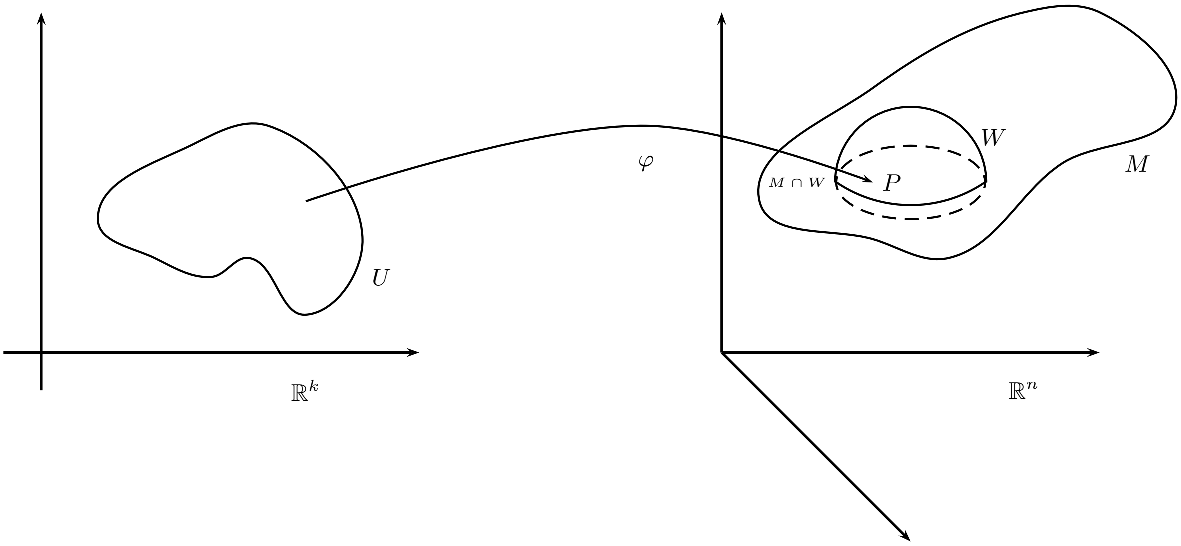
Since det is continuous it follows that $\mathrm{rk}(D\varphi(x)) = k$ in a neighborhood of x_0.

(iii) In particular for $k = 1$, $\varphi : U \subseteq \mathbb{R} \to \mathbb{R}^n$ is an immersion if $\varphi'(t) \neq 0\ \forall t$, i.e., if φ is a regular curve.

2.1.5 Definition. *A subset M of $\mathbb{R}^n$ is called a k-dimensional submanifold of $\mathbb{R}^n$ ($k \le n$) if*

$$(P) \begin{cases} \textit{For each } p \in M \textit{ there exists an open neighborhood } W \textit{ of } p \textit{ in } \mathbb{R}^n, \\ \textit{an open subset } U \textit{ of } \mathbb{R}^k \textit{ and an immersion } \varphi : U \to \mathbb{R}^n \textit{ such that} \\ \varphi : U \to \varphi(U) \textit{ is a homeomorphism and } \varphi(U) = M \cap W. \end{cases}$$

Such a φ is called a local parametrization *of M.*



Thus φ is regular and identifies U and $\varphi(U) = M \cap W$ topologically ($\varphi(U) = M \cap W$ carries the trace topology of $\mathbb{R}^n$). The following result gives an alternative criterion which is sometimes used in the definition of submanifolds of $\mathbb{R}^n$.

2.1.6 Proposition. *For each $M \subseteq \mathbb{R}^n$, property (P) is equivalent to*

$$(P') \begin{cases} \textit{For each } p \in M \textit{ there exists a smooth map } \varphi : U \to \mathbb{R}^n, \textit{ where } U \\ \textit{is an open neighborhood of } 0 \textit{ in } \mathbb{R}^k, \varphi(0) = p \textit{ and } \varphi \textit{ is regular at } 0 \\ \textit{(i.e., } D\varphi(0) \textit{ is injective) and such that for any open neighborhood} \\ U_1 \subseteq U \textit{ of } 0 \textit{ there exists an open neighborhood } W_1 \textit{ of } p \textit{ in } \mathbb{R}^n \textit{ with} \\ \varphi(U_1) = W_1 \cap M. \end{cases}$$

Proof. Obviously (P) implies (P'). Conversely, we first note that by 2.1.4 we may without loss of generality suppose that φ is an immersion on all of U. By assumption, φ is continuous. To establish (P) we will show that there exists an open subset U_1 of U such that $\varphi|_{U_1}$ is a homeomorphism onto its image, equipped with the trace topology from M.

Since $D\varphi(0)$ is injective there exists a left inverse linear map $A : \mathbb{R}^n \to \mathbb{R}^k$, i.e., $\mathrm{id}_{\mathbb{R}^k} = A \cdot D\varphi(0) = D(A \cdot \varphi)(0)$. [Let $B := D\varphi(0)$, then $B : \mathbb{R}^k \to \mathrm{im}(B)$ is bijective. Call $\tilde{A}$ the inverse of this map. Then we may take $A := \tilde{A} \circ \mathrm{pr}_{\mathrm{im}(B)}$.] By 2.1.1 the map $x \mapsto A \cdot \varphi(x)$ is a local diffeomorphism on $\mathbb{R}^k$, so there exist open neighborhoods $U_1 \subseteq U$ of 0 and U_2 of $A(p)$ such that $h := (A \circ \varphi)^{-1} : U_2 \to U_1$ is smooth.

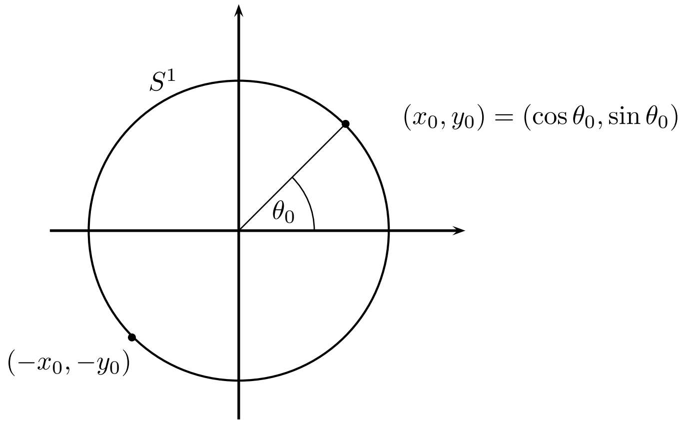
Now set $\psi := h \circ A : A^{-1}(U_2) \to U_1$. Then ψ is smooth and

$$\psi \circ \varphi(x) = (A \circ \varphi)^{-1} \circ A \circ \varphi(x) = x \qquad \forall x \in U_1\,,$$

so ψ is a left-inverse of $\varphi|_{U_1}$. In particular, $\varphi|_{U_1}$ is injective. Thus $\varphi : U_1 \to \varphi(U_1) = W_1 \cap M$ is bijective with inverse $\psi|_{W_1 \cap M}$. The latter map is continuous w.r.t. the trace topology, so $\varphi : U_1 \to \varphi(U_1)$ is a homeomorphism. $\square$

2.1.7 Examples.

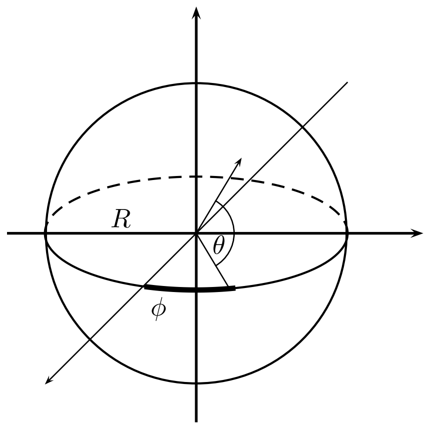
(i) The unit circle S^1.
Let $\varphi : \theta \mapsto (\cos\theta, \sin\theta)$. Then for all $(x_0, y_0) = (\cos\theta_0, \sin\theta_0)$, $\varphi : (\theta_0 - \pi, \theta_0 + \pi) \to \mathbb{R}^2$ is a parametrization of S^1 around (x_0, y_0). Here W can be taken, e.g., as $\mathbb{R}^2 \backslash \{(-x_0, -y_0)\}$. Hence S^1 is a 1-dimensional submanifold of $\mathbb{R}^2$. Note that no single parametrization can be used for all of S^1! (There is no homeomorphism from some open subset of $\mathbb{R}$ onto S^1 since S^1 is compact).



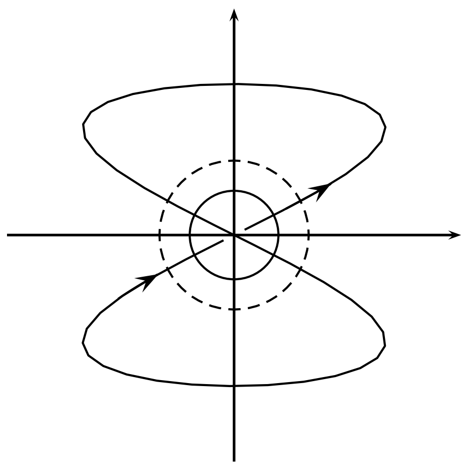
(ii) The 2-sphere S^2 in $\mathbb{R}^3$.
Let $\varphi(\phi, \theta) = (\cos\phi\cos\theta, \sin\phi\cos\theta, \sin\theta)$. Then

$$D\varphi = \begin{pmatrix} -\sin\phi\cos\theta & -\cos\phi\sin\theta \\ \cos\phi\cos\theta & -\sin\phi\sin\theta \\ 0 & \cos\theta \end{pmatrix}$$

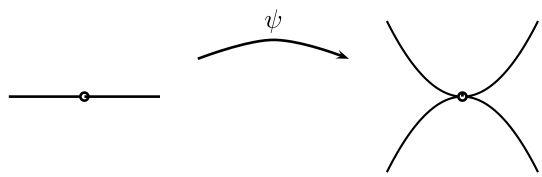
φ is a parametrization of S^2 e.g. on $(0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$. In fact, on this domain φ is injective and $\mathrm{rk}(D\varphi) = 2$, since $\cos\theta \neq 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Again, more than one parametrization is needed to cover S^2.



(iii) Figure eight manifold.
Let $M := \{(\sin 2s, \sin s) | s \in (0, 2\pi)\}$. The map $\varphi : s \mapsto (\sin 2s, \sin s)$ is an injective immersion: indeed, $D\varphi(s) = \varphi'(s) = (2\cos 2s, \cos s) \neq (0,0)$ on $(0, 2\pi)$.

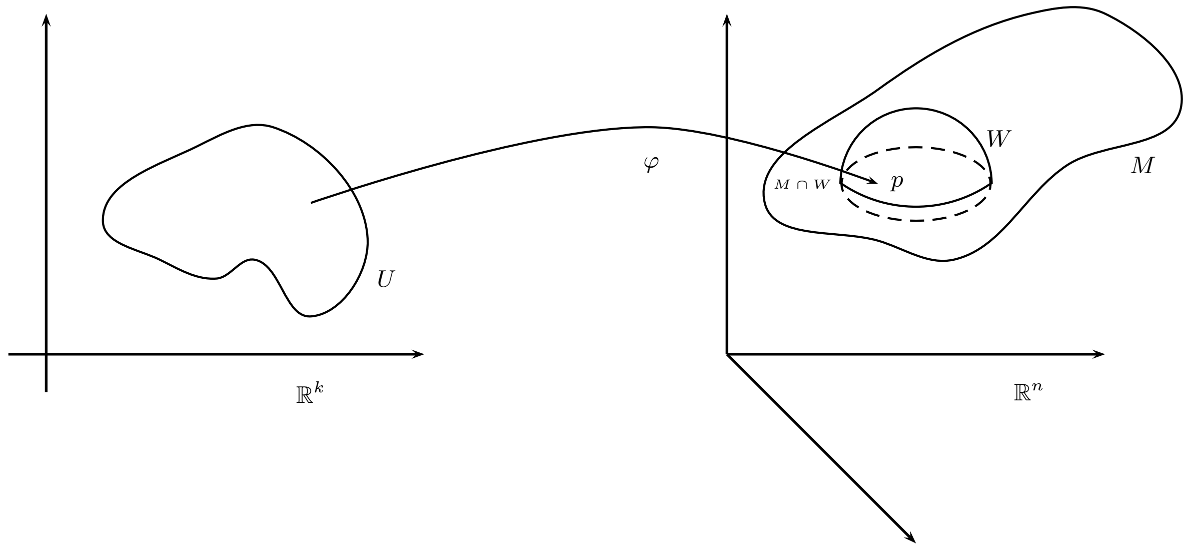


However, M is not a submanifold of $\mathbb{R}^2$! In fact, suppose that there exists a parametrization $\psi : (-\varepsilon, \varepsilon) \to B_r(0,0)$ of M around $p = (0,0)$ with $r < 1$ such that $\psi : (-\varepsilon, \varepsilon) \to B_r(0,0) \cap M$ is a homeomorphism. Then since $(-\varepsilon, \varepsilon) \setminus \{0\}$ has two connected components, while $(M \cap B_r(0,0)) \setminus (0,0)$ has four, we arrive at a contradiction. M is what is usually called an *immersive submanifold* of $\mathbb{R}^2$. In what follows, we will restrict our attention to submanifolds in the sense of 2.1.5.



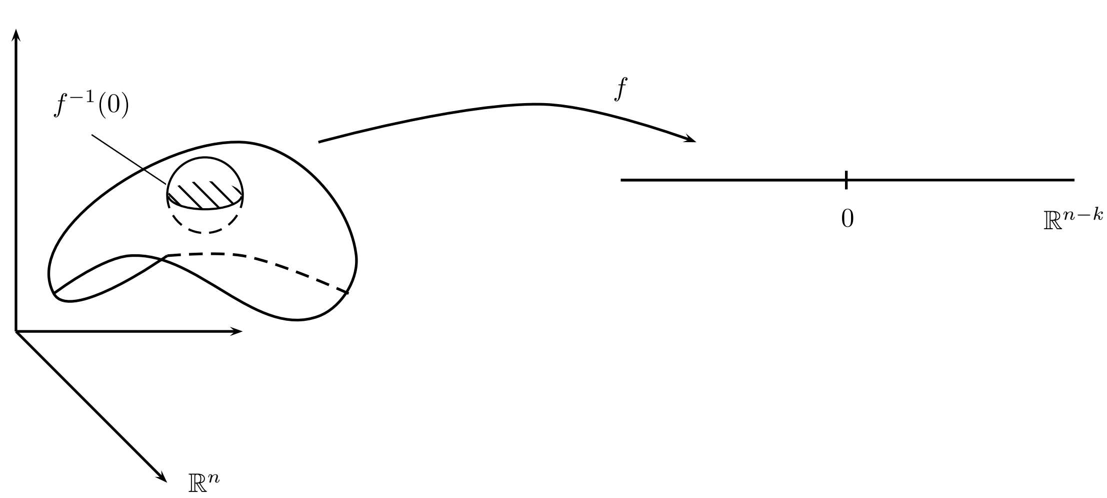
2.1.8 Theorem. *Let $M \subseteq \mathbb{R}^n$. The following are equivalent:*

(P) (Local Parametrization) M is a k-dimensional submanifold of $\mathbb{R}^n$.



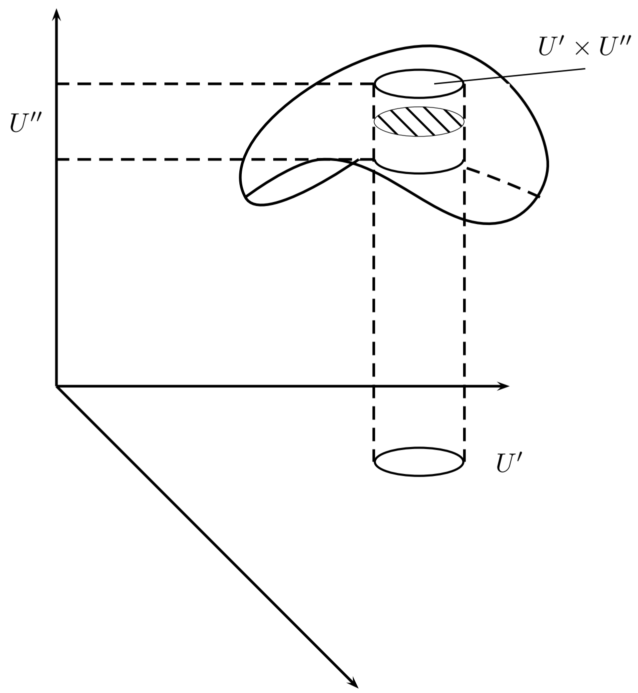
(Z) (Local Zero Set) For every $p \in M$ there exist an open neighborhood W of p in $\mathbb{R}^n$ and a C^∞-map $f : W \to \mathbb{R}^{n-k}$ which is regular (i.e., $\mathrm{rk} Df(q) = n - k$ for all $q \in W$) satisfying

$$M \cap W = f^{-1}(0) = \{x \in W \mid f(x) = 0\}.$$



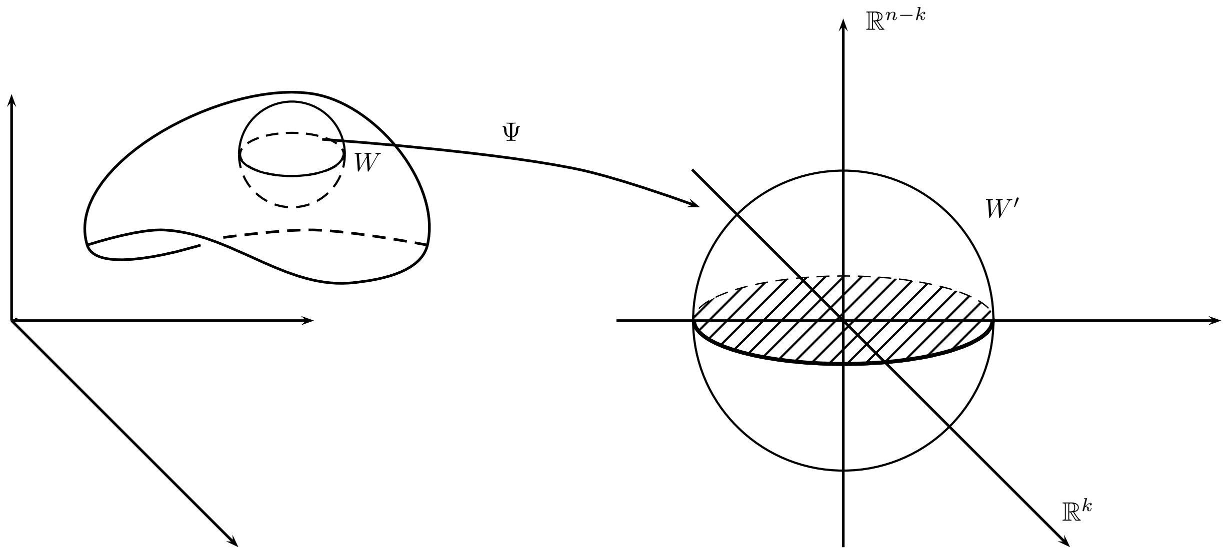
(G) (Local Graph) For each $p \in M$ there exist (after re-numbering the coordinates if necessary) open neighborhoods $U' \subseteq \mathbb{R}^k$ of $p' := (p_1, \ldots, p_k)$ and $U'' \subseteq \mathbb{R}^{n-k}$ of $p'' := (p_{k+1}, \ldots, p_n)$ and a C^∞-map $g : U' \to U''$ such that

$$M \cap (U' \times U'') = \{(x', x'') \in U' \times U'' | x'' = g(x')\} = \operatorname{graph}(g)$$



(T) (Local Trivialization) For each $p \in M$ there exist an open neighborhood W of p in $\mathbb{R}^n$, an open set W' in $\mathbb{R}^n \cong \mathbb{R}^k \times \mathbb{R}^{n-k}$ and a diffeomorphism $\Psi : W \to W'$ such that

$$\Psi(M \cap W) = W' \cap (\mathbb{R}^k \times \{0\}) \subseteq \mathbb{R}^k \times \{0\} \cong \mathbb{R}^k.$$



Proof. $(P) \Rightarrow (G)$:
Without loss of generality we may suppose that $\varphi(0) = p$ and $\det \frac{\partial(\varphi_1,\dots,\varphi_k)}{\partial(x_1,\dots,x_k)}(0) \neq 0$. By 2.1.1 there exists some open neighborhood $U_1 \subseteq U$ of 0 and some open $V_1 \subseteq \mathbb{R}^k$ such that $\varphi' := (\varphi_1, \dots, \varphi_k)$ is a diffeomorphism. Let $\psi : V_1 \to U_1$ be the inverse of φ' and $G := \varphi \circ \psi : V_1 \to \mathbb{R}^n$. Then with $\varphi'' := (\varphi_{k+1}, \dots, \varphi_n)$ we have

$$G(x_1, \dots, x_k) := (\underbrace{\varphi' \circ \psi(x_1, \dots, x_k)}_{=\underbrace{(x_1,\dots,x_k)}_{x'}}, \underbrace{\varphi'' \circ \psi}_{=:g}(x_1, \dots, x_k)) = (x', g(x'))$$

with $g : V_1 \to \mathbb{R}^{n-k}$ smooth. Since φ is a homeomorphism, $\varphi(U_1)$ is open in M, i.e., there exists some W_1 open in $\mathbb{R}^n$ such that $\varphi(U_1) = M \cap W_1$. Hence

$$M \cap W_1 = \varphi(\underbrace{\psi(V_1)}_{U_1}) = G(V_1) = \{(x', g(x'))|x' \in V_1\}.$$

We now choose open sets $U' \subseteq V_1$ and $U'' \subseteq \mathbb{R}^{n-k}$ such that $p \in U' \times U'' \subseteq W_1$. Then

$$\begin{aligned} M \cap (U' \times U'') &= M \cap W_1 \cap (U' \times U'') = \{(x', g(x'))|x' \in V_1\} \cap (U' \times U'') \\ &= \{(x', x'') \in U' \times U''|g(x') = x''\} \end{aligned}$$

$(G) \Rightarrow (Z)$:
Set $W := U' \times U''$ and $f : W \to \mathbb{R}^{n-k}$,

$$f_j(x_1, \dots, x_n) := x_{k+j} - g_j(x_1, \dots, x_k) \qquad (1 \leq j \leq n-k)$$

Then $f \in \mathcal{C}^\infty$ and $\frac{\partial(f_1,\dots,f_{n-k})}{\partial(x_{k+1},\dots,x_n)} = I_{n-k}$, so f is regular. Moreover

$$f^{-1}(0) = \{(x', x'') \in U' \times U''|g(x') = x''\} = M \cap (U' \times U'') = M \cap W.$$

$(Z) \Rightarrow (T)$:
Without loss of generality we may suppose that $\det \frac{\partial(f_1,\dots,f_{n-k})}{\partial(x_{k+1},\dots,x_n)}(p) \neq 0$. Let $\Psi(x) := (x', f(x)) = (x_1, \dots, x_k, f_1(x), \dots, f_{n-k}(x))$. Then

$$D\Psi(p) = \begin{pmatrix} I_k & 0 \\ * & \frac{\partial(f_1,\dots,f_{n-k})}{\partial(x_{k+1},\dots,x_n)}(p) \end{pmatrix}$$

is invertible.
By 2.1.1, there exists an open neighborhood $W_1 \subseteq W$ of p in $\mathbb{R}^n$, and some W' open in $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, such that $\Psi : W_1 \to W'$ is a diffeomorphism. We show that $\Psi(M \cap W_1) = (\mathbb{R}^k \times \{0\}) \cap W'$:

$\subseteq$: $\Psi(M \cap W_1) \subseteq \Psi(W_1) = W'$ and $x \in M \cap W_1 \Rightarrow f(x) = 0$
$\Rightarrow \Psi(x) = (x', f(x)) = (x', 0) \in \mathbb{R}^k \times \{0\}$.

$\supseteq$:

$$\left.\begin{array}{l} y \in W' \Rightarrow y = \Psi(x) = (x', f(x)) \text{ with } x \in W_1 \\ f(x) = 0 \Rightarrow x \in f^{-1}(0) = W \cap M \end{array}\right\} \Rightarrow x \in W_1 \cap M$$

$\Rightarrow y = \Psi(x) \in \Psi(M \cap W_1)$.

Moreover, $\psi := \Psi|_{W_1 \cap M} : W_1 \cap M \to W' \cap (\mathbb{R}^k \times \{0\})$ is a homeomorphism: it is clearly continuous and bijective, and $\psi^{-1} = \Psi^{-1}|_{(W' \cap (\mathbb{R}^k \times \{0\}))}$ is continuous.
$(T) \Rightarrow (P)$:
Let $\Phi : W' \to W$ be the inverse of Ψ and denote by $\varphi : (\mathbb{R}^k \times \{0\}) \cap W' =: U \subseteq$

$\mathbb{R}^k \times \{0\} \cong \mathbb{R}^k \to \mathbb{R}^n$ the map $(x_1, \dots, x_k) \mapsto \Phi(x_1, \dots, x_k, 0, \dots, 0)$, i.e., $\varphi = \Phi \circ i$ with $i : \mathbb{R}^k \hookrightarrow \mathbb{R}^n$. Then φ is an immersion since $D\varphi = D\Phi \circ Di$ is injective. Moreover,

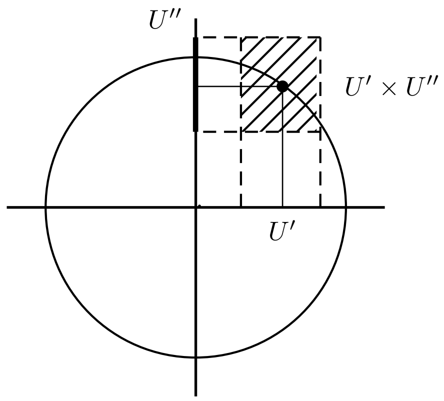
$$\varphi(U) = \Phi((\mathbb{R}^k \times \{0\}) \cap W') = \Psi^{-1}((\mathbb{R}^k \times \{0\}) \cap W') = M \cap W.$$

Finally, $\varphi : (\mathbb{R}^k \times \{0\}) \cap W' \to M \cap W$ is a homeomorphism, since it is bijective, continuous, and: $\varphi^{-1} = \Psi|_{M \cap W}$ is continuous. $\square$

2.1.9 Examples. (cf. 2.1.7!)

(i) Circle $M = \{(x, y) \mid x^2 + y^2 = R^2\}$

- *Local Zero Set*: $W := \mathbb{R}^2 \setminus \{(0, 0)\}$, $f : W \to \mathbb{R}$, $f(x, y) = x^2 + y^2 - R^2$, $M \cap W = f^{-1}(0)$.
- *Local Graph*: $M \cap (U' \times U'') = graph(g)$, $g : x \mapsto \sqrt{R^2 - x^2}$.



- *Local Trivialization*: $\Psi : (x, y) = (r \cos\varphi, r \sin\varphi) \mapsto (\varphi, r - R)$. Then locally $\psi := \Psi|_{W \cap M} = (R \cos\varphi, R \sin\varphi) \mapsto (\varphi, 0)$ (with suitable W).

(ii) Sphere in $\mathbb{R}^3$

- *Local Zero Set*: $x^2 + y^2 + z^2 = R^2$.
- *Local Graph*: $(x, y) \mapsto \sqrt{R^2 - x^2 - y^2}$
- *Local Trivialization*: Inverse spherical coordinates (with fixed radius).

(iii) Let $U \subseteq \mathbb{R}^n$ be open. Then U is a submanifold of $\mathbb{R}^n$ with local parametrization $\mathrm{id} : U \to U$.

For example, $\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n^2} \mid \det A \neq 0\}$ is open in $\mathbb{R}^{n^2}$ since $\det : \mathbb{R}^{n^2} \to \mathbb{R}$ is continuous (even $\mathcal{C}^\infty$) $\Rightarrow \mathrm{GL}(n, \mathbb{R})$ is an n^2-dimensional submanifold of $\mathbb{R}^{n^2}$.

(iv) An example of a matrix group as a submanifold.
Let $\mathrm{SL}(n, \mathbb{R}) := \{A \in \mathbb{R}^{n^2} \mid \det A = 1\} \subseteq \mathrm{GL}(n, \mathbb{R})$. Hence $\mathrm{SL}(n, \mathbb{R})$ is given as the zero set of the smooth map $f(A) = \det A - 1$. By 2.1.8 (Z) it therefore remains to show that f is regular in any $A \in \mathrm{SL}(n, \mathbb{R})$ (note that if a map is regular in one point then it is regular in a whole neighborhood of that point

since a sub-determinant of the Jacobian is nonzero in the point, hence in a neighborhood by continuity). Thus let $A \in \mathrm{SL}(n, \mathbb{R})$. Then

$$\begin{aligned} Df(A) \cdot A &= \frac{d}{dt}\Big|_0 f((1+t)A) = \frac{d}{dt}\Big|_0 (\det(1+t)A - 1) \\ &= n(1+t)^{n-1} \det A\big|_{t=0} = n \det A \neq 0, \end{aligned}$$

so for all $r \in \mathbb{R}$ we have $Df(A)(\frac{r}{n \det A} A) = r$, i.e., f is regular near A. By 2.1.8, $\mathrm{SL}(n, \mathbb{R})$ is a submanifold of $\mathbb{R}^{n^2}$ of dimension $n^2 - 1$ (in fact $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SL}(n; \mathbb{R})$ are examples of *Lie groups*).

Our next aim is to do analysis on submanifolds of $\mathbb{R}^n$. We begin by introducing the notion of smooth map on submanifolds:

2.1.10 Definition. *Let $M \subseteq \mathbb{R}^m$ and $N \subseteq \mathbb{R}^n$ be submanifolds. A map $f : M \to N$ is called smooth (or $\mathcal{C}^\infty$), if for all $p \in M$ there exists some open neighborhood U_p of p in $\mathbb{R}^m$ and some smooth map $\tilde{f} : U_p \to \mathbb{R}^n$ with $\tilde{f}|_{M \cap U_p} = f|_{M \cap U_p}$.*
If f is bijective and both f and f^{-1} are smooth, then f is called diffeomorphism.

2.1.11 Remark.

(i) The case $N = \mathbb{R}^n$ is included as a special case of the above definition.

(ii) The composition of smooth maps is smooth: Let $f_1 : M_1 \to M_2, f_2 : M_2 \to M_3$ be smooth, $p \in M_1$, and $\tilde{f}_1 : U_p \to \mathbb{R}^{m_2}$, $\tilde{f}_2 : U_{f_1(p)} \to \mathbb{R}^{m_3}$ smooth extensions. Then (since $\tilde{f}_1$ is smooth, hence continuous): $\tilde{f}_1^{-1}(U_{f_1(p)}) \cap U_p$ is an open neighborhood of p and $\tilde{f}_2 \circ \tilde{f}_1 : \tilde{f}_1^{-1}(U_{f_1(p)}) \cap U_p \to \mathbb{R}^{m_3}$ is a smooth extension of $f_2 \circ f_1$.

2.1.12 Definition. *Let M be a k-dimensional submanifold of $\mathbb{R}^n$. A chart (ψ, V) of M is a diffeomorphism of an open set $V \subseteq M$ onto an open subset of $\mathbb{R}^k$.*

Charts are the inverses of local parametrizations in the following sense:

2.1.13 Proposition. *Let M be a k-dimensional submanifold of $\mathbb{R}^n$.*

(i) Let $\varphi : U \subseteq \mathbb{R}^k \to \mathbb{R}^n$ (U open) be a local parametrization of M, $\varphi(U) = W \cap M$ ($W \subseteq \mathbb{R}^n$ open). Then $\psi := \varphi^{-1} : W \cap M \to U$ is a chart of M.

(ii) Conversely, if $\psi : V \to U \subseteq \mathbb{R}^k$ is a chart of M, then $\varphi := \mathrm{id}_{M \hookrightarrow \mathbb{R}^n} \circ \psi^{-1} : U \to \mathbb{R}^n$ is a local parametrization of M.

Proof.

(i) By 2.1.10, φ is a smooth map from U to $W \cap M$. Also, φ is bijective. It remains to prove that $\psi = \varphi^{-1} : W \cap M \to U$ is smooth in the sense of 2.1.10, i.e., possesses a smooth extension to some neighborhood of any given point of $W \cap M$.

Let $p \in W \cap M$ and set $x_0' := \psi(p) \in U$. Here we employ the notations of 2.1.8: $x' := (x_1, \ldots, x_k)$, $x'' := (x_{k+1}, \ldots, x_n)$, $\varphi' := (\varphi_1, \ldots, \varphi_k)$, $\varphi'' := (\varphi_{k+1}, \ldots, \varphi_n)$. φ is an immersion, so without loss of generality we may suppose that $\frac{\partial(\varphi_1, \ldots, \varphi_k)}{\partial(x_1, \ldots, x_k)}(x_0')$ is invertible.

Let $\Phi : U \times \mathbb{R}^{n-k} \to \mathbb{R}^n, \Phi(x', x'') := (\varphi'(x'), \varphi''(x') + x'') = \varphi(x') + (0, x'')$. In particular: $\Phi(x', 0) = \varphi(x')$. Then

$$D\Phi(x_0', 0) = \begin{pmatrix} D\varphi'(x_0') & 0 \\ D\varphi''(x_0') & I_{n-k} \end{pmatrix}$$

is invertible. By 2.1.1, Φ is a local diffeomorphism of $U_1 \times U_2$ onto some W_1, where U_1, U_2 are open neighborhoods of x_0' in U respectively of 0 in $\mathbb{R}^{n-k}$. Since $p = \Phi(x_0', 0) \in W_1$ we may w.l.o.g. suppose that $W_1 \subseteq W$.

We have $\varphi(U_1) = \Phi(U_1 \times \{0\}) \subseteq W_1 \subseteq W$. Since φ is a homeomorphism there exists some open subset W_2 of $\mathbb{R}^n$ with $\varphi(U_1) = W_2 \cap M$. W.l.o.g. we may suppose that $W_2 \subseteq W_1$ (otherwise replace W_2 by $W_2 \cap W_1$). Let $\Psi : W_1 \to U_1 \times U_2$ be the inverse of Φ.

Then for $q \in W_2 \cap M$ we have $q = \varphi(x') = \Phi(x', 0)$ for some $x' \in U_1$. Since $(x', 0) \in U_1 \times U_2$ we get $\psi(q) = \varphi^{-1}(q) = x' = \mathrm{pr}_1 \circ \Psi(q)$. Hence $\mathrm{pr}_1 \circ \Psi$ is a smooth extension of ψ to the neighborhood W_2 of p, so ψ is smooth at p, as claimed.

(ii) Let $\psi : V \to U \subseteq \mathbb{R}^k$ be a chart, and set $\varphi := \mathrm{id}_{M \hookrightarrow \mathbb{R}^n} \circ \psi^{-1} : U \to \mathbb{R}^n$. Then φ is smooth and $\varphi : U \to V$ is a homeomorphism (since $\psi : V \to U$ is).

Finally, φ is an immersion: let $\tilde{\psi}$ be a smooth extension of ψ (to some open neighborhood), then $\tilde{\psi} \circ \varphi = \psi \circ \varphi = \mathrm{id}_U$, so $D\tilde{\psi}(\varphi(x)) \cdot D\varphi(x) = \mathrm{id}_U(x)\ \forall x \in U$, so $D\varphi(x)$ is injective.

□

2.1.14 Remark. If Ψ is a trivialization as in 2.1.8 (T), $\Psi : W \to W'$, $\Psi(W \cap M) = W' \cap (\mathbb{R}^k \times \{0\})$, then $\psi := \Psi|_{M \cap W}$ is a chart of M (cf. the proof of 2.1.8, (T)⇒(P) and 2.1.13 (i)).

If M is a k-dimensional submanifold of $\mathbb{R}^n$ and (ψ, V) is a chart of M, then for $p \in V$ we may write $\psi(p) = (\psi_1(p), \dots, \psi_k(p)) = (x_1, \dots, x_k)$. The smooth functions $\psi_i = \mathrm{pr}_i \circ \psi$ are called *local coordinate functions*, the x_i are called *local coordinates* of p.
Let M^m, N^n be submanifolds, $f : M \to N$, $p \in M$, φ a chart of M around p and ψ a chart of N around $f(p)$. Then $\psi \circ f \circ \varphi^{-1}$ is called local representation of f. We have

$$\psi \circ f \circ \varphi^{-1} : (x_1, \dots, x_m) \mapsto (\underbrace{\psi_1(f(\varphi^{-1}}_{=:f_1}(x)), \dots, \underbrace{\psi_n(f(\varphi^{-1}}_{=:f_n}(x)))).$$

The f_i are called coordinate functions of f with respect to φ, ψ.
By means of charts, smoothness of maps can be characterized without resorting to the surrounding Euclidean space, hence intrinsically:

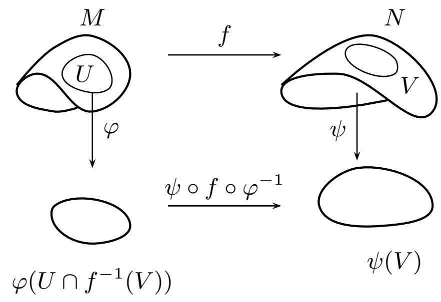
2.1.15 Proposition. *Let $M^m \subseteq \mathbb{R}^s$, $N^n \subseteq \mathbb{R}^t$ be submanifolds and $f : M \to N$. TFAE:*

(i) f is smooth.

(ii) For all $p \in M$ there exist charts (φ, U) of M at p, (ψ, V) of N at $f(p)$ such that the domain $\varphi(U \cap f^{-1}(V))$ of the local representation $\psi \circ f \circ \varphi^{-1}$ is open and $\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \psi(V)$ is smooth.

(iii) f is continuous and for all $p \in M$ there exist charts (φ, U) of M at p, (ψ, V) of N at $f(p)$ such that the local representation $\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \psi(V)$ is smooth.

(iv) f is continuous and for all $p \in M$, all charts (φ, U) of M at p and all charts (ψ, V) of N at $f(p)$, the local representation $\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \psi(V)$ is smooth.

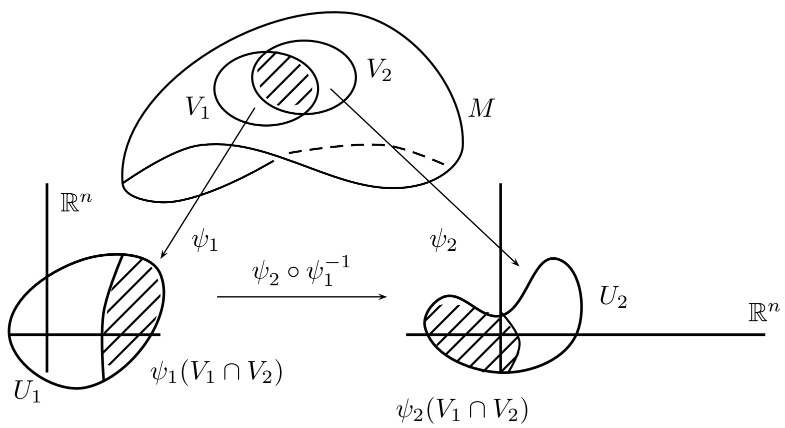


Proof. (i)⇒(iv): f is continuous since around any point it is the restriction of a continuous map. Hence $f^{-1}(V)$ and therefore also $\varphi(U \cap f^{-1}(V))$ is open. By 2.1.11 (ii), the map $\psi \circ f \circ \varphi^{-1}$ (whose domain of definition is $\varphi(U \cap f^{-1}(V))$) is smooth.
(iv)⇒(iii), and (iii)⇒(ii) are clear.
(ii)⇒(i): On the open neighborhood $U \cap f^{-1}(V)$ of p we have $f = \psi^{-1} \circ (\psi \circ f \circ \varphi^{-1}) \circ \varphi$, so f is smooth by 2.1.11 (ii). □

2.2 Abstract Manifolds

In what follows we want to extend the concept of differentiable manifolds to sets which a priori are not realized as subsets of some $\mathbb{R}^n$. The key to this generalization of the notion of submanifold of $\mathbb{R}^n$ is the formulation of the properties we derived in the previous section in terms of charts. These will allow to dispense with the surrounding Euclidean space.

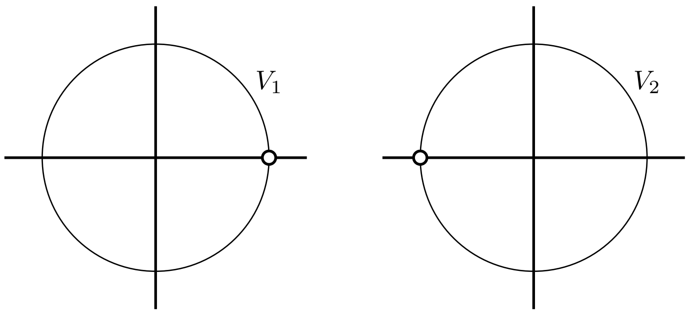
2.2.1 Definition. *Let M be a set. A* chart (ψ, V) *of M is a bijective map ψ of $V \subseteq M$ onto an open subset U of $\mathbb{R}^n$, $\psi : V \to U$. Two charts (ψ_1, V_1), (ψ_2, V_2) are called* ($\mathcal{C}^\infty-$) compatible *if $\psi_1(V_1 \cap V_2)$ and $\psi_2(V_1 \cap V_2)$ are open in $\mathbb{R}^n$ and the* change of charts $\psi_2 \circ \psi_1^{-1} : \psi_1(V_1 \cap V_2) \to \psi_2(V_1 \cap V_2)$ *is a $\mathcal{C}^\infty$-diffeomorphism (note that this condition is symmetric in ψ_1, ψ_2).*



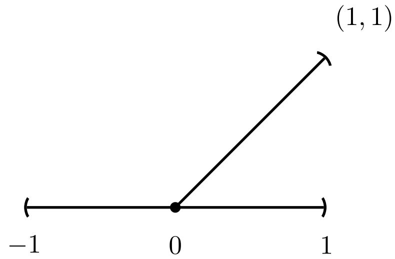
A $\mathcal{C}^\infty$-atlas *of M is a family $\mathcal{A} = \{(\psi_\alpha, V_\alpha) \mid \alpha \in A\}$ of pairwise compatible charts such that $M = \bigcup_{\alpha \in A} V_\alpha$. Two atlasses $\mathcal{A}_1, \mathcal{A}_2$ are called equivalent if $\mathcal{A}_1 \cup \mathcal{A}_2$ itself is an atlas of M, i.e., if all charts of $\mathcal{A}_1 \cup \mathcal{A}_2$ are compatible. An (abstract)* differentiable manifold *is a set M together with an equivalence class of atlasses. Such an equivalence structure is called a differentiable (or $\mathcal{C}^\infty$-)structure on M.*

2.2.2 Examples.

(i) Let$S^1 = \{(x, y) \mid x^2+y^2 = 1\} \subseteq \mathbb{R}^2$ and set $V_1 := \{(\cos\varphi, \sin\varphi) \mid 0 < \varphi < 2\pi\}$ and $\psi_1 : V_1 \to (0, 2\pi),\ (\cos\varphi, \sin\varphi) \mapsto \varphi$. Let $V_2 := \{(\cos\varphi, \sin\varphi) \mid -\pi < \varphi < \pi\},\ \psi_2 : V_2 \to (-\pi, \pi),\ (\cos\varphi, \sin\varphi) \mapsto \varphi$. Then (ψ_1, V_1) and (ψ_2, V_2) are charts for S^1 and $S^1 = V_1 \cup V_2$. Moreover, ψ_1 and ψ_2 are compatible. In fact, $\psi_1(V_1 \cap V_2) = (0, \pi) \cup (\pi, 2\pi)$ and $\psi_2 \circ \psi^{-1}\big|_{(0,\pi)} = \varphi \mapsto \varphi$. We have $\psi_2 \circ \psi_1^{-1}|_{(\pi,2\pi)} = \varphi \mapsto \varphi - 2\pi$, so the change of charts $\psi_2 \circ \psi_1^{-1} : \psi_1(V_1 \cap V_2) \to \psi_2(V_1 \cap V_2)$ is a diffeomorphism. Hence $\mathcal{A} := \{(\psi_1, V_1), (\psi_2, V_2)\}$ is an atlas of S^1.



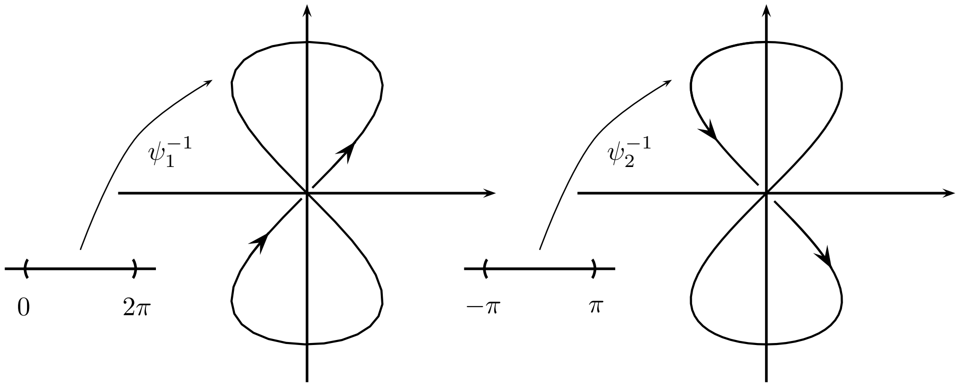
(ii) Let M be the following subset of $\mathbb{R}^n$: Let $V_1 := \{(s, 0) | -1 < s < 1\},\ \psi_1 : V_1 \to (-1, 1),\ \psi(s, 0) = s$. Further, let $V_2 := \{(s, 0) | -1 < s \le 0\} \cup \{(s, s) | 0 < s < 1\},\ \psi_2 : V_2 \to (-1, 1),\ \psi_2(s, 0) = s,\ \psi_2(s, s) = s$.



Then ψ_1, ψ_2 are bijective, hence charts, and $\psi_2 \circ \psi_1^{-1} = s \mapsto s$.

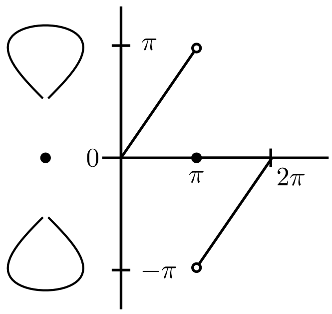
However, $\psi_1(V_1 \cap V_2) = (-1, 0]$ is not open, so ψ_1, ψ_2 are *not* compatible. In fact M also can't be a submanifold of $\mathbb{R}^n$ (same argument as in 2.1.7(iii)).

(iii) As in 2.1.7 (iii) let $M := \{(\sin 2s, \sin s) | s \in \mathbb{R}\}$ be the figure eight manifold. Let $V_1 = M,\ \psi_1 : V_1 \to (0, 2\pi),\ \psi(\sin 2s, \sin s) = s$. Then ψ_1 is a chart and $\mathcal{A}_1 := \{(\psi_1, V_1)\}$ is an atlas defining a $\mathcal{C}^\infty$-structure on M.



On the other hand, let $V_2 = M$, $\psi_2 : V_2 \to (-\pi, \pi)$, $\psi_2(\sin 2s, \sin s) = s$. Then also $\mathcal{A}_2 := \{(\psi_2, V_2)\}$ is an atlas. However, $\mathcal{A}_1$ and $\mathcal{A}_2$ are *not* equivalent: $\psi_2 \circ \psi^{-1} : (0, 2\pi) \to (-\pi, \pi)$,

$$\psi_2 \circ \psi^{-1}(s) = \begin{cases} s & 0 < s < \pi & \text{upper loop} \\ s - \pi & s = \pi & \text{origin} \\ s - 2\pi & \pi < s < 2\pi & \text{lower loop} \end{cases}$$



Hence $\psi_2 \circ \psi_1^{-1}$ is not even continuous.

Thus M can be endowed with different $\mathcal{C}^\infty$-structures. With any such structure, M is an example of a $\mathcal{C}^\infty$-manifold which is not a submanifold of $\mathbb{R}^2$ (cf. 2.1.7 (iii)!).

(iv) One can show that for $n \neq 4$, up to diffeomorphism there is precisely one $\mathcal{C}^\infty$-structure on $\mathbb{R}^n$. On $\mathbb{R}^4$ however, there are uncountably many inequivalent smooth structures!

An atlas for a manifold is called maximal if it is not contained in any strictly larger atlas.

2.2.3 Proposition. *Let M be a $\mathcal{C}^\infty$-manifold with atlas $\mathcal{A}$. Then there is a unique maximal atlas on M which contains $\mathcal{A}$.*

Proof. Let $\tilde{\mathcal{A}} := \{\varphi | \varphi$ is a chart of M and φ is compatible with every $\psi \in \mathcal{A}\}$. Then $\tilde{\mathcal{A}} \supseteq \mathcal{A}$ and we show that $\tilde{\mathcal{A}}$ itself is an atlas.
Let $(\varphi_1, W_1),\ (\varphi_2, W_2) \in \tilde{\mathcal{A}}$ with $W_1 \cap W_2 \neq \emptyset$. Then since φ_1, φ_2 are bijective, so is $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(W_1 \cap W_2) \to \varphi_2(W_1 \cap W_2)$. It remains to show that $\varphi_2 \circ \varphi_1^{-1}$ is

a diffeomorphism whose domain $\varphi_1(W_1 \cap W_2)$ is open. Let $x \in \varphi_1(W_1 \cap W_2)$ and (ψ, V) a chart in $\mathcal{A}$ with $\varphi_1^{-1}(x) \in V$. By definition of $\tilde{\mathcal{A}}$, $\varphi_2 \circ \psi^{-1} : \psi(W_2 \cap V) \to \varphi_2(W_2 \cap V)$ and $\psi \circ \varphi_1^{-1} : \varphi_1(W_1 \cap V) \to \psi(W_1 \cap V)$ are diffeomorphisms between open subsets of $\mathbb{R}^n$. Therefore, $(\varphi_2 \circ \psi^{-1}) \circ (\psi \circ \varphi_1^{-1})$ is a diffeomorphism with domain $(\psi \circ \varphi_1^{-1})^{-1}(\psi(W_2 \cap V)) = \varphi_1(W_1 \cap W_2 \cap V)$.
Note that

$$\varphi_1(W_1 \cap W_2 \cap V) = \varphi_1 \circ \psi^{-1}(\psi(V \cap W_1 \cap W_2)) = \varphi_1 \circ \psi^{-1}(\psi(V \cap W_1) \cap \psi(V \cap W_2))$$

is open. Summing up, for all $x \in \varphi_1(W_1 \cap W_2)$ there exists an open neighborhood $\varphi_1(W_1 \cap W_2 \cap V) \subseteq \varphi_1(W_1 \cap W_2)$, on which $\varphi_2 \circ \varphi_1^{-1}$ is a diffeomorphism. Moreover, $\varphi_2 \circ \varphi_1^{-1}$ is bijective on the open set $\varphi_1(W_1 \cap W_2)$. Thus $\varphi_2 \circ \varphi_1^{-1}$ is a diffeomorphism, so φ_1 and φ_2 are compatible.
Maximality and uniqueness of $\tilde{\mathcal{A}}$ are clear. $\square$

From now on, whenever a smooth manifold M is given, by a chart of M we mean an element of the maximal atlas of M.
Next we want to equip any smooth manifold with a natural topology induced by its charts. We will make use of the following auxiliary result:

2.2.4 Lemma. *Let M be a smooth manifold, (ψ, V) a chart of M and $W \subseteq V$ such that $\psi(W)$ is open in $\mathbb{R}^n$. Then also $(\psi|_W, W)$ is a chart of M.*

Proof. $\psi|_W : W \to \psi(W)$ is bijective. Let (φ, U) be another chart of M. We have to show that $\psi|_W$ and φ are compatible. Now $\psi|_W \circ \varphi^{-1} : \varphi(U \cap W) \to \psi(U \cap W)$ is bijective and is the restriction of the diffeomorphism $\psi \circ \varphi^{-1}$ to $\varphi(U \cap W)$. Also,

$$\varphi(U \cap W) = \varphi \circ \psi^{-1}(\psi(U \cap W)) = \varphi \circ \psi^{-1}(\psi(U \cap V) \cap \psi(W))$$

is open. Thus $\psi|_W \circ \varphi^{-1}$ itself is a diffeomorphism, so $\psi|_W \in \mathcal{A}$. $\square$

2.2.5 Proposition. *Let M be a manifold with maximal atlas $\mathcal{A} = \{(\psi_\alpha, V_\alpha) | \alpha \in A\}$. Then $\mathcal{B} := \{V_\alpha | \alpha \in A\}$ is the basis of a topology, the so-called* natural *or* manifold topology *of M.*

Proof. Clearly $\bigcup_{\alpha \in A} V_\alpha = M$. For $\alpha, \beta \in A$, $\psi_\alpha(V_\alpha \cap V_\beta)$ is open in $\mathbb{R}^n$ (since ψ_α and ψ_β are compatible), hence by 2.2.4, $(\psi_\alpha|_{V_\alpha \cap V_\beta}, V_\alpha \cap V_\beta)$ itself is an element of $\mathcal{A}$. Therefore, $V_\alpha \cap V_\beta \in \mathcal{B}$ and so $\mathcal{B}$ is the basis of a uniquely defined topology. $\square$

2.2.6 Proposition. *With respect to the manifold topology of M, any chart (ψ, V) is a homeomorphism of the open subset V of M onto the open subset $\psi(V)$ of $\mathbb{R}^n$.*

Proof. Let $\psi : V \to U$ be a chart of M. Then by 2.2.5, V is open in M. We first show that ψ is continuous. Let $U_1 \subseteq U$ be open and $W_1 := \psi^{-1}(U_1)$. By 2.2.4, $(\psi|_{W_1}, W_1)$ is a chart of M, so $W_1 \in \mathcal{B}$, hence open in M. It remains to show that ψ is open (so that ψ^{-1} is continuous). To this end it suffices to show that ψ maps any $W \in \mathcal{B}$ with $W \subseteq V$ to an open subset of $\mathbb{R}^n$.
For such a W, by 2.2.5 there exists a chart φ with domain W. Hence $\varphi \circ \psi^{-1} : \psi(W \cap V) \to \varphi(W \cap V)$ is a diffeomorphism. In particular, $\psi(W \cap V) = \psi(W)$ is open. $\square$

2.2.7 Lemma. *Let M be a set, $\mathcal{A}$ a $\mathcal{C}^\infty$-atlas of M, τ the manifold topology induced by $\mathcal{A}$ and τ' another topology on M. TFAE:*

(i) $\tau = \tau'$

(ii) *If* $(\psi, V) \in \mathcal{A}$*, then* $V \in \tau'$ *and* $\psi : V \to \psi(V)$ *is a homeomorphism with respect to* τ'*.*

Proof. $(i) \Rightarrow (ii)$ is immediate from 2.2.6.
$(ii) \Rightarrow (i)$: Let $p \in M$, $(\psi, V) \in \mathcal{A}$ with $p \in V$. Let $\mathcal{U}$ be a basis of neighborhoods of $\psi(p)$ in $\psi(V) \subseteq \mathbb{R}^n$. Then $(\psi^{-1}(U))_{U \in \mathcal{U}}$ is a basis neighborhoods of p with respect to τ and also with respect to τ'. It follows that every $p \in M$ has the same neighborhoods with respect to τ and τ', so $\tau = \tau'$. □

After these preparations we are now in a position to completely clarify the relationship between submanifolds of $\mathbb{R}^n$ and abstract manifolds.

2.2.8 Theorem. *Let* M *be an* m*-dimensional submanifold of* $\mathbb{R}^n$*. Then* M *is an* m*-dimensional* $\mathcal{C}^\infty$*-manifold in the sense of 2.2.1. The manifold topology of* M *coincides with the trace topology of* $\mathbb{R}^n$ *on* M*.*

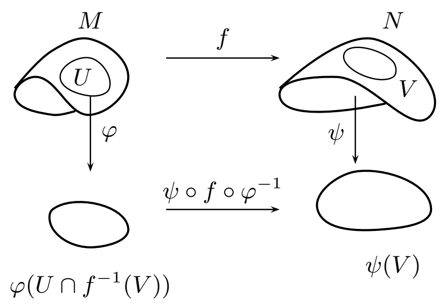
Proof. As an atlas of M we pick the family of all $\psi = \varphi^{-1}$, where φ is a local parametrization. By 2.1.13 these are precisely the charts in the sense of 2.1.12. By 2.1.15 (iii) (with $f = \mathrm{id}_M$) all changes of charts are diffeomorphisms, so M is a smooth manifold in the sense of 2.2.1. According to 2.1.5, every φ is a homeomorphism with respect to the trace topology of $\mathbb{R}^n$ on M. Hence by 2.2.7 the trace topology of $\mathbb{R}^n$ is precisely the manifold topology. □

From 2.1.15 we may distill an appropriate definition of smoothness for mappings between abstract manifolds:

2.2.9 Definition. *Let* M*,* N *be* $\mathcal{C}^\infty$*-manifolds and* $f : M \to N$ *a map.* f *is called smooth (*$\mathcal{C}^\infty$*) if it is continuous and for all* $p \in M$ *there exists a chart* φ *of* M *around* p *and a chart* ψ *of* N *around* $f(p)$ *such that* $\psi \circ f \circ \varphi^{-1}$ *is smooth.* f *is called diffeomorphism if it is bijective and* f *and* f^{-1} *are smooth.*

2.2.10 Remark.

(i)



Let (φ, U), (ψ, V) be charts as above. Then the domain of definition of $\psi \circ f \circ \varphi^{-1}$ is $\varphi(U \cap f^{-1}(V))$. This set is open since f is continuous and φ is a homeomorphism.

Conversely, if $f : M \to N$ is some map such that for all $p \in M$ there exists a chart φ of M around p and a chart ψ of N around $f(p)$ such that $\varphi(U \cap f^{-1}(V))$ is open and $\psi \circ f \circ \varphi^{-1}$ is smooth, then f is smooth. In fact, f is continuous since $f = \psi^{-1} \circ (\psi \circ f \circ \varphi^{-1}) \circ \varphi$ on the open set $U \cap f^{-1}(V)$ (cf. also 2.1.15(ii)).

(ii) If $(\tilde\varphi, \tilde U)$, $(\tilde\psi, \tilde V)$ are further charts around p resp. $f(p)$, then also $\tilde\psi \circ f \circ \tilde\varphi^{-1}$ is smooth: near p we have

$$\tilde\psi \circ f \circ \tilde\varphi^{-1} = (\underbrace{\tilde\psi \circ \psi^{-1}}_{\mathcal{C}^\infty}) \circ (\underbrace{\psi \circ f \circ \varphi^{-1}}_{\mathcal{C}^\infty}) \circ (\underbrace{\varphi \circ \tilde\varphi^{-1}}_{\mathcal{C}^\infty}).$$

Since p was arbitrary, $\tilde\psi \circ f \circ \tilde\varphi^{-1}$ is smooth on its entire domain of definition.

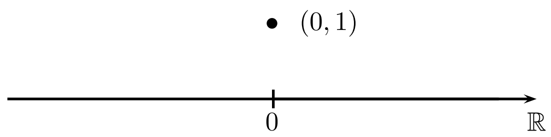
(iii) Obviously the composition of smooth maps is smooth.

2.3 Topological Properties of Manifolds

2.3.1 Proposition. *Every manifold M satisfies the separation axiom T_1.*

Proof. Let $p_1 \neq p_2 \in M$. If there exists a chart (ψ, V) with $p_1,\ p_2 \in V$ then there exist U_1, U_2 open in $\psi(V)$ such that $\psi(p_1) \in U_1,\ \psi(p_2) \in U_2,\ U_1 \cap U_2 = \emptyset$. Hence $\psi^{-1}(U_1)$ and $\psi^{-1}(U_2)$ are disjoint neighborhoods of p_1 resp. p_2. Otherwise there exists a chart (ψ_1, V_1) with $p_1 \in V_1$ and $p_2 \notin V_1$ and vice versa. $\square$

2.3.2 Example. The natural topology of a manifold is *not* automatically T_2 (Hausdorff): Let M be the following set:



Let $V_1 = \{(s,0) | s \in \mathbb{R}\}$, $V_2 := \{(s,0) | s \neq 0\} \cup \{(0,1)\}$, $\psi_1 : V_1 \to \mathbb{R}$, $\psi_1(s,0) = s$, $\psi_2 : V_2 \to \mathbb{R}$, $\psi_2(s,0) = s\ (s \neq 0)$, $\psi_2(0,1) = 0$. Then $\psi_2 \circ \psi_1^{-1} : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$, $s \mapsto s$. Therefore $\mathcal{A} := \{\psi_1, \psi_2\}$ is a $\mathcal{C}^\infty$-atlas for M. However, M is not T_2 since $(0,0)$ and $(0,1)$ cannot be separated by open sets in M. In fact, let V, W be open in M, $(0,0) \in V$, $(0,1) \in W$. Then $\psi_1(V_1 \cap V)$, $\psi_2(V_2 \cap W)$ are open in $\mathbb{R}$ and contain 0. Hence they contain some $a \neq 0$, so $\psi_1^{-1}(a) = (a,0) = \psi_2^{-1}(a) \in V_1 \cap V \cap V_2 \cap W \subseteq V \cap W$. Thus $V \cap W \neq \emptyset$, so M is not Hausdorff.

2.3.3 Proposition. *Every manifold satisfies the first axiom of countability, i.e., each of its points possesses a countable basis of neighborhoods.*

Proof. Let $p \in M$, and (ψ, V) a chart around p. Then there exists a countable basis of neighborhoods $(U_m)_{m\in\mathbb{N}}$ of $\psi(p)$ in $\psi(V)$. Hence $(\psi^{-1}(U_m))_{m\in\mathbb{N}}$ is a countable basis of neighborhoods of p in M. $\square$

2.3.4 Proposition. *Every manifold is locally pathwise connected.*

Proof. Let $p \in M$ and (ψ, V) a chart around p such that $\psi(V)$ is pathwise connected (e.g., $\psi(V)$ a ball in $\mathbb{R}^n$, cf. 2.2.4). For $q \in V$ there exists a continuous map $c : [0,1] \to \psi(V)$ with $c(0) = \psi(p)$, $c(1) = \psi(q)$, hence $\tilde c := \psi^{-1} \circ c : [0,1] \to M$, $\tilde c(0) = p$, $\tilde c(1) = q$. $\square$

2.3.5 Corollary. *Every connected manifold is pathwise connected.*

2.3.6 Proposition. *Every Hausdorff manifold is locally compact.*

Proof. Let $p \in M$ and let (ψ, V) be a chart around p. Let B be a closed ball with center $\psi(p)$ in $\mathbb{R}^n$ and $B \subseteq \psi(V)$. Then since ψ is a homeomorphism, $\psi^{-1}(B)$ is a compact neighborhood of p in M. $\square$

2.3.7 Proposition. *Let M be a manifold. TFAE:*

(i) M satisfies the second axiom of countability (i.e., M possesses a countable basis of its topology, or: M is second countable).

(ii) M possesses a countable atlas.

Proof. (i)$\Rightarrow$(ii): Let $\mathcal{B}$ be a countable basis of the topology of M and let $\mathcal{A} = \{(\psi_\alpha, V_\alpha)|\alpha \in A\}$ be an atlas of M. Then by 2.2.4, $\tilde{\mathcal{A}} := \{(\psi_\alpha|_B, B)|B \in \mathcal{B},\ B \subseteq V_\alpha$ for some $\alpha \in A\}$ is a countable atlas of M.
(ii)$\Rightarrow$(i): Let $\mathcal{A} = \{(\psi_\alpha, V_\alpha)|\alpha \in \mathbb{N}\}$ be a countable atlas of M. Every $U_\alpha = \psi_\alpha(V_\alpha)$ is open in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is second countable there are open sets U_{α_i} $(i \in \mathbb{N})$ in $\mathbb{R}^n$ such that $\{U_{\alpha_i}|i \in \mathbb{N}\}$ is a basis of U_α. Hence every open subset V of V_α is the union of certain $\psi_\alpha^{-1}(U_{\alpha_i})$. Since any open $W \subseteq M$ is the union of certain $W \cap V_\alpha$, $\{V_{\alpha_i}|\alpha \in \mathbb{N}, i \in \mathbb{N}\}$ is a countable basis of the manifold topology of M. $\square$

2.3.8 Corollary. *Every compact manifold is second countable.*

Proof. We may even select a finite atlas from any given atlas. $\square$

In differential geometry and analysis on manifolds one frequently encounters problems which can easily be solved locally (in a chart domain). To obtain global statements, one has to 'patch together' these local constructions. The most important tool in this context are the so-called *partitions of unity*:

2.3.9 Definition. *Let M be a manifold. The* support *of any $f : M \to \mathbb{R}$ is defined as the set* $\mathrm{supp}(f) := \overline{\{p \in M|f(p) \neq 0\}}$. *A family $\mathcal{V}$ of subsets of M is called* locally finite *if every $p \in M$ possesses a neighborhood which intersects only finitely many $V \in \mathcal{V}$. Let $\mathcal{U}$ be an open cover of M. A* partition of unity *subordinate to $\mathcal{U}$ is a family $\{\chi_\alpha|\alpha \in A\}$ of smooth maps $\chi_\alpha : M \to \mathbb{R}^+$ such that:*

(i) $\{\mathrm{supp}\chi_\alpha|\alpha \in A\}$ *is locally finite.*

(ii) For all $\alpha \in A$ there exists some $U \in \mathcal{U}$ such that $\mathrm{supp}(\chi_\alpha) \subseteq U$.

(iii) For all $p \in M$, $\sum_{\alpha \in A} \chi_\alpha(p) = 1$

Note that by (i) the sum in (iii) is finite for any $p \in M$.
Our next goal is to prove the following result:

2.3.10 Theorem. *Let M be a second countable Hausdorff manifold. Then for any open cover $\mathcal{U}$ of M there exists a partition of unity $\{\chi_j|j \in \mathbb{N}\}$ subordinate to $\mathcal{U}$ such that, for all j,* $\mathrm{supp}\chi_j$ *is compact and contained in a chart domain.*

To prepare the proof we need several auxilliary results. To begin with, we show that there exist smooth functions on $\mathbb{R}$ of arbitrarily small support:

2.3.11 Lemma. *Let $f : \mathbb{R} \to \mathbb{R}$,*

$$f(x) := \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x}} & x > 0 \end{cases}$$

Then f is smooth.

Proof. By induction we obtain that

$$f^{(n)}(x) := \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x}} P_n(\frac{1}{x}) & x > 0 \end{cases}$$

where P_n is a polynomial. Hence $\lim_{x \nearrow 0} f^{(n)}(x) = \lim_{x \searrow 0} f^{(n)}(x) = 0$ for all n. $\square$

2.3.12 Lemma. *Let M be a Hausdorff manifold, U an open subset of M and $p \in U$. Then there exists a chart neighborhood V of p and a $\mathcal{C}^\infty$-function $\chi : M \to \mathbb{R}^+$ such that $\overline{V}$ is compact, $\overline{V} \subseteq U$, $\chi > 0$ on V and $\chi \equiv 0$ on $M \backslash V$.*

Proof. Choose a chart (ψ, W) around p such that $W \subseteq U$ and $\psi(p) = 0$. Let $r > 0$ such that for the open ball $B_r(0)$ around 0 we have $\overline{B_r(0)} \subseteq \psi(W)$. Then $V := \psi^{-1}(B_r(0))$ is a neighborhood of p, and $\overline{V} = \psi^{-1}(\overline{B_r(0)})$ is a compact subset of W. Choose f as in 2.3.11 and let $g : \mathbb{R}^n \to \mathbb{R}^+$, $g(x) := f(r^2 - |x|^2)$. Then g is smooth, $g > 0$ on $B_r(0)$, and $g = 0$ on $\mathbb{R}^n \setminus B_r(0)$. Now let

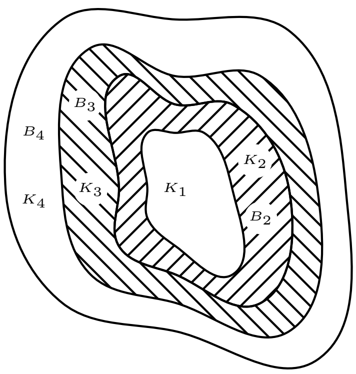
$$\chi(q) := \begin{cases} g \circ \psi(q) & q \in W \\ 0 & q \in M \setminus \overline{V} \end{cases}$$

Now W and $M \setminus \overline{V}$ are open, cover M and χ is smooth on both sets, hence on M. It follows that χ has the desired properties. $\square$

2.3.13 Lemma. *Let M be a second countable Hausdorff manifold. Then M possesses an exhaustion by compact sets: $\exists (K_j)_{j \in \mathbb{N}}$, $K_j \subset\subset M$, $K_j \subseteq K_{j+1}^\circ$ $\forall j$ and $M = \bigcup_{j \in \mathbb{N}} K_j$.*

Proof. Since M is locally compact, there exists a cover $\mathcal{V}$ of M consisting of open sets whose closure is compact. By second countability, we may extract from this a countable cover $(V_j)_{j \in \mathbb{N}}$ of M. (Let $\mathcal{B}$ be a countable basis of the topology and $\mathcal{B}' := \{B \in \mathcal{B} | \exists V_B \in \mathcal{V}$ with $B \subseteq V_B\}$. Then $\{V_B | B \in \mathcal{B}'\}$ fulfills this purpose.)
Let $K_1 := \overline{V_1} \subset\subset M$. Choose $r_2 > 1$ such that $K_1 \subseteq \bigcup_{i=1}^{r_2} V_i$ (possible since K_1 is compact). Let $W_2 := \bigcup_{i=1}^{r_2} V_i$ and $K_2 = \overline{W_2} = \bigcup_{i=1}^{r_2} \overline{V_i} \subset\subset M$. Then K_2 is compact and $K_1 \subseteq K_2^\circ$. For $j \geq 2$, suppose that $K_j = \overline{W_j}$ has already been defined. Denote by r_{j+1} the first index with $K_j \subseteq \bigcup_{i=1}^{r_{j+1}} V_i$ and set $W_{j+1} = \bigcup_{i=1}^{\max(r_{j+1}, j+1)} V_i$, $K_{j+1} := \overline{W_{j+1}} = \bigcup_{i=1}^{\max(r_{j+1}, j+1)} \overline{V_i}$. Then $K_{j+1} \subset\subset M$, $K_j \subseteq K_{j+1}^\circ$ and $\bigcup_{j=1}^\infty K_j \supseteq \bigcup_{j=1}^\infty V_j = M$. $\square$

Proof of 2.3.10 Let $(K_i)_{i \in \mathbb{N}}$ be as in 2.3.13.



Set $K_{-1} = K_0 = \emptyset$ and $B_j := K_j \setminus K_{j-1}^\circ$, so $B_j \subset\subset M$. For each $p \in B_j$ there exists a $U \in \mathcal{U}$ with $p \in U$ and (by 2.3.12) a chart neighborhood V of p with $\overline{V}$ compact, $\overline{V} \subseteq U \cap M \setminus K_{j-2} = U \setminus K_{j-2}$. Moreover, there exists $\tilde{\chi} \in \mathcal{C}^\infty(M)$ with $\tilde{\chi} > 0$ on V and $\tilde{\chi} \equiv 0$ on $M \setminus V$.
Since B_j is compact it is contained in a finite union of such V. Carrying out this construction for each $j \in \mathbb{N}$ we obtain a countable cover $(V_k)_{k\in\mathbb{N}}$ of M with corresponding $\mathcal{C}^\infty$-functions $(\chi_j)_{j\in\mathbb{N}}$. The family $(\overline{V_k})_{k\in\mathbb{N}}$ is locally finite. In fact, those $\overline{V}_k$ coming from the cover of B_j are disjoint from K_{j-2}, hence disjoint from K_l for $l \leq j-2$. Hence every $p \in M$ possesses an open neighborhood K_l° which intersects only finitely many $\overline{V_k}$. Now let $\chi_j : M \to \mathbb{R}$,

$$\chi_j := \frac{\tilde{\chi}_j}{\sum_{i\in\mathbb{N}} \tilde{\chi}_i}.$$

Then χ_j is well-defined since $\sum_{i\in\mathbb{N}} \tilde{\chi}_i > 0$ (the $(V_j)_{j\in\mathbb{N}}$ form a cover of M, and $\tilde{\chi}_j|_{V_j} > 0$). Summing up, $\chi_j \in \mathcal{C}^\infty(M, \mathbb{R}^+)$, and $\sum_{j\in\mathbb{N}} \chi_j = \frac{\sum_{j\in\mathbb{N}} \tilde{\chi}_j}{\sum_{i\in\mathbb{N}} \tilde{\chi}_i} = 1$, so $(\chi_j)_{j\in\mathbb{N}}$ is the desired partition of unity subordinate to $\mathcal{U}$. □

2.3.14 Corollary. *Let M be a second countable Hausdorff manifold and $\mathcal{U} = \{U_\alpha | \alpha \in A\}$ an open cover of M. Then there exists a partition of unity $\{\chi_\alpha | \alpha \in A\}$ with* $\mathrm{supp}\chi_\alpha \subseteq U_\alpha$ $\forall \alpha \in A$. *(The χ_α will not have compact support in general).*

Proof. Choose $\{\chi_j | j \in \mathbb{N}\}$ as in 2.3.10, subordinate to $\mathcal{U}$. Then $\forall j \in \mathbb{N}\ \exists\ \alpha_j$ with $\mathrm{supp}\chi_j \subseteq U_{\alpha_j}$. Let $\chi_\alpha = \sum_{\{j|\alpha_j=\alpha\}} \chi_j$. Then by 2.3.9 (i),

$$\mathrm{supp}\chi_\alpha = \overline{\{p \mid \chi_\alpha(p) \neq 0\}} \subseteq \overline{\bigcup_{\alpha_j=\alpha} \mathrm{supp}\chi_j} = \bigcup_{\alpha_j=\alpha} \overline{\mathrm{supp}\chi_j} = \bigcup_{\alpha_j=\alpha} \mathrm{supp}\chi_j \subseteq U_\alpha.$$

□

2.3.15 Remark. More generally, one can show: for any manifold M, the following are equivalent:

(a) For each cover $\mathcal{U}$, M possesses a partition of unity subordinate to $\mathcal{U}$.

(b) M is Hausdorff and every connected component of M is second countable.

(c) M is metrizable.

(d) M is Hausdorff and paracompact.

Convention: From now on, by a smooth manifold we will always mean a manifold (in the above sense) whose natural topology is Hausdorff and second countable.
Note that, in particular, every submanifold of $\mathbb{R}^n$ is a smooth manifold in this sense (by 2.2.8 it carries the trace topology of $\mathbb{R}^n$, hence is Hausdorff and second countable).

2.4 Differentiation, Tangent Space

After the topological interlude of the previous section we now turn to a study of analysis on manifolds. From 2.2.9 and 2.2.10 we know what smooth maps between manifolds are. However, so far we have not given a definition of the derivative of a smooth map. In $\mathbb{R}^n$, the derivative of a map is the optimal linear approximation to the map. This terminology only makes sense in the vector space setting. Manifolds, on the other hand, in general do not carry a vector space structure. Differentiation on manifolds therefore can be viewed (heuristically) as a two-step approximation process: first, in any given point the manifold is approximated by a vector space (the tangent space, corresponding to the tangent plane of a surface). The derivative itself is then defined as a linear map on this tangent space. To motivate this general procedure we first have a look at the special case of submanifolds of $\mathbb{R}^n$.

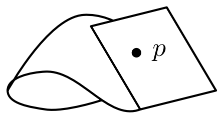
2.4.1 Theorem. *Let M be a submanifold of $\mathbb{R}^n$ and $p \in M$. Then the following subsets of $\mathbb{R}^n$ coincide:*

(i) $\mathrm{im} D\varphi(0)$ *where φ is a local parametrization of M with $\varphi(0) = p$.*

(ii) $\{c'(0) \mid c : I \to M\ \mathcal{C}^\infty, I \subseteq \mathbb{R}$ *an interval*$, c(0) = p\}$

(iii) $\ker Df(p)$, *where, locally around p, M is the zero set of the regular map $f : \mathbb{R}^n \to \mathbb{R}^{n-k}$ (with $k = \dim M$).*

(iv) $\mathrm{graph}(Dg(p'))$, *where, locally around p, M is the graph of the smooth map g and $p = (p', g(p'))$.*



Proof. $(i) \subseteq (ii)$: Given $D\varphi(0) \cdot v \in \mathrm{im} D\varphi(0)$, let $c(t) := \varphi(t \cdot v)$. Then for a suitable interval I, $c : I \to M$ is smooth, $c(0) = \varphi(0) = p$ and $c'(0) = \frac{d}{dt}\big|_0 \varphi(t \cdot v) = D\varphi(0)v \in (ii)$.
$(ii) \subseteq (iii)$: Let $c'(0) \in (ii)$, $c : I \to M$ and f as in (iii). Then locally around 0 we have $f \circ c(t) = 0$. Hence

$$0 = \frac{d}{dt}\bigg|_0 f(c(t)) = Df(\underbrace{c(0)}_{=p})c'(0) \Rightarrow c'(0) \in \ker Df(p)$$

$(iii) \subseteq (i)$: Since $(i) \subseteq (iii)$ it suffices to prove that $\dim(\mathrm{im} D\varphi(0)) = \dim \ker Df(p)$. Since φ is an immersion, $\dim(\mathrm{im} D\varphi(0)) = k = \dim M$. Moreover, $\dim(\mathrm{im} Df(p)) = n - k$, so $\dim \ker Df(p) = n - (n - k) = k$.
$(iii) = (iv)$: Let g as in (iv) (cf. 2.1.8, (Gr)$\Rightarrow$(Z)), and $f_j(x_1, \dots, x_n) := x_{k+j} - g_j(x')$ $(j = 1, \dots, n-k)$. Then locally around p, M is the zero set of f and $\ker(Df(p)) = \ker(q \mapsto q'' - Dg(p')q') = \{(q', Dg(p')q') | q' \in \mathbb{R}^k\} = \mathrm{graph}(Dg(p'))$.
□

2.4.2 Definition. *Let M be a submanifold of $\mathbb{R}^n$ and $p \in M$. The linear subspace of $\mathbb{R}^n$ characterized in 2.4.1 is called the tangent space of M at p and is denoted by T_pM ($\dim T_pM = k = \dim M$). The elements of T_pM are called tangent vectors of M at p.*
If N is a submanifold of $\mathbb{R}^{n'}$ and $f : M \to N$ is smooth, then let $T_pf : T_pM \to T_{f(p)}N$, $c'(0) \mapsto (f \circ c)'(0)$. T_pf is called the tangent map of f at p.

T_pf is well-defined: let $c_1, c_2 : I \to M$, $c_1(0) = p = c_2(0)$ be smooth with $c_1'(0) = c_2'(0)$. Since f is smooth, locally around p there exists some $\tilde{f} : U \to \mathbb{R}^{n'}$ (U open in $\mathbb{R}^n$) with $\tilde{f}|_{U \cap M} = f|_{U \cap M}$. Then $\tilde{f} \circ c_i = f \circ c_i$ $(i = 1, 2)$, so

$$(f \circ c_1)'(0) = (\tilde{f} \circ c_1)'(0) = D\tilde{f}(p)c_1'(0) = D\tilde{f}(p)c_2'(0) = \cdots = (f \circ c_2)'(0).$$

Moreover, we conclude that $T_pf(c'(0)) = D\tilde{f}(p)c'(0)$, so T_pf is linear.

2.4.3 Lemma. *(Chain Rule) Let M, N, P be submanifolds, $f : M \to N,\ g : N \to P$ $\mathcal{C}^\infty, p \in M$. Then*
$$T_p(g \circ f) = T_{f(p)} g \circ T_p f$$

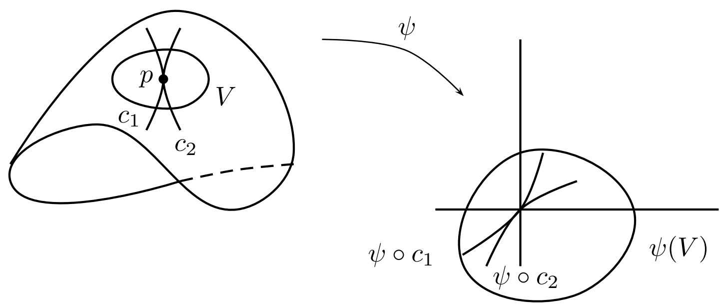
Proof. Let $\tilde g$ and $\tilde f$ be smooth extensions of g and f. Then $\tilde g \circ \tilde f$ is a smooth extension of $g \circ f$ and

$$\begin{aligned} T_p(g \circ f)(c'(0)) &= (\tilde g \circ \tilde f \circ c)'(0) = D\tilde g(f \circ c(0))((\tilde f \circ c)'(0)) = \\ &= T_{f(p)} g(D\tilde f(p)c'(0)) = T_{f(p)} g \circ T_p f(c'(0)) \end{aligned}$$

□

Next we want to extend the concept of tangent space also to abstract manifolds. However, for M an abstract manifold and $c : I \to M$ smooth, the derivative $c'(0)$ at the moment does not make sense due to the lack of a surrounding Euclidean space. Instead, we will resort to charts:

2.4.4 Definition. *Let M be a manifold, $p \in M$ and (ψ, V) a chart around p. Two $\mathcal{C}^\infty$-curves $c_1,\ c_2 : I \to M$ with $c_1(0) = p = c_2(0)$ are called tangential at p with respect to ψ if $(\psi \circ c_1)'(0) = (\psi \circ c_2)'(0)$.*



2.4.5 Lemma. *The notion of being tangent at a point is independent of the chart used in 2.4.4*

Proof. Let $c_1,\ c_2$ be smooth curves at p with c_1 tangent to c_2 with respect to the chart ψ_1. Let ψ_2 be another chart around p. Then locally around 0 we have $\psi_2 \circ c_i = (\psi_2 \circ \psi_1^{-1}) \circ (\psi_1 \circ c_i)$ $(i = 1, 2)$, so

$$(\psi_2 \circ c_1)'(0) = D(\psi_2 \circ \psi_1^{-1})(\psi_1(p)) \underbrace{(\psi_1 \circ c_1)'(0)}_{=(\psi_1 \circ c_2)'(0)} = (\psi_2 \circ c_2)'(0).$$

□

On the space of smooth curves at p we define an equivalence relation by $c_1 \sim c_2 :\Leftrightarrow$ c_1 tangential to c_2 at p with respect to one (hence any) chart. For $c : I \to M,\ c(0) = p$ we denote by $[c]_p$ the equivalence class of c with respect to $\sim$. Then $[c]_p$ is called a tangent vector at p.

2.4.6 Definition. *The* tangent space *of a manifold M at $p \in M$ is $T_pM = \{[c]_p \mid c : I \to M\ \mathcal{C}^\infty,\ I$ interval in $\mathbb{R},\ c(0) = p\}$.*

We first note that for submanifolds of $\mathbb{R}^n$ this definition reduces to 2.4.2 since in this case the map $c'(0) \mapsto [c]_p$ gives a bijection between 'old' and 'new' tangent space. In fact, with $\tilde\psi$ a local smooth extension of ψ,

$$[c_1]_p = [c_2]_p \Rightarrow \underbrace{(\psi \circ c_1)'(0)}_{D\tilde\psi(p)c_1'(0)} = \underbrace{(\psi \circ c_2)'(0)}_{D\tilde\psi(p)c_2'(0)},$$

so $c_1'(0) = c_2'(0)$. Hence the map $c'(0) \mapsto [c]_p$ is injective. Also, it obviously is surjective.

2.4.7 Definition. *Let M, N be manifolds and $f : M \to N$ a smooth map. Then we call*

$$\begin{aligned} T_pf : T_pM &\to T_{f(p)}N \\ [c]_p &\mapsto [f \circ c]_{f(p)} \end{aligned}$$

the tangent map *of f at p.*

2.4.8 Remark.

(i) T_pf is well-defined:
Let φ be a chart of M at p, ψ a chart of N at $f(p)$, $c_1,\ c_2 : I \to M$ curves through p with $c_1 \sim c_2$. Then

$$\begin{aligned} (\psi \circ f \circ c_1)'(0) &= ((\psi \circ f \circ \varphi^{-1}) \circ (\varphi \circ c_1))'(0) \\ &= D(\psi \circ f \circ \varphi^{-1})(\varphi(p))\, \underbrace{(\varphi \circ c_1)'(0)}_{=(\varphi \circ c_2)'(0)} \\ &= \cdots = (\psi \circ f \circ c_2)'(0), \end{aligned}$$

so $f \circ c_1 \sim_{f(p)} f \circ c_2$, i.e., $[f \circ c_1]_{f(p)} = [f \circ c_2]_{f(p)}$.

(ii) In the particular case where $M,\ N$ are submanifolds, T_pf is precisely the map from 2.4.2 in the sense of the above identification ($c'(0) \leftrightarrow [c]_p$).

$$\underbrace{c'(0)}_{\substack{\updownarrow \\ [c]_p}} \mapsto \underbrace{(f \circ c)'(0)}_{\substack{\updownarrow \\ [f\circ c]_{f(p)}}}$$

2.4.9 Proposition. *(Chain Rule) Let M, N, P be manifolds, $f : M \to N$ and $g : N \to P$ smooth, and $p \in M$. Then*

$$T_p(g \circ f) = T_{f(p)}g \circ T_pf$$

Moreover, since $T_p(\mathrm{id}_M) = \mathrm{id}_{T_pM}$, for any diffeomorphism $f : M \to N$, T_pf is bijective and $(T_pf)^{-1} = T_{f(p)}f^{-1}$.

Proof. Let c be a curve through p. Then

$$T_p(g \circ f)([c]_p) = [(g \circ f) \circ c]_{g(f(p))} = T_{f(p)}g([f \circ c]_{f(p)}) = T_{f(p)}g \circ T_pf([c]_p).$$

□

So far we did not endow T_pM with a vector space structure. In order to do this we first analyze the local situation in more detail.

2.4.10 Lemma. *Let $U \subseteq \mathbb{R}^n$ be open and $p \in U$. Then $i : T_pU \to \mathbb{R}^n,\ i([c]_p) := c'(0)$ is bijective, so T_pU can be identified with $\mathbb{R}^n$. In terms of this identification, for any smooth map $f : U \to V$ with $V \subseteq \mathbb{R}^m$ open we have $T_pf = Df(p)$.*

Proof. The map i is well-defined (choose the chart $\psi = \mathrm{id}_U$) and injective ($c_1'(0) = c_2'(0) \Rightarrow (\psi \circ c_1)'(0) = (\psi \circ c_2)'(0)$ for any chart ψ). Also, i is surjetive: Let $v \in \mathbb{R}^n$ and $c : t \mapsto p + t \cdot v$. Then $c'(0) = v$. Now let $f : U \to V$ be smooth and consider

$$\begin{array}{ccc} T_pU & \xrightarrow{T_pf} & T_{f(p)}V \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ \mathbb{R}^n & \xrightarrow{Df(p)} & \mathbb{R}^m \end{array}$$

The diagram commutes since

$$i \circ T_p f([c]_p) = i([f \circ c]_{f(p)}) = (f \circ c)'(0) = Df(p) \cdot c'(0) = Df(p) \circ i([c]_p).$$

$\square$

2.4.11 Proposition. *Let M be a manifold, $p \in M$, and (ψ, V) a chart around p. The vector space structure induced on T_pM by the bijection $T_p\psi : T_pM \to T_{\psi(p)}\psi(V) \cong \mathbb{R}^n$ is independent of the chosen chart (ψ, V).*

Proof. By definition, $T_pV = T_pM$, so $T_p\psi : T_pM \to T_{\psi(p)}\psi(V) \cong \mathbb{R}^n$ (by 2.4.10). Also, $T_p\psi$ is bijective by 2.4.9. Let $[c_1]_p, [c_2]_p \in T_pM$, $\alpha, \beta \in \mathbb{R}$ and φ another chart at p, w.l.o.g. with the same domain V. Then

$$\begin{aligned}
\alpha[c_1]_p + \beta[c_2]_p &:= (T_p\psi)^{-1}(\alpha T_p\psi([c_1]_p) + \beta T_p\psi([c_2]_p)) \\
&\overset{2.4.10}{=} (T_p\psi)^{-1}(\alpha(\psi \circ c_1)'(0) + \beta(\psi \circ c_2)'(0)) \\
&= (T_p\psi)^{-1}(\alpha(\psi \circ \varphi^{-1} \circ \varphi \circ c_1)'(0) + \beta(\psi \circ \varphi^{-1} \circ \varphi \circ c_2)'(0)) \\
&= (T_p\psi)^{-1}(D(\psi \circ \varphi^{-1})(\varphi(p))(\alpha(\varphi \circ c_1)'(0) + \beta(\varphi \circ c_2)'(0))) \\
&\overset{2.4.10}{=} (T_p\psi)^{-1}(T_{\varphi(p)}(\psi \circ \varphi^{-1}))(\alpha(\varphi \circ c_1)'(0) + \beta(\varphi \circ c_2)'(0)) \\
&\overset{2.4.9}{=} (T_p\varphi)^{-1}(\alpha T_p\varphi([c_1]_p) + \beta T_p\varphi([c_2]_p)),
\end{aligned}$$

which establishes our claim. $\square$

In this way, T_pM is endowed with an intrinsic (chart independent) vector space structure. Moreover, if $f : M \to N$ is smooth, then $T_pf : T_pM \to T_{f(p)}N$ is linear with respect to the corresponding vector space structures on T_pM, $T_{f(p)}N$: it suffices to show that $T_{f(p)}\psi \circ T_pf \circ T_{\varphi(p)}\varphi^{-1}$ is linear for any charts φ of M at p and ψ of N at $f(p)$. This map is given by

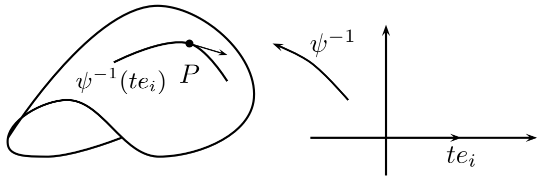
$$T_{\varphi(p)}(\psi \circ f \circ \varphi^{-1}) \overset{2.4.10}{=} D(\psi \circ f \circ \varphi^{-1})(\varphi(p)),$$

hence is indeed linear.
Any chart of M allows to pick a particular basis of T_pM: Let (ψ, V) be a chart of M at p, and let $\psi(p) = (x^1(p), \dots, x^n(p))$ (the x^i are called coordinate functions of ψ). For $1 \le i \le n$ let e_i denote the i-th standard unit vector of $\mathbb{R}^n$. Let $\psi(p) = 0$. Then we set

$$\left.\frac{\partial}{\partial x^i}\right|_p := (T_p\psi)^{-1}(e_i) \in T_pM.$$

More precisely, in the sense of 2.4.10 we have

$$\left.\frac{\partial}{\partial x^i}\right|_p = (T_p\psi)^{-1}([t \mapsto te_i]_0) = [t \mapsto \psi^{-1}(te_i)]_p.$$



Hence $\frac{\partial}{\partial x^i}\big|_p$ results from transporting the tangent vector of the coordinate line $t \mapsto te_i$ to M via the chart ψ. Since $T_p\psi$ is a linear isomorphism, $\{\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p\}$ indeed forms a basis of T_pM.
If, in particular, M is a submanifold of $\mathbb{R}^n$, and φ is a local parametrization of p (with $\varphi(0) = p$), then $\psi = \varphi^{-1}$ is a chart at p (cf. 2.1.13(i)) and we have

$$\frac{\partial}{\partial x^i}\bigg|_p = T_0\varphi(e_i) = (\varphi \circ (t \cdot e_i))'(0) = D\varphi(0)e_i.$$

Thus $\frac{\partial}{\partial x^i}\big|_p$ is precisely the i-th column of the Jacobian of φ at $\psi(p) = 0$.
The notation $\frac{\partial}{\partial x^i}\big|_p$ already suggests another interpretation of tangent vectors, namely as directional derivatives. In fact, any tangent vector can be viewed as a directional derivative in the following sense:
Let $v = [c]_p \in T_pM$. Let $f \in \mathcal{C}^\infty(M, \mathbb{R})$ (or $\mathcal{C}^\infty(M)$, for short), the space of smooth maps from M to $\mathbb{R}$. Then define $\partial_v : \mathcal{C}^\infty(M, \mathbb{R}) \to \mathbb{R}$ by $\partial_v f := T_pf(v)$. Since we use the identification 2.4.10 we have:

$$\partial_v(f) = T_pf(v) = T_pf([c]_p) = [f \circ c]_{f(p)} = (f \circ c)'(0), \tag{2.4.1}$$

which corresponds to differentiation in the direction v.
In particular, for $v = \frac{\partial}{\partial x^i}\big|_p$ we have (writing v instead of ∂_v):

$$\frac{\partial}{\partial x^i}\bigg|_p (f) = (f \circ \psi^{-1}(t \mapsto te_i))'(0) = D_i(f \circ \psi^{-1})(\psi(p)), \tag{2.4.2}$$

so $\frac{\partial}{\partial x^i}\big|_p$ corresponds to partial differentiation in the chart ψ.

2.4.12 Definition. *A map $\partial : \mathcal{C}^\infty(M) \to \mathbb{R}$ is called* derivation *at $p \in M$ if ∂ is linear and satisfies the Leibniz-rule:*

(i) $\partial(f + \alpha g) = \partial f + \alpha \partial g$

(ii) $\partial(f \cdot g) = \partial f \cdot g(p) + f(p) \cdot \partial g$

for all $f, g \in \mathcal{C}^\infty(M)$ and all $\alpha \in \mathbb{R}$. The vector space of all derivations at p is denoted by $\mathrm{Der}_p(\mathcal{C}^\infty(M), \mathbb{R})$.

The following theorem shows that in fact, the tangent space T_pM can be identified with the space $\mathrm{Der}_p(\mathcal{C}^\infty(M), \mathbb{R})$ of derivations at p.

2.4.13 Theorem. *The map*

$$\begin{aligned} A : T_pM &\to \mathrm{Der}_p(\mathcal{C}^\infty(M), \mathbb{R}) \\ v &\mapsto \partial_v \end{aligned}$$

is a linear isomorphism.

Proof. To begin with we show that any ∂_v is a derivation at p: Linearity is obvious $(\partial_v(f + \alpha g) = T_p(f + \alpha g)(v) = (T_pf + \alpha T_pg)(v))$ and letting $v = [c]_p$ we have

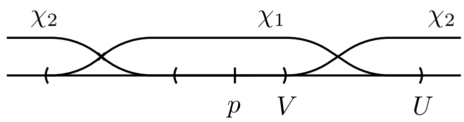
$$\begin{aligned} \partial_v(f \cdot g) &= ((f \cdot g) \circ c)'(0) = ((f \circ c) \cdot (g \circ c))'(0) \\ &= f(c(0)) \cdot (g \circ c)'(0) + g(c(0)) \cdot (f \circ c)'(0) \\ &= f(p)\partial_v(g) + \partial_v(f)g(p) \end{aligned}$$

A is linear:

$$(A(v_1 + \alpha v_2))(f) = T_pf(v_1 + \alpha v_2) = T_pf(v_1) + \alpha T_pf(v_2) = (A(v_1) + \alpha A(v_2))(f).$$

A is injective:
We first show that any derivation ∂ at p only 'feels' values of f near p. More precisely, if U is an open neighborhood of p and $f_1,\ f_2 \in \mathcal{C}^\infty(M)$ are such that $f_1|_U = f_2|_U$, then $\partial(f_1) = \partial(f_2)$. In fact, let $f := f_1 - f_2$. Then $f|_U = 0$ and we want to show that $\partial(f)|_U = 0$.
Choose a neighborhood V of p such that $\overline{V} \subseteq U$ (cf. 2.3.6). Then by 2.3.14 there is a partition of unity $\{\chi_1,\ \chi_2\}$ subordinate to $\{U, M \setminus \overline{V}\}$. Then

$$0 = \partial(0) = \partial(\chi_1 \cdot f) = \underbrace{\chi_1(p)}_{=1} \cdot \partial(f) + \partial(\chi_1)\underbrace{f(p)}_{=0} = \partial(f).$$



Since in this way any $\mathcal{C}^\infty$-function defined locally at p can be extended to M it follows that in fact any derivation at p is a map from all local $\mathcal{C}^\infty$-functions at p (the so called *germs* of smooth functions at p) into $\mathbb{R}$.
Suppose that $A(v) = 0$, where $v = [c]_p$, i.e., $\partial_v f = 0$ for all smooth functions f locally defined at p. Let ψ be a chart at p with $\psi(p) = 0$ and set $f := x^i$ (where $\psi = (x^1, \dots, x^n)$). Then $0 = \partial_v f = T_p f(v) = T_p f([c]_p) = (x^i \circ c)'(0)$, so $(\psi \circ c)'(0) = 0$. By 2.4.10, then, $i(T_p\psi(v)) = (\psi \circ c)'(0) = 0$ and therefore $v = 0$ since $T_p\psi$ is a linear isomorphism by 2.4.11.
A is surjective:
Let $\partial \in \mathrm{Der}_p(\mathcal{C}^\infty(M), \mathbb{R})$. We first note that ∂ vanishes on any constant function $f \equiv k$:

$$\partial(k) = \partial(1 \cdot k) = 1 \cdot \partial(k) + k \cdot \partial(1) = 2\partial(k) \Rightarrow \partial(k) = 0.$$

Let $\psi : V \to U$ be a chart of M at p, $\psi(p) = 0$, $\psi = (x^1, \dots, x^n)$ and $B_1(0) \subseteq U$. Let $f \in \mathcal{C}^\infty(M)$ and $g := f \circ \psi^{-1}$. Then for $x \in B_1(0)$ we have:

$$\begin{aligned} g(x) - g(0) &= \int_0^1 \frac{d}{dt} g(tx)dt = \int_0^1 Dg(tx)x dt = \int_0^1 \sum_{i=1}^n D_i g(tx) \cdot x^i dt \\ &= \sum_{i=1}^n x^i \underbrace{\int_0^1 D_i g(tx)dt}_{=:h_i(x)}. \end{aligned}$$

Hence, on V,

$$f(q) = g(\psi(q)) = g(0) + \sum_{i=1}^n \psi^i(q) \underbrace{h_i(\psi(q))}_{=:\tilde{h}_i(q)}.$$

Since ∂ acts locally, we conclude:

$$\partial(f) = 0 + \sum_{i=1}^n [\partial(\psi^i)\tilde{h}_i(p) + \underbrace{\psi^i(p)}_{=0} \partial(\tilde{h}_i)].$$

Now

$$\tilde{h}_i(p) = h_i(0) = \int_0^1 D_i g(0)dt = D_i g(0) = D_i(f \circ \psi^{-1})(\psi(p)) = \left.\frac{\partial}{\partial x^i}\right|_p (f)$$

Summing up, we get

$$\partial(f) = \partial_v(f) \qquad \forall f \in \mathcal{C}^\infty(M)$$

where $v = \sum_{i=1}^n \partial(\psi^i) \, \frac{\partial}{\partial x^i}\big|_p$, establishing that A is surjective. $\square$

Due to this result we will henceforth identify T_pM and $\mathrm{Der}_p(\mathcal{C}^\infty(M), \mathbb{R})$. In fact, in the literature it is quite common to *define* T_pM as $\mathrm{Der}_p(\mathcal{C}^\infty(M), \mathbb{R})$. One of the reasons for this approach is that formal manipulations become particularly simple: let $\partial \in \mathrm{Der}_p(\mathcal{C}^\infty(M), \mathbb{R})$, $f \in \mathcal{C}^\infty(M)$. Then $\partial = \partial_v$ for some $v \in T_pM$. Therefore,

$$T_pf(\partial) = T_pf(\partial_v) \overset{(2.4.1)}{=} \partial_v(f) = \partial(f),$$

and we obtain:

$$T_pf(\partial) = \partial(f) \tag{2.4.3}$$

Now let $f \in \mathcal{C}^\infty(M, N)$. Then the tangent map of f in the derivation picture is computed as follows:

$$\begin{array}{rcl} T_pf : \mathrm{Der}_p(\mathcal{C}^\infty(M), \mathbb{R}) & \to & \mathrm{Der}_{f(p)}(\mathcal{C}^\infty(N), \mathbb{R}) \\ \partial & \mapsto & (g \mapsto \partial(g \circ f)) \end{array} \tag{2.4.4}$$

In fact, by (2.4.3) we have

$$(T_pf(\partial))(g) \overset{(2.4.3)}{=} T_{f(p)}g(T_pf(\partial)) \overset{2.4.9}{=} T_p(g \circ f)(\partial) \overset{(2.4.3)}{=} \partial(g \circ f)$$

2.4.14 Proposition. *Let M^m, N^n be $\mathcal{C}^\infty$-manifolds, $f \in \mathcal{C}^\infty(M, N)$, $p \in M$, $\varphi = (x^1, \dots, x^m)$ a chart of M around p, $\psi = (y^1, \dots, y^n)$ a chart of N around $f(p)$. Then the matrix representation of the linear map $T_pf : T_pM \to T_{f(p)}N$ with respect to the bases $\mathcal{B}_{T_pM} = \{ \frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^m}\big|_p\}$ and $\mathcal{B}_{T_{f(p)}N} = \{ \frac{\partial}{\partial y^1}\Big|_{f(p)}, \dots, \frac{\partial}{\partial y^n}\Big|_{f(p)}\}$ is precisely the Jacobian of the local representation $f_{\psi\varphi} := \psi \circ f \circ \varphi^{-1}$ of f. Thus,*

$$T_pf(\left.\frac{\partial}{\partial x^i}\right|_p) = \sum_{k=1}^n D_i(\psi^k \circ f \circ \varphi^{-1})(\varphi(p)) \left.\frac{\partial}{\partial y^k}\right|_{f(p)} = \sum_{k=1}^n \frac{\partial f^k_{\psi\varphi}}{\partial x^i} \frac{\partial}{\partial y^k} \tag{2.4.5}$$

Proof. The i-th column of $[T_pf]_{\mathcal{B}_{T_pM}, \mathcal{B}_{T_{f(p)}N}}$ is $[T_pf(\frac{\partial}{\partial x^i}\big|_p)]_{\mathcal{B}_{T_{f(p)}N}}$. Hence we want to write $T_pf(\frac{\partial}{\partial x^i}\big|_p)$ in the basis $\{ \frac{\partial}{\partial y^1}\Big|_{f(p)}, \dots, \frac{\partial}{\partial y^n}\Big|_{f(p)}\}$. We have

$$Df_{\psi\varphi}(\varphi(p)) \overset{2.4.10}{=} T_{\varphi(p)}(\psi \circ f \circ \varphi^{-1}) \overset{2.4.9}{=} T_{f(p)}\psi \circ T_pf \circ (T_p\varphi)^{-1}.$$

Let $J_{ki} := D_i(f^k_{\psi\varphi})(\varphi(p)) = D_i(\psi^k \circ f \circ \varphi^{-1})(\varphi(p))$. Then

$$\begin{aligned} T_pf(\left.\frac{\partial}{\partial x^i}\right|_p) &= T_pf((T_p\varphi)^{-1}(e^i)) = (T_{f(p)}\psi)^{-1}(Df_{\psi\varphi}(\varphi(p))e^i) = \\ &= \sum_{k=1}^n J_{ki}(T_{f(p)}\psi)^{-1}(e^k) = \sum_{k=1}^n J_{ki} \left.\frac{\partial}{\partial y^k}\right|_{f(p)} \end{aligned}$$

$\square$

2.4.15 Corollary. *Let M^n be a manifold, $p \in M$ and let $\varphi = (x^1, \dots, x^n)$ and $\psi = (y^1, \dots, y^n)$ be charts around p. Then*

$$\left.\frac{\partial}{\partial x^i}\right|_p = \sum_{k=1}^n D_i(\psi^k \circ \varphi^{-1})(\varphi(p)) \left.\frac{\partial}{\partial y^k}\right|_p = \sum_{k=1}^n \frac{\partial y^k}{\partial x^i} \frac{\partial}{\partial y^k} \tag{2.4.6}$$

Proof. Set $f = \mathrm{id}_M$ in 2.4.14. $\square$

2.5 Tangent bundle, Vector Fields

A vector field on an open subset U of $\mathbb{R}^n$ is an assignment $p \mapsto X_p$ of a vector $X_p \in \mathbb{R}^n \cong T_pU$ to each $p \in U$. To analyze, e.g., differential equations with right hand side X (i.e., $\dot{c}(t) = X(c(t))$) one will typically assume X to be smooth (at least C^1). We want to extend such notions to the manifold setting. Thus we are looking for maps X mapping points in a manifold M to vectors in T_pM. At the moment, however, we do not have a concept of smoothness for such maps: the individual tangent spaces are not yet bundled together into one manifold. Our first aim therefore is to remedy this deficiency.

2.5.1 Definition. *Let M be a smooth manifold. The tangent bundle (or tangent space) of M is defined as the disjoint union of the vector spaces T_pM $(p \in M)$:*

$$TM := \bigsqcup_{p\in M} T_pM := \bigcup_{p\in M} \{p\} \times T_pM$$

The map $\pi_M : TM \to M$, $(p, v) \mapsto p$ is called the canonical projection. If $f : M \to N$ is smooth, then the tangent map Tf of f is defined as $Tf(p, v) = (f(p), T_pf(v))$.

2.5.2 Lemma. *(Chain Rule) Let $f : M \to N$, $g : N \to P$ be smooth. Then $T(g \circ f) = Tg \circ Tf$. Moreover, $T(\mathrm{id}_M) = \mathrm{id}_{TM}$, so for any diffeomorphism $f : M \to N$ we have $(Tf)^{-1} = T(f^{-1})$.*

Proof. By 2.4.9,

$$\begin{aligned} T(g \circ f)(p, v) &= (g(f(p)), T_p(g \circ f)(v)) = (g(f(p)), T_{f(p)}g \circ T_pf(v))) \\ &= Tg(f(p), T_pf(v)) = (Tg \circ Tf)(p, v) \end{aligned}$$

and

$$T(\mathrm{id}_M)(p, v) = (p, T_p\mathrm{id}_M(v)) = (p, v) = \mathrm{id}_{TM}(p, v)$$

□

In order to turn TM into a smooth manifold we have to endow it with a $\mathcal{C}^\infty$-atlas. Natural candidates for the charts of TM are the tangent maps $T\psi$ of charts (ψ, V) of M:

$$T\psi : TV = \bigcup_{p\in V} \{p\} \times T_pV = \bigcup_{p\in V} \{p\} \times T_pM =: TM|_V \to T(\psi(V)) = \psi(V) \times \mathbb{R}^n$$

Here, $T(\psi(V)) = \bigcup_{x\in\psi(V)}\{x\} \times \underbrace{T_x(\psi(V))}_{=\mathbb{R}^n} = \psi(V) \times \mathbb{R}^n$. Any such $T\psi$ is bijective.

2.5.3 Proposition. *Let M^n be a smooth manifold with atlas $\mathcal{A} = \{(\psi_\alpha, V_\alpha) \mid \alpha \in A\}$. Then $\tilde{\mathcal{A}} := \{(T\psi_\alpha, TM|_{V_\alpha}) \mid \alpha \in A\}$ is a $\mathcal{C}^\infty$-atlas for TM. The natural manifold topology of TM is Hausdorff and second countable, hence TM is a smooth manifold of dimension $2n$.*

Proof. The TV_α cover TM and any $T\psi_\alpha : TV_\alpha \to \psi_\alpha(V_\alpha) \times \mathbb{R}^n$ is bijective. Let $TM|_{V_\alpha} \cap TM|_{V_\beta} \neq \emptyset$, i.e., $V_\alpha \cap V_\beta \neq \emptyset$. Then:

$$\begin{aligned} T\psi_\beta \circ (T\psi_\alpha)^{-1} &= T(\psi_\beta \circ \psi_\alpha^{-1}) : \underbrace{T(\psi_\alpha(V_\alpha \cap V_\beta))}_{=\psi_\alpha(V_\alpha\cap V_\beta)\times\mathbb{R}^n} \to \underbrace{T(\psi_\beta(V_\alpha \cap V_\beta))}_{=\psi_\beta(V_\alpha\cap V\beta)\times\mathbb{R}^n} \\ T(\psi_\beta \circ \psi_\alpha^{-1})(x, w) &= (\psi_\beta \circ \psi_\alpha^{-1}(x), T_x(\psi_\beta \circ \psi_\alpha^{-1}) \cdot w) \\ &\overset{2.4.10}{=} (\psi_\beta \circ \psi_\alpha^{-1}(x), D(\psi_\beta \circ \psi_\alpha^{-1})(x) \cdot w), \end{aligned} \tag{2.5.1}$$

Since any such map is smooth, TM is a $\mathcal{C}^\infty$-manifold of dimension $2n$ if we additionally verify that it is Hausdorff and second countable.
TM is Hausdorff: Let $(p_1, v_1) \neq (p_2, v_2) \in TM$. Then there are two possibilities.

1.) $p_1 \neq p_2$. Then since M is Hausdorff there exist chart neighborhoods V_1, V_2 of p_1, p_2 with $V_1 \cap V_2 = \emptyset$. Then TV_1, TV_2 are neighborhoods of (p_1, v_1), (p_2, v_2) in the natural manifold topology of TM with $TV_1 \cap TV_2 = \emptyset$.

2.) $p_1 = p_2$: Choose a chart (ψ, V) around p and separate $T\psi(p_1, v_1)$, $T\psi(p_2, v_2)$ in $T\psi(TV) = \psi(V) \times \mathbb{R}^n$. Since $T\psi$ is a homeomorphism this gives the desired separation in TM.

TM is second countable: By 2.3.7 there exists a countable atlas $\{(\psi_m, V_m) \mid m \in \mathbb{N}\}$ of M. Then $\{(T\psi_m, TV_m) \mid m \in \mathbb{N}\}$ is a countable atlas of TM, so, again by 2.3.7, the claim follows. □

2.5.4 Remark.

(i) If $f : M^m \to N^n$ is smooth, then so is $Tf : TM \to TN$. In fact, for (ψ, V) a chart of N, and (φ, U) a chart of M we have

$$\begin{aligned} T\psi \circ Tf \circ T\varphi^{-1}(x, w) &= T(\psi \circ f \circ \varphi^{-1})(x, w) \\ &= (\psi \circ f \circ \varphi^{-1}(x), D(\psi \circ f \circ \varphi^{-1})(x) \cdot w), \end{aligned}$$

which is smooth on its open domain $\varphi(U \cap f^{-1}(V)) \times \mathbb{R}^m = T(\varphi(U \cap f^{-1}(V)))$. This gives the result by 2.2.10 (ii).

(ii) $\pi_M : TM \to M$ is smooth. In fact, locally π_M is a projection: let (ψ, V) be a chart of M^n. Then

$$\begin{array}{ccc} TM|_V & \xrightarrow{\pi_M} & V \subseteq M \\ {\scriptstyle T\psi}\big\downarrow & & \big\downarrow{\scriptstyle \psi} \\ T(\psi(V)) = \psi(V) \times \mathbb{R}^n & \xrightarrow{\mathrm{pr}_1} & \psi(V) \end{array}$$

$$\begin{aligned} \psi \circ \pi_M \circ T\psi^{-1}(x, w) &= \psi \circ \pi_M(\psi^{-1}(x), T_x\psi^{-1}(w)) \\ &= \psi(\psi^{-1}(x)) = x = \mathrm{pr}_1(x, w). \end{aligned}$$

On closer examination it turns out that TM in fact has more structure than a 'pure' manifold: the images of the charts $T\psi_\alpha(TV_\alpha) = \psi_\alpha(V_\alpha) \times \mathbb{R}^n$ are cartesian products of open subsets of $\mathbb{R}^n$ with vector spaces. The changes of charts (2.5.1) respect this structure, as they are of the form $(x, w) \mapsto (\varphi_1(x), \varphi_2(x) \cdot w)$ with $\varphi_2(x)$ a linear map for each x. Thus TM furnishes our first example of a *vector bundle* in the sense of the following definition.

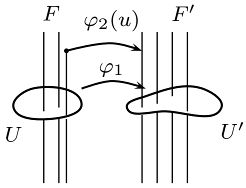
2.5.5 Definition.

(i) Local vector bundles*: Let E, F be (finite dimensional, real) vector spaces, and $U \subseteq E$ open. Then $U \times F$ is called a local vector bundle with base U. We identify U with $U \times \{0\}$. For $u \in U$ we call $\{u\} \times F$ the* fiber *over u. The fiber is equipped with the vector space structure of F. The map $\pi : U \times F \to U$, $(u, f) \mapsto u$ is called the projection of $U \times F$. Then the fiber over u is precisely $\pi^{-1}(u)$.*

A map $\varphi : U \times F \to U' \times F'$ of local vector bundles is called a local vector bundle homomorphism (resp. a local vector bundle isomorphism) if φ is smooth (resp. a diffeomorphism) and has the form

$$\varphi(u, f) = (\varphi_1(u), \varphi_2(u) \cdot f),$$

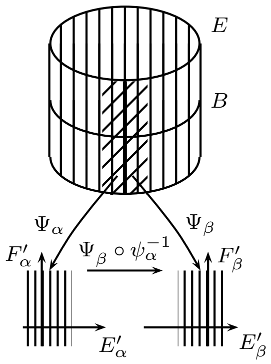
where $\varphi_2(u)$ is linear (resp. a linear isomorphism) from F to (resp. onto) F' for each $u \in U$.



(ii) Vector bundles*: Let E be a set. A local vector bundle chart (or vb-chart) of E is a pair (Ψ, W), where $W \subseteq E$ and $\Psi : W \to W' \times F'$ is a bijection onto a local vector bundle $W' \times F'$ (with W', F' depending on Ψ). A vector bundle atlas is a family $\mathcal{A} = \{(\Psi_\alpha, W_\alpha) \mid \alpha \in A\}$ of local vector bundle charts such that the W_α cover E and any two vector bundle charts (Ψ_α, W_α), (Ψ_β, W_β) in $\mathcal{A}$ with $W_\alpha \cap W_\beta \neq \emptyset$ are compatible in the sense that*

$$\Psi_\beta \circ \Psi_\alpha^{-1} : \Psi_\alpha(W_\alpha \cap W_\beta) \to \Psi_\beta(W_\alpha \cap W_\beta)$$

is a local vector bundle isomorphism (in particular, $\Psi_\alpha(W_\alpha \cap W_\beta)$, $\Psi_\beta(W_\alpha \cap W_\beta)$ are supposed to be local vector bundles).



Two vector bundle atlasses $\mathcal{A}_1$, $\mathcal{A}_2$ are called equivalent if $\mathcal{A}_1 \cup \mathcal{A}_2$ is again a vector bundle atlas. A vector bundle structure $\mathcal{V}$ is an equivalence class of vector bundle atlasses. A vector bundle is a set E together with a vector bundle structure. Since any vector bundle atlas is, in particular, a $\mathcal{C}^\infty$-atlas, E is automatically a $\mathcal{C}^\infty$-manifold. Again we require that the natural manifold topology of E is Hausdorff and second countable.

2.5.6 Remark.

(i) In any vector bundle E there exists a distinguished subset B, the basis of E, defined by:

$$B := \{e \in E \mid \exists \text{ vb-chart } (\Psi, W) \text{ s.t. } e = \Psi^{-1}(w', 0) \text{ for some } w' \in W'\}.$$

B is independent of the vector bundle charts used in the definition since any change of vector bundle charts is linear in the second component (so 0 is mapped to 0). If $\mathcal{A} = \{(\Psi_\alpha, W_\alpha) \mid \alpha \in A\}$ is a vector bundle atlas for E, then $\mathcal{A}' = \{(\Psi_\alpha|_{W_\alpha \cap B}, W_\alpha \cap B) \mid \alpha \in A\}$ is a $\mathcal{C}^\infty$-atlas for B. Thus B is a smooth manifold. In fact, if $\Psi_\beta \circ \Psi_\alpha^{-1}(w', f') = (\psi^{(1)}_{\beta\alpha}(w'), \psi^{(2)}_{\beta\alpha}(w') \cdot f')$, then $\Psi_\beta|_{W_\beta \cap B} \circ (\Psi_\alpha|_{W_\alpha \cap B})^{-1}(w', 0) = (\psi^{(1)}_{\beta\alpha}(w'), 0)$, which is smooth. Thus the changes of charts in B are exactly the $\psi^{(1)}_{\beta\alpha}$, if we identify $W' \times \{0\}$ with W'.

There is a well-defined projection $\pi : E \to B$: let $e \in E$, Ψ_α a vector bundle chart around e and $\Psi_\alpha(e) = (w', f')$ ($\Psi_\alpha : W \to W' \times F'$). Then let $\pi(e) := \Psi_\alpha^{-1}(w', 0)$. This definition is independent of Ψ_α: Let (Ψ_β, W_β) be another vector bundle chart around e, $\Psi_\beta(e) = (w'', f'')$. Then

$$\Psi_\beta \circ \Psi_\alpha^{-1}(w', f') = (\psi_{\beta\alpha}^{(1)}(w'), \psi_{\beta\alpha}^{(2)}(w') \cdot f') = (w'', f''),$$

so $w'' = \psi_{\beta\alpha}^{(1)}(w')$ and therefore $\Psi_\beta \circ \Psi_\alpha^{-1}(w', 0) = (w'', 0)$. Hence $\pi(e) = \Psi_\alpha^{-1}(w', 0) = \Psi_\beta^{-1}(w'', 0)$. Obviously, π is surjective. Moreover, π is smooth:

$$\begin{array}{ccc} E & \xrightarrow{\pi} & B \\ {\scriptstyle \Psi_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle \Psi_\alpha|_B} \\ W' \times F' & \xrightarrow{\mathrm{pr}_1(\times 0)} & W' \times \{0\} \end{array}$$

Since pr_1 is smooth, so is π.

For $b \in B$ we call $\pi^{-1}(b)$ the fiber over b. It carries a vector space structure induced by the vector bundle charts: Let $e_1, e_2 \in \pi^{-1}(b)$, Ψ_α a vector bundle chart around b, $\Psi_\alpha(e_i) = (w', f'_i)$ ($i = 1, 2$). Then let $e_1 + \lambda e_2 := \Psi_\alpha^{-1}(w', f'_1 + \lambda f'_2)$. This is independent of the chosen vector bundle chart: Let Ψ_β be another vector bundle chart, $\Psi_\beta(e_i) = (v', g'_i)$ ($i = 1, 2$). Then $\Psi_\beta \circ \Psi_\alpha^{-1}(w', f'_i) = (\psi_{\beta\alpha}^{(1)}(w'), \psi_{\beta\alpha}^{(2)}(w') f'_i) = (v', g'_i)$, so $\Psi_\beta \circ \Psi_\alpha^{-1}(w', f'_1 + \lambda f'_2) = (\psi_{\beta\alpha}^{(1)}(w'), \psi_{\beta\alpha}^{(2)}(w') \cdot f'_1 + \lambda \psi_{\beta\alpha}^{(2)}(w') \cdot f'_2) = (v', g'_1 + \lambda g'_2)$. Thus $e_1 + \lambda e_2 = \Psi_\alpha^{-1}(w'_1, f'_1 + \lambda f'_2) = \Psi_\beta^{-1}(w', g'_1 + \lambda g'_2)$.

For $U \subseteq B$ open let $E|_U := \bigcup_{b \in U} \{b\} \times E_b$.

(ii) From (i) we may extract the following alternative description of vector bundles which often is used as a definition:

A vector bundle is a triple (E, B, π) consisting of two $\mathcal{C}^\infty$-manifolds E, B and a smooth surjection $\pi : E \to B$ such that for all $b \in B$ we have:

- The fiber $\pi^{-1}(b) =: E_b$ is a vector space.
- There exists an open neighborhood V of b in B and a diffeomorphism $\tilde{\Psi} : W := \pi^{-1}(V) \to V \times F'$, which is fiberwise linear (i.e., $\tilde{\Psi}\Big|_{\pi^{-1}(b)}$ is linear $\forall b \in V$) and such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(V) & \xrightarrow{\tilde{\Psi}} & V \times F' \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_1} \\ V & \xrightarrow{\mathrm{id}} & V \end{array}$$

(In our approach, $\tilde{\Psi} := ((\Psi|_B)^{-1} \times \mathrm{id}_{F'}) \circ \Psi$).

2.5.7 Example. (TM, M, π_M) is a vector bundle:
Let $\mathcal{A} = \{(\psi_\alpha, V_\alpha) \mid \alpha \in A\}$ be an atlas of M. By 2.5.3, with $\Psi_\alpha := T\psi_\alpha$, $W_\alpha := TV_\alpha$ the family $\mathcal{A}' := \{(\Psi_\alpha, W_\alpha) \mid \alpha \in A\}$ is a vector bundle atlas of TM. By 2.4.11, the fibers $\pi_M^{-1}(p) = \{p\} \times T_pM \cong T_pM$ carry the vector space structure induced by Ψ_α. Hence, locally TM has a product structure: $T\psi_\alpha : TM|_{V_\alpha} = TV_\alpha \to \psi_\alpha(V_\alpha) \times \mathbb{R}^n$ and we obtain the following commutative diagram:

$$\begin{array}{ccccc} TV_\alpha = \pi^{-1}(V_\alpha) & \xrightarrow{T\psi_\alpha} & \psi_\alpha(V_\alpha) \times \mathbb{R}^n & \xrightarrow{\psi_\alpha^{-1} \times \mathrm{id}} & V_\alpha \times \mathbb{R}^n \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_1} & & \big\downarrow{\scriptstyle \mathrm{pr}_1} \\ V_\alpha & \xrightarrow{\psi_\alpha} & \psi_\alpha(V_\alpha) & \xrightarrow{\psi_\alpha^{-1}} & V_\alpha \end{array}$$

After this clarification of the underlying structures we return to our original task of defining vector fields on manifolds. Thus we are looking for maps which smoothly assign to each $p \in M$ an element $X_p = X(p)$ of T_pM.

2.5.8 Definition. *Let (E, B, π) be a vector bundle. A map $X : B \to E$ is called a* section *of E (more precisely: of $\pi : E \to B$), if $\pi \circ X = \mathrm{id}_B$. The set of all smooth sections of E is denoted by $\Gamma(B, E)$ (or $\Gamma(E)$).*

Thus a vector field is a section of TM ($\pi(X_p) = p\ \forall p$). If $(\psi, V), \psi = (x^1, \dots, x^n)$ is a chart of M then for any $p \in V$ the $\frac{\partial}{\partial x^i}\big|_p$ form a basis of T_pM. Since $X_p \in T_pM$, for each p there exist uniquely determined $X^i(p) \in \mathbb{R}$ such that $X_p = \sum_{i=1}^n X^i(p)\, \frac{\partial}{\partial x^i}\big|_p$. This is called the local representation of X on V.

2.5.9 Proposition. *Let X be a vector field on a manifold M. TFAE:*

(i) $X : M \to TM$ is smooth, i.e., $X \in \Gamma(TM)$.

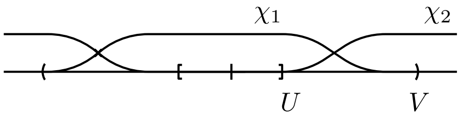
(ii) For every $f \in \mathcal{C}^\infty(M)$, $p \mapsto X_p(f) : M \to \mathbb{R}$ is smooth.

(iii) For every chart (ψ, V) of M, $\psi = (x^1, \dots, x^n)$ we have: in the local representation

$$X(p) = \sum_{i=1}^n X^i(p)\, \frac{\partial}{\partial x^i}\bigg|_p,$$

$X^i \in \mathcal{C}^\infty(V, \mathbb{R})$ for all $i = 1, \dots, n$.

Proof. (i)⇒(ii): $X : M \to TM$ is smooth by assumption. Also, if $f \in \mathcal{C}^\infty(M)$, then $Tf : TM \to T\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$ is smooth by 2.5.4(i). Hence $p \mapsto Tf(X_p) = (f(p), T_pf(X_p)) = (f(p), X_p(f))$, and therefore also $p \mapsto X_p(f)$ is smooth by 2.4.3.
(ii)⇒(iii): Let $p_0 \in V$ and let U be an open neighborhood of p_0 such that $\overline{U} \subseteq V$. By 2.3.14 we may choose a partition of unity $\{\chi_1, \chi_2\}$ subordinate to $\{V, M \setminus \overline{U}\}$.



Let $1 \le j \le n$ and set $f := \chi_1 x^j$ (extended by 0 outside of V). Then $f \in \mathcal{C}^\infty(M)$ and $f|_U = x^j\big|_U$. For $p \in U$ we obtain:

$$\begin{aligned} X_p(f) &= \sum_{i=1}^n X^i(p)\, \frac{\partial}{\partial x^i}\bigg|_p (x^j) = \\ &= \sum_{i=1}^n X^i(p) D_i(\underbrace{x^j \circ \psi^{-1}}_{=\mathrm{pr}_j \circ \psi \circ \psi^{-1}})(\psi(p)) = \\ &= \sum_{i=1}^n X^i(p)\delta_{i,j} = X^j(p) \end{aligned}$$

Therefore, each $X^j\big|_U$ is smooth. Since p_0 was an arbitrary point in V, each X^j is smooth on V ($1 \le j \le n$).
(iii)⇒ (i): Let (ψ, V) be a chart at $p \in M$. By 2.2.10 (i), it suffices to show that $T\psi \circ X \circ \psi^{-1}$ is smooth (on its open domain $\psi(V)$). Now

$$\begin{aligned} T\psi \circ X(p) &= T\psi(\sum_{i=1}^n X^i(p)\, \frac{\partial}{\partial x^i}\bigg|_p) = T\psi(\sum_{i=1}^n X^i(p)(T_p\psi)^{-1}(e_i)) \\ &= (\psi(p), T_p\psi(\sum_{i=1}^n X^i(p)(T_p\psi)^{-1}(e_i)) = (\psi(p), \sum_{i=1}^n X^i(p)e_i), \end{aligned}$$

so, finally,

$$T\psi \circ X \circ \psi^{-1}(x) = (x, \sum_{i=1}^{n} X^i(\psi^{-1}(x))e_i) \tag{2.5.2}$$

is smooth, as claimed. $\square$

2.5.10 Definition. *The space of smooth vector fields on M is denoted by $\mathfrak{X}(M)$.*

2.5.11 Examples.

(i) Vector fields on $\mathbb{R}^n$:

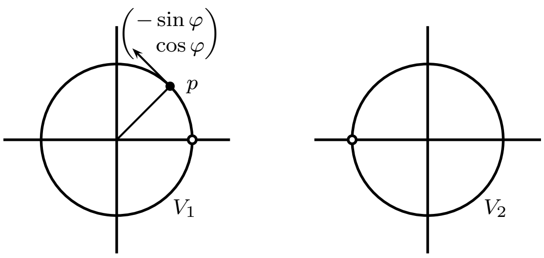
Let $U \subseteq \mathbb{R}^n$ be open. From our analysis course we know: a vector field is a $\mathcal{C}^\infty$-map $X : U \to \mathbb{R}^n$, $X(p) = (X^1(p), \dots, X^n(p)) = \sum_{i=1}^n x^i(p)e_i$. How does this fit into the above framework?

U is a manifold with the single chart $\psi = \mathrm{id}_U$ and the corresponding atlas $\mathcal{A} = \{(id_U, U)\}$. By 2.4.10 we have $T_p\psi = D\psi(p) = \mathrm{id}$, so $\frac{\partial}{\partial x^i}\big|_p = (T_p\psi)^{-1}(e_i) = e_i$. As a derivation, according to (2.4.2), $\frac{\partial}{\partial x^i}\big|_p$ acts as follows:

$$\left.\frac{\partial}{\partial x^i}\right|_p (f) = D_i(f \circ \mathrm{id}^{-1})(\mathrm{id}(p)) = D_i f(p) = \frac{\partial f}{\partial x^i}(p).$$

Hence $X_p = \sum_{i=1}^n X^i(p)e_i$ resp. $X_p = \sum_{i=1}^n X^i(p) \frac{\partial}{\partial x^i}\big|_p$ correspond to viewing X as a vector or as a differential operator (directional derivative in the direction $(X^1(p), \dots, X^n(p))$), respectively.

(ii) As in 2.2.2, let $M = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, and set $V_1 = \{(\cos\varphi, \sin\varphi) \mid \varphi \in (0, 2\pi)\}$, $\psi_1 : V_1 \to (0, 2\pi)$, $\psi_1(\cos\varphi, \sin\varphi) = \varphi$, and $V_2 = \{(\cos\tilde{\varphi}, \sin\tilde{\varphi}) \mid \tilde{\varphi} \in (-\pi, \pi)\}$, $\psi_2 : V_2 \to (-\pi, \pi)$, $\psi_2(\cos\tilde{\varphi}, \sin\tilde{\varphi}) = \tilde{\varphi}$.



With respect to the chart ψ_1, at $p = (\cos\varphi, \sin\varphi)$ the vector field $\frac{\partial}{\partial\varphi}$ is given by

$$\left.\frac{\partial}{\partial\varphi}\right|_p = (T_p\psi_1)^{-1}(e_1) = T_\varphi\psi_1^{-1}(1) = D\psi_1^{-1}(\varphi)\cdot 1 = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}$$

Analogously, with respect to ψ_2 we have:

$$\left.\frac{\partial}{\partial\tilde{\varphi}}\right|_p = \begin{pmatrix}-\sin\tilde{\varphi}\\ \cos\tilde{\varphi}\end{pmatrix}$$

at $p = (\cos\tilde{\varphi}, \sin\tilde{\varphi})$. By (2.4.6), on $V_1 \cap V_2$ we have

$$\left.\frac{\partial}{\partial\varphi}\right|_p = \frac{\partial\tilde{\varphi}}{\partial\varphi}\left.\frac{\partial}{\partial\tilde{\varphi}}\right|_p$$

and

$$\frac{\partial\tilde{\varphi}}{\partial\varphi} = D(\psi_2 \circ \psi_1^{-1})(\psi_1(p)) = 1$$

since

$$\psi_2 \circ \psi_1^{-1} = \varphi \mapsto \begin{cases} \varphi & \varphi \in (0,\pi) \\ \varphi - 2\pi & \varphi \in (\pi, 2\pi) \end{cases}$$

Therefore, $\frac{\partial}{\partial\varphi} = \frac{\partial}{\partial\tilde{\varphi}}$ on $V_1 \cap V_2$ and we conclude that

$$X := \begin{cases} \frac{\partial}{\partial\varphi} & \text{on } V_1 \\ \frac{\partial}{\partial\tilde{\varphi}} & \text{on } V_2 \end{cases}$$

is a well-defined vector field on S^1. Often one simply writes $X = \frac{\partial}{\partial\varphi}$.

Let $f : S^1 \to \mathbb{R}$ be a smooth function. By (2.4.2) we have (for $\tilde{f}$ a local smooth extension of f):

$$\begin{aligned} (Xf)(p) &= \frac{\partial}{\partial\varphi}\Big|_p (f) = D(f \circ \psi_1^{-1})(\underbrace{\psi_1(p)}_{=\varphi}) = \frac{\partial}{\partial\varphi}\tilde{f}(\cos\varphi, \sin\varphi) = \\ &= \frac{\partial\tilde{f}}{\partial x}(\cos\varphi, \sin\varphi)\cdot(-\sin\varphi) + \frac{\partial\tilde{f}}{\partial y}(\cos\varphi, \sin\varphi)\cdot\cos\varphi = \\ &= (-\sin\varphi \cdot \frac{\partial}{\partial x} + \cos\varphi \cdot \frac{\partial}{\partial y})\tilde{f}. \end{aligned}$$

It follows that $\frac{\partial}{\partial\varphi} = -\sin\varphi \cdot \frac{\partial}{\partial x} + \cos\varphi \cdot \frac{\partial}{\partial y}$ in the basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\} \cong \{e_1, e_2\}$ of $\mathbb{R}^2$.

In 2.4.13 we identified T_pM with the space of derivations $\mathrm{Der}_p(\mathcal{C}^\infty(M), \mathbb{R})$ at p. Thus for any $X \in \mathfrak{X}(M)$ and any $p \in M$, X_p is a derivation at p. The map $\mathcal{C}^\infty(M) \ni f \mapsto X(f)$, where $X(f) := p \mapsto X_p(f)$ is linear and satisfies

$$X(f \cdot g) = X(f)\cdot g + f \cdot X(g)$$

In fact, for $p \in M$ we have:

$$\begin{aligned} (X(f\cdot g))(p) &= X_p(f\cdot g) \\ &= f(p)X_p(g) + g(p)X_p(f) \\ &= (f\cdot X(g) + g \cdot X(f))(p). \end{aligned}$$

Consequently, X is a derivation in the following sense:

2.5.12 Definition. *An $\mathbb{R}$-linear map $D : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is called a derivation of the algebra $\mathcal{C}^\infty(M)$ if it satisfies the following product rule:*

$$D(f\cdot g) = f \cdot D(g) + g \cdot D(f).$$

The space of derivations on $\mathcal{C}^\infty(M)$ is denoted by $\mathrm{Der}(\mathcal{C}^\infty(M))$.

2.5.13 Theorem. *The derivations on $\mathcal{C}^\infty(M)$ are precisely the smooth vector fields on M:* $\mathrm{Der}(\mathcal{C}^\infty(M)) = \mathfrak{X}(M)$. *More precisely, every smooth vector field is a derivation on $\mathcal{C}^\infty(M)$, and, conversely, every derivation on $\mathcal{C}^\infty(M)$ is given by the action of a smooth vector field.*

Proof. $\mathfrak{X}(M) \subseteq \mathrm{Der}(\mathcal{C}^\infty(M))$ by 2.5.9 (ii) and the above considerations. Conversely, let $D \in \mathrm{Der}(\mathcal{C}^\infty(M))$. Then for any $p \in M$ the map $\mathcal{C}^\infty(M) \ni f \mapsto (D(f))(p)$ is a derivation at p:

$$\begin{aligned}(D(f\cdot g))(p) &= (D(f)\cdot g + f\cdot D(g))(p) = \\ &= (D(f))(p)\cdot g(p) + f(p)\cdot D(g)(p).\end{aligned}$$

By 2.4.13 it follows that there exists a unique $X_p \in T_pM$ with $X_p(f) = (D(f))(p)$. Hence $p \mapsto X_p$ is a vector field on M with $X(f) = D(f)\ \forall f \in \mathcal{C}^\infty(M)$. X is smooth by 2.5.9 (ii). □

2.5.14 Definition. *Let $X,\ Y \in \mathfrak{X}(M)$. The* Lie bracket *of X and Y is defined as*

$$[X,Y](f) := X(Yf) - Y(Xf) \qquad (f \in \mathcal{C}^\infty(M))$$

It follows that $[X,Y] : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is linear and satisfies the product rule, so by 2.5.13, $[X,Y] \in \mathfrak{X}(M)$.

2.5.15 Proposition. *(Properties of the Lie bracket) Let $X,\ Y,\ Z \in \mathfrak{X}(M)$, $f,\ g \in \mathcal{C}^\infty(M)$. Then:*

(i) $(X,Y) \mapsto [X,Y]$ *is $\mathbb{R}$-bilinear.*

(ii) $[X,Y] = -[Y,X]$ $\quad$ ($[\ ,\ ]$ *is skew-symmetric).*

(iii) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ *(Jacobi-identity).*

(iv) $[fX,gY] = fg[X,Y] + fX(g)Y - gY(f)X$.

(v) $[\ ,\ ]$ *is local: If V is open in M, then* $[X,Y]|_V = [X|_V, Y|_V]$.

(vi) Local representation: If (ψ,V) is a chart, $\psi = (x^1,\dots,x^n)$, $X|_V = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$, $Y|_V = \sum_{i=1}^n Y^i \frac{\partial}{\partial x^i}$, then:

$$[X,Y]|_V = \sum_{i=1}^n (\sum_{k=1}^n (X^k \frac{\partial Y^i}{\partial x^k} - Y^k \frac{\partial X^i}{\partial x^k}))\frac{\partial}{\partial x^i}$$

Proof. (i), (ii) are immediate from the definition.
(iii) We calculate:

$$\begin{aligned}[X,[Y,Z]]f &= X(Y(Zf) - X(Z(Yf)) - [Y,Z](Xf) = \\ &= X(Y(Zf)) - X(Z(Yf)) - Y(Z(Xf)) + Z(Y(Xf)) \\ [Y,[Z,X]]f &= Y(Z(Xf)) - Y(X(Zf)) - Z(X(Yf)) + X(Z(Yf)) \\ [Z,[X,Y]]f &= Z(X(Yf)) - Z(Y(Xf)) - X(Y(Zf)) + Y(X(Zf)),\end{aligned}$$

which sums to 0.
(iv) Let $h \in \mathcal{C}^\infty(M)$. Then

$$\begin{aligned}[fX,gY]h &= (fX)(gY(h)) - (gY)(fX(h)) = \\ &= fX(g)\cdot Y(h) + \underbrace{f\cdot g\cdot X(Y(h)) - f\cdot g\cdot Y(X(h))}_{=fg[X,Y](h)} - gY(f)X(h).\end{aligned}$$

(v) Let $f \in \mathcal{C}^\infty(V)$. Then $X_p(f)$ is well-defined for all $p \in V$ (cf. the proof of 2.4.13). Thus the map $p \mapsto X_p(f)$ is defined on V and coincides with $X|_V(f)$. An analogous statement holds for Y. For $p \in V$ we therefore have:

$$\begin{aligned}[X,Y]_p(f) &= X_p(Yf) - Y_p(Xf) = X_p(Y|_V(f)) - Y_p(X|_V(f)) = \\ &= (X|_V)_p(Y|_V(f)) - (Y|_V)_p(X|_V(f)) = \\ &= [X|_V, Y|_V]_p(f).\end{aligned}$$

(vi) Let $f \in \mathcal{C}^\infty(V, \mathbb{R})$. Then:

$$[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}]_p f = \frac{\partial}{\partial x^i}\Big|_p (\frac{\partial}{\partial x^j} f) - \frac{\partial}{\partial x^j}\Big|_p (\frac{\partial}{\partial x^i} f)$$

Now

$$\begin{aligned}
\frac{\partial}{\partial x^i}\Big|_p (\frac{\partial f}{\partial x^j}) &\overset{(2.4.2)}{=} \frac{\partial}{\partial x^i}\Big|_p (q \mapsto \underbrace{D_j(f \circ \psi^{-1})(\psi(q))}_{=:g_j(q)}) = \\
&\overset{(2.4.2)}{=} \underbrace{D_i(g_j \circ \psi^{-1})}_{D_i D_j(f\circ\psi^{-1})}(\psi(p)) = \\
&= D_j D_i(f \circ \psi^{-1})(\psi(p)) = \\
&= \frac{\partial}{\partial x^j}\Big|_p (\frac{\partial}{\partial x^i} f),
\end{aligned}$$

so $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0\ \forall i, j$. Hence

$$\begin{aligned}
[X, Y]|_V &\overset{(v)}{=} [X|_V, Y|_V] = \\
&= [\sum_{i=1}^n X^i \frac{\partial}{\partial x^i}, \sum_{k=1}^n Y^k \frac{\partial}{\partial x^k}] = \\
&\overset{(i),(iv)}{=} \sum_{i,k=1}^n (X^i Y^k \underbrace{[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}]}_{=0} + X^i \frac{\partial Y^k}{\partial x^i}\frac{\partial}{\partial x^k} - Y^k \frac{\partial X^i}{\partial x^k}\frac{\partial}{\partial x^i}) = \\
&= \sum_{i=1}^n (\sum_{k=1}^n (X^k \frac{\partial Y^i}{\partial x^k} - Y^k \frac{\partial X^i}{\partial x^k}))\frac{\partial}{\partial x^i}
\end{aligned}$$

□

In the theory of dynamical systems one analyzes solutions of autonomous ODEs $\dot{c}(t) = X(c(t))$, where X is a vector field. In applications, X is often not defined on an open subset of $\mathbb{R}^n$. For example, c might be subject to certain 'constraints', i.e., be constrained by some regular equation. By 2.1.8 this means that X is in fact defined on some differentiable manifold M. Thus we are interested in the ODE

$$\dot{c}(t) = X(c(t)) \tag{2.5.3}$$

with $X \in \mathfrak{X}(M)$.
To begin with we have to clarify what we mean by $\dot{c}(t)$. For $c \in \mathcal{C}^\infty(I, \mathbb{R}^n)$, $\dot{c}(t)$ is given by the vector $Dc(t) \cdot 1$ (where $1 = e_1 \in \mathbb{R}$). For $c \in \mathcal{C}^\infty(I, M)$ we analogously set

$$\dot{c}(t) = T_t c(1) \overset{2.5.11}{=} T_t c(\frac{\partial}{\partial t}\Big|_t).$$

Since differentiation is a local operation we may write (2.5.3) in local coordinates: let (ψ, V) be a chart in M. The local representation of X with respect to $\psi = (x^1, \dots, x^n)$ is $\psi_* X := T\psi \circ X \circ \psi^{-1}$:

$$\begin{array}{ccc}
TM & \xrightarrow{T\psi} & \psi(V) \times \mathbb{R}^n \\
{\scriptstyle X}\uparrow & & \uparrow{\scriptstyle \psi_* X} \\
M \supseteq V & \xrightarrow{\psi} & \psi(V)
\end{array}$$

Here, $\psi_* X$ is called push-forward of X under ψ. By (2.5.2), $\psi_* X$ is the map

$$x \mapsto (x, \sum_{i=1}^{n} X^i(\psi^{-1}(x))e_i)$$

(for $X|_V = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$). One often drops the first component in this formula. Hence locally X is a vector field with components $(X^1 \circ \psi^{-1}, \dots, X^n \circ \psi^{-1})$. We also localize $\dot{c}$, i.e., we write $\dot{c}(t)$ in the chart ψ: $\dot{c}$ is the second component of Tc, applied to 1 ($\cong \frac{\partial}{\partial t}$). An application of $T\psi$ gives

$$(T\psi \circ Tc)(t, 1) = T(\psi \circ c)(t, 1) = (\psi \circ c(t), D(\psi \circ c)(t) \cdot 1).$$

Now $D(\psi \circ c)(t) \cdot 1 = (\psi \circ c)^{\boldsymbol{\cdot}}(t)$. Thus with respect to the chart ψ, (2.5.3) reads:

$$(\psi \circ c)^{\boldsymbol{\cdot}}(t) = (\psi_* X)(\psi \circ c(t)), \tag{2.5.4}$$

so locally we obtain the autonomous ODE

$$(x^i \circ c)^{\boldsymbol{\cdot}}(t) = (X^i \circ \psi^{-1})(\psi \circ c(t)) \qquad (1 \le i \le n) \tag{2.5.5}$$

or, with $\tilde{c}^i = x^i \circ c$, $\tilde{X}^i = X^i \circ \psi^{-1}$

$$\dot{\tilde{c}}^i(t) = \tilde{X}^i(\tilde{c}(t)).$$

To study the global behavior of the solutions of (2.5.3) (the so-called *integral curves* of X) we will need the following fundamental existence and uniqueness result for ODEs:

2.5.16 Theorem. *Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be smooth. Then there exists an open interval I of $0 \in \mathbb{R}$ and an open ball U around $0 \in \mathbb{R}^n$ such that for each $x \in U$ there is a unique solution $c_x : I \to \mathbb{R}^n$ of the initial value problem*

$$\begin{aligned} \dot{c}_x(t) &= F(t, c_x(t)) \\ c_x(0) &= x \end{aligned}$$

The map $(t, x) \mapsto c_x(t)$, $I \times U \to \mathbb{R}^n$ is smooth.

Proof. See your favorite ODE-course or Dieudonne, Vol. 1, 10.8.1, 10.8.2. □

Based on this result we establish the following fundamental theorem on ODEs on manifolds.

2.5.17 Theorem. *Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. Then*

(i) Any $p \in M$ is contained in a unique maximal integral curve of X, i.e., there is a unique smooth solution c_p of (2.5.3) with $c_p(0) = p$ and maximal domain of definition (t^p_-, t^p_+).

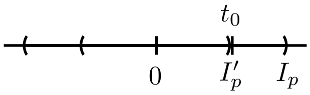
(ii) If $t^p_+ < \infty$, then $\lim_{t \to t^p_+} c_p(t) = \infty$. That is to say, for $t \to t^p_+$, the curve $c_p(t)$ leaves every compact subset of M (and analogously for $t^p_- > -\infty$).

(iii) The set $U = \{(t, p) \mid t^p_- < t < t^p_+\}$ is an open neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$. The flow *of X, defined by $\mathrm{Fl}^X : U \to M$, $(t, p) \mapsto c_p(t)$ is smooth (U is the maximal domain of definition of Fl^X). For every $p \in M$ the map $t \mapsto \mathrm{Fl}^X(t, p) \equiv \mathrm{Fl}^X_t(p)$ satisfies the following semi-group property:*

$$\mathrm{Fl}^X_{t+s}(p) = \mathrm{Fl}^X_t(\mathrm{Fl}^X_s(p))$$

whenever the right hand side of this equation exists.

Proof. (i) As we have seen in (2.5.4), in every chart domain (2.5.3) can be transformed into a local autonomous ODE. Thus 2.5.16 implies the existence of smooth solutions of (2.5.3), i.e., of integral curves of X. Moreover, again by 2.5.16 these solutions are locally unique, i.e., if two solutions coincide in a t-value t_0 then they in fact coincide on a neighborhood of t_0.
Let $p \in M$, and $c_1 : I_1 \to M$, $c_2 : I_2 \to M$ two integral curves of X with $c_1(0) = p = c_2(0)$. Then $J := \{t \in I_1 \cap I_2 \mid c_1(t) = c_2(t)\}$ is nonempty (since $0 \in J$) and closed in $I_1 \cap I_2$. By the above J is also open in $I_1 \cap I_2$, so $J = I_1 \cap I_2$. Thus c_1 and c_2 can be combined into a single integral curve on $I_1 \cup I_2$. The maximal integral curve c_p through p therefore is defined on $(t_-^p, t_+^p) = \bigcup\{I \mid \exists$ integral curve $c : I \to M$ with $c(0) = p\}$.
(iii) Since $0 \in (t_-^p, t_+^p)$ for all $p \in M$ it follows that $\{0\} \times M \subseteq U$. Moreover, $\mathrm{Fl}^X(0, p) = c_p(0) = p$. Suppose that $\mathrm{Fl}_t^X(\mathrm{Fl}_s^X(p)))$ exists, i.e., $t \mapsto \mathrm{Fl}_t^X(\mathrm{Fl}_s^X(p))$ is the maximal integral curve of X through $\mathrm{Fl}_s^X(p)$. Since also $t \mapsto \mathrm{Fl}_{s+t}^X(p)$ is an integral curve of X with initial value $\mathrm{Fl}_s^X(p)$, it follows that $\mathrm{Fl}_{s+t}^X(p) = \mathrm{Fl}_t^X(\mathrm{Fl}_s^X(p))$. By 2.5.16, Fl^X is defined and smooth on a neighborhood of $\{0\} \times M$. For $p \in M$ let $I_p := (t_-^p, t_+^p)$ and $I_p' := \{t \in \mathbb{R} \mid \mathrm{Fl}^X$ is defined and smooth in a neighborhood of $[0, t] \times \{p\}$ (for $t \geq 0$) resp. of $[t, 0] \times \{p\}$ (for $t < 0$)$\}$
Then $I_p' \subseteq I_p$, $0 \in I_p'$ and I_p' is an open interval. We will show that $I_p' = I_p$. Suppose to the contrary that $I_p' \subsetneq I_p$.



Without loss of generality we may suppose that $t_0 := \inf\{t > 0 \mid t \in I_p \setminus I_p'\} > 0$. Note that $t_0 \notin I_p'$ since I_p' is open.
We know that Fl^X is defined and smooth on a neighborhood W of $(0, \mathrm{Fl}_{t_0}^X(p)) \in \mathbb{R} \times M$. We choose some δ with $0 < \delta < t_0$, and a neighborhood V of p in M such that

$$(-\delta, 2\delta) \times \mathrm{Fl}_{t_0-\delta}^X(V) \subseteq W$$

(which is possible since $(s, q) \mapsto \mathrm{Fl}_s^X(q)$ is continuous) and such that $q \mapsto \mathrm{Fl}_{t_0-\delta}^X(q)$ is smooth on V. Then the map

$$(s, q) \mapsto \mathrm{Fl}_s^X(\mathrm{Fl}_{t_0-\delta}^X(q)) = \mathrm{Fl}_{s+t_0-\delta}^X(q)$$

is smooth on the neighborhood $(-\delta, 2\delta) \times V$ of $[0, \delta] \times \{p\}$, so Fl^X is smooth on the neighborhood $(t_0 - 2\delta, t_0 + \delta) \times V$ of $[t_0 - \delta, t_0] \times \{p\}$. Moreover (by definition of t_0) $t_0 - \delta \in I_p'$, so Fl^X is smooth on a neighborhood of $[0, t_0 - \delta] \times \{p\}$. Summing up, Fl^X is smooth on a neighborhood of $([0, t_0 - \delta] \cup [t_0 - \delta, t_0]) \times \{p\} = [0, t_0] \times \{p\}$. But according to the definition of I_p' this means that $t_0 \in I_p'$, contradicting the definition of t_0, which establishes $I_p = I_p'$.
Hence $U = \{(t, p) \mid t \in I_p\} = \{(t, p) \mid t \in I_p'\}$ is open and Fl^X is smooth on U (both according to the definition of I_p').
(ii) Let $p \in M$, $t_+^p < \infty$ and K a compact subset of M. We want to show that $c_p(t) \notin K$ for t sufficiently close to t_+^p. Suppose to the contrary that there exists a sequence (t_n) with $t_n \nearrow t_+^p$ and $c_p(t_n) \in K$ for all n. Since K is compact, a subsequence of $(c_p(t_n))$, w.l.o.g. $(c_p(t_n))$ itself, converges to some $p' \in K$.
There exists some $\varepsilon > 0$ and some neighborhood V of p' such that Fl^X is smooth on $(-\varepsilon, \varepsilon) \times V$. Choose n_0 such that $c_p(t_n) \in V$ $\forall n \geq n_0$. Since

$$\mathrm{Fl}_t^X(c_p(t_n)) = \mathrm{Fl}_t^X(\mathrm{Fl}_{t_n}^X(p)) = \mathrm{Fl}_{t+t_n}^X(p) = c_p(t + t_n),$$

$c_p(t + t_n)$ exists for all $|t| < \varepsilon$ and all $n \geq n_0$. Thus $c_p(s)$ is defined for $s \in (t_n - \varepsilon, t_n + \varepsilon)$ $\forall n \geq n_0$. Choose $n \geq n_0$ such that $t_n > t_+^p - \frac{\varepsilon}{2}$. Then $c_p(s)$ is exists up to $t_+^p - \frac{\varepsilon}{2} + \varepsilon = t_+^p + \frac{\varepsilon}{2} > t_+^p$, contradicting the definition of t_+^p. $\square$

2.5.18 Definition. *Let M be a manifold and let $X \in \mathfrak{X}(M)$. X is called* complete, *if Fl^X is defined on all of $\mathbb{R} \times M$ (i.e., $U = \mathbb{R} \times M$).*

Completeness of X therefore means that each integral curve of X exists for all times. From 2.5.17 (ii) we conclude:

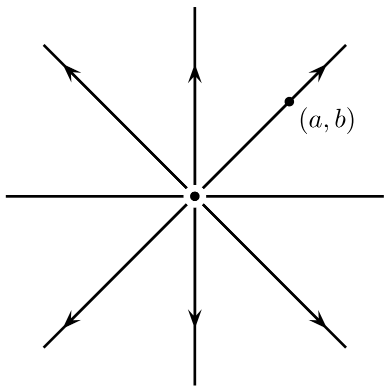
2.5.19 Corollary. *Every vector field on a compact manifold is complete.*

2.5.20 Examples.

(*i*) Let $M = \mathbb{R}^2$, and $X = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2}$. To determine the integral curves of X we have to solve the ODE $\dot{c}(t) = X(c(t))$. Applying (2.5.5) with $\psi = \mathrm{id}_{\mathbb{R}^2}$ gives: $\tilde{c} = c$, $\tilde{X} = X$. Hence we consider

$$\begin{aligned} \dot{c}^1(t) &= x^1(c(t)) = c^1(t) \\ \dot{c}^2(t) &= x^2(c(t)) = c^2(t) \\ c(0) &= (a,b) \in \mathbb{R}^2 \end{aligned}$$

Thus $c(t) = (ae^t, be^t) = \mathrm{Fl}_t^X(a,b)$. Obviously, $\mathrm{Fl}_{s+t}^X(a,b) = \mathrm{Fl}_s^X(\mathrm{Fl}_t^X(a,b))$. For $(a,b) = (0,0)$ it follows that $c(t) \equiv 0$ since $X(0,0) = (0,0)$. $(0,0)$ is called a *critical point* of X (i.e., zero of X).



Every integral curve of X is defined on all of $\mathbb{R}$, so X is complete.

(ii) Let $M = \mathbb{R}^2$, and $X = e^{-x^1} \frac{\partial}{\partial x^1}$. Using the chart $\psi = \mathrm{id}_{\mathbb{R}^2}$ we obtain

$$\begin{aligned} \dot{c}^1(t) &= e^{-c^1(t)} \\ \dot{c}^2(t) &= 0 \\ c(0) &= (a,b) \end{aligned}$$

Thus $c(t) = (\log(t + \exp a), b) = \mathrm{Fl}_t^X(a,b)$ (it is easily verified that the flow property $\mathrm{Fl}_{t+s}^X(a,b) = \mathrm{Fl}_t^X(\mathrm{Fl}_s^X(a,b))$ holds). c is defined on $(-e^a, \infty) \subsetneq \mathbb{R}$, so X is not complete.

(iii) (cf. 2.1.7 (ii)).

Let $\tilde{M} = S^2$, $\psi : (x,y,z) = (\cos\phi\cos\theta, \sin\phi\cos\theta, \sin\theta) \mapsto (\phi,\theta) = (\psi^1, \psi^2)$, and $M := \psi^{-1}((0,2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})) \overset{\text{open}}{\subseteq} \tilde{M}$. Let X on M be given, with respect to ψ, by

$$X = \phi \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \theta}$$

In (2.5.5) we have $\tilde{X}^1(\phi,\theta) = \phi$, $\tilde{X}^2(\phi,\theta) = 1$, $\tilde{c}(t) = (\phi(t),\theta(t))$. (Note that X cannot be extended smoothly to S^2 since ϕ has a jump.) Hence (2.5.5) reads:

$$\begin{aligned}
\dot{\phi}(t) &= \phi(t) \\
\dot{\theta}(t) &= 1 \\
(\phi(0),\theta(0)) &= (\phi_0,\theta_0)
\end{aligned}$$

Thus $\tilde{c}(t) = (\phi(t),\theta(t)) = (\phi_0 e^t, t+\theta_0)$, so

$$c(t) = \psi^{-1}\circ\tilde{c}(t) = (\cos(\phi_0 e^t)\cos(t+\theta_0), \sin(\phi_0 e^t)\cos(t+\theta_0), \sin(t+\theta_0)).$$

$X \in \mathfrak{X}(M)$ is not complete.

2.5.21 Remark. Let M^k be a k-dimensional submanifold of $\mathbb{R}^n$. Then $\mathfrak{X}(M) = \{X : M \to \mathbb{R}^n \mid X\ \ \mathcal{C}^\infty \text{ and } X_p \in T_pM\ \forall p \in M\}$.

Proof. Let $X \in \mathfrak{X}(M)$. Then $X_p \in T_pM\ \forall p$. Locally (with respect to a parametrization (φ, U)), X is given by

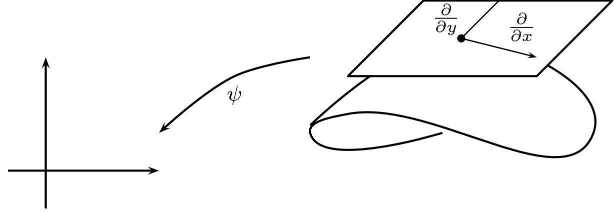
$$\begin{aligned}
X(\varphi(x)) &= \sum_{i=1}^k X^i(\varphi(x))\,\frac{\partial}{\partial x^i}\Big|_{\varphi(x)} \\
&= \sum_{i=1}^k X^i(\varphi(x)) D_i\varphi(x)
\end{aligned}$$

(cf. the remark preceding 2.4.12). Hence $X\circ\varphi$ is smooth since X^i and φ are. But then X is smooth by 2.1.13 (i) and 2.1.15.
Conversely, let $X : M \to \mathbb{R}^n$ be smooth and suppose that $X_p \in T_pM$ for all $p \in M$. Then X is a section of TM and it remains to show that X is smooth. To this end we employ 2.5.9 (ii): let $f \in \mathcal{C}^\infty(M)$ with local smooth extension $\tilde{f}$. Then $X(f)$ is locally given by $p \mapsto X_p(f) = T_pf(X_p) = D\tilde{f}(p)X_p$ which clearly is smooth on M. $\square$

Caution: Note that the $X^1,\dots,X^k$ should not be confused with the n components of X as a vector in $\mathbb{R}^n$!

2.6 Tensors

Heuristically, if we want to determine the area of a curved surface, or, more generally, the volume of some submanifold, we first have to approximate the surface 'infinitesimally' by its tangent space, then determine the area of these approximating spaces and then sum (resp. integrate) up the results.



Thus we first need a way of assigning volumes to parallelepipeds in vector spaces. A map ω which assigns a volume to a parallelepiped with edges u, v, w should possess the following properties:

(i) $\omega(\alpha u, v, w, \dots) = \omega(u, \alpha v, w, \dots) = \dots = \alpha\cdot\omega(u, v, w, \dots)$

(ii) $\omega(u_1 + u_2, v, w, \dots) = \omega(u_1, v, w \dots) + \omega(u_2, v, w, \dots)$, and analogously for $v, w, \dots$

(iii) $\omega(u, u, w, \dots) = \omega(u, v, v, \dots) = \dots = 0$

Since $0 = \omega(u+v, u+v, w, \dots) = \omega(u, v, w, \dots) + \omega(v, u, w, \dots)$, (iii) is equivalent to ω being antisymmetric (or skew-symmetric).
Due to (i),(ii) we have to consider multilinear mappings on vector spaces (in particular, on T_pM). The skew-symmetry (iii) will be taken into account in the following section. We therefore begin this section with a crash-course in multilinear algebra.
In what follows let $E_1, \dots, E_k, E, F$ be finite-dimensional vector spaces. Then by $L^k(E_1, \dots, E_k; F)$ we denote the space of multilinear maps from $E_1 \times \dots \times E_k$ to F. An important special case is ($k = 1$): $L(E, \mathbb{R}) = E^*$, the dual space of E, i.e., the vector space of linear functionals on E. If $\mathcal{B}_E = \{e_1, \dots, e_n\}$ is a basis of E, then the functionals defined by

$$\alpha^j(e_i) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

($j = 1, \dots, n$) form a basis of E^*, the *dual basis* of $\mathcal{B}_E$. For each $e \in E$ we have $e = \sum_{i=1}^n \alpha^i(e) e_i$ and for each $\alpha \in E^*$ we get $\alpha = \sum_{i=1}^n \alpha(e_i)\alpha^i$. The bidual space $E^{**} = (E^*)^*$ is canonically isomorphic to E: the map

$$\begin{aligned} i : E &\to E^{**} \\ i(e) &= \underbrace{\alpha}_{\in E^*} \mapsto \alpha(e) \end{aligned}$$

is a linear isomorphism.

2.6.1 Definition. *Let E be a vector space. Then*

$$T^r_s(E) := L^{r+s}(\underbrace{E^*, \dots, E^*}_{r}, \underbrace{E, \dots, E}_{s}; \mathbb{R})$$

is called the space of r-times contra- and s-times covariant tensors, or, for short, $\binom{r}{s}$-tensors. The elements of $T^r_s(E)$ are called tensors of type $\binom{r}{s}$.
For $t_1 \in T^{r_1}_{s_1}(E)$, $t_2 \in T^{r_2}_{s_2}(E)$, the tensor product $t_1 \otimes t_2 \in T^{r_1+r_2}_{s_1+s_2}(E)$ is defined by:

$$\begin{aligned} &t_1 \otimes t_2(\beta^1, \dots, \beta^{r_1}, \gamma^1, \dots, \gamma^{r_2}, f_1, \dots, f_{s_1}, g_1, \dots, g_{s_2}) \\ &:= t_1(\beta^1, \dots, \beta^{r_1}, f_1, \dots, f_{s_1}) \cdot t_2(\gamma^1, \dots, \gamma^{r_2}, g_1, \dots, g_{s_2}) \end{aligned}$$

($\beta^j, \gamma^j \in E^*, f_j, g_j \in E$).

Clearly, $\otimes$ is associative and bilinear.

2.6.2 Example.

(i) By definition, $T^0_1(E) = L(E, \mathbb{R}) = E^*$ and $T^1_0(E) = L(E^*, \mathbb{R}) = E^{**} = E$. Elements of E (vectors) therefore are $\binom{1}{0}$-tensors, elements of E^* (often called co-vectors) are $\binom{0}{1}$-tensors.

(ii) Let E be a vector space with scalar product $g(e, f) = \langle e, f \rangle$. Then g is a bilinear map $g : E \times E \to \mathbb{R}$, i.e., a $\binom{0}{2}$-tensor.

2.6.3 Proposition. *Let* $\dim(E) = n$. *Then* $\dim(T^r_s(E)) = n^{r+s}$. *If $\{e_1, \dots, e_n\}$ is a basis of E and $\{\alpha^1, \dots, \alpha^n\}$ is the corresponding dual basis, then*

$$\mathcal{B}^r_s := \{e_{i_1} \otimes \dots \otimes e_{i_r} \otimes \alpha^{j_1} \otimes \dots \otimes \alpha^{j_s} \mid 1 \leq i_k, j_k \leq n\}$$

is a basis of $T^r_s(E)$.

Proof. $\mathcal{B}^r_s$ is linearly independent: let

$$\sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}} \underbrace{t^{i_1\dots i_r}_{j_1\dots j_s}}_{\in\mathbb{R}} e_{i_1}\otimes\dots\otimes e_{i_r}\otimes\alpha^{j_1}\otimes\dots\otimes\alpha^{j_s} = 0$$

Inserting $(\alpha^{k_1},\dots,\alpha^{k_r},e_{l_1},\dots,e_{l_s})$, then since $\alpha^i(e_j) = e_j(\alpha^i) = \delta_{ij}$ it follows that all $t^{i_1\dots i_r}_{j_1\dots j_s}$ vanish.
$\mathcal{B}^r_s$ generates $T^r_s(E)$: each $t \in T^r_s(E)$ can be written as follows:

$$t = \sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}} t(\alpha^{i_1},\dots,\alpha^{i_r},e_{j_1},\dots,e_{j_s}) e_{i_1}\otimes\dots\otimes e_{i_r}\otimes\alpha^{j_1}\otimes\dots\otimes\alpha^{j_s}.$$

To see this, it suffices to show that both sides of this equation define the same multilinear map. Let $\beta^1 = \sum\lambda^1_{i_1}\alpha^{i_1},\dots,\beta^r = \sum\lambda^r_{i_r}\alpha^{i_r} \in E^*$, and $x_1 = \sum\mu^{j_1}_1 e_{j_1},\dots,$ $x_s = \sum\mu^{j_s}_s e_{j_s} \in E$. Then

$$\begin{aligned}t(\beta^1,\dots,\beta^r,x_1,\dots,x_s) &= \sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}} \lambda^1_{i_1}\dots\lambda^r_{i_r}\mu^{j_1}_1\dots\mu^{j_s}_s t(\alpha^{i_1},\dots,\alpha^{i_r},e_{j_1},\dots,e_{j_s})\\ &= \sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}} t(\alpha^{i_1},\dots,\alpha^{i_r},e_{j_1},\dots,e_{j_s}) e_{i_1}\otimes\dots\otimes e_{i_r}\otimes\alpha^{j_1}\otimes\dots\otimes\alpha^{j_s}(\beta^1,\dots,x_s)\end{aligned}$$

□

Every linear map $\varphi : E \to F$ possesses an adjoint map $\varphi^* \in L(F^*,E^*)$: for $\beta \in F^*$, $e \in E$ one sets $\varphi^*(\beta)(e) := \beta(\varphi(e))$. If A is the matrix of φ with respect to bases of E resp. F, then A^t is the matrix of φ^* with respect to the corresponding dual bases of F^* resp. E^*.
More generally, we now want to assign to any $\varphi \in L(E,F)$ a linear map $\varphi^r_s \in L(T^r_s(E),T^r_s(F))$. If φ is a linear isomorphism we may combine such a map from φ and φ^*:

2.6.4 Definition. *Let $\varphi \in L(E,F)$ be bijective. Then $T^r_s(\varphi) \equiv \varphi^r_s \in L(T^r_s E, T^r_s F)$ is defined as*

$$(\varphi^r_s(t))(\beta^1,\dots,\beta^r,f_1,\dots,f_s) := t(\varphi^*(\beta^1),\dots,\varphi^*(\beta^r),\varphi^{-1}(f_1),\dots,\varphi^{-1}(f_s))$$

for $t \in T^r_s(E)$, $\beta^1,\dots,\beta^r \in F^*$, $f_1,\dots,f_s \in F$.

2.6.5 Example. $\varphi^1_0 : E = T^1_0(E) \to T^1_0(F) = F$, $\varphi^1_0(e)(\beta) = e(\varphi^*(\beta)) = \varphi(e)(\beta)$. Thus we may identify φ^1_0 with φ.
$\varphi^0_1 : E^* = T^0_1(E) \to T^0_1(F) = F^*$, $\varphi^0_1(\alpha)(f) = \alpha(\varphi^{-1}(f)) = (\varphi^{-1})^*(\alpha)(f)$, so we may identify φ^0_1 with $(\varphi^{-1})^*$.
It follows that $T^r_s\varphi = \varphi^r_s$ is a simultaneous extension of φ and $(\varphi^{-1})^*$ to general tensor spaces.

2.6.6 Proposition. *Let $\varphi : E \to F$, $\psi : F \to G$ be linear isomorphisms. Then:*

(i) $(\psi\circ\varphi)^r_s = \psi^r_s\circ\varphi^r_s$

(ii) $(\mathrm{id}_E)^r_s = \mathrm{id}_{T^r_s(E)}$

(iii) *$\varphi^r_s : T^r_s E \to T^r_s F$ is a linear isomorphism, and $(\varphi^r_s)^{-1} = (\varphi^{-1})^r_s$.*

(iv) *If $t_1 \in T^{r_1}_{s_1}(E)$, $t_2 \in T^{r_2}_{s_2}(E)$, then $\varphi^{r_1+r_2}_{s_1+s_2}(t_1\otimes t_2) = \varphi^{r_1}_{s_1}(t_1)\otimes\varphi^{r_2}_{s_2}(t_2)$.*

Proof. (i) We first note that for $\gamma \in G^*$, $e \in E$ we have

$$(\varphi^* \circ \psi^*)(\gamma)(e) = (\varphi^*(\psi^*(\gamma)))(e) = \psi^*(\gamma)(\varphi(e)) = \gamma(\psi(\varphi(e))) = (\psi \circ \varphi)^*(\gamma)(e).$$

Now let $\gamma^1, \dots, \gamma^r \in G^*, g_1, \dots, g_s \in G$ and $t \in T^r_s(E)$. Then

$$\begin{aligned}
&(\psi^r_s(\varphi^r_s(t)))(\gamma^1, \dots, \gamma^r, g_1, \dots, g_s) \\
&\qquad = (\varphi^r_s(t))(\psi^*\gamma^1, \dots, \psi^*\gamma^r, \psi^{-1}(g_1), \dots, \psi^{-1}(g_s)) \\
&\qquad = t(\underbrace{\varphi^*(\psi^*\gamma^1)}_{(\psi\circ\varphi)^*\gamma^1}, \dots, \varphi^*(\psi^*\gamma^r), \underbrace{\varphi^{-1}(\psi^{-1})(g_1)}_{(\psi\circ\varphi)^{-1}(g_1)}, \dots, \varphi^{-1}(\psi^{-1}(g_s))) \\
&\qquad = ((\psi \circ \varphi)^r_s(t))(\gamma^1, \dots, \gamma^r, g_1, \dots, g_s).
\end{aligned}$$

(ii) Since $\mathrm{id}_E^{-1} = \mathrm{id}_E$ and $\mathrm{id}_E^* = \mathrm{id}_{E^*}$ this is immediate from the definitions.
(iii) follows from (i) and (ii).
(iv)

$$\begin{aligned}
&\varphi^{r_1+r_2}_{s_1+s_2}(t_1 \otimes t_2)(\beta^1, \dots, \beta^{r_1+r_2}, f_1, \dots, f_{s_1+s_2}) \\
&\qquad = (t_1 \otimes t_2)(\varphi^*\beta^1, \dots, \varphi^*\beta^{r_1+r_2}, \varphi^{-1}(f_1), \dots, \varphi^{-1}(f_{s_1+s_2})) \\
&\qquad = t_1(\varphi^*\beta^1, \dots, \varphi^*\beta^{r_1}, \varphi^{-1}(f_1), \dots, \varphi^{-1}(f_{s_1})) \cdot \\
&\qquad\quad\ t_2(\varphi^*\beta^{r_1+1}, \dots, \varphi^*\beta^{r_1+r_2}, \varphi^{-1}(f_{s_1+1}), \dots, \varphi^{-1}(f_{s_1+s_2})) \\
&\qquad = (\varphi^{r_1}_{s_1}t_1) \otimes (\varphi^{r_2}_{s_2}t_2)(\beta^1, \dots, \beta^{r_1+r_2}, f_1, \dots, f_{s_1+s_2}).
\end{aligned}$$

□

To simplify notations, in what follows we will employ Einstein's *summation convention*: for every index which appears both as an upper and as a lower index, summation is carried out over its entire set of values. Thus, instead of

$$\sum_{\substack{i_1,\dots,i_r \\ j_1,\dots,j_s}} t^{i_1,\dots,i_r}_{j_1,\dots,j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \alpha^{j_1} \otimes \cdots \otimes \alpha^{j_s}$$

we simply write $t^{i_1,\dots,i_r}_{j_1,\dots,j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \alpha^{j_1} \otimes \cdots \otimes \alpha^{j_s}$. Deviations from this convention will be mentioned explicitly.
Our next aim is to extend the above constructions of multilinear algebra to tangent vectors, i.e., to elements of certain vector bundles. To carry out this transfer we first consider the case of local vector bundles.

2.6.7 Definition. *Let $\varphi : U \times F \to U' \times F'$, $\varphi(u, f) = (\varphi_1(u), \varphi_2(u)f)$ be a local vector bundle isomorphism (cf.* 2.5.5 *(i)). Then define $\varphi^r_s : U \times T^r_s F \to U' \times T^r_s F'$ by*

$$\varphi^r_s(u, t) = (\varphi_1(u), (\varphi_2(u))^r_s(t)) \qquad (t \in T^r_s F)$$

Note that $\varphi_2(u)$ is an isomorphism for each u, so $(\varphi_2(u))^r_s$ is well-defined.

2.6.8 Lemma. *Under the assumptions of 2.6.7, $\varphi^r_s : U \times T^r_s F \to U' \times T^r_s F'$ is a local vector bundle isomorphism.*

Proof. By 2.6.6 (iii), every $(\varphi_2(u))^r_s$ is a linear isomorphism. Hence φ^r_s is bijective and it remains to show that $(u, t) \mapsto \varphi^r_s(u, t)$ is smooth (it then follows that also $(\varphi^r_s)^{-1} = (\varphi^{-1})^r_s$ is smooth). Clearly, φ_1 is smooth.
Concerning φ_2 we first note that on the space $L(F, F')$ of linear maps (i.e., matrices) the map $\varphi \mapsto \varphi^*$ ($= A \mapsto A^t$) is linear, hence smooth. Moreover, the space of invertible matrices $\mathrm{GL}(F, F')$ is open in $L(F, F')$ (since $\mathrm{GL}(F, F') = \{A \in L(F, F') \mid \det(A) \neq 0\}$) and $\varphi \mapsto \varphi^{-1}$ (corresponding to $A \mapsto A^{-1}$)) is smooth on $\mathrm{GL}(F, F')$

by the inversion formula for matrices. Thus the maps $u \mapsto \varphi_2(u)^*$ and $u \mapsto \varphi_2(u)^{-1}$ are smooth. Moreover, the maps $i_k,\ i'_k : (\beta^1, \dots \beta^r, f_1, \dots, f_s) \mapsto \beta^k$ resp. $\mapsto f_k$ are linear, hence smooth as well. Summing up,

$$
\begin{array}{rcl}
(u,t) & \mapsto & (\varphi_2(u))^r_s(t) = \\
(u,t) & \mapsto & (t, \varphi_2(u)^*, \dots, \varphi_2(u)^*, \varphi_2(u)^{-1}, \dots, \varphi_2(u)^{-1}) \\
 & \mapsto & t \circ (\varphi_2(u)^* \circ i_1, \dots, \varphi_2(u)^* \circ i_r, \varphi_2(u)^{-1} \circ i'_1, \dots, \varphi_2(u)^{-1} \circ i'_s)
\end{array}
$$

is smooth since also the last of the above maps is multilinear, hence $\mathcal{C}^\infty$. $\square$

After these preparations we may now assign to any vector bundle E the corresponding $\binom{r}{s}$ tensor bundle, which has precisely the $(E_b)^r_s$ as fibers:

2.6.9 Definition. *Let (E, B, π) be a vector bundle, with $E_b = \pi^{-1}(b)$ the fiber over b. Then let*

$$T^r_s(E) := \bigsqcup_{b \in B} T^r_s(E_b) = \bigcup_{b \in B} \{b\} \times (E_b)^r_s$$

be the $\binom{r}{s}$-tensor bundle over E. Let $\pi^r_s : T^r_s(E) \to B,\ \pi^r_s(e) = b$ for $e \in T^r_s(E_b)$ denote the canonical projection. For $A \subseteq B$ let $T^r_s(E)|_A := \bigsqcup_{b \in A} T^r_s(E_b)$.

We wish to turn $T^r_s(E)$ itself into a vector bundle with basis B. To this end we will produce vector bundle charts for $T^r_s(E)$ from those of E, according to the following pattern:

2.6.10 Definition. *Let $E,\ E'$ be vector bundles and $f : E \to E'$. f is called a vector bundle homomorphism, if for each $e \in E$ there exists a vector bundle chart (Ψ, W) around e and a vector bundle chart (Ψ', W') around $f(e)$, such that $f(W) \subseteq W'$ and $f_{\Psi'\Psi} := \Psi' \circ f \circ \Psi^{-1}$ is a local vector bundle homomorphism (cf. 2.5.5 (i)). If f in addition is a diffeomorphism and $f|_{E_b} : E_b \to E'_{f(b)}$ is a linear isomorphism for all $b \in B$ then f is called a vector bundle isomorphism. In this case we define $f^r_s : T^r_s E \to T^r_s E'$ by*

$$f^r_s|_{T^r_s(E_b)} := (f|_{E_b})^r_s\ \forall b \in B$$

It is straightforward to check that a smooth map $f : E \to E'$ is a vector bundle homomorphism if and only if f is fiber-linear, i.e., if and only if $f|_{E_b} : E_b \to E'_{f(b)}$ is linear for each $b \in B$.

2.6.11 Examples.

(i) Let $M,\ N$ be manifolds and $f : M \to N$ smooth. Then $Tf : TM \to TN$ is a vector bundle homomorphism. In fact, by 2.5.4 we have:

$$
\begin{aligned}
T\psi \circ Tf \circ T\varphi^{-1}(x,w) &= T(\psi \circ f \circ \varphi^{-1})(x,w) \\
&= (\psi \circ f \circ \varphi^{-1}(x), D(\psi \circ f \circ \varphi^{-1})(x)w).
\end{aligned}
$$

If f is a diffeomorphism then $Tf : TM \to TN$ is a vector bundle isomorphism.

(ii) Let E be a vector bundle, and (Ψ, W) a vector bundle chart of E. Then $\Psi : W \to U \times \mathbb{R}^n$ is a vector bundle isomorphism. This holds, in particular, for $E = TM$ and $\Psi = T\psi$, where ψ is any chart of M.

2.6.12 Theorem. *Let (E, B, π) be a vector bundle with vector bundle atlas $\mathcal{A} = \{(\Psi_\alpha, W_\alpha) \mid \alpha \in A\}$. Then $(T^r_s E, B, \pi^r_s)$ is a vector bundle with vector bundle atlas $\mathcal{A}^r_s = \{((\Psi_\alpha)^r_s, (T^r_s E)|_{W_\alpha \cap B}) \mid \alpha \in A\}$. $(T^r_s E, B, \pi^r_s)$ is called the tensor bundle of type $\binom{r}{s}$ over E.*

Proof. Clearly the $(T^r_s E)|_{W_\alpha \cap B}$ form a covering of $T^r_s E$. Let Ψ_α, Ψ_β be vector bundle charts from $\mathcal{A}$ with $W_{\alpha\beta} := W_\alpha \cap W_\beta \neq \emptyset$. Since $E|_{W_{\alpha\beta}\cap B} \cong \bigcup_{b\in W_{\alpha\beta}\cap B}\{b\}\times E_b$, it follows that Ψ_α is of the form $\Psi_\alpha(b,e) = (\psi_{\alpha 1}(b), \psi_{\alpha 2}(b)\cdot e)$, with $b \in B$, $e \in E_b$, and $\psi_{\alpha 2}(b)$ linear for each b. Therefore, $(\Psi_\alpha)^r_s$ can be defined as $(b,t) \mapsto (\psi_{\alpha 1}(b), (\psi_{\alpha 2}(b))^r_s t)$ $(t \in T^r_s(E_b))$.
Then

$$\begin{aligned}
\Psi_\beta \circ \Psi_\alpha^{-1}(x,w) &= \Psi_\beta(\underbrace{\psi_{\alpha 1}^{-1}(x)}_{=:b}, \psi_{\alpha 2}(b)^{-1}w) \\
&= (\psi_{\beta 1}(\psi_{\alpha 1}^{-1}(x)), \psi_{\beta 2}(b)\psi_{\alpha 2}(b)^{-1}w) \\
&=: (\psi_{\beta\alpha 1}(x), \psi_{\beta\alpha 2}(x)\cdot w).
\end{aligned}$$

Hence by 2.6.6 (i) and 2.6.7,

$$\begin{aligned}
(\Psi_\beta)^r_s \circ ((\Psi_\alpha)^r_s)^{-1}(x,t') &= (\Psi_\beta)^r_s(\psi_{\alpha 1}^{-1}(x), (\psi_{\alpha 2}(b)^{-1})^r_s(t')) \\
&= (\psi_{\beta 1}(\psi_{\alpha 1}^{-1}(x)), (\psi_{\beta 2}(b)\psi_{\alpha 2}(b)^{-1})^r_s(t')) \\
&= (\psi_{\beta\alpha 1}(x), (\psi_{\beta\alpha 2}(x))^r_s(t')) = (\Psi_\beta \circ \Psi_\alpha^{-1})^r_s(x,t'),
\end{aligned}$$

which, by 2.6.8, is a local vector bundle isomorphism. Thus $T^r_s(E)$ is a vector bundle. As in the proof of 2.5.3 (for TM) it follows that $T^r_s(E)$ is Hausdorff and second countable. □

For us the most important special case of this construction is $E = TM$:

2.6.13 Definition. *Let M be a manifold. Then $T^r_s(M) := T^r_s(TM)$ is called the bundle of r-times contra- and s-times covariant tensors on M (resp. of tensors of type $\binom{r}{s}$). $T^*M := T^0_1(M)$ is called the cotangent bundle of M.*

By 2.6.5 we have $T^1_0(M) = TM$: in fact, $\pi^{-1}(p) = T_pM$ $\forall p$ and $T^1_0(T_pM) = T_pM$.
For each chart ψ of M, $(T\psi)^1_0 = T\psi$.
If $\mathcal{A} = \{(\psi_\alpha, V_\alpha) \mid \alpha \in A\}$ is an atlas of M, then by 2.6.12,

$$\mathcal{A}^r_s = \{((T\psi_\alpha)^r_s, (T^r_s M)|_{V_\alpha}) \mid \alpha \in A\}$$

is a vector bundle atlas of $T^r_s M$.

2.6.14 Definition. *Smooth sections of $T^r_s M$ (i.e., smooth maps $t : M \to T^r_s M$ with $\pi^r_s \circ t = \mathrm{id}_M$) are called $\binom{r}{s}$-tensors (resp. $\binom{r}{s}$-tensor fields) on M. The space $\Gamma(M, T^r_s M)$ of $\binom{r}{s}$-tensor fields is denoted by $\mathcal{T}^r_s(M)$. In particular, $\mathcal{T}^1_0(M) = \mathfrak{X}(M)$. We also write $\Omega^1(M)$ instead of $\mathcal{T}^0_1(M)$. The elements of $\Omega^1(M)$ are called differential forms of order 1 (1-forms, covector fields).*

If $t \in \mathcal{T}^r_s(M)$ and $f \in \mathcal{C}^\infty(M)$, then $ft : p \mapsto f(p)t(p) \in (T_pM)^r_s$ is a tensor field on M. Then $\mathcal{T}^r_s(M)$ with the pointwise operations $+$, $f\cdot$ is a $\mathcal{C}^\infty(M)$-module.
As in the case of $\mathfrak{X}(M) = \mathcal{T}^1_0(M)$ we also want to derive local representations of general tensor fields in charts. We first consider the special case $\Omega^1(M) = \mathcal{T}^0_1(M) = \Gamma(M, T^0_1 M)$. As a set,

$$T^0_1 M = \bigsqcup_{p\in M} (T_pM)^* = \bigcup_{p\in M} \{p\} \times (T_pM)^*.$$

The vector bundle charts of $T^0_1 M = T^*M$ are of the form $(T\psi)^0_1 : T^0_1 M\big|_V \to \psi(V) \times (\mathbb{R}^n)^0_1 = \psi(V) \times (\mathbb{R}^n)^*$ for any chart (ψ, V) of M. As in the case of $TM = T^1_0 M$ we want to use the vector bundle charts to define a basis of $(T_pM)^*$. Recall that for T_pM in this way we derived the basis $\{\frac{\partial}{\partial x^i}\big|_p \mid 1 \le i \le n\}$, where $\frac{\partial}{\partial x^i}\big|_p = (T_p\psi)^{-1}(e_i)$, i.e., $\frac{\partial}{\partial x^i} = p \mapsto (T\psi)^{-1}(\psi(p), e_i)$.

In the case of $T^0_1 M$ let $\{\alpha^j \mid 1 \le j \le n\}$ be the dual basis of $\{e_i \mid 1 \le i \le n\}$ in $(\mathbb{R}^n)^*$. Then for any $p \in V$ the family

$$dx^i\,|_p := [(T\psi)^0_1]^{-1}(\psi(p), \alpha^i) \qquad (1 \le i \le n)$$

is a basis of $(T_pM)^*$. We have

$$\begin{aligned}
dx^i\big|_p &= [(T\psi)^0_1]^{-1}(\psi(p), \alpha^i) = \\
&= (p, [(T_p\psi)^0_1]^{-1}(\alpha^i)) \overset{2.6.5}{=} (p, (((T_p\psi)^{-1})^*)^{-1}(\alpha^i)) \\
&= (p, (T_p\psi)^*(\alpha^i)). \qquad (2.6.1)
\end{aligned}$$

Since $dx^j\big|_p \in (T_pM)^*$ and $\frac{\partial}{\partial x^i}\big|_p \in T_pM$, we can apply $dx^j\big|_p$ to $\frac{\partial}{\partial x^i}\big|_p$:

$$\begin{aligned}
dx^j\big|_p\Big(\frac{\partial}{\partial x^i}\Big|_p\Big) &= (T_p\psi)^*(\alpha^j)((T_p\psi)^{-1}(e_i)) = \\
&= \alpha^j(T_p\psi((T_p\psi)^{-1}(e_i))) = \\
&= \alpha^j(e_i) = \delta_{ij}
\end{aligned}$$

It follows that $\{\,dx^j\big|_p \mid 1 \le j \le n\}$ is precisely the dual basis of $\{\,\frac{\partial}{\partial x^i}\big|_p \mid 1 \le i \le n\}$ in $(T_pM)^*$.
Another way of looking at dx^i results from the following definition:

2.6.15 Definition. *Let $f \in \mathcal{C}^\infty(M)$. Then $df : M \to T^*M$, $p \mapsto T_pf$ is called the exterior derivative of f.*

2.6.16 Remark. (i) $df \in \mathcal{T}^0_1(M)$. In fact, for any $p \in M$, $T_pf \in L(T_pM, \mathbb{R}) = (T_pM)^*$. Moreover, df is smooth since for any chart ψ around p we have (setting $\psi(p) = x$):

$$\begin{aligned}
(T\psi)^0_1 \circ df \circ \psi^{-1}(x) &= (x, ((T_p\psi)^{-1})^* \circ T_pf) = (x, T_pf \circ (T_p\psi)^{-1}) \\
&= (x, T_x(f \circ \psi^{-1})) = (x, D(f \circ \psi^{-1})(x))
\end{aligned}$$

$$\begin{array}{ccc}
T^0_1 M & \xrightarrow{(T\psi)^0_1} & \psi(V) \times (\mathbb{R}^n)^* \\
{\scriptstyle df}\Big\uparrow & & \Big\uparrow{\scriptstyle \mathrm{id} \times D(f\circ\psi^{-1})} \\
M \supseteq V & \xrightarrow{\psi} & \psi(V)
\end{array}$$

(ii) If $f \in \mathcal{C}^\infty(M)$ and $X \in \mathfrak{X}(M)$, then for all $p \in M$, $X_p \in T_pM$ and $df|_p \in T_pM^*$, so $df(X) := p \mapsto df|_p\,(X_p) : M \to \mathbb{R}$ is well-defined. We have:

$$df|_p\,(X_p) = T_pf(X_p) = X(f)|_p\,.$$

Thus $df(X) = X(f)$. In particular, $df(X) \in \mathcal{C}^\infty(M)$.
(iii) Let (ψ, V) be a chart, $\psi = (x^1, \dots, x^n)$. Then $d(x^i)$ in the sense of 2.6.15 is precisely the above dx^i. Indeed, by (ii) we have

$$d(x^j)\Big(\frac{\partial}{\partial x^i}\Big) = \frac{\partial}{\partial x^i}(x^j) = \delta_{ij},$$

i.e., $\{d(x^j)\,|_p \mid 1 \le j \le n\}$ is precisely the dual basis of $\{\,\frac{\partial}{\partial x^i}\big|_p \mid 1 \le i \le n\}$ for all $p \in V$.

If (ψ, V) is a chart of M, $\psi = (x^1, \dots, x^n)$, then for all $p \in M$ the tuple $\{ \frac{\partial}{\partial x^i}\big|_p \mid 1 \le i \le n\}$ is a basis of T_pM and $\{ dx^j\big|_p \mid 1 \le j \le n\}$ is the corresponding dual basis of T_pM^*. Thus, by 2.6.3, for any $p \in M$ the tuple:

$$\{ \frac{\partial}{\partial x^{i_1}}\bigg|_p \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}}\bigg|_p \otimes dx^{j_1}\big|_p \otimes \cdots \otimes dx^{j_s}\big|_p \mid 1 \le i_k, j_k \le n\}$$

is a basis of $(T_pM)^r_s$. Hence if t is a section of T^r_sM then there are uniquely determined functions $t^{i_1 \dots i_r}_{j_1 \dots j_s}$ on V such that

$$t|_V = t^{i_1 \dots i_r}_{j_1 \dots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s} \tag{2.6.2}$$

(cf. the special case of vector fields in 2.5.8: $X|_V = X^i \frac{\partial}{\partial x^i}$.) As for vector fields we also have a characterization of smoothness for tensor fields in terms of local coordinates:

2.6.17 Proposition. *Let t be a section of the bundle $T^r_s(M)$. TFAE:*

(i) t is smooth, i.e., $t \in \mathcal{T}^r_s(M)$.

(ii) In every chart representation (2.6.2) *all coefficient functions $t^{i_1 \dots i_r}_{j_1 \dots j_s}$ are smooth.*

Proof. Let (ψ, V) be a chart of M. Then $(T\psi)^r_s$ is a vector bundle chart of T^r_sM. By definition, t is smooth if and only if the push-forward $\psi_* t := (T\psi)^r_s \circ t \circ \psi^{-1} : \psi(V) \to \psi(V) \times (\mathbb{R}^n)^r_s$ is smooth for every chart ψ.

$$\begin{array}{ccc} T^r_sM & \xrightarrow{(T\psi)^r_s} & \psi(V) \times (\mathbb{R}^n)^r_s \\ {\scriptstyle t}\uparrow & & \uparrow{\scriptstyle \psi_* t} \\ M \supseteq V & \xrightarrow{\psi} & \psi(V) \end{array}$$

For $x \in \psi(V)$ we have (setting $p := \psi^{-1}(x)$):

$$\begin{aligned} (T\psi)^r_s \circ t \circ \psi^{-1}(x) &\overset{(2.6.2)}{=} (T\psi)^r_s (t^{i_1 \dots i_r}_{j_1 \dots j_s}(p) \frac{\partial}{\partial x^{i_1}}\bigg|_p \otimes \cdots \otimes dx^{j_s}\big|_p) \\ &= (x, (T_p\psi)^r_s (t^{i_1 \dots i_r}_{j_1 \dots j_s}(p) \frac{\partial}{\partial x^{i_1}}\bigg|_p \otimes \cdots \otimes dx^{j_s}\big|_p)) \\ &\overset{2.6.6(iv)}{=} (x, t^{i_1 \dots i_r}_{j_1 \dots j_s}(p) \underbrace{(T_p\psi)^1_0}_{=T_p\psi} (\frac{\partial}{\partial x^{i_1}}\bigg|_p) \otimes \cdots \otimes \underbrace{(T_p\psi)^0_1}_{((T_p\psi)^*)^{-1}} (dx^{j_s}\big|_p)) \\ &= (x, t^{i_1 \dots i_r}_{j_1 \dots j_s}(\psi^{-1}(x)) \underbrace{e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \alpha^{j_1} \otimes \cdots \otimes \alpha^{j_s}}_{\in (\mathbb{R}^n)^r_s}) \end{aligned}$$

This map is smooth if and only if all $t^{i_1 \dots i_r}_{j_1 \dots j_s} \circ \psi^{-1}$ are smooth, i.e., if and only if all $t^{i_1 \dots i_r}_{j_1 \dots j_s}$ are smooth on V. $\square$

If $t \in \mathcal{T}^r_s(M)$ and $\alpha^1, \dots, \alpha^r \in \Omega^1(M), X_1, \dots, X_s \in \mathfrak{X}(M)$, then

$$p \mapsto t(p)(\alpha^1(p), \dots, \alpha^r(p), X_1(p), \dots, X_s(p))$$

is a well-defined function $M \to \mathbb{R}$ which we denote by $t(\alpha^1, \dots, \alpha^r, X_1, \dots, X_s)$. For $f \in \mathcal{C}^\infty(M)$ we have

$$t(f\alpha^1, \alpha^2, \dots) = t(\alpha^1, f\alpha^2, \dots) = \cdots = t(\alpha^1, \dots, fX_s) = ft(\alpha^1, \dots, X_s).$$

Thus $(\alpha^1, \dots, \alpha^r, X_1, \dots, X_s) \mapsto t(\alpha^1, \dots, X_s)$ is $\mathcal{C}^\infty(M)$-multilinear.

2.6.18 Proposition. *Let t be a section of the bundle $T^r_s(M)$. TFAE:*

(i) t *is smooth (i.e.,* $t \in \mathcal{T}^r_s(M)$*).*

(ii) $\forall \alpha^1, \dots, \alpha^r \in \Omega^1(M), \forall X_1, \dots, X_s \in \mathfrak{X}(M)$*, the map* $t(\alpha^1, \dots, \alpha^r, X_1, \dots, X_s)$ *is in* $\mathcal{C}^\infty(M)$*.*

Proof. Let (ψ, V) be a chart in M, $\psi = (x^1, \dots, x^n)$.
(i)$\Rightarrow$(ii): Let $X_k = X_k^{a_k} \frac{\partial}{\partial x^{a_k}}$ $(1 \le k \le s)$, $\alpha^m = \alpha^m_{b_m} dx^{b_m}$ $(1 \le m \le r)$ be the local representations with respect to ψ. By 2.6.17, all coefficient functions $X_j^{a_j}$, $\alpha^i_{b_i}$, $t^{b_1 \dots b_r}_{a_1 \dots a_s}$ are smooth on V. Hence so is

$$\begin{aligned} t(\alpha^1, \dots, X_s) &= \alpha^1_{b_1} \dots \alpha^r_{b_r} X_1^{a_1} \dots X_s^{a_s} t(dx^{b_1}, \dots, dx^{b_r}, \frac{\partial}{\partial x^{a_1}}, \dots, \frac{\partial}{\partial x^{a_s}}) \\ &\overset{(2.6.2)}{=} \alpha^1_{b_1} \dots \alpha^r_{b_r} X_1^{a_1} \dots X_s^{a_s} t^{b_1 \dots b_r}_{a_1 \dots a_s}. \end{aligned}$$

(ii)$\Rightarrow$ (i): By 2.6.17 we have to show that $t^{i_1 \dots i_r}_{j_1 \dots j_s}$ is smooth on V for all $i_1, \dots, i_r$, $j_1, \dots, j_s$. As in the proof of 2.5.9, (ii)$\Rightarrow$(iii), we extend $dx^{i_1}, \dots, \frac{\partial}{\partial x^{j_s}}$ to elements of $\Omega^1(M)$ resp. $\mathfrak{X}(M)$. Then $t^{i_1 \dots i_r}_{j_1 \dots j_s} = t(dx^{i_1}, \dots, \frac{\partial}{\partial x^{j_s}})$ is smooth by (ii). $\square$

The above observations lead to the following algebraic characterization of smooth tensor fields. Let

$$L^{r+s}_{\mathcal{C}^\infty(M)}(\underbrace{\Omega^1(M) \times \dots \times \Omega^1(M)}_{r} \times \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{s}, \mathcal{C}^\infty(M))$$

be the space of $\mathcal{C}^\infty(M)$-multilinear maps from $\Omega^1(M) \times \dots \times \Omega^1(M) \times \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)$ to $\mathcal{C}^\infty(M)$. Then we have:

2.6.19 Theorem. *The map*

$$\begin{aligned} A : \mathcal{T}^r_s(M) &\to L^{r+s}_{\mathcal{C}^\infty(M)}(\Omega^1(M) \times \dots \times \mathfrak{X}(M), \mathcal{C}^\infty(M)) \\ t &\mapsto [(\alpha^1, \dots, \alpha^r, X_1, \dots, X_s) \mapsto t(\alpha^1, \dots, X_s)] \end{aligned}$$

is a $\mathcal{C}^\infty(M)$*-linear isomorphism.*

Proof. By 2.6.18, for all $t \in \mathcal{T}^r_s(M)$, $A(t) \in L^{r+s}_{\mathcal{C}^\infty(M)}(\Omega^1(M) \times \dots \times \mathfrak{X}(M), \mathcal{C}^\infty(M))$ and clearly A is $\mathcal{C}^\infty(M)$-linear.
A is injective: If $A(t) = 0$ then for all $p \in M$ and all $\alpha^1, \dots, X_s$ we have $t(p)(\alpha^1(p), \dots, X_s(p)) = 0$. All elements of T_pM resp. of $(T_pM)^*$ can be written in this way (i.e., are of the form $X_i(p)$ resp. $\alpha^j(p)$ for certain smooth fields X_i, α^j: this is seen by extending any given constant (co-)vector to a smooth field using a partition of unity, cf. the proof of 2.5.9, (ii)$\Rightarrow$(iii)). Hence it follows that $t(p) = 0$ $\forall p$, i.e., $t = 0$.
A is surjective: Let $\Phi \in L^{r+s}_{\mathcal{C}^\infty(M)}(\Omega^1(M) \times \dots \times \mathfrak{X}(M), \mathcal{C}^\infty(M))$. We have to show that there exists some $t \in \mathcal{T}^r_s(M)$ with $A(t) = \Phi$. To this end we first demonstrate that $\Phi(\alpha^1, \dots, X_s)\big|_p$ depends only on $\alpha^1(p), \dots, X_s(p)$. It suffices to show that $\alpha^1(p) = 0$ implies $\Phi(\alpha^1, \dots, X_s)\big|_p = 0$ (and analogously for $\alpha^2, \dots, X_s$). This we do in two steps

1) If V is an open neighborhood of p and $\alpha^1\big|_V = 0$, then $\Phi(\alpha^1, \dots, X_s)\big|_p = 0$ (i.e., Φ depends only on the local behavior of α^1). Choose an open neighborhood U of p such that $\overline{U} \subseteq V$. By 2.3.14 there exists a partition of unity (χ_1, χ_2) subordinate to $\{V, M \setminus \overline{U}\}$. Then $\alpha^1 = \chi_2 \cdot \alpha^1$, and therefore

$$\Phi(\alpha^1, \dots, X_s)\big|_p = \Phi(\chi_2 \alpha^1, \alpha^2, \dots, X_s)\big|_p = \underbrace{\chi_2(p)}_{=0} \Phi(\alpha^1, \alpha^2, \dots, X_s)\big|_p = 0$$

2) Now let $\alpha^1(p) = 0$, let V be a chart neighborhood of p, and write $\alpha^1\big|_V = \alpha^1_j dx^j$. Then by 1),

$$\begin{aligned}\Phi(\alpha^1, \dots, X_s)\big|_p &= \Phi(\alpha^1_j dx^j, \alpha^2, \dots, X_s)\big|_p \\ &= \alpha^1_j(p)\ \Phi(dx^j, \alpha^2, \dots, X_s)\big|_p = 0\end{aligned}$$

Therefore, for each $p \in M$ we may define some $t(p) \in (T_pM)^r_s$ by

$$t(p)(\alpha^1(p), \dots, X_s(p)) := \Phi(\alpha^1, \dots, X_s)\big|_p .$$

(Recall that all elements of $T_pM^* \times \cdots \times T_pM$ can be written in this way, as was demonstrated above). Thus t is a section of T^r_sM. By construction, for all $\alpha^i \in \Omega^1(M)$ and all $X_j \in \mathfrak{X}(M)$ we have $t(\alpha^1, \dots, X_s) = \Phi(\alpha^1, \dots, X_s) \in \mathcal{C}^\infty(M)$, so $t \in \mathcal{T}^r_s(M)$ by 2.6.18. Obviously, $A(t) = \Phi$, so A is onto.

□

All standard operations of multilinear algebra can be transferred fiber-wise to tensor fields. We have already encountered the following:

- $f \in \mathcal{C}^\infty(M),\ t \in \mathcal{T}^r_s(M) \Rightarrow\ f \cdot t := p \mapsto f(p) \cdot t(p) \in \mathcal{T}^r_s(M)$
- $t \in \mathcal{T}^r_s(M),\ \alpha^1, \dots, \alpha^r \in \Omega^1(M),\ X_1, \dots, X_s \in \mathfrak{X}(M) \Rightarrow\ t(\alpha^1, \dots, X_s) \in \mathcal{C}^\infty(M)$

Moreover, for $t_1 \in \mathcal{T}^{r_1}_{s_1}(M),\ t_2 \in \mathcal{T}^{r_2}_{s_2}(M)$ the tensor product $t_1 \otimes t_2 \in \mathcal{T}^{r_1+r_2}_{s_1+s_2}(M)$ is defined as

$$t_1 \otimes t_2 : p \mapsto t_1(p) \otimes t_2(p)$$

$t_1 \otimes t_2$ is smooth by 2.6.17 or also by 2.6.18.

2.7 Differential Forms

In this section we wish to study alternating multilinear forms, first in the vector space setting and later on vector bundles.

2.7.1 Definition. *Let E be a finite dimensional vector space and $\Lambda^k E^* := L^k_{\mathrm{alt}}(E, \mathbb{R})$ the space of all multilinear alternating maps from $E^k = E \times \cdots \times E$ to $\mathbb{R}$.*

2.7.2 Remark.

(i) $t \in L^k(E, \mathbb{R})$ is called alternating if

$$t(f_1, \dots, f_i, \dots, f_j, \dots, f_k) = -t(f_1 \dots, f_j, \dots, f_i, \dots, f_k)\ (1 \le i < j \le k).$$

Let $S_k := \{\varphi : \{1, \dots, k\} \to \{1, \dots, k\} \mid \varphi \text{ bijective }\}$ be the permutation group of order k. Then for $\sigma \in S_k$ and $t \in \Lambda^k E^*$ we have

$$t(f_{\sigma(1)}, \dots, f_{\sigma(k)}) = \mathrm{sgn}(\sigma) t(f_1, \dots, f_k)$$

For $\sigma, \tau \in S_k$, $\mathrm{sgn}(\sigma \circ \tau) = \mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(\tau)$. Since S_k is a group, for all $\tau_0 \in S_k$ the map $\tau \mapsto \tau \circ \tau_0 : S_k \to S_k$ is a bijection.

(ii) We set $\Lambda^0 E^* = \mathbb{R}$. Moreover, $\Lambda^1 E^* = L^1_{\mathrm{alt}}(E, \mathbb{R}) = L(E, \mathbb{R}) = E^*$.

(iii) $\Lambda^k E^*$ is a subspace of $T^0_k(E)$, the space of all multilinear maps $E \times \cdots \times E \to \mathbb{R}$.

2.7.3 Definition. *The map* Alt $: T_k^0(E) \to T_k^0(E)$,

$$\mathrm{Alt}(t)(f_1,\ldots,f_k) := \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)t(f_{\sigma(1)},\ldots,f_{\sigma(k)})$$

is called alternator.

2.7.4 Lemma. Alt *is a linear projection of* $T_k^0(E)$ *onto* $\Lambda^k E^*$*, i.e.,*

(i) Alt *is linear,* $\mathrm{Alt}(T_k^0(E)) \subseteq \Lambda^k E^*$.

(ii) $\mathrm{Alt}|_{\Lambda^k E^*} = \mathrm{id}_{\Lambda^k E^*}$.

(iii) $\mathrm{Alt}\circ\mathrm{Alt} = \mathrm{Alt}$.

(iv) $\mathrm{Alt}(T_k^0(E)) = \Lambda^k E^*$.

Proof. (i) Clearly, Alt is linear. Let $t \in T_k^0(E),\ \tau \in S_k$. Then

$$\begin{aligned}
\mathrm{Alt}(t)(f_{\tau(1)},\ldots,f_{\tau(k)}) &= \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)t(f_{\tau\sigma(1)},\ldots,f_{\tau\sigma(k)})\\
&= \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\tau)\mathrm{sgn}(\tau\circ\sigma)t(f_{\tau\sigma(1)},\ldots,f_{\tau\sigma(k)})\\
&= \mathrm{sgn}(\tau)\mathrm{Alt}(t)(f_1,\ldots,f_k).
\end{aligned}$$

(ii) If $t \in \Lambda^k E^*$, then

$$\mathrm{Alt}(t)(f_1,\ldots,f_k) = \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)t(f_{\sigma(1)},\ldots,f_{\sigma(k)}) = t(f_1,\ldots,f_k).$$

(iii) and (iv) follow from (i) and (ii). □

2.7.5 Definition. *Let* $\alpha \in T_k^0(E),\ \beta \in T_l^0(E)$*. The exterior product (or wedge product) of* α *and* β *is defined as*

$$\alpha\wedge\beta := \frac{(k+l)!}{k!l!}\mathrm{Alt}(\alpha\otimes\beta)$$

For $\alpha \in T_0^0 E = \Lambda^0 E^* = \mathbb{R}$ *we set* $\alpha\wedge\beta = \beta\wedge\alpha = \alpha\cdot\beta$.

2.7.6 Example. Let $\alpha,\beta \in \Lambda^1 E^* = T_1^0(E) = E^*$. Then

$$\begin{aligned}
(\alpha\wedge\beta)(f_1,f_2) &= \frac{2!}{1!1!}\frac{1}{2!}\sum_{\sigma\in S_2}\mathrm{sgn}(\sigma)(\alpha\otimes\beta)(f_{\sigma(1)},f_{\sigma(2)})\\
&= (\alpha\otimes\beta)(f_1,f_2) - (\alpha\otimes\beta)(f_2,f_1) = (\alpha\otimes\beta - \beta\otimes\alpha)(f_1,f_2).
\end{aligned}$$

2.7.7 Proposition. *Let* $\alpha \in T_k^0(E),\ \beta \in T_l^0(E)$*, and* $\gamma \in T_m^0(E)$*. Then:*

(i) $\alpha\wedge\beta = \mathrm{Alt}(\alpha)\wedge\beta = \alpha\wedge\mathrm{Alt}(\beta)$.

(ii) $\wedge$ *is bilinear.*

(iii) $\alpha\wedge\beta = (-1)^{k\cdot l}\beta\wedge\alpha$.

(iv) $\wedge$ *is associative:* $\alpha\wedge(\beta\wedge\gamma) = (\alpha\wedge\beta)\wedge\gamma$.

Proof. (i) For $\tau \in S_k$ and $\alpha \in T_k^0(E)$ let $(\tau\alpha)(f_1,\dots,f_k) := \alpha(f_{\tau(1)},\dots,f_{\tau(k)})$. Then

$$\begin{aligned}\mathrm{Alt}(\tau\alpha)(f_1,\dots,f_k) &= \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\alpha(f_{\sigma\tau(1)},\dots,f_{\sigma\tau(k)})\\ &= \mathrm{sgn}(\tau)\frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma\tau)\alpha(f_{\sigma\tau(1)},\dots,f_{\sigma\tau(k)})\\ &= \mathrm{sgn}(\tau)\mathrm{Alt}(\alpha)(f_1,\dots,f_k).\end{aligned}$$

Therefore,

$$\mathrm{Alt}(\tau\alpha) = \mathrm{sgn}(\tau)\cdot\mathrm{Alt}(\alpha). \tag{2.7.1}$$

Using this, we obtain

$$\begin{aligned}\mathrm{Alt}(\mathrm{Alt}(\alpha)\otimes\beta)(f_1,\dots,f_{k+l}) &= \mathrm{Alt}(\mathrm{Alt}(\alpha)(f_1,\dots,f_k)\beta(f_{k+1},\dots,f_{k+l}))\\ &= \mathrm{Alt}(\frac{1}{k!}\sum_{\tau\in S_k}\mathrm{sgn}(\tau)\alpha(f_{\tau(1)},\dots,f_{\tau(k)})\beta(f_{k+1},\dots,f_{k+l}))\\ &= \mathrm{Alt}(\frac{1}{k!}\sum_{\tau\in S_k}\mathrm{sgn}(\tau)((\tau\alpha)\otimes\beta)(f_1,\dots,f_{k+l}))\\ &= \frac{1}{k!}\sum_{\tau\in S_k}\mathrm{sgn}(\tau)\mathrm{Alt}((\tau\alpha)\otimes\beta)(f_1,\dots,f_{k+l}) = (*)\end{aligned}$$

We define $\tau' \in S_{k+l}$ by

$$\tau'(1,\dots,k+l) := (\tau(1),\dots,\tau(k),k+1,\dots,k+l).$$

Then $\mathrm{sgn}(\tau') = \mathrm{sgn}(\tau)$ and $(\tau\alpha)\otimes\beta = \tau'(\alpha\otimes\beta)$. Therefore,

$$\begin{aligned}(*) &= \frac{1}{k!}\sum_{\tau\in S_k}\mathrm{sgn}(\tau')\mathrm{Alt}(\tau'(\alpha\otimes\beta))(f_1,\dots,f_{k+l})\\ &\overset{(2.7.1)}{=} \mathrm{Alt}(\alpha\otimes\beta)(f_1,\dots,f_{k+l}),\end{aligned}$$

so $\mathrm{Alt}(\mathrm{Alt}(\alpha)\otimes\beta) = \mathrm{Alt}(\alpha\otimes\beta)$, and $\mathrm{Alt}(\alpha)\wedge\beta = \alpha\wedge\beta$. The second equation in (i) follows in the same way.
(ii) is clear since $\otimes$ is bilinear and Alt is linear.
(iii) Let $\sigma_0 \in S_{k+l}$ be given by $\sigma_0(1,\dots,k+l) := (k+1,\dots,k+l,1,\dots,k)$. Then $\mathrm{sgn}(\sigma_0) = (-1)^{k\cdot l}$ and $\alpha\otimes\beta(f_1,\dots,f_{k+l}) = \beta\otimes\alpha(f_{\sigma_0(1)},\dots,f_{\sigma_0(k+l)})$. By (2.7.1) this entails

$$\alpha\wedge\beta = \frac{(k+l)!}{k!l!}\mathrm{Alt}(\alpha\otimes\beta) = \frac{(k+l)!}{k!l!}\mathrm{Alt}(\sigma_0(\beta\otimes\alpha)) = (-1)^{kl}\beta\wedge\alpha.$$

(iv)

$$\begin{aligned}\alpha\wedge(\beta\wedge\gamma) &= \frac{(l+m)!}{l!m!}\alpha\wedge\mathrm{Alt}(\beta\otimes\gamma) \overset{(i)}{=} \frac{(l+m)!}{l!m!}\alpha\wedge(\beta\otimes\gamma)\\ &= \frac{(l+m)!}{l!m!}\frac{(k+l+m)!}{k!(l+m)!}\mathrm{Alt}(\underbrace{\alpha\otimes(\beta\otimes\gamma)}_{(\alpha\otimes\beta)\otimes\gamma})\\ &= \frac{(k+l+m)!}{k!l!m!}\frac{(k+l)!m!}{(k+l+m)!}(\alpha\otimes\beta)\wedge\gamma\\ &\overset{(i)}{=} \underbrace{\frac{(k+l)!}{k!l!}\mathrm{Alt}(\alpha\otimes\beta)}_{=\alpha\wedge\beta}\wedge\gamma = (\alpha\wedge\beta)\wedge\gamma.\end{aligned}$$

$\square$

2.7.8 Definition. $\Lambda E^* := \bigoplus_{k=0}^{\infty} \Lambda^k E^*$ *with the operations* $+$, $\lambda\cdot$ *and* $\wedge$ *is called the* exterior algebra *(or* Grassmann algebra*) of* E.

As we shall demonstrate in a moment, $\Lambda^k E^* = \{0\}$ for $k > n$, so in fact

$$\Lambda E^* = \bigoplus_{k=0}^{n} \Lambda^k E^*.$$

To prove this we need the following auxilliary result:

2.7.9 Lemma. *Let* $\alpha^1, \dots, \alpha^k \in \Lambda^1 E^* = E^*$ *and* $f_1, \dots, f_k \in E$. *Then*

$$(\alpha^1 \wedge \cdots \wedge \alpha^k)(f_1, \dots, f_k) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \alpha^1(f_{\sigma(1)}) \cdot \cdots \cdot \alpha^k(f_{\sigma(k)})$$

Proof. We have to show that

$$\alpha^1 \wedge \cdots \wedge \alpha^k = k! \cdot \operatorname{Alt}(\alpha^1 \otimes \cdots \otimes \alpha^k)$$

This we do by induction, the case $k = 1$ being obvious. For $k - 1 \to k$ we calculate:

$$\begin{aligned}
\alpha^1 \wedge \cdots \wedge \alpha^k &\overset{2.7.7(iv)}{=} \alpha^1 \wedge (\alpha^2 \wedge \cdots \wedge \alpha^k) = \\
&\overset{\text{ind.hyp.}}{=} (k-1)! \alpha^1 \wedge \operatorname{Alt}(\alpha^2 \otimes \cdots \otimes \alpha^k) = \\
&\overset{2.7.7(i)}{=} (k-1)! \alpha^1 \wedge (\alpha^2 \otimes \cdots \otimes \alpha^k) = \\
&= (k-1)! \frac{(k-1+1)!}{(k-1)!1!} \operatorname{Alt}(\alpha^1 \otimes \cdots \otimes \alpha^k)
\end{aligned}$$

$\square$

2.7.10 Proposition. *Let* $n = \dim(E)$. *Then* $\dim(\Lambda^k E^*) = \binom{n}{k}$ *for* $0 \le k \le n$. *For* $k > n$, $\Lambda^k E^* = \{0\}$. *Therefore,* $\dim(\Lambda E^*) = \sum_{k=0}^{n} \binom{n}{k} = 2^n$. *If* $\{e_1, \dots, e_n\}$ *is a basis of* E *and* $\{\alpha^1, \dots, \alpha^n\}$ *is the corresponding dual basis, then* $\mathcal{B} := \{\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} \mid 1 \le i_1 < \cdots < i_k \le n\}$ *is a basis of* $\Lambda^k E^*$.

Proof. $\mathcal{B}$ spans $\Lambda^k E^*$: Let $t \in \Lambda^k E^* \subseteq T_k^0(E)$. By 2.6.3, t is of the form

$$t = t(e_{i_1}, \dots, e_{i_k}) \alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}.$$

By 2.7.4 (ii) and 2.7.9,

$$t = \operatorname{Alt}(t) = t(e_{i_1}, \dots, e_{i_k}) \operatorname{Alt}(\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}) = \frac{1}{k!} t(e_{i_1}, \dots, e_{i_k}) \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}.$$

Since t is antisymmetric, all terms in this sum where two indices coincide vanish. In particular, $t = 0$ for $k > n$ (so $\Lambda^k E^* = \{0\}$ $\forall k > n$). If all i_j are distinct, then for any $\sigma \in S_k$ we have

$$t(e_{i_1}, \dots, e_{i_k}) \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} = \operatorname{sgn}(\sigma)^2 t(e_{i_{\sigma(1)}}, \dots, e_{i_{\sigma(k)}}) \alpha^{i_{\sigma(1)}} \wedge \cdots \wedge \alpha^{i_{\sigma(k)}}$$

There are $k!$ such terms, so:

$$t = \sum_{1 \le i_1 < \cdots < i_k \le n} t(e_{i_1}, \dots, e_{i_k}) \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}$$

$\mathcal{B}$ is linearly independent: let

$$\sum_{1\le i_1<\cdots<i_k\le n} t_{i_1\dots i_k}\alpha^{i_1}\wedge\cdots\wedge\alpha^{i_k}=0.$$

We have to show that all $t_{i_1\dots i_k}$ vanish. Let $1\le i'_1<\cdots<i'_k\le n$ be some fixed k-tuple and choose $j'_{k+1}<\cdots<j'_n$ such that $\{i'_1,\dots,i'_k\}\cup\{j'_{k+1},\dots,j'_n\}=\{1,\dots,n\}$. Then by 2.7.7,

$$\begin{aligned} 0 &= \sum_{1\le i_1<\cdots<i_k\le n} t_{i_1\dots i_k}\alpha^{i_1}\wedge\cdots\wedge\alpha^{i_k}\wedge\alpha^{j'_{k+1}}\wedge\cdots\wedge\alpha^{j'_n} = \\ &= \pm t_{i'_1\dots i'_k}\alpha^1\wedge\cdots\wedge\alpha^n. \end{aligned}$$

Since $\alpha^1\wedge\cdots\wedge\alpha^n\neq 0$ (by 2.7.9, $\alpha^1\wedge\cdots\wedge\alpha^n(e_1,\dots,e_n)=1$), it follows that $t_{i'_1\dots i'_k}=0$. □

2.7.11 Definition. *Let* $\dim(E)=n$, $\omega\in\Lambda^nE^*$, $\omega\neq 0$. *Then* ω *is called a volume element on* E. *Two volume elements* ω_1, ω_2 *are called equivalent if* $\omega_1=c\cdot\omega_2$ *for some* $c>0$ *(recall that* $\dim(\Lambda^nE^*)=1$*). An equivalence class of volume elements on* E *is called an orientation of* E.

2.7.12 Proposition. *Let* $\dim(E)=n$, *and* $\varphi\in L(E,E)$. *Then there is a unique number* $\det\varphi\in\mathbb{R}$, *the determinant of* φ, *such that for the pullback-map*

$$\begin{aligned} \varphi^*:\Lambda^nE^* &\to \Lambda^nE^* \\ (\varphi^*\omega)(f_1,\dots,f_n) &:= \omega(\varphi(f_1),\dots,\varphi(f_n)) \end{aligned}$$

we have $\varphi^*\omega=\det\varphi\cdot\omega$, *for all* $\omega\in\Lambda^nE^*$.

Proof. Obviously, the map φ^* is linear: $\Lambda^nE^*\to\Lambda^nE^*$. By 2.7.10, $\dim(\Lambda^nE^*)=1$. Thus with respect to any basis $\{\omega_0\}$ of Λ^nE^*, φ^* is given by a 1×1-matrix $c\in\mathbb{R}$. Hence for any $\omega=a\cdot\omega_0$ we have $\varphi^*\omega=c\cdot a\cdot\omega_0$, and we obtain $\det\varphi:=c$. □

2.7.13 Remark. The determinant in the sense of 2.7.12 is precisely the homonymous number from linear algebra: let $\mathcal{B}:=\{e_1,\dots,e_n\}$ be a basis of E, $\{\alpha^1,\dots,\alpha^n\}$ the corresponding dual basis, and set $\omega:=\alpha^1\wedge\cdots\wedge\alpha^n$. Then

$$\begin{aligned} \det\varphi &= \det\varphi\,\omega(e_1,\dots,e_n)=\varphi^*\omega(e_1,\dots,e_n)=\omega(\varphi(e_1),\dots,\varphi(e_n)) \\ &\overset{2.7.9}{=} \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\alpha^1(\varphi(e_{\sigma(1)}))\cdot\dots\cdot\alpha^n(\varphi(e_{\sigma(n)})) \\ &= \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\varphi_{1\sigma(1)}\dots\varphi_{n\sigma(n)} \end{aligned}$$

where $(\varphi_{ij})_{i,j}$ is the matrix representation of φ with respect to $\mathcal{B}$. □

2.7.14 Definition. *Let* $\varphi\in L(E,F)$, $\alpha\in T^0_k(F)$. *The* pullback *of* α *under* φ *is defined as* $\varphi^*:T^0_k(F)\to T^0_k(E)$,

$$\varphi^*(\alpha)(e_1,\dots,e_k):=\alpha(\varphi(e_1),\dots,\varphi(e_k))\qquad (e_1,\dots,e_k\in E).$$

If φ *is bijective, then the* push-forward $\varphi_*:T^0_k(E)\to T^0_k(F)$ *is defined as* $\varphi_*:=(\varphi^{-1})^*$. *Thus, for* $\alpha\in T^0_k(E)$,

$$\varphi_*(\alpha)(f_1,\dots,f_k)=\alpha(\varphi^{-1}(f_1),\dots,\varphi^{-1}(f_k))\qquad (f_1,\dots,f_k\in F)$$

Then $\varphi_*=\varphi^0_k$ *in the sense of 2.6.4.*

2.7.15 Proposition. *Let $\varphi \in L(E,F)$, $\psi \in L(F,G)$. Then:*

(i) $\varphi^* : T^0_k(F) \to T^0_k(E)$ *is linear and* $\varphi^*(\Lambda^k F^*) \subseteq \Lambda^k E^*$.

(ii) $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

(iii) If $\varphi = \mathrm{id}_E$, *then* $\varphi^* = \mathrm{id}_{T^0_k(E)}$.

(iv) If φ *is bijective, then so is* φ^* *and* $(\varphi^*)^{-1} = (\varphi^{-1})^*$.

(v) If φ *is bijective, then so is* φ_* *and* $(\varphi_*)^{-1} = (\varphi^{-1})_*$. *If* ψ *is bijective, then* $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

(vi) If $\alpha \in \Lambda^k F^*$, $\beta \in \Lambda^l F^*$, *then* $\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta$.

Proof. (i) and (iii) are clear.
(ii) $(\psi \circ \varphi)^*\alpha(e_1, \dots, e_k) = \alpha(\psi(\varphi(e_1)), \dots, \psi(\varphi(e_k))) = (\psi^*\alpha)(\varphi(e_1), \dots, \varphi(e_k)) = \varphi^*(\psi^*\alpha)(e_1, \dots, e_k)$.
(iv) follows from (ii) and (iii).
(v) $(\varphi_*)^{-1} = ((\varphi^{-1})^*)^{-1} \overset{(iv)}{=} ((\varphi^*)^{-1})^{-1} = \varphi^*$.
(vi) $\varphi^*(\alpha \wedge \beta)(e_1, \dots, e_{k+l}) = (\alpha \wedge \beta)(\varphi(e_1), \dots, \varphi(e_{k+l})) \overset{2.7.3, 2.7.5}{=\!=\!=\!=} ((\varphi^*\alpha) \wedge (\varphi^*\beta))\,(e_1, \dots, e_{k+l})$. □

We are now going to transfer the above constructions to the vector bundle setting, starting with the case of local vector bundles.

2.7.16 Definition. *Let $\varphi : U \times F \to U' \times F'$ be a local vector bundle isomorphism, $\varphi(u,f) = (\varphi_1(u), \varphi_2(u) \cdot f)$. Then let $\varphi_* : U \times \Lambda^k F^* \to U' \times \Lambda^k F'^*$,*

$$(u, \omega) \mapsto (\varphi_1(u), \varphi_2(u)_*(\omega)).$$

2.7.17 Remark. Since $\varphi_* = \varphi^0_k$, by 2.6.8 (and 2.7.15) φ_* is a local vector bundle isomorphism.

2.7.18 Definition. *Let (E, B, π) be a vector bundle, $E_b = \pi^{-1}(b)$ the fiber over $b \in B$. Then set*

$$\Lambda^k E^* := \bigsqcup_{b \in B} \Lambda^k E_b^* = \bigcup_{b \in B} \{b\} \times \Lambda^k E_b^*$$

and $\pi^0_k : \Lambda^k E^* \to B$, $\pi^0_k(e) = b$ for $e \in \Lambda^k E^*_b$. For $A \subseteq B$ set $\Lambda^k E^*\big|_A = \bigsqcup_{b \in A} \Lambda^k E^*_b$.

2.7.19 Theorem. *Let (E, B, π) be a vector bundle with atlas $\mathcal{A} = \{(\Psi_\alpha, W_\alpha) \mid \alpha \in A\}$. Then $(\Lambda^k E^*, B, \pi^0_k)$ is a vector bundle with atlas $\tilde{\mathcal{A}} := \{((\Psi_\alpha)_*, \Lambda^k E^*\big|_{W_\alpha \cap B}) \mid \alpha \in A\}$, where $(\Psi_\alpha)_* = (\Psi_\alpha)^0_k$ (cf. 2.6.10), i.e., $(\Psi_\alpha)_*|_{\Lambda^k E^*_b} = (\Psi_\alpha|_{E_b})_*$.*

Proof. Clearly the $\Lambda^k E^*\big|_{W_\alpha \cap B}$ cover $\Lambda^k E^*$. By 2.7.15 (v), the $(\Psi_\alpha)_*|_{\Lambda^k E^*_b}$ are linear isomorphisms with image $\{\psi_{\alpha 1}(b)\} \times \Lambda^k(\mathbb{R}^n)^*$. The changes of vector bundle chart are local vector bundle isomorphisms according to 2.6.12 and 2.7.15. In fact, $(\Psi_\alpha)_* = (\Psi_\alpha)^0_k$. That $\Lambda^k E^*$ is Hausdorff and second countable follows again as in 2.5.3. □

Again our main interest is in the case $E = TM$:

2.7.20 Definition. *Let M be a manifold. Then $\Lambda^k T^*M := \Lambda^k(TM)^*$ is called the vector bundle of exterior k-forms on TM. The space of smooth sections of $\Lambda^k T^*M$ is denoted by $\Omega^k(M)$. Its elements are called* differential forms of order k *or (exterior) k-forms on M.*

Note that $\Omega^0(M) = \mathcal{C}^\infty(M)$ and $\Omega^1(M)$ is the space of 1-forms (cf. 2.6.14).

2.7.21 Remark.

(i) Due to 2.7.2 (iii), every fiber of $\Lambda^k T^*M$, i.e., every $\Lambda^k T_p^*M$ is precisely the subspace of $(T_pM)^0_k$ consisting of the alternating $\binom{0}{k}$-tensors. Thus, sections of $\Lambda^k T^*M$ are certain $\binom{0}{k}$-tensor fields, namely those which in every $p \in M$ are alternating multilinear maps.

(ii) Let (ψ, V) be a chart of M, $\psi = (x^1, \dots, x^n)$. By 2.7.10, for every $p \in V$ the tuples $\{ dx^{i_1}\big|_p \wedge \dots \wedge dx^{i_k}\big|_p \mid 1 \le i_1 < \dots < i_k \le n\}$ form a basis of $\Lambda^k T_pM^*$. Hence every section ω of $\Lambda^k T^*M$ can locally be written as

$$\omega|_V = \sum_{1 \le i_1 < \dots < i_k \le n} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \tag{2.7.2}$$

with $\omega_{i_1 \dots i_k} = \omega(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}})$. Since the vector bundle charts of $\Lambda^k T^*M$ are of the form $(T\psi)^0_k$, ω is smooth if and only if for every chart (ψ, V) the map $\psi_*\omega = (T\psi)^0_k \circ \omega \circ \psi^{-1} = (T\psi)_* \circ \omega \circ \psi^{-1}$ is smooth. As in the proof of 2.6.17 (only using 2.7.15 (vi) instead of 2.6.6 (iv)) it follows that

$$\psi_*\omega = \sum_{1 \le i_1 < \dots < i_k \le n} \omega_{i_1 \dots i_k} \circ \psi^{-1} \alpha^{i_1} \wedge \dots \wedge \alpha^{i_k}.$$

Hence ω is smooth if and only if for every chart all local components $\omega_{i_1 \dots i_k}$ are smooth.

(iii) By (i) and 2.6.18, a section ω of $\Lambda^k T^*M$ is smooth if and only if for all vector fields $X_1, \dots, X_k \in \mathfrak{X}(M)$, $\omega(X_1, \dots, X_k) \in \mathcal{C}^\infty(M)$.

(iv) By (i) and 2.6.19, $\Omega^k(M)$ is precisely the space of all $\mathcal{C}^\infty(M)$-multilinear and alternating maps $(\mathfrak{X}(M))^k \to \mathcal{C}^\infty(M)$.

(v) Apart from the operations $+$, $f\cdot$ and $\otimes$ studied so far, for differential forms also the exterior product is available: let $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$. Then set $\alpha \wedge \beta := p \mapsto \alpha(p) \wedge \beta(p) \in \Lambda^{k+l} T_pM^*$. It follows that $\alpha \wedge \beta \in \Omega^{k+l}(M)$ (smoothness follows from (ii) or (iii)).

$\Omega(M) := \bigoplus_{k=0}^n \Omega^k(M)$ with these operations is called the algebra of differential forms on M.

In 2.6.15, 2.6.16 we introduced the exterior derivative df of a smooth function f. We now wish to extend this operation from $\Omega^0(M)$ to general $\Omega^k(M)$.

2.7.22 Theorem. *Let M be a manifold. For every open $U \subseteq M$ there exists a uniquely determined family of maps $d^k(U) : \Omega^k(U) \to \Omega^{k+1}(U)$, denoted simply by d, such that:*

(i) d is $\mathbb{R}$-linear and for $\alpha \in \Omega^k(U)$, $\beta \in \Omega^l(U)$ we have:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

(ii) For $f \in \Omega^0(U) = \mathcal{C}^\infty(U)$, df is the exterior derivative from 2.6.15.

(iii) $d \circ d = 0$.

(iv) If U, V are open, $U \subseteq V \subseteq M$ and $\alpha \in \Omega^k(V)$, then $d(\alpha|_U) = (d\alpha)|_U$, i.e.,

$$\begin{array}{ccc} \Omega^k(V) & \xrightarrow{|_U} & \Omega^k(U) \\ {\scriptstyle d}\big\downarrow & & \big\downarrow{\scriptstyle d} \\ \Omega^{k+1}(V) & \xrightarrow{|_U} & \Omega^{k+1}(U) \end{array}$$

d is called exterior derivative.

Proof. *Uniqueness:* By (iv) it suffices to show that d is uniquely determined on any chart (ψ, U). Thus let $\omega \in \Omega^k(U)$. By 2.7.21 (ii),

$$\omega = \sum_{1\le i_1<\cdots<i_k\le n} \omega_{i_1\dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

Hence due to (i), (ii), (iii) we necessarily have:

$$\begin{aligned} d\omega &= \sum_{1\le i_1<\cdots<i_k\le n} d(\omega_{i_1\dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\ &= \sum_{1\le i_1<\cdots<i_k\le n} d\omega_{i_1\dots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \qquad (*) \\ &\quad + \sum_{1\le i_1<\cdots<i_k\le n} \omega_{i_1\dots i_k} \underbrace{d(dx^{i_1})}_{=0} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \\ &\quad +0 + \cdots + 0, \end{aligned}$$

and uniqueness follows.
Existence: For any chart domain we define d by $(*)$ above. We first show that this d has the claimed properties (i)–(iv):
(i): Linearity being obvious, it suffices to calculate $d(\alpha\wedge\beta)$ for $\alpha = f_0 df_1 \wedge\cdots\wedge df_k$, $\beta = g_0 dg_1 \wedge \cdots \wedge dg_l$. We first note that for any $X \in \mathfrak{X}(U)$ we have $d(f_0 g_0)(X) = X(f_0 g_0) = X(f_0)\cdot g_0 + X(g_0)\cdot f_0 = (g_0 df_0 + f_0 dg_0)(X)$, so $d(f_0 g_0) = g_0 df_0 + f_0 dg_0$. Thus

$$\begin{aligned} d(\alpha\wedge\beta) &= d(f_0 g_0 df_1 \wedge \cdots \wedge df_k \wedge dg_1 \wedge \cdots \wedge dg_l) \\ &\overset{(*)}{=} d(f_0 g_0) \wedge df_1 \wedge \cdots \wedge df_k \wedge dg_1 \wedge \cdots \wedge dg_l \\ &= g_0 df_0 \wedge df_1 \wedge \cdots \wedge dg_l + f_0 dg_0 \wedge df_1 \wedge \cdots \wedge \cdots \wedge dg_l \\ &= d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta \end{aligned}$$

(ii) and(iv) are obvious.
(iii): It suffices to show that $d(df) = 0$ for all $f \in \mathcal{C}^\infty(U)$. By (2.6.2), $df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$. Hence,

$$d(df) = \sum_{i=1}^n d(\frac{\partial f}{\partial x^i}) \wedge dx^i = \sum_{i,j} \underbrace{\frac{\partial}{\partial x^j}(\frac{\partial f}{\partial x^i})}_{\text{symm. in } i,j} \underbrace{dx^j \wedge dx^i}_{\text{antisymm}} = 0.$$

It remains to show that the above gives a well-defined global object on M. To this end, let $\tilde\psi = (y^1, \dots, y^n)$ be another chart, w.l.o.g. with the same domain U. Define $\tilde d$ by $(*)$ (with $x \leftrightarrow y$). By the proof of uniqueness, it follows that

$$\tilde d\omega = \sum_{1\le i_1<\cdots<i_k\le n} \underbrace{\tilde d\omega_{i_1\dots i_k}}_{\overset{(ii)}{=} d\omega_{i_1\dots i_k}} \wedge \underbrace{\tilde d x^{i_1}}_{=dx^{i_1}} \wedge \cdots \wedge \underbrace{\tilde d x^{i_k}}_{=dx^{i_k}} = d\omega.$$

Thus d looks the same in any chart, hence is globally well-defined. $\square$

2.7.23 Example.

(i) Let $\omega = P(x,y)dx + Q(x,y)dy$ be a 1-form on $\mathbb{R}^2$. Then

$$\begin{aligned} d\omega &= dP \wedge dx + dQ \wedge dy = \\ &= (\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy) \wedge dx + (\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy) \wedge dy = \\ &= (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dx \wedge dy. \end{aligned}$$

(ii) Let $\omega = P(x,y,z)dy \wedge dz + Q(x,y,z)dz \wedge dx + R(x,y,z)dx \wedge dy$. Then

$$d\omega = (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z})dx \wedge dy \wedge dz.$$

2.7.24 Definition. *Let M, N be manifolds, and $F : M \to N$ smooth. For $\omega \in \mathcal{T}^0_k(N)$, the pullback of ω under F is defined as $F^*\omega(p) := (T_pF)^*(\omega(F(p)))$ (cf. 2.7.14). For $X_1, \dots, X_k \in T_pM$ we therefore have*

$$F^*\omega(p)(X_1, \dots X_k) = \omega(F(p))(T_pF(X_1), \dots, T_pF(X_k))$$

In particular, $F^*f = f \circ F$ for $f \in \mathcal{C}^\infty(N) = \Omega^0(N)$.

2.7.25 Lemma. *Let $F : M \to N$, $G : N \to P$ be smooth. Then*

(i) $F^* : \mathcal{T}^0_k(N) \to \mathcal{T}^0_k(M)$, $F^* : \Omega^k(N) \to \Omega^k(M)$.

(ii) $(G \circ F)^* = F^* \circ G^*$.

(iii) $\mathrm{id}^*_M = \mathrm{id}_{\Omega^k(M)}$ *(resp.* $= \mathrm{id}_{\mathcal{T}^0_k(M)}$*).*

(iv) If F is a diffeomorphism, then F^* is a linear isomorphism and $(F^*)^{-1} = (F^{-1})^*$.

Proof. (i) By 2.7.15 (i), $(T_pF)^*(\omega(F(p))) \in T^0_k(T_pM)$ resp. $\in \Lambda^k(T_pM)^*$. Thus we only have to show that $F^*\omega$ is smooth. To this end, let (φ, U), (ψ, V) be charts of M resp. N with $F(U) \subseteq V$. Then both $F_{\psi\varphi} = \psi \circ F \circ \varphi^{-1}$ and $\psi_*\omega = (T\psi)_* \circ \omega \circ \psi^{-1}$ are smooth (see 2.6.17, 2.7.21 (ii)).
By 2.7.15 (ii) we get (setting $p = \varphi^{-1}(x)$):

$$\begin{aligned} (DF_{\psi\varphi}(x))^* &= (T_xF_{\psi\varphi})^* \\ &= (T_{F(p)}\psi \circ T_pF \circ (T_p\varphi)^{-1})^* \\ &= \underbrace{((T_p\varphi)^{-1})^*}_{\overset{2.7.15(v)}{=}(T_p\varphi)_*} \circ (T_pF)^* \circ (T_{F(p)}\psi)^* \qquad (*) \end{aligned}$$

Hence, by 2.6.17, 2.7.21 (ii), the local representation $\varphi_*(F^*\omega)(x)$ of $F^*\omega$ with respect to φ is given by

$$\begin{aligned} & (T\varphi)_* \circ F^*\omega \circ \varphi^{-1}(x) \\ = \;& (T_p\varphi)_* \circ (T_pF)^*(\omega \circ F \circ \varphi^{-1}(x)) \\ = \;& (T_p\varphi)_* \circ (T_pF)^* \circ (T_{F(p)}\psi)^* \big((T_{F(p)}\psi)_* \circ \omega \circ \psi^{-1} \circ \psi \circ F \circ \varphi^{-1}(x)\big) \\ \overset{(*)}{=} \;& \underbrace{(DF_{\psi\varphi}(x))^*}_{\mathcal{C}^\infty}(\underbrace{(\psi_*\omega)}_{\mathcal{C}^\infty} \circ \underbrace{F_{\psi\varphi}}_{\mathcal{C}^\infty}(x)) \end{aligned}$$

which is smooth by the chain rule.

(ii)

$$\begin{aligned}
(G\circ F)^*(\omega)(p) &= (T_p(G\circ F))^*(\omega(G\circ F(p))) = \\
&= (T_{F(p)}G\circ T_pF)^*(\omega(G\circ F(p))) = \\
&\overset{2.7.15(ii)}{=} (T_pF)^*\circ (T_{F(p)}G)^*(\omega(G(F(p)))) = \\
&= (T_pF)^*(G^*\omega(F(p))) = F^*(G^*\omega)(p)
\end{aligned}$$

(iii) Obvious.
(iv) Follows from (ii) and (iii). □

2.7.26 Theorem. *Let $F : M \to N$ be smooth. Then:*

(i) $F^* : \Omega(N) \to \Omega(M)$ is an algebra homomorphism, i.e., it is linear and $F^*(\alpha\wedge\beta) = (F^*\alpha)\wedge(F^*\beta)$.

(ii) For all $\omega\in\Omega(N)$, $F^*(d\omega) = d(F^*\omega)$.

Proof. (i) To begin with, let $\alpha = f \in \Omega^0(N) = \mathcal{C}^\infty(N)$. Then

$$\begin{aligned}
F^*(f\wedge\beta)(p) &= F^*(f\cdot\beta)(p) = \\
&= (T_pF)^*(f(F(p))\beta(F(p))) = \\
&= \underbrace{f(F(p))}_{F^*f(p)}\underbrace{(T_pF)^*(\beta(F(p)))}_{F^*\beta(p)} \\
&= (F^*f\wedge F^*\beta)(p).
\end{aligned}$$

In the general case we have

$$\begin{aligned}
F^*(\alpha\wedge\beta)(p) &= (T_pF)^*(\alpha(F(p))\wedge\beta(F(p))) = \\
&\overset{2.7.15(vi)}{=} (T_pF)^*(\alpha(F(p)))\wedge(T_pF)^*(\beta(F(p))) = \\
&= ((F^*\alpha)\wedge(F^*\beta))(p).
\end{aligned}$$

(ii) By definition of F^* and 2.7.22 (iv) it suffices to show that every $p\in M$ has a neighborhood U with $d(F^*\omega|_U) = (F^*d\omega)|_U$ for all $\omega\in\Omega(N)$. Let (ψ, V) be a chart around $F(p)$, and U a neighborhood of p with $F(U)\subseteq V$. Then for $\omega\in\Omega^k(V)$ we have

$$\begin{aligned}
\omega &= \sum_{1\le i_1<\dots<i_k\le n} \omega_{i_1\dots i_k}\, dx^{i_1}\wedge\dots\wedge dx^{i_k} \\
d\omega &= \sum_{1\le i_1<\dots<i_k\le n} d\omega_{i_1\dots i_k}\wedge dx^{i_1}\wedge\dots\wedge dx^{i_k}
\end{aligned}$$

By (i),

$$F^*\omega|_U = \sum F^*\omega_{i_1\dots i_k} F^*(dx^{i_i})\wedge\dots\wedge F^*(dx^{i_k}) \qquad (*)$$

In general, for $f\in\mathcal{C}^\infty(N)$, $F^*(df) = d(F^*f)$. In fact, if $X\in T_pM$, then

$$\begin{aligned}
F^*(df)(p)(X) &= df(F(p))(T_pF(X)) = T_{F(p)}f(T_pF(X)) \\
&= T_p(f\circ F)(X) = d(\underbrace{f\circ F}_{=F^*f})(p)(X).
\end{aligned}$$

Thus, from (*) we conclude that

$$\begin{aligned}
d(F^*\omega|_U) &= d(\sum F^*\omega_{i_1\ldots i_k} d(F^*x^{i_1}) \wedge \cdots \wedge d(F^*x^{i_k})) \\
&= \sum d(F^*\omega_{i_1\ldots i_k}) \wedge d(F^*x^{i_1}) \wedge \cdots \wedge d(F^*x^{i_k}) \\
&= \sum F^*(d\omega_{i_1\ldots i_k}) \wedge F^*(dx^{i_1}) \wedge \cdots \wedge F^*(dx^{i_k}) \\
&= F^*(\sum d\omega_{i_1\ldots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\
&= (F^*d\omega)|_U\,.
\end{aligned}$$

□

2.7.27 Proposition. *Let M be a manifold, $p \in M$, (φ, V), (ψ, V) charts around p, $\varphi = (x^1, \ldots, x^n)$, $\psi = (y^1, \ldots, y^n)$. Then:*

(i) $dx^i\big|_p = \sum_{k=1}^n D_k(\varphi^i \circ \psi^{-1})(\psi(p))\, dy^k\big|_p = \sum_{k=1}^n \frac{\partial x^i}{\partial y^k}\Big|_p\, dy^k\big|_p$

(ii) $dx^1 \wedge \cdots \wedge dx^n\big|_p = \det D(\varphi \circ \psi^{-1})(\psi(p))\, dy^1 \wedge \cdots \wedge dy^n\big|_p$

(iii) If $\omega \in \Omega^n(M)$, $\varphi_*\omega = \omega_\varphi \alpha^1 \wedge \cdots \wedge \alpha^n$, $\psi_*\omega = \omega_\psi \alpha^1 \wedge \cdots \wedge \alpha^n$ ($\alpha^1, \ldots, \alpha^n$ the standard basis of $(\mathbb{R}^n)^*$), then:

$$\omega_\psi(y) = \omega_\varphi(\varphi \circ \psi^{-1}(y)) \cdot \det D(\varphi \circ \psi^{-1})(y) \qquad \forall y \in \psi(V)$$

Proof. (i) Since $\{dx^i\big|_p \mid 1 \le i \le n\}$ is the dual basis of $\{\frac{\partial}{\partial x^j}\big|_p \mid 1 \le j \le n\}$ it suffices to show that

$$\left(\sum_{k=1}^n \frac{\partial x^i}{\partial y^k}\Big|_p\, dy^k\big|_p\right)\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \delta_{ij}.$$

In fact,

$$\sum_{k=1}^n \frac{\partial x^i}{\partial y^k}\Big|_p \underbrace{dy^k\big|_p\left(\frac{\partial}{\partial x^j}\Big|_p\right)}_{=\frac{\partial y^k}{\partial x^j}\big|_p} = \sum_{k=1}^n \underbrace{D_k(\varphi^i \circ \psi^{-1})(\psi(p))}_{[D(\varphi\circ\psi^{-1})]_{ik}} \cdot \underbrace{D_j(\psi^k \circ \varphi^{-1})(\varphi(p))}_{[D(\psi\circ\varphi^{-1})]_{kj}} = \delta_{ij}.$$

(ii) By (i) we obtain (recall the summation convention!):

$$\begin{aligned}
dx^1 \wedge \cdots \wedge dx^n\big|_p &= \left(\frac{\partial x^1}{\partial y^{\sigma_1}}\Big|_p\, dy^{\sigma_1}\big|_p\right) \wedge \cdots \wedge \left(\frac{\partial x^n}{\partial y^{\sigma_n}}\Big|_p\, dy^{\sigma_n}\big|_p\right) = \\
&= \frac{\partial x^1}{\partial y^{\sigma_1}}\Big|_p \cdots \frac{\partial x^n}{\partial y^{\sigma_n}}\Big|_p \underbrace{dy^{\sigma_1} \wedge \cdots \wedge dy^{\sigma_n}\big|_p}_{=\begin{cases} \operatorname{sgn}(\sigma)\, dy^1\wedge\cdots\wedge dy^n|_p & \sigma\in S_n \\ 0 & \text{else}\end{cases}} \\
&= \underbrace{\left(\sum_{\sigma \in S_n} \frac{\partial x^1}{\partial y^{\sigma_1}}\Big|_p \cdot \ldots \cdot \frac{\partial x^n}{\partial y^{\sigma_n}}\Big|_p \cdot \operatorname{sgn}(\sigma)\right)}_{=\det(D(\varphi\circ\psi^{-1})(\psi(p)))} \cdot dy^1 \wedge \cdots \wedge dy^n\big|_p
\end{aligned}$$

(iii) Let $\omega = f\,dx^1 \wedge \cdots \wedge dx^n = g\,dy^1 \wedge \cdots \wedge dy^n$. Then by 2.7.21 (ii), $\omega_\varphi = f \circ \varphi^{-1}$, $\omega_\psi = g \circ \psi^{-1}$. Thus (ii) gives

$$\begin{aligned} f(p)\; dx^1 \wedge \cdots \wedge dx^n\big|_p &= f(p) \det D(\varphi \circ \psi^{-1})(\psi(p))\; dy^1 \wedge \cdots \wedge dy^n\big|_p = \\ &= g(p)\; dy^1 \wedge \cdots \wedge dy^n\big|_p . \end{aligned}$$

Hence,

$$\begin{aligned} \omega_\psi(y) &= g(\psi^{-1}(y)) = f(\psi^{-1}(y)) \det D(\varphi \circ \psi^{-1})(y) \\ &= \omega_\varphi(\varphi \circ \psi^{-1}(y)) \det D(\varphi \circ \psi^{-1})(y) \end{aligned}$$

□

2.7.28 Remark. A direct proof of 2.7.27 (iii) can be based on 2.7.12: Let $\psi_*\omega = \omega_\psi \alpha^1 \wedge \cdots \wedge \alpha^n$, $\varphi_*\omega = \omega_\varphi \alpha^1 \wedge \cdots \wedge \alpha^n$. Then

$$\begin{aligned} \omega_\psi(y)\alpha^1 \wedge \cdots \wedge \alpha^n &= (\psi^{-1})^* \circ \varphi^*(\omega_\varphi \alpha^1 \wedge \cdots \wedge \alpha^n)(y) \\ &= (T_y(\varphi \circ \psi^{-1}))^*(\omega_\varphi(\varphi \circ \psi^{-1}(y))\alpha^1 \wedge \cdots \wedge \alpha^n) \\ &\overset{2.7.12}{=} \det(D(\varphi \circ \psi^{-1}))(y)\omega_\varphi(\varphi \circ \psi^{-1}(y))\alpha^1 \wedge \cdots \wedge \alpha^n, \end{aligned}$$

so $\omega_\psi(y) = \omega_\varphi(\varphi \circ \psi^{-1}(y)) \cdot \det D(\varphi \circ \psi^{-1})(y)$.

2.8 Integration, Stokes' Theorem

Our aim in this section is to develop a theory of integrating differential forms on manifolds. Based on this we will prove Stokes' theorem, which provides a far-reaching generalization of the classical integration theorems of analysis (Gauss, Stokes, Green). As a fundamental tool we will need the transformation rule for integrals:

2.8.1 Theorem. *Let $U, V \subseteq \mathbb{R}^n$ be open, $\Phi : U \to V$ a diffeomorphism, $f \in \mathcal{C}(V)$,* supp$f$ *compact. Then:*

$$\int_U f(\Phi(x))|\det D\Phi(x)|d^n x = \int_V f(y) d^n y \tag{2.8.1}$$

Our strategy for defining $\int_M \omega$ for $\omega \in \Omega^n_c(V)$, (Ω^n_c denoting the space of compactly supported n-forms, V a chart neighborhood) will be to set

$$\int_M \omega := \int_{\varphi(V)} \omega_\varphi(x) d^n x.$$

To make this a well-defined expression it should be independent of the chosen chart. The transformation behavior of ω_φ according to 2.7.27 (iii), however, differs from (2.8.1) (no absolute value of $\det D(\varphi \circ \psi^{-1})$). We therefore consider manifolds with distinguished atlasses:

2.8.2 Definition. *A manifold M is called* orientable *if it possesses an* oriented atlas $\mathcal{A} = \{(\psi_\alpha, V_\alpha) \mid \alpha \in A\}$ *such that* $\det D(\psi_\beta \circ \psi_\alpha^{-1})(x) > 0$ $\forall x \in \psi_\alpha(V_\alpha \cap V_\beta)$ $\forall\alpha$ $\forall\beta$. *As in the case of smooth manifolds, also for oriented manifolds one can define corresponding $\mathcal{C}^\infty$-structures (allowing only oriented atlasses). Charts in an oriented atlas are called* positively oriented. *A manifold M together with an oriented atlas is called* oriented.

2.8.3 Remark.

(i) Not every manifold is orientable. The most famous example of a non-orientable manifold is the Möbius strip.

(ii) One can show that the following are equivalent:

- M is orientable.
- $\exists \omega \in \Omega^n(M)$ with $\omega(p) \neq 0$ $\forall p \in M$. Such an ω is called *volume form* on M (cf. 2.7.11).
- The $\mathcal{C}^\infty(M)$-module $\Omega^n(M)$ is one-dimensional (every volume form provides a basis).

In the special case $M = \mathbb{R}^n$ we proceed as follows: For $\omega = a(x_1, \dots, x_n)dx^1 \wedge \dots \wedge dx^n$ with compact support $K \subseteq U$, U open in $\mathbb{R}^n$, let $\int_U \omega := \int_K a(x)d^n x$. To extend this definition to general manifolds we first consider the case $\omega \in \Omega^n_c(M)$ such that $\mathrm{supp}(\omega) \subseteq U$, where (φ, U) is a chart of M. Then put

$$\int_{(\varphi)} \omega := \int \varphi_*(\omega|_U) = \int_{\varphi(U)} \omega_\varphi(x) d^n x$$

2.8.4 Lemma. *Let M be an oriented manifold, $\omega \in \Omega^n_c(M)$, (φ, U), (ψ, V) positively oriented charts and* $\mathrm{supp}(\omega) \subseteq U \cap V$. *Then $\int_{(\varphi)} \omega = \int_{(\psi)} \omega$. Thus we may simply write $\int \omega$ for this common value.*

Proof. Let $\varphi_* \omega = \omega_\varphi \alpha^1 \wedge \dots \wedge \alpha^n$, $\psi_* \omega = \omega_\psi \alpha^1 \wedge \dots \wedge \alpha^n$. Then

$$\begin{aligned}
\int_{(\psi)} \omega \quad &= \quad \int_{\psi(V)} \omega_\psi(y) d^n y = \int_{\psi(U \cap V)} \omega_\psi(y) d^n y = \\
&\overset{2.7.17(iii)}{=} \int_{\psi(U\cap V)} \omega_\varphi(\varphi \circ \psi^{-1}(y)) \underbrace{\det D(\varphi \circ \psi^{-1})(y)}_{=|\det D(\varphi\circ\psi^{-1})(y)|} d^n y = \\
&= \quad \int_{\varphi(U \cap V)} \omega_\varphi(x) d^n x = \int_{\varphi(U)} \omega_\varphi(x) d^n x = \int_{(\varphi)} \omega.
\end{aligned}$$

$\square$

2.8.5 Definition. *Let M be an oriented manifold and $\mathcal{A} = \{(\psi_\alpha, V_\alpha) \mid \alpha \in A\}$ an oriented atlas. Let $\{\chi_\alpha \mid \alpha \in A\}$ be a partition of unity subordinate to $\{V_\alpha \mid \alpha \in A\}$. Let $\omega \in \Omega^n_c(M)$ and $\omega_\alpha := \chi_\alpha \cdot \omega$ (hence* $\mathrm{supp}(\omega_\alpha)$ *is compact and contained in V_α). Then let*

$$\int_M \omega := \sum_{\alpha \in A} \int \omega_\alpha.$$

2.8.6 Proposition.

(*i*) *The sum in 2.8.5 contains only finitely many non-vanishing terms.*

(*ii*) *Definition 2.8.5 is independent of the chosen oriented atlas (in the given oriented $\mathcal{C}^\infty$-structure) and partition of unity.*

Proof. (i) Since $\{\mathrm{supp}\chi_\alpha \mid \alpha \in A\}$ is locally finite, only finitely many $\mathrm{supp}\chi_\alpha$ intersect the compact set $\mathrm{supp}(\omega)$ (every $p \in \mathrm{supp}(\omega)$ has a neighborhood intersecting only finitely many $\mathrm{supp}\chi_\alpha$, finitely many such neighborhoods cover $\mathrm{supp}(\omega)$).

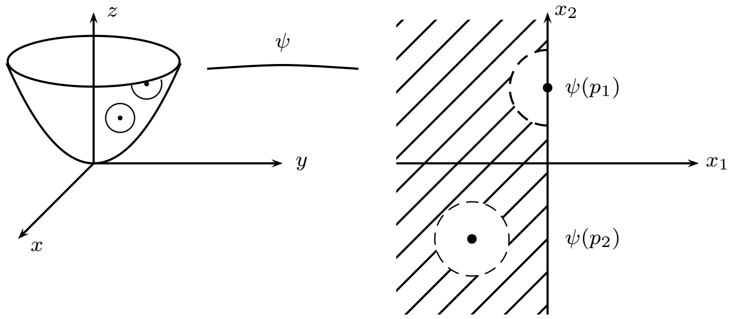
(ii) Let $\mathcal{A}' = \{(\varphi_\beta, U_\beta) \mid \beta \in B\}$ be another oriented atlas in the same oriented $\mathcal{C}^\infty$-structure, $\{\mu_\beta \mid \beta \in B\}$ a partition of unity subordinate to $\{U_\beta \mid \beta \in B\}$. Then

$$\sum_{\alpha \in A} \int \omega_\alpha \overset{\sum_\beta \mu_\beta = 1}{=} \sum_{\alpha \in A} \int \sum_{\beta \in B} \mu_\beta \chi_\alpha \omega = \sum_{\alpha,\beta} \int \mu_\beta \chi_\alpha \omega = \cdots = \sum_{\beta \in B} \int \mu_\beta \omega.$$

□

In the integral theorems of vector analysis, typical domains of integration are n-dimensional domains with boundary, where the boundary itself forms an $(n-1)$-dimensional domain of integration. Such domains are currently not covered by our notion of manifold:

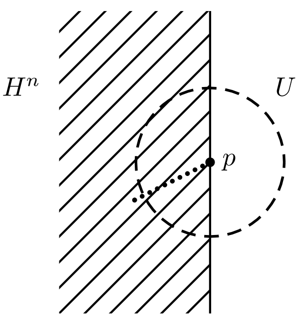
2.8.7 Example. Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2, z \le z_0, z_0 > 0\}$.



M is not a manifold since points like p_1 do not have open neighborhoods in M which are homeomorphic to $\mathbb{R}^2$. On the other hand it is quite obvious that M has charts which are homeomorphic to relatively open subsets of a suitable half-space. Points like p_1 form the boundary (but not in the topological sense!) of M, which itself is a 1-dimensional manifold (without boundary).

We now want to make precise these observations in the following definition.

2.8.8 Definition. *Let the half-space $H^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^1 \le 0\}$ be equipped with the trace topology of $\mathbb{R}^n$ (i.e., $V \subseteq H^n$ is open $\Leftrightarrow \exists U \subseteq \mathbb{R}^n$ open s.t. $U \cap H^n = V$). Let $V \subseteq H^n$ be open. Then $f : V \to \mathbb{R}^m$ is called smooth on V if there exists an open subset $U \supseteq V$ of $\mathbb{R}^n$ and a smooth extension $\tilde{f}$ of f to U. For any $p \in V$ we then set $Df(p) := D\tilde{f}(p)$.*



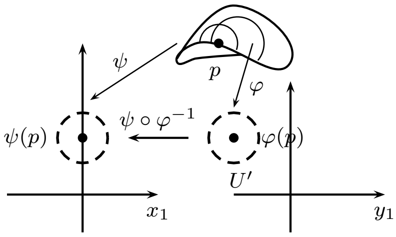
We have to check that $Df(p)$ is independent of $\tilde{f}$: This is clear if $V \subseteq (H^n)^\circ$. Thus let $p = (0, x^2, \ldots, x^n)$ and $\tilde{f}$, $\tilde{\tilde{f}}$ be two extensions of f to an open neighborhood U of p in $\mathbb{R}^n$. Set $g := \tilde{f} - \tilde{\tilde{f}}$. We have to show that $Dg(p) = 0$. To this end, pick a sequence of points $p_m \in (H^n)^\circ$ with $p_m \to p$. Then $Dg(p_m) = 0$ for all m, so also $Dg(p) = \lim_{m\to\infty} Dg(p_m) = 0$.

2.8.9 Definition. *A* manifold with boundary *is a set M together with an atlas $\mathcal{A} = \{(\psi_\alpha, V_\alpha) \mid \alpha \in A\}$ of bijective maps $\psi_\alpha : V_\alpha \to \psi_\alpha(V_\alpha) \subseteq H^n$ (relatively) open, such that $\bigcup_{\alpha \in A} V_\alpha = M$ and for all α, β with $V_\alpha \cap V_\beta \ne \emptyset$ we have $\psi_\beta \circ \psi_\alpha^{-1}$:*

$\psi_\alpha(V_\alpha \cap V_\beta) \to \psi_\beta(V_\alpha \cap V_\beta)$ is smooth in the sense of 2.8.8. As in the case of manifolds without boundary we require M with its natural topology (induced by the charts) to be Hausdorff and second countable.

2.8.10 Lemma. *Let M be a manifold with boundary. A point $p \in M$ is called* boundary point *of M if there exists a chart $(\psi = (x^1, \ldots, x^n), V)$ with $x^1(p) = 0$. If p is a boundary point, denoted by $p \in \partial M$ then for any chart $(\varphi = (y^1, \ldots, y^n), U)$ around p we have $y^1(p) = 0$.*

Proof. Suppose to the contrary that there would exist a chart $\varphi = (y^1, \ldots, y^n)$ with $y^1(p) < 0$.



Choose a neighborhood U' of $\varphi(p)$ which is open in $\mathbb{R}^n$ and contained in $\varphi(U \cap V) \subseteq H^n$. Since $\det(D(\psi \circ \varphi^{-1}))(\varphi(p)) \neq 0$, by 2.1.1, $\psi \circ \varphi^{-1}$ is a diffeomorphism onto a neighborhood of $\psi \circ \varphi^{-1}(\varphi(p)) = \psi(p)$ which is open in $\mathbb{R}^n$. This neighborhood must therefore be contained in H^n, contradicting $\psi^1(p) = x^1(p) = 0$. $\square$

All constructions we already know for manifolds without boundary like tangent space, tensors, differential forms, orientability, etc. work out completely analogously for manifolds with boundary. The next result shows that ∂M itself is a manifold (without boundary).

2.8.11 Proposition. *Let M be an n-dimensional manifold with boundary. Then ∂M is an $(n-1)$-dimensional manifold (without boundary). If M is oriented then the orientation of M induces an orientation of ∂M.*

Proof. Let $\mathcal{A} = \{(\psi_\alpha, V_\alpha) \mid \alpha \in A\}$ be an atlas of M. Set $A' := \{\alpha \in A \mid V_\alpha \cap \partial M \neq \emptyset\}$, $\mathcal{A}' := \{(\psi_\alpha|_{V_\alpha \cap \partial M}, V_\alpha \cap \partial M) \mid \alpha \in A'\}$. We show that $\mathcal{A}'$ is an atlas for ∂M. Set $\tilde{V}_\alpha := V_\alpha \cap \partial M$, $\tilde{\psi}_\alpha := \psi_\alpha|_{\tilde{V}_\alpha}$. Then $\tilde{\psi}_\alpha : \tilde{V}_\alpha \to \psi_\alpha(\tilde{V}_\alpha)$ is bijective and by 2.8.10 it follows that $\tilde{\psi}_\alpha(\tilde{V}_\alpha) = \psi_\alpha(V_\alpha) \cap \{x^1 = 0\}$. Clearly, $\bigcup_{\alpha \in A'} \tilde{V}_\alpha = \partial M$. Now let α, $\beta \in A'$ such that $\tilde{V}_\alpha \cap \tilde{V}_\beta \neq \emptyset$. Since $\psi_\alpha(V_\alpha \cap V_\beta) \subseteq H^n$ is open, $\tilde{\psi}_\alpha(\tilde{V}_\alpha \cap \tilde{V}_\beta) = \psi_\alpha(V_\alpha \cap V_\beta) \cap \{x^1 = 0\}$ is open in $\{x^1 = 0\} \cong \mathbb{R}^{n-1}$. Moreover, $\tilde{\psi}_\beta \circ \tilde{\psi}_\alpha^{-1}$ is smooth on $\psi_\alpha(\tilde{V}_\alpha \cap \tilde{V}_\beta)$ as a restriction of the smooth map $\psi_\beta \circ \psi_\alpha^{-1}$. Suppose now that $\mathcal{A}$, in addition, is oriented, i.e., that $\det D(\psi_\beta \circ \psi_\alpha^{-1}) > 0$ for all α, β with $V_\alpha \cap V_\beta \neq \emptyset$. Let $\psi_\alpha = (x_\alpha^1, \ldots, x_\alpha^n)$, $\psi_\beta = (x_\beta^1, \ldots, x_\beta^n)$. Then for every $(0, x_\alpha^2, \ldots, x_\alpha^n) \in \tilde{\psi}_\alpha(\tilde{V}_\alpha \cap \tilde{V}_\beta)$, $\psi_\beta \circ \psi_\alpha^{-1}(0, \underbrace{x_\alpha^2, \ldots, x_\alpha^n}_{=: x'_\alpha}) = (0, \tilde{\psi}_\beta \circ \tilde{\psi}_\alpha^{-1}(x'_\alpha))$.

Therefore,

$$D(\psi_\beta \circ \psi_\alpha^{-1})(0, x'_\alpha) = \left. \left(\begin{array}{c|ccc} \frac{\partial(\psi_\beta^1 \circ \psi_\alpha^{-1})}{\partial x^1} & 0 & \ldots & 0 \\ \hline * & & & \\ \vdots & & D(\tilde{\psi}_\beta \circ \tilde{\psi}_\alpha^{-1}) & \\ * & & & \end{array} \right) \right|_{(0, x'_\alpha)}$$

$$\Rightarrow \ \det D(\psi_\beta \circ \psi_\alpha^{-1})(0, x'_\alpha) = \left.\frac{\partial(\psi_\beta^1 \circ \psi_\alpha^{-1})}{\partial x^1}\right|_{(0,x'_\alpha)} \det D(\tilde\psi_\beta \circ \tilde\psi_\alpha^{-1})(0, x'_\alpha) \quad (*)$$

Now $\psi_\beta^1 \circ \psi_\alpha^{-1}(0, x'_\alpha) = 0$ and $\psi_\beta^1 \circ \psi_\alpha^{-1}(x^1, x'_\alpha) < 0$ for $x^1 < 0$ (since $\psi_\beta \circ \psi_\alpha^{-1} : H^n \to H^n$). Therefore, $\frac{\partial(\psi_\beta^1 \circ \psi_\alpha^{-1})}{\partial x^1} \geq 0$ and $\neq 0$ (by $(*)$), hence > 0. Again by $(*)$ it follows that $\det D(\tilde\psi_\beta \circ \tilde\psi_\alpha^{-1}) > 0$, so $\mathcal{A}'$ is oriented. $\square$

As the final ingredient for Stokes' theorem we consider the restriction of differential forms defined on M to ∂M: Let $i : \partial M \hookrightarrow M$ be the natural inclusion. We first note that i is smooth since for any chart $\psi = (x^1, \ldots, x^n)$ of M we have:

$$\begin{array}{ccc} \partial M & \xrightarrow{\ i\ } & M \\ {\scriptstyle\tilde\psi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ \tilde\psi(\tilde V) & \xrightarrow{\ j\ } & \psi(V) \end{array}$$

where $j : (x^2, \ldots, x^n) \mapsto (0, x^2, \ldots, x^n)$. This is obviously smooth.
The restriction of any $\omega \in \Omega^k(M)$ is defined as $i^*\omega \in \Omega^k(\partial M)$. As in (2.7.2), the local representation of ω with respect to ψ can be written as

$$\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Then $\psi_*\omega$ is given by

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \omega_{i_1 \ldots i_k} \circ \psi^{-1} \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} =: \sum_{1 \leq i_1 < \cdots < i_k \leq n} \omega^\psi_{i_1 \ldots i_k} \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}.$$

The local representation of $i^*\omega$ with respect to $\tilde\psi$ therefore is

$$\begin{aligned} \tilde\psi_*(i^*\omega) &= (\tilde\psi^{-1})^*(i^*\omega) = (i \circ \tilde\psi^{-1})^*\omega = (\psi^{-1} \circ j)^*\omega = j^*((\psi^{-1})^*\omega) = \\ &= j^*(\psi_*\omega) \overset{2.7.26(i)}{=} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \omega^\psi_{i_1 \ldots i_k} \circ j \ j^*(\alpha^{i_1}) \wedge \cdots \wedge j^*(\alpha^{i_k}). \end{aligned}$$

Observing now that

$$j^*(\alpha^k)(v)\big|_x \overset{2.7.24}{=} \alpha^k(\underbrace{Dj(x)}_{=j \text{ by linearity}}(v)) = \alpha^k(j(v)) = \begin{cases} 0 & k = 1 \\ v_k = \alpha^k(v) & k \neq 1 \end{cases}$$

we finally arrive at

$$\tilde\psi_*(i^*\omega) = \sum_{1 < i_1 < \cdots < i_k \leq n} \omega^\psi_{i_1 \ldots i_k} \circ j \ \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} \tag{2.8.2}$$

2.8.12 Theorem. *(Stokes' theorem) Let M be an oriented manifold with boundary, $\omega \in \Omega_c^{n-1}(M)$, and $i : \partial M \hookrightarrow M$. Then:*

$$\int_{\partial M} i^*\omega = \int_M d\omega$$

Proof. Denote by K the compact support of ω. We consider the following two cases:

1.) There exists a chart $(\psi = (x^1, \dots, x^n), V)$ with $K \subseteq V$. Since $\omega \in \Omega^{n-1}(M)$, the local representation of ω with respect to ψ reads

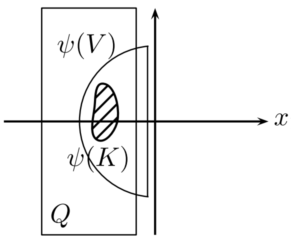
$$\omega = \sum_{k=1}^{n} \omega_k dx^1 \wedge \dots \wedge dx^{k-1} \wedge dx^{k+1} \wedge \dots \wedge dx^n \tag{2.8.3}$$

where $\omega_j \in \mathcal{C}^\infty(V)$ for all j. Hence

$$d\omega = (\sum_{k=1}^{n} (-1)^{k-1} \frac{\partial \omega_k}{\partial x^k}) dx^1 \wedge \dots \wedge dx^n \tag{2.8.4}$$

with $\frac{\partial \omega_k}{\partial x^k} = D_k(\omega_k \circ \psi^{-1})(\psi(.))$. We now distinguish the following sub-cases:
1a) $V \cap \partial M = \emptyset$. Then $i^*\omega = 0$ (cf., e.g., (2.8.2)), hence $\int_{\partial M} i^*\omega = 0$ and we have to show that also

$$\int_M d\omega \overset{2.8.4}{=} \int_{\psi(V)} \psi_*(d\omega) \overset{(2.8.4),2.7.21(ii)}{=} \int_{\psi(V)} \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial \omega_k^\psi}{\partial x^k} dx^1 \dots dx^n = 0.$$

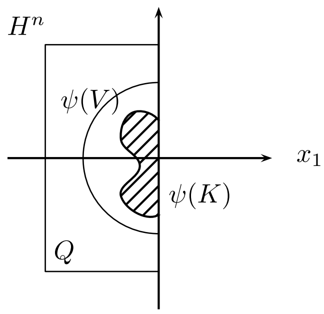


We now choose a parallelepiped $Q \subsetneq H^n$ of the form $Q = \{(x^1, \dots, x^n \mid a^k \leq x^k \leq b^k \ (1 \leq k \leq n)\}$ such that $\psi(K)$ lies in the interior of Q. Then if we extend the (compactly supported) ω_k^ψ by 0 to all of H^n, we obtain (applying the fundamental theorem of calculus):

$$\begin{aligned}
\int_{\psi(V)} \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial \omega_k^\psi}{\partial x^k} dx^1 \dots dx^n &= \sum_{k=1}^{n} (-1)^{k-1} \int_Q \frac{\partial \omega_k^\psi}{\partial x^k} dx^1 \dots dx^n \\
&= \sum_{k=1}^{n} (-1)^{k-1} \int (\underbrace{\omega_k^\psi(x^1, \dots, x^{k-1}, b^k, x^{k+1}, \dots, x^n)}_{=0} \\
&\quad - \underbrace{\omega_k^\psi(x^1, \dots, x^{k-1}, a^k, x^{k+1}, \dots, x^n)}_{=0}) dx^1 \dots dx^{k-1} dx^{k+1} \dots dx^n \\
&= 0
\end{aligned}$$

1b) Now suppose that $V \cap \partial M \neq \emptyset$. Then

$$\begin{aligned}
\int_{\partial M} i^*\omega &\overset{2.8.4}{=} \int_{\tilde{\psi}(V \cap \partial M)} \tilde{\psi}_*(i^*\omega) \\
&\overset{(2.8.3),(2.8.2)}{=} \underbrace{\int_{\tilde{\psi}(V \cap \partial M)} \omega_1^\psi(0, x^2, \dots, x^n) dx^2 \dots dx^n}_{=\int_{\psi(K) \cap \{x^1 = 0\}}}
\end{aligned} \tag{2.8.5}$$



Again we extend the ω_k^ψ by 0 to all of H^n and choose a parallelepiped $Q \subseteq H^n$, this time of the form $Q = [a^1, 0] \times [a^2, b^2] \times \cdots \times [a^n, b^n]$ such that $\psi(K) \subseteq Q^\circ \cup \{x^1 = 0\}$. Then as in the previous case we obtain:

$$\begin{aligned}
\int_M d\omega &= \sum_{k=1}^n (-1)^{k-1} \int_Q \frac{\partial \omega_k^\psi}{\partial x^k} dx^1 \dots dx^n \\
&= \int_{[a^2,b^2]\times\cdots\times[a^n,b^n]\ (=Q\cap\{x^1=0\})} \big(\omega_1^\psi(0, x^2, \dots, x^n) - \underbrace{\omega_1^\psi(a^1, x^2, \dots, x^n)}_{=0}\big) dx^2 \dots dx^n \\
&\quad + \sum_{k=2}^n (-1)^{k-1} \int \big(\underbrace{\omega_k^\psi(x^1, \dots, b^k, \dots, x^n)}_{=0} \\
&\quad - \underbrace{\omega_k^\psi(x^1, \dots, a^k, \dots, x^n)}_{=0}\big) dx^1 \dots dx^{k-1} dx^{k+1} \dots dx^n \\
&= \int_{\psi(K)\cap\{x^1=0\}} \omega_1^\psi(0, x^2, \dots, x^n) dx^2 \dots dx^n \\
&\overset{(2.8.5)}{=} \int_{\partial M} i^*\omega
\end{aligned}$$

2.) The general case: Let $\{(\psi_\alpha, V_\alpha) \mid \alpha \in A\}$ be an oriented atlas, $\{\chi_\alpha \mid \alpha \in A\}$ a subordinate partition of unity. Then the $\omega_\alpha := \chi_\alpha \cdot \omega$ satisfy the assumptions of case 1.). Also, $\sum_\alpha d\chi_\alpha = d(\sum_\alpha \chi_\alpha) = d(1) = 0$. Thus $\omega = \sum_\alpha \omega_\alpha$ and

$$\sum_\alpha d\omega_\alpha = \sum_\alpha d(\chi_\alpha \cdot \omega) = \sum_\alpha (d\chi_\alpha)\omega + \sum_\alpha \chi_\alpha d\omega = d\omega.$$

From this we finally obtain

$$\int_M d\omega = \sum_\alpha \int_M d\omega_\alpha \overset{1.)}{=} \sum_\alpha \int_{\partial M} i^*\omega_\alpha = \int_{\partial M} i^*(\sum_\alpha \omega_\alpha) = \int_{\partial M} i^*\omega.$$

□

2.8.13 Examples.

(i) Applying 2.8.12 to the ω from 2.7.23 (i), we obtain Green's theorem in the plane:

$$\int_{\partial M} Pdx + Qdy = \int_M (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dxdy$$

(ii) From 2.7.23 (ii) and 2.8.12 we derive Gauss' divergence theorem (in $\mathbb{R}^3$):

$$\int_M (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}) dxdydz = \int_{\partial M} Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$$

Chapter 3

Hypersurfaces

As an application of the concepts introduced in chapters 1 and 2 we now turn to a study of hypersurfaces in $\mathbb{R}^n$.

3.1 Curvature of Hypersurfaces

3.1.1 Definition. *A hypersurface of $\mathbb{R}^n$ is an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$.*

Locally a hypersurface is given by one of the equivalent descriptions in 2.1.8, e.g. as the zero set of a regular map $f : \mathbb{R}^n \to \mathbb{R}$. Since $\mathbb{R}^n$ is equipped with the standard scalar product $\langle v, w\rangle = \sum_{i=1}^n v_i w_i$, every tangent space T_pM possesses a one-dimensional orthogonal complement. This provides two possible unit normals of M. To be able to pick one of these, we will suppose M to be oriented.

3.1.2 Lemma. *Let M^m be an (abstract) oriented manifold and $p \in M$. A basis $\{v^1, \dots, v^m\}$ of T_pM is called positively oriented, if in some positively oriented chart $\varphi = (x^1, \dots, x^m)$ of M around p we have: $(dx^1 \wedge \dots \wedge dx^m)\big|_p (v^1, \dots, v^m) > 0$. This notion is independent of the chosen chart.*

Proof. Let $\psi = (y^1, \dots, y^m)$ be another positively oriented chart. Since by 2.7.27,

$$(dy^1 \wedge \dots \wedge dy^m)\big|_p = \underbrace{\det D(\psi \circ \varphi^{-1})(\varphi(p))}_{>0} \cdot (dx^1 \wedge \dots \wedge dx^m)\big|_p ,$$

the claim follows. $\square$

3.1.3 Definition. *Let M be an oriented hypersurface of $\mathbb{R}^n$. The* Gauss map *$p \mapsto \nu_p$ assigns to every $p \in M$ the unit normal vector ν_p for which $\det(\nu_p, e^1, \dots, e^{n-1}) > 0$ for any positively oriented basis $\{e^1, \dots, e^{n-1}\}$ of T_pM.*

3.1.4 Remark.

(i) ν_p is well-defined: Let $\{f^1, \dots, f^{n-1}\}$ be any positively oriented basis of T_pM and let $\psi = (x^1, \dots, x^{n-1})$ be any positively oriented chart of M. Set $e^j := \frac{\partial}{\partial x^j}\big|_p$ $(1 \le j \le n-1)$ and $f^k = \sum_{i_k=1}^{n-1} a_{k i_k} e^{i_k}$ $(1 \le k \le n-1)$. Then

$$\begin{aligned} 1 &= \operatorname{sgn}(dx^1 \wedge \dots \wedge dx^{n-1})\big|_p (f^1, \dots, f^{n-1}) = \\ &= \operatorname{sgn}\Big(\sum_{i_1, \dots, i_{n-1}} a_{1 i_1} \dots a_{(n-1) i_{n-1}} \underbrace{(dx^1 \wedge \dots \wedge dx^{n-1})\big|_p (e^{i_1}, \dots, e^{i_{n-1}})}_{=\operatorname{sgn}(i_1, \dots, i_{n-1})}\Big) \quad (*) \end{aligned}$$

Therefore,

$$\det(\nu_p, f^1, \dots, f^{n-1}) =$$

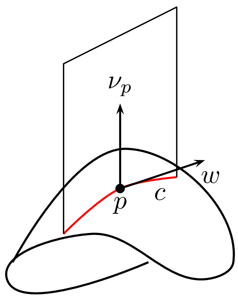
$$= \underbrace{\sum_{i_1,\dots,i_{n-1}} \operatorname{sgn}(i_1, \dots, i_{n-1}) a_{1i_1} \dots a_{(n-1)i_{n-1}}}_{>0 \text{ by } (*)} \det(\nu_p, e^1, \dots, e^{n-1}).$$

(ii) The Gauss map $p \mapsto \nu_p$ is smooth. Let $(\psi = (x^1, \dots, x^{n-1}), V)$ be a positively oriented chart. Then $\{\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^{n-1}}\big|_p\}$ is a positively oriented basis of T_pM since $(dx^1 \wedge \dots \wedge dx^{n-1})\big|_p (\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^{n-1}}\big|_p) = 1$ for all $p \in V$. By 2.1.8 and 2.4.1, locally around any $p_0 \in V$, M is given as the zero set of some regular map f and for the gradient $\operatorname{grad}(f)(p) = Df(p)$ we get: $\langle \operatorname{grad} f(p), v\rangle = Df(p)(v) = 0\ \forall v \in T_pM$. Thus $\operatorname{grad} f(p) \perp T_pM$ for all p near p_0. W.l.o.g. we may suppose $\det(\operatorname{grad} f(p_0), \frac{\partial}{\partial x^1}\big|_{p_0}, \dots, \frac{\partial}{\partial x^{n-1}}\big|_{p_0}) > 0$ (otherwise replace f by $-f$). By continuity, $\det(\operatorname{grad} f(p), \frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^{n-1}}\big|_p) > 0$ for p near p_0. Hence locally around p_0 the Gauss map is given by

$$\nu_p = \frac{\operatorname{grad} f(p)}{\|\operatorname{grad} f(p)\|}$$

which clearly is smooth.

To define the curvature of a hypersurface M in a point $p \in M$ we reduce this question to the curvature of curves as considered in chapter 1. Let $w \in T_pM$, $\|w\| = 1$. We consider the curve determined by the intersection of M with the plane spanned by w and ν_p.



This plane is given by: $(t, s) \mapsto p + t\nu_p + sw$. The curvature of the intersection curve c will be called the normal curvature of M in the direction w. Let M locally be given as the zero set of the regular map f and recall that we suppose M to be oriented. Then c is given implicitly by

$$f(p + t\nu_p + sw) = 0 \quad \forall t,\ s \qquad (*)$$

By 3.1.4 (ii), $\nu_p = \frac{\operatorname{grad} f(p)}{\|\operatorname{grad} f(p)\|}$. Hence

$$\frac{\partial}{\partial t}\bigg|_{(t,s)=(0,0)} f(p + t\nu_p + sw) = Df(p)\nu_p = \langle \operatorname{grad} f(p), \nu_p\rangle \neq 0.$$

Therefore, locally we can solve $(*)$ for t as a function of s (cf. 2.1.2). We obtain a curve $c : s \mapsto p + t(s)\nu_p + sw$. Then $c(0) = p$, and

$$\underbrace{c'(0)}_{\in T_pM} = \underbrace{t'(0) \cdot \nu_p}_{\in T_pM^\perp \Rightarrow = 0} + \underbrace{w}_{\in T_pM} = w.$$

W.l.o.g. we may suppose that c is parametrized by arclength. To calculate the curvature of c we need to determine the accompanying frame. As $\{\nu_p, w\}$ is positively oriented, $e_1 = c'(0) = w$, $e_2 = -\nu_p$. Since $c'(s) \in T_{c(s)}M$, it follows that $\langle c'(s), \nu_{c(s)}\rangle = 0\ \forall s$. Consequently,

$$0 = \frac{d}{ds}\Big|_0 \langle c'(s), \nu \circ c(s)\rangle = \langle c''(0), \nu_p\rangle + \langle \underbrace{c'(0)}_{=w}, T_p\nu \cdot \underbrace{c'(0)}_{=w}\rangle,$$

and so the normal curvature $\kappa(w)$ of M in the direction w is given by:

$$\kappa(w) = \kappa_c(0) \overset{(1.2.1)}{=} \langle c''(0), e_2\rangle = -\langle c''(0), \nu_p\rangle = \langle w, T_p\nu \cdot w\rangle \tag{3.1.1}$$

3.1.5 Definition. *The* Weingarten map *(the* shape-operator*)* L_p *is defined as:*

$$L_p := T_p\nu : T_pM \to T_{\nu_p}S^{n-1} = \nu_p^\perp = T_pM.$$

Thus L_pw is the infinitesimal change of the normal vector ν in the direction w.
To further analyze the Weingarten map we need the following observation on vector fields on submanifolds of $\mathbb{R}^n$.

3.1.6 Lemma. *Let M^k be a k-dimensional submanifold of $\mathbb{R}^n$. Then:*

(i) For any $X \in \mathfrak{X}(M)$ and any $p \in M$ there exists a neighborhood U of p in $\mathbb{R}^n$ and a map $\tilde{X} \in \mathcal{C}^\infty(U, \mathbb{R}^n)$ such that $\tilde{X}\,|_{U\cap M} = X|_{U\cap M}$.

(ii) If $X, Y \in \mathfrak{X}(M)$ and $\tilde{X}$, $\tilde{Y}$ are smooth extensions as in (i), then

$$[X, Y]_p = D\tilde{Y}(p)\tilde{X}(p) - D\tilde{X}(p)(\tilde{Y}(p)) \quad \forall p \in M.$$

Proof. (i) By 2.5.21, $X \in \mathcal{C}^\infty(M, \mathbb{R}^n)$. Hence by 2.1.10 each of the n components of X can be extended smoothly to a neighborhood of p in $\mathbb{R}^n$. This gives the desired extension $\tilde{X}$ (in a non-unique way).
(ii) Let $\tilde{f} \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ and $f := \tilde{f}\,|_M$, so $f \in \mathcal{C}^\infty(M, \mathbb{R})$. Then $T_pf = D\tilde{f}(p)\,|_{T_pM}$ (cf. the calculation preceding 2.4.3). Hence $X(f)(p) = T_pf(X_p) = D\tilde{f}(p)(\tilde{X}_p)$, so

$$Y(X(f))|_p = D(q \mapsto D\tilde{f}(q)(\tilde{X}_q))|_p(\tilde{Y}_p) = D^2\tilde{f}(p)(\tilde{X}_p, \tilde{Y}_p) + D\tilde{f}(p)D\tilde{X}(p)\tilde{Y}_p,$$

and analogously for $X(Y(f))$. Since $D^2\tilde{f}(p)$ is symmetric, we conclude that

$$\begin{aligned} D\tilde{f}(p)([X,Y]_p) &= [X,Y]_p(f) = (X(Y(f)) - Y(X(f)))_p \\ &= D\tilde{f}(p)(D\tilde{Y}(p)\tilde{X}(p) - D\tilde{X}(p)\tilde{Y}(p)). \end{aligned}$$

Inserting in particular $\tilde{f} = \mathrm{pr}_i : \mathbb{R}^n \to \mathbb{R}$, it follows that $D\tilde{f}(p) = \mathrm{pr}_i$, so $[X,Y]_p = D\tilde{Y}(p)\tilde{X}(p) - D\tilde{X}(p)\tilde{Y}(p)$, as claimed. □

3.1.7 Proposition. *The Weingarten map $L_p : T_pM \to T_pM$ is symmetric, i.e.,*

$$\langle L_pv, w\rangle = \langle v, L_pw\rangle \qquad \forall v, w \in T_pM.$$

Proof. Choose $X, Y \in \mathfrak{X}(M)$ with $X_p = v$, $Y_p = w$ and choose smooth extensions as in 3.1.6 (i). To also extend ν to a neighborhood of p in $\mathbb{R}^n$ we note that ν is locally given as $\nu_p = \frac{\mathrm{grad} f(p)}{\|\mathrm{grad} f(p)\|}$ (where $f = 0$ is a local representation of M by a regular map). Thus ν is a $\mathcal{C}^\infty$-map from M to $\mathbb{R}^n$ which, by 2.1.10, can locally

be extended to a neighborhood of p in $\mathbb{R}^n$. Denote by $\tilde{\nu}$ such an extension. Then according to the calculation preceding 2.4.3, $T_p\nu = D\tilde{\nu}(p)|_{T_pM}$.
Since $q \mapsto \langle X_q, \nu_q\rangle \equiv 0$, also $T_p(q \mapsto \langle X_q, \nu_q\rangle) = 0$. Therefore,

$$\begin{aligned} 0 &= T_p(q \mapsto \langle X_q, \nu_q\rangle)Y_p = D(q \mapsto \langle \tilde{X}_q, \tilde{\nu}_q\rangle)\tilde{Y}_p \\ &= \langle D\tilde{X}(p)\tilde{Y}_p, \tilde{\nu}_p\rangle + \langle \underbrace{\tilde{X}_p}_{=X_p}, \underbrace{D\tilde{\nu}(p)}_{=L_p} \underbrace{\tilde{Y}_p}_{Y_p} \rangle \end{aligned}$$

and analogously for $q \mapsto \langle Y_q, \nu_q\rangle$. Summing up, we obtain

$$\langle X_p, L_pY_p\rangle - \langle L_pX_p, Y_p\rangle = \langle \underbrace{D\tilde{Y}(p)\tilde{X}(p) - D\tilde{X}(p)\tilde{Y}(p)}_{\overset{3.1.6(ii)}{=}[X,Y]_p \in T_pM}, \underbrace{\nu_p}_{\in T_pM^\perp} \rangle = 0.$$

□

3.1.8 Definition. *Let M be an oriented hypersurface in $\mathbb{R}^n$.*

(i) The Riemannian metric *g, or first fundamental form I, is the $\binom{0}{2}$-tensor field*

$$T_pM \times T_pM \ni (v, w) \mapsto g_p(v, w) := \langle v, w\rangle \ (= I_p(v, w))$$

(ii) The second fundamental form II is the $\binom{0}{2}$-tensor field

$$T_pM \times T_pM \ni (v, w) \mapsto II_p(v, w) := \langle v, L_pw\rangle$$

3.1.9 Remark.

(i) g is a section of the bundle $T^0_2(M)$ since every g_p is a bilinear map from $T_pM \times T_pM$ to $\mathbb{R}$, i.e., $g_p \in T^0_2(T_pM)$ for all $p \in M$. Moreover, g is smooth by 2.6.18: Let $X, Y \in \mathfrak{X}(M)$, i.e., $X, Y : M \to \mathbb{R}^n$ $\mathcal{C}^\infty$, $X_p, Y_p \in T_pM$ $\forall p$ (cf. 2.5.21). Then also $p \mapsto \langle X_p, Y_p\rangle = g_p(X_p, Y_p)$ is smooth.

g_p is precisely the restriction of the standard scalar product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$ to $T_pM \times T_pM$. It allows to measure lengths and angles in T_pM.

If φ is a local parametrization of M around p and $\varphi^{-1} = (x^1, \ldots, x^{n-1})$ then, by (2.6.2), g locally is of the form

$$g(p) = g_{ij}(p)\ dx^i\big|_p \otimes dx^j\big|_p \tag{3.1.2}$$

where

$$\begin{aligned} g_{ij}(p) &= g_p\Big(\frac{\partial}{\partial x^i}\Big|_p, \frac{\partial}{\partial x^j}\Big|_p\Big) \\ &= \Big\langle \frac{\partial}{\partial x^i}\Big|_p, \frac{\partial}{\partial x^j}\Big|_p \Big\rangle = \langle \underbrace{\frac{\partial\varphi}{\partial x^i}(\varphi^{-1}(p))}_{D_i\varphi(\varphi^{-1}(p))}, \underbrace{\frac{\partial\varphi}{\partial x^j}(\varphi^{-1}(p))}_{D_j\varphi(\varphi^{-1}(p))}\rangle. \end{aligned} \tag{3.1.3}$$

Since g is symmetric, $g_{ij} = g_{ji}$ $\forall i, j$.

(ii) For every $p \in M$, the second fundamental form II_p is a symmetric bilinear map $T_pM \times T_pM \to \mathbb{R}$, so II is a section of T^0_2M. Smoothness again follows from 2.6.18: Let $X, \tilde{X}, Y, \tilde{Y}, \nu, \tilde{\nu}$ be as in the proof of 3.1.7. Then

$$p \mapsto II_p(X_p, Y_p) = p \mapsto -\langle D\tilde{X}(p)\tilde{Y}_p, \tilde{\nu}_p\rangle$$

is smooth.

(iii) In the classical differential geometry of surfaces in $\mathbb{R}^3$, the relevant special case of (i) is that of a parametrization $\varphi : (t,s) \mapsto (\varphi_1(t,s), \varphi_2(t,s), \varphi_3(t,s))$. Setting $E := \langle \varphi_t, \varphi_t \rangle$, $F := \langle \varphi_t, \varphi_s \rangle$, $G := \langle \varphi_s, \varphi_s \rangle$ (with φ_s, φ_t the partial derivatives of φ), then g is given by the matrix

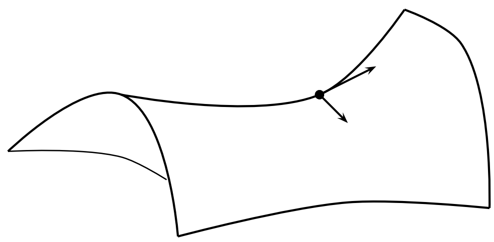
$$[I] = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

with respect to the basis $\{\varphi_t, \varphi_s\}$ of $T_{\varphi(t,s)}M$. If $v = v_1 \cdot \varphi_t + v_2 \cdot \varphi_s$, $w = w_1\varphi_t + w_2\varphi_s$ are vectors in $T_{\varphi(t,s)}M$, then

$$\begin{aligned} g_{\varphi(t,s)}(v,w) &= (Edt \otimes dt + Fdt \otimes ds + Fds \otimes dt + Gds \otimes ds) \\ &\quad (v_1\tfrac{\partial}{\partial t} + v_2\tfrac{\partial}{\partial s}, w_1\tfrac{\partial}{\partial t} + w_2\tfrac{\partial}{\partial s}) \\ &= Ev_1w_1 + Fv_1w_2 + Fv_2w_1 + Gv_2w_2 \\ &= (v_1, v_2)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \end{aligned}$$

3.1.10 Example. Let $M = S^1 \times \mathbb{R}$ be a cylinder over the unit circle. A parametrization of M is given by $\varphi : (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$, $\varphi(t,s) = (\cos t, \sin t, s)$. Then $\varphi_t = (-\sin t, \cos t, 0)$, $\varphi_s = (0,0,1)$, so $E = 1$, $F = 0$, $G = 1$. Therefore, $[I]_{\varphi(t,s)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

We now come back to the problem of determining the curvature of a hypersurface. We are looking for those directions in which the normal curvatures become extremal:



By (3.1.1) we are therefore looking for the critical points of the map $w \mapsto \langle w, L_p w \rangle = \kappa(w)$ for $w \in S^{n-1}$.

3.1.11 Theorem. *(Rodriguez) The critical points of the normal curvature in $p \in M$ are precisely the eigenvectors of the symmetric linear map L_p. If w is such an eigenvector, then the corresponding eigenvalue λ is given by $\kappa(w)$.*

Proof. Let $w \in T_pM$ with $\|w\| = 1$. w is a critical point of κ if and only if $v \mapsto \kappa(v) : S^{n-1} \to \mathbb{R}$ has a critical point in w. This we determine using the method of Lagrange multipliers. Let $g : v \mapsto \langle v, v \rangle - 1$. Our problem then is to find an extremal of κ on $\{g = 0\}$.
Therefore, $D\kappa(w) = \lambda Dg(w)$ has to be satisfied for some Lagrange multiplier λ (on $\{g = 0\}$). By (3.1.1),

$$D\kappa(w)(v) = \langle v, L_p w \rangle + \langle w, L_p v \rangle \overset{3.1.7}{=} 2\langle v, L_p w \rangle$$

and $Dg(w)(v) = 2\langle v, w \rangle$. Thus the equation $2\langle v, L_p w \rangle = 2\lambda \langle v, w \rangle$ has to hold for all v, i.e., $L_p w = \lambda w$ and $g(w) = 0$ ($\Leftrightarrow \|w\| = 1$). Thus $w \in S^{n-1}$ is a critical point iff w is an eigenvector of L_p. Furthermore,

$$\lambda = \lambda\langle w, w\rangle = \langle w, L_p w\rangle = \kappa(w)$$

$\square$

3.1.12 Definition. *Let M be a hypersurface in $\mathbb{R}^n$ and $p \in M$. The eigenvalues of L_p are called* principal curvatures. *The corresponding eigenvectors are called* principal curvature directions. *Since L_p is symmetric, all principal curvatures are real and there exists an orthonormal basis of T_pM $\{w_i \mid 1 \leq i \leq n-1\}$ consisting of principal curvature directions. A curve c is called a* line of curvature *if $c'(t)$ is a principal curvature direction for all t. The* Gaussian curvature *K of M is defined as the product of the principal curvatures, i.e., $K = \prod_{i=1}^{n-1} \kappa_i$. The* mean curvature *of M in p is the arithmetic mean of the principal curvatures, i.e., $\frac{1}{n-1}\mathrm{tr}(L_p)$ (with $\mathrm{tr}(L_p)$ denoting the trace of L_p). p is called* umbilic, *if all principal curvatures coincide in p, i.e., if $L_p = \lambda \cdot \mathrm{id}_{T_pM}$. An umbilic point is called* level point, *if, in addition, $\lambda = 0$, i.e., if $L_p = 0$.*

3.1.13 Examples.

(i) Let $M = \nu^\perp$ be a hypersurface of $\mathbb{R}^n$. Then $\nu_p = \nu$ for all p and $L_p = T_p\nu = 0$ for all p. Thus all principal curvatures vanish and every point p of M is a level point.

(ii) Let $M = S^{n-1}$. For any $p \in M$, p itself is a normal vector to M, so $\nu = \mathrm{id}$ and $L_p = \mathrm{id}_{T_pM}$. Therefore, all points of M are umbilic and all tangent vectors are principal curvature directions.

Since the inception of differential geometry the distinction between intrinsic quantities which are determined entirely by M itself (which, in other words, are accessible to the inhabitants of M) and extrinsic quantities, where additional information is needed, has been a central object of study. As a rule, those quantities which can be formulated for abstract manifolds are intrinsic, whereas extrinsic quantities directly refer to the surrounding space, like, for example the Gauss map $p \mapsto \nu_p$. Although we have defined the Riemannian metric on hypersurfaces of $\mathbb{R}^n$ by using the scalar product of the surrounding $\mathbb{R}^n$, one can define Riemannian metrics also for abstract manifolds:

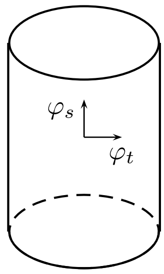
3.1.14 Definition. *Let M be an abstract manifold. A smooth $\binom{0}{2}$-tensor field $g \in \mathcal{T}^0_2(M)$ is called a* Riemannian metric *on M if $g_p : T_pM \times T_pM \to \mathbb{R}$ is a (positive definite) scalar product for all $p \in M$. (M, g) is then called a* Riemannian manifold. *If $f : (M, g) \to (N, h)$ is a (local) diffeomorphism of Riemannian manifolds such that $f^*h = g$, then f is called a (local)* isometry. *Two Riemannian manifolds are called (locally)* isometric *if there exists a (local) isometry $f : M \to N$.*

By 2.7.24, a (local) diffeomorphism f is a (local) isometry if and only if for all $p \in M$:

$$h_{f(p)}(T_pf \cdot v, T_pf \cdot w) = g_p(v, w) \qquad \forall v, w \in T_pM \tag{3.1.4}$$

Thus if we transport tangent vectors v, w by means of f (more precisely, T_pf) from M to N, then their lengths and their angle remain unchanged. For any given Riemannian metric, quantities like length and angles are intrinsic. We now pose the question which of the curvatures introduced so far are intrinsic. By the above, intrinsic notions have to remain unchanged under the action of local isometries.

3.1.15 Example. Let M be the cylinder from 3.1.10 and $\varphi : (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$, $\varphi(t, s) = (\cos t, \sin t, s)$. We consider $U := (0, 2\pi) \times \mathbb{R} \subseteq \mathbb{R}^2$ as a Riemannian manifold with the standard scalar product $g \equiv \langle\ ,\ \rangle$ of $\mathbb{R}^2$ and M as a Riemannian manifold with $h = I$ as in 3.1.10.



Then by 3.1.9 (iii) and 3.1.10, $\varphi : (U, g) \to (M, I)$ is a local isometry:

$$\begin{aligned} I_{\varphi(t,s)}(\underbrace{T_{(t,s)}\varphi}_{=(\varphi_t\ \varphi_s)} \cdot v, T_{(t,s)}\varphi \cdot w) &= I_{\varphi(t,s)}(v_1\varphi_t + v_2\varphi_s, w_1\varphi_t + w_2\varphi_s) = \\ &= (v_1, v_2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \binom{w_1}{w_2} \\ &= \langle \binom{v_1}{v_2}, \binom{w_1}{w_2} \rangle = \\ &= g_{(t,s)}(v, w) \end{aligned}$$

Hence M and U are locally isometric.
If $(x, s) \in M = S^1 \times \mathbb{R}$, then $\nu_p = (x, 0)$ and $T_pM = \{(v, s) \mid v \perp x\}$. Hence the Gauss map in p is given by $\nu_p = \mathrm{id} \times 0|_{S^1 \times \mathbb{R}}$ and therefore $L_p = \mathrm{id} \times 0$. Thus one principal curvature is 1 and the other is 0. Since on U (due to ν =const, hence $L \equiv 0$) all curvatures vanish it follows that neither the normal curvatures nor the mean curvature are intrinsic. The only remaining candidate for an intrinsic curvature therefore is the Gauss curvature K, which vanishes for both manifolds.

The Theorema Egregium of Gauss states that the Gauss curvature in fact is intrinsic. To prove this theorem we are going to derive a formula for K (in the case $M \subseteq \mathbb{R}^3$) which depends exclusively on the quantities E, F, G and their derivatives.

3.1.16 Lemma. *Let V be a vector space with basis $\mathcal{B} = \{g_1, \dots, g_m\}$, let $\langle\ ,\ \rangle$ be a scalar product on V and let $T : V \to V$ be linear. Denote by G the matrix with entries $\langle g_i, g_j \rangle$, by $[T]$ the matrix of T with respect to $\mathcal{B}$, and by A the matrix with entries $\langle Tg_i, g_j \rangle$. Then $[T] = (AG^{-1})^t$.*

Proof. We first show that G is invertible: let $\mathcal{B}^* = \{g^1, \dots, g^m\}$ be the dual basis to $\mathcal{B}$ and $\Phi : V \to V^*$ the linear isomorphism $v \mapsto \langle v, . \rangle$. Then $\Phi(g_i)(g_j) = \langle g_i, g_j \rangle =: g_{ij}$, so $\Phi(g_i) = \sum_j g_{ij} g^j$, and therefore G (which is symmetric) is the matrix of Φ with respect to $\mathcal{B}$ and $\mathcal{B}^*$. Due to $Tg_j = \sum_i T_{ij} g_i$ we have:

$$A_{jk} = \langle Tg_j, g_k \rangle = \sum_i T_{ij} \langle g_i, g_k \rangle = \sum_i (T^t)_{ji} g_{ik} = (T^t G)_{jk}$$

It follows that $A = T^t G$, so $[T] = (AG^{-1})^t$, as claimed. $\square$

3.1.17 Proposition. *Let φ be a local parametrization of a hypersurface M in $\mathbb{R}^3$. Let $\varphi_t = \frac{\partial\varphi}{\partial t}$, $\varphi_s = \frac{\partial\varphi}{\partial s}$, $\varphi_{tt} = \frac{\partial^2\varphi}{\partial t^2}$, etc. Set $E = \langle\varphi_t,\varphi_t\rangle$, $F = \langle\varphi_t,\varphi_s\rangle$, $G = \langle\varphi_s,\varphi_s\rangle$, $e := -\langle\nu,\varphi_{tt}\rangle$, $f := -\langle\nu,\varphi_{ts}\rangle$, $g = -\langle\nu,\varphi_{ss}\rangle$. Then the following matrix representations are valid with respect to the basis $\{\varphi_t,\varphi_s\}$ of $T_{\varphi(t,s)}M$:*

$$[I] = \begin{pmatrix} E & F \\ F & G \end{pmatrix},\ II = \begin{pmatrix} e & f \\ f & g \end{pmatrix},\ [L] = \tfrac{1}{EG-F^2}\begin{pmatrix} eG-fF & fE-eF \\ fG-gF & gE-fF \end{pmatrix}$$

Finally, the Gauss curvature K of M is given by $K = \frac{eg-f^2}{EG-F^2}$.

Proof. The matrix representation of I was already derived in 3.1.9 (iii). Concerning L, for any hypersurface in $\mathbb{R}^n$ we have

$$0 = \langle \underbrace{\nu\circ\varphi}_{\in T_pM^\perp}, \underbrace{\frac{\partial\varphi}{\partial x^i}}_{\in T_pM} \rangle \stackrel{\frac{\partial}{\partial x^j}}{\Rightarrow} 0 = \langle L\cdot\frac{\partial\varphi}{\partial x^j},\frac{\partial\varphi}{\partial x^i}\rangle + \langle\nu\circ\varphi,\frac{\partial^2\varphi}{\partial x^i\partial x^j}\rangle$$

$$\Rightarrow \langle L\frac{\partial\varphi}{\partial x^j},\frac{\partial\varphi}{\partial x^i}\rangle = -\langle\nu\circ\varphi,\frac{\partial^2\varphi}{\partial x^i\partial x^j}\rangle. \tag{3.1.5}$$

Since $L = L^t$ it follows from 3.1.16 that

$$[L] = -(\langle\nu\circ\varphi,\frac{\partial^2\varphi}{\partial x^i\partial x^j}\rangle)\cdot(\langle\frac{\partial\varphi}{\partial x^i},\frac{\partial\varphi}{\partial x^j}\rangle)^{-1} \tag{3.1.6}$$

In our case ($n = 3$) it follows from (3.1.5) that $e = \langle L\varphi_t,\varphi_t\rangle$, $f = \langle L\varphi_s,\varphi_t\rangle$, and $g = \langle L\varphi_s,\varphi_s\rangle$. Thus if $v = v_1\varphi_t + v_2\varphi_s$, $w = w_1\varphi_t + w_2\varphi_s \in T_pM$, then

$$\begin{aligned} II(v,w) &= \langle Lv,w\rangle \\ &= v_1w_1\langle L\varphi_t,\varphi_t\rangle + v_1w_2\langle L\varphi_t,\varphi_s\rangle + v_2w_1\langle L\varphi_s,\varphi_t\rangle + v_2w_2\langle L\varphi_s,\varphi_s\rangle = \\ &= (v_1,v_2)\begin{pmatrix} e & f \\ f & g\end{pmatrix}\begin{pmatrix} w_1 \\ w_2\end{pmatrix} \Rightarrow [II] = \begin{pmatrix} e & f \\ f & g\end{pmatrix} \end{aligned}$$

By (3.1.6) we conclude that

$$[L] = \begin{pmatrix} e & f \\ f & g\end{pmatrix}\begin{pmatrix} E & F \\ F & G\end{pmatrix}^{-1} = \dots = \frac{1}{EG-F^2}\begin{pmatrix} eG-fF & fE-eF \\ fG-gF & gE-fF \end{pmatrix},$$

so $K = \det L = \dots = \frac{eg-f^2}{EG-F^2}$. □

In the case $n = 3$, using the vector product, we may calculate ν directly:

$$\nu = \frac{\varphi_t\times\varphi_s}{\|\varphi_t\times\varphi_s\|}.$$

Then $\|\varphi_t\times\varphi_s\| = \sqrt{EG-F^2}$, implying:

$$e = -\langle\nu,\varphi_{tt}\rangle = -\frac{1}{\sqrt{EG-F^2}}\langle\varphi_t\times\varphi_s,\varphi_{tt}\rangle = -\frac{1}{\sqrt{EG-F^2}}\det(\varphi_t,\varphi_s,\varphi_{tt}).$$

Analogously,

$$f = -\frac{1}{\sqrt{EG-F^2}}\det(\varphi_t,\varphi_s,\varphi_{ts}),\ g = -\frac{1}{\sqrt{EG-F^2}}\det(\varphi_t,\varphi_s,\varphi_{ss}).$$

Set $D := \sqrt{EG - F^2}$. Then

$$
\begin{aligned}
K \cdot D^4 &\overset{3.1.17}{=} D^2(eg - f^2) \\
&= (-eD)(-gD) - (-fD)^2 \\
&= \det(\varphi_t, \varphi_s, \varphi_{tt}) \det(\varphi_t, \varphi_s, \varphi_{ss}) - \det(\varphi_t, \varphi_s, \varphi_{ts})^2 \\
&= \det((\varphi_t, \varphi_s, \varphi_{tt})^t) \det(\varphi_t, \varphi_s, \varphi_{ss}) \\
&\qquad\qquad - \det((\varphi_t, \varphi_s, \varphi_{ts})^t) \det(\varphi_t, \varphi_s, \varphi_{ts}) \\
&= \det \begin{pmatrix} \overbrace{\langle \varphi_t, \varphi_t \rangle}^{E} & \overbrace{\langle \varphi_t, \varphi_s \rangle}^{F} & \langle \varphi_t, \varphi_{ss} \rangle \\ \underbrace{\langle \varphi_s, \varphi_t \rangle}_{F} & \underbrace{\langle \varphi_s, \varphi_s \rangle}_{G} & \langle \varphi_s, \varphi_{ss} \rangle \\ \langle \varphi_{tt}, \varphi_t \rangle & \langle \varphi_{tt}, \varphi_s \rangle & \langle \varphi_{tt}, \varphi_{ss} \rangle \end{pmatrix} \\
&\qquad\qquad - \det \begin{pmatrix} \overbrace{\langle \varphi_t, \varphi_t \rangle}^{E} & \overbrace{\langle \varphi_t, \varphi_s \rangle}^{F} & \langle \varphi_t, \varphi_{ts} \rangle \\ \underbrace{\langle \varphi_s, \varphi_t \rangle}_{F} & \underbrace{\langle \varphi_s, \varphi_s \rangle}_{G} & \langle \varphi_s, \varphi_{ts} \rangle \\ \langle \varphi_{ts}, \varphi_t \rangle & \langle \varphi_{ts}, \varphi_s \rangle & \langle \varphi_{ts}, \varphi_{ts} \rangle \end{pmatrix} \\
&= \det \begin{pmatrix} E & F & \langle \varphi_t, \varphi_{ss} \rangle \\ F & G & \langle \varphi_s, \varphi_{ss} \rangle \\ \langle \varphi_{tt}, \varphi_t \rangle & \langle \varphi_{tt}, \varphi_s \rangle & \langle \varphi_{tt}, \varphi_{ss} \rangle - \langle \varphi_{ts}, \varphi_{ts} \rangle \end{pmatrix} \\
&\qquad\qquad - \det \begin{pmatrix} E & F & \langle \varphi_t, \varphi_{ts} \rangle \\ F & G & \langle \varphi_s, \varphi_{ts} \rangle \\ \langle \varphi_{ts}, \varphi_t \rangle & \langle \varphi_{ts}, \varphi_s \rangle & 0 \end{pmatrix}
\end{aligned}
$$

where we have developed with respect to the third row in the last step. Now

$$
\begin{array}{lll}
E_t = 2\langle \varphi_{tt}, \varphi_t \rangle & F_t = \langle \varphi_{tt}, \varphi_s \rangle + \langle \varphi_t, \varphi_{st} \rangle & G_t = 2\langle \varphi_{st}, \varphi_s \rangle \\
E_s = 2\langle \varphi_{ts}, \varphi_t \rangle & F_s = \langle \varphi_{ts}, \varphi_s \rangle + \langle \varphi_t, \varphi_{ss} \rangle & G_s = 2\langle \varphi_{ss}, \varphi_s \rangle \\
E_{ss} = 2(\langle \varphi_{tss}, \varphi_t \rangle + \langle \varphi_{ts}, \varphi_{ts} \rangle) & F_{ts} = \langle \varphi_{tst}, \varphi_s \rangle + \langle \varphi_{tt}, \varphi_{ss} \rangle + \frac{1}{2} E_{ss} & \\
F_{ts} - \frac{1}{2}(E_{ss} + G_{tt}) = \langle \varphi_{tt}, \varphi_{ss} \rangle - \langle \varphi_{st}, \varphi_{st} \rangle & &
\end{array}
$$

Summing up, we obtain :

$$
KD^4 = \det \begin{pmatrix} E & F & F_s - \frac{1}{2} G_t \\ F & G & \frac{1}{2} G_s \\ \frac{1}{2} E_t & F_t - \frac{1}{2} E_s & F_{ts} - \frac{1}{2}(E_{ss} + G_{tt}) \end{pmatrix} - \det \begin{pmatrix} E & F & \frac{1}{2} E_s \\ F & G & \frac{1}{2} G_t \\ \frac{1}{2} E_s & \frac{1}{2} G_t & 0 \end{pmatrix}
$$

Hence,

$$
K \text{ is a function of } E,\ F,\ G \text{ and their derivatives (up to order 2).} \tag{3.1.7}
$$

Based on this we are finally in a position to prove

3.1.18 Theorem. *(Theorema Egregium, Gauss, 1827) The Gauss curvature K is intrinsic. Locally isometric hypersurfaces in $\mathbb{R}^3$ have the same Gauss curvature in corresponding points.*

Proof. K is intrinsic by (3.1.7). Let $A : M \to N$ be a local isometry of hypersurfaces M, N in $\mathbb{R}^3$, $p_0 \in M$ and φ a local parametrization of M around p_0. Then $\psi := A \circ \varphi$ is a local parametrization of N around $A(p_0)$. Since A is a local isometry, $\langle T_pA \cdot v, T_pA \cdot w \rangle = \langle v, w \rangle\ \forall v, w \in T_pM$ (cf. (3.1.4)). In particular, let

$v = T_{(t,s)}\varphi(e_1) = \varphi_t$, $w = T_{(t,s)}\varphi(e_2) = \varphi_s$, $p = \varphi(t,s)$. Then:

$$\begin{aligned}\langle\varphi_t,\varphi_t\rangle &= \langle v,v\rangle = \langle T_pAv, T_pAv\rangle \\ &= \langle \underbrace{T_{\varphi(t,s)}A\circ T_{(t,s)}\varphi}_{=T_{(t,s)}(A\circ\varphi)=T_{(t,s)}\psi}(e_1), T_{\varphi(t,s)}A\circ T_{(t,s)}\varphi(e_1)\rangle \\ &= \langle\psi_t,\psi_t\rangle,\end{aligned}$$

so $E^\varphi\big|_{\varphi(t,s)} = E^\psi\big|_{\psi(t,s)}$ and analogously for F and G. Hence by (3.1.7), $K(p) = K(A(p))$. □

3.2 Covariant Derivatives

Throughout this section we will assume M to be an oriented hypersurface in $\mathbb{R}^n$. The directional derivative of a smooth map $f : U \to \mathbb{R}$ ($U \subseteq \mathbb{R}^n$ open) in the direction $v \in \mathbb{R}^n$ (the rate of change of f in the direction of v) is

$$D_vf(x) = \lim_{t\to 0}\frac{1}{t}(f(x+tv) - f(x)) = \frac{d}{dt}\Big|_0 f(x+tv) = Df(x)\cdot v.$$

Let M be a manifold, $f \in \mathcal{C}^\infty(M)$, $v \in T_pM$. Then analogously (cf. (2.4.3)):

$$\partial_v f = v(f) = T_pf(v).$$

In particular, if M is a submanifold of $\mathbb{R}^n$ and $f \in \mathcal{C}^\infty(M)$, we may choose a smooth extension $\tilde f$ of f to a neighborhood of p in $\mathbb{R}^n$. Then

$$\partial_v f = T_pf(v) = D\tilde f(p)\cdot v =: D_vf(p). \tag{3.2.1}$$

Again we call D_vf the directional derivative of f in the direction v.
Analogously we want to study the rate of change of a vector field in the direction of a tangent vector. Let M be a submanifold of $\mathbb{R}^n$, $Y \in \mathfrak{X}(M)$, and $v \in T_pM$. By 2.5.21, $Y : M \to \mathbb{R}^n$ is smooth and by 3.1.6 there exists a smooth extension $\tilde Y$ of Y to a neighborhood in $\mathbb{R}^n$ of any given point of M. We set

$$D_vY(p) := T_pY(v) = \lim_{t\to 0}\frac{1}{t}(\tilde Y(p+tv) - \tilde Y(p)) = \frac{d}{dt}\Big|_0 \tilde Y(p+tv). \tag{3.2.2}$$

D_vY is called the directional derivative of Y in the direction v. If $\tilde Y = (\tilde Y^1,\dots,\tilde Y^n)$, then $D\tilde Y(p) = (D\tilde Y^1(p),\dots,D\tilde Y^n(p))^t$, so by 2.4.2

$$D_vY(p) = D\tilde Y(p)\cdot v = (\underbrace{D\tilde Y^1(p)\cdot v}_{=T_pY^1(v)},\dots,\underbrace{D\tilde Y^n(p)\cdot v}_{T_pY^n(v)}) = (v(Y^1),\dots,v(Y^n)). \tag{3.2.3}$$

D_vY is the rate of change of Y in the direction v. Note, however, that $D_vY(p) \notin T_pM$ in general! If $X \in \mathfrak{X}(M)$, then let

$$D_XY := p \mapsto D_{X_p}Y(p) \tag{3.2.4}$$

be the directional derivative of Y in the direction X. By (3.2.3) we have

$$D_XY(p) = D\tilde Y(p)X(p),$$

so $D_XY \in \mathcal{C}^\infty(M,\mathbb{R}^n)$. In general, however, $D_XY \notin \mathfrak{X}(M)$ (since $D_XY(p) \notin T_pM$ in general).

3.2.1 Example. Let φ be a local parametrization of M and $\psi = (x^1, \dots, x^{n-1}) = \varphi^{-1}$ the corresponding chart. Then (with $x := \varphi^{-1}(p)$), $\frac{\partial}{\partial x^j}\big|_p = D_j\varphi(x)$. With Φ as in 2.1.8, $(T) \Rightarrow (P)$, $\big(\frac{\partial}{\partial x^j}\big)^{\sim} := q \mapsto D_j\Phi(\Phi^{-1}(q))$ is a smooth extension of $\frac{\partial}{\partial x^j}$ to a neighborhood of p in $\mathbb{R}^n$. Hence, recalling that $D_i\varphi(x) = D_i\Phi(x) = D\Phi(x)\cdot e_i$, we obtain

$$\begin{aligned} D_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} &= \frac{d}{dt}\Big|_0 D_j\Phi(\Phi^{-1}(p + tD_i\varphi(x))) = D(D_j\Phi)(x)\cdot D\Phi^{-1}(p)\cdot D\Phi(x)\cdot e_i \\ &= D(D_j\Phi)(x)\cdot e_i = D_{ij}\Phi(x) = \frac{\partial^2}{\partial x^i\partial x^j}\varphi(x). \end{aligned}$$

To obtain an intrinsic quantity from D_XY, we project it orthogonally onto T_pM:

3.2.2 Definition. *Let M be an oriented hypersurface in $\mathbb{R}^n$, $X,\ Y \in \mathfrak{X}(M)$. The* covariant derivative *of Y in the direction X is defined as the tangent part of D_XY:*

$$\nabla_XY := (D_XY)^{tang} = D_XY - \langle D_XY, \nu\rangle\nu.$$

For $f \in \mathcal{C}^\infty(M)$ we set $\nabla_Xf := D_Xf$.

3.2.3 Proposition. *Let M be an oriented hypersurface in $\mathbb{R}^n$, $X,\ Y \in \mathfrak{X}(M)$. Then:*

(i) $\nabla_XY \in \mathfrak{X}(M)$.

(ii) The normal part of D_XY is $(D_XY)^{nor} = \langle D_XY, \nu\rangle\cdot\nu = -II(X,Y)\cdot\nu$.

(iii) $D_XY = \nabla_XY - II(X,Y)\cdot\nu$ (Gauss equation).

Proof. (i) The smoothness of $\nabla_XY : M \to \mathbb{R}^n$ follows from that of D_XY and ν. Since clearly $\nabla_XY(p) \in T_pM$ for all p, $\nabla_XY \in \mathfrak{X}(M)$ by 2.5.21.
(ii) Let φ be a local parametrization of M and

$$f(x) := \langle \underbrace{Y\circ\varphi(x)}_{\in T_pM}, \underbrace{\nu\circ\varphi(x)}_{\in T_pM^\perp}\rangle \equiv 0.$$

Let $v \in T_pM$. Then by 2.4.1 (i) there exists some $w \in \mathbb{R}^{n-1}$ with $v = D\varphi(x)\cdot w$ ($\varphi(x) = p$). Hence

$$\begin{aligned} 0 &= Df(x)\cdot w = \langle D(Y\circ\varphi)(x)\cdot w, \nu\circ\varphi(x)\rangle + \langle Y\circ\varphi(x), D(\nu\circ\varphi)(x)\cdot w\rangle = \\ &= \langle D\tilde{Y}(p)\underbrace{D\varphi(x)\cdot w}_{=v}, \nu(p)\rangle + \langle Y(p), \underbrace{D\tilde{\nu}(p)}_{=L_p}\cdot\underbrace{D\varphi(x)\cdot w}_{=v}\rangle = \\ &= \langle D_vY(p), \nu_p\rangle + \langle Y_p, L_pv\rangle. \end{aligned}$$

In particular, for $v = X_p$ we conclude:

$$\langle D_{X_p}Y(p), \nu_p\rangle = -\langle Y_p, L_pX_p\rangle = -II(X_p, Y_p).$$

(iii) is immediate from 3.2.2 and (ii). □

3.2.4 Lemma. *Let $X,\ Y,\ X_i,\ Y_i \in \mathfrak{X}(M)$, M an oriented hypersurface in $\mathbb{R}^n$. Let $f \in \mathcal{C}^\infty(M)$, $\alpha \in \mathbb{R}$. Then:*

(i) $D_{fX_1+X_2}Y = fD_{X_1}Y + D_{X_2}Y$
$\nabla_{fX_1+X_2}Y = f\nabla_{X_1}Y + \nabla_{X_2}Y$

(ii) $D_X(\alpha Y_1 + Y_2) = \alpha D_X Y_1 + D_X Y_2$
$\nabla_X(\alpha Y_1 + Y_2) = \alpha \nabla_X Y_1 + \nabla_X Y_2$

(iii) $D_X(fY) = f D_X Y + D_X f \cdot Y$
$\nabla_X(fY) = f \nabla_X Y + \nabla_X f \cdot Y$

(iv) $D_X \langle Y_1, Y_2\rangle = \langle D_X Y_1, Y_2\rangle + \langle Y_1, D_X Y_2\rangle$
$\nabla_X \langle Y_1, Y_2\rangle = \langle \nabla_X Y_1, Y_2\rangle + \langle Y_1, \nabla_X Y_2\rangle$

Proof. Since $D_X Y(p) = D\tilde{Y}(p) \cdot X_p$ and $D_X f(p) = D\tilde{f}(p) \cdot X_p$, (i)–(iv) for D follow directly from the usual rules of differentiation. By means of 3.2.2, (i)–(iii) for ∇ follow from the corresponding properties of D. Finally,

$$\begin{aligned}\nabla_X \langle Y_1, Y_2\rangle &= D_X\langle Y_1, Y_2\rangle \\ &= \langle D_X Y_1, Y_2\rangle + \langle Y_1, D_X Y_2\rangle \\ &\overset{3.2.2,\ \nu\perp Y_1,Y_2}{=} \langle \nabla_X Y_1, Y_2\rangle + \langle Y_1, \nabla_X Y_2\rangle.\end{aligned}$$

□

3.2.5 Proposition. *Let M be a hypersurface, $X,\ Y \in \mathfrak{X}(M)$. Then:*

$$[X, Y] = D_X Y - D_Y X = \nabla_X Y - \nabla_Y X$$

Proof. $[X, Y] = D_X Y - D_Y X$ follows from 3.1.6 (ii) and (3.2.4). Moreover,

$$\nabla_X Y - \nabla_Y X = D_X Y - D_Y X \underbrace{-\langle D_X Y, \nu\rangle \cdot \nu + \langle D_Y X, \nu\rangle \cdot \nu}_{=0 \text{ by} 3.2.3(ii), \text{ since } II \text{ symm.}}.$$

□

To show that the covariant derivative ∇ is intrinsic it suffices to show that it can be expressed solely by means of the first fundamental form, i.e., the Riemannian metric. To do this we first derive some local formulas. Let φ be a local parametrization of M, and $(\psi = \varphi^{-1} = (x^1, \dots, x^{n-1}), V)$ the corresponding chart. Then (with $x = \varphi^{-1}(p)$), $\frac{\partial}{\partial x^i}\big|_p = D_i\varphi(x)$ and by (3.1.3) we have:

$$g_{ij}(p) = \langle \frac{\partial}{\partial x^i}\bigg|_p, \frac{\partial}{\partial x^j}\bigg|_p \rangle = \langle D_i\varphi(x), D_j\varphi(x)\rangle.$$

Let $X,\ Y \in \mathfrak{X}(M)$ with local representations $X = \sum_{i=1}^{n-1} X^i D_i\varphi$, $Y = \sum_{j=1}^{n-1} Y^j D_j\varphi$. Then according to 3.2.4,

$$\begin{aligned}\nabla_X Y &= \sum_{i=1}^{n-1} X^i \nabla_{\frac{\partial}{\partial x^i}} Y = \sum_{i,j=1}^{n-1} X^i \nabla_{\frac{\partial}{\partial x^i}} (Y^j \frac{\partial}{\partial x^j}) \\ &= \sum_{i,j=1}^{n-1} X^i (\frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} + Y^j \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}).\end{aligned}$$

$\nabla_X Y$ is uniquely determined by all scalar products $\langle \nabla_X Y, \frac{\partial}{\partial x^k}\rangle$ (cf. 3.1.16). It therefore suffices to show that all

$$\langle \nabla_X Y, \frac{\partial}{\partial x^k}\rangle = \sum_{i,j=1}^{n-1} X^i (\frac{\partial Y^j}{\partial x^i} g_{jk} + Y^j \langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\rangle)$$

are intrinsic, i.e., depend exclusively on g. In this expression, the

$$\Gamma_{ij,k} := \langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \rangle \tag{3.2.5}$$

are called *Christoffel symbols of the first kind*. Since $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$ $\forall i, j$ (cf. 2.5.15) and using 3.2.5,

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}, \qquad (*)$$

so $\Gamma_{ij,k} = \Gamma_{ji,k}$ $\forall i, j, k$. Since $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \in \mathfrak{X}(V)$ there exist smooth functions Γ^k_{ij} such that

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^{n-1} \Gamma^k_{ij} \frac{\partial}{\partial x^k}. \tag{3.2.6}$$

The Γ^k_{ij} are called *Christoffel symbols of the second kind*. They, too, are symmetric in i, j by $(*)$: $\Gamma^k_{ij} = \Gamma^k_{ji}$ $\forall i, j, k$. By (3.2.5) and (3.2.6) it follows that

$$\Gamma_{ij,k} = \sum_{m=1}^{n-1} \Gamma^m_{ij} g_{mk}. \tag{3.2.7}$$

It remains to show that the $\Gamma_{ij,k}$ are intrinsic, i.e., depend only on g. We have:

$$\begin{aligned}
\frac{\partial}{\partial x^k} g_{ij} \quad &= \quad D_{\frac{\partial}{\partial x^k}} g_{ij} = \nabla_{\frac{\partial}{\partial x^k}} g_{ij} = \nabla_{\frac{\partial}{\partial x^k}} \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle \\
&\overset{3.2.4(iv)}{=} \langle \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle + \langle \frac{\partial}{\partial x^i}, \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j} \rangle \overset{(3.2.5)}{=} \Gamma_{ik,j} + \Gamma_{jk,i}.
\end{aligned}$$

Cyclic permutation gives

$$\begin{aligned}
\frac{\partial}{\partial x^i} g_{jk} &= \Gamma_{ji,k} + \Gamma_{ki,j} \\
\frac{\partial}{\partial x^j} g_{ki} &= \Gamma_{kj,i} + \Gamma_{ij,k},
\end{aligned}$$

from which by adding resp. subtracting we obtain

$$\Gamma_{ij,k} = \frac{1}{2}\left(-\frac{\partial}{\partial x^k} g_{ij} + \frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ki}\right).$$

This expression in fact depends exclusively on g. Thus we have proved:

3.2.6 Theorem. *The covariant derivative is an intrinsic quantity.*

□

3.2.7 Remark. Keeping the above notations, let $II(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) =: h_{ij}$. Then with $p = \varphi(x)$ we have:

$$\begin{aligned}
D_{ij}\varphi(x) \quad &\overset{3.2.1}{=} \quad D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}(p) \overset{3.2.3(iii)}{=} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\Big|_p - h_{ij}|_p \nu_p \\
&\overset{(3.2.6)}{=} \sum_k \Gamma^k_{ij}(p) \frac{\partial}{\partial x^k}\Big|_p - h_{ij}\big|_p \cdot \nu_p
\end{aligned} \tag{3.2.8}$$

3.3 Geodesics

A vector field Y on $\mathbb{R}^n$ is constant iff $DY = 0$, iff $D_X Y = 0$ for all vector fields X on $\mathbb{R}^n$. Since $Y_p \in T_p\mathbb{R}^n \cong \mathbb{R}^n$ this is equivalent to all Y_p being parallel (and of equal length). For hypersurfaces we analogously define:

3.3.1 Definition. *Let M be a hypersurface in $\mathbb{R}^n$, $Y \in \mathfrak{X}(M)$. Y is called* parallel*, if $\nabla_X Y = 0$ for all $X \in \mathfrak{X}(M)$.*

Geodesics in $\mathbb{R}^n$, i.e., straight lines, have the property that their tangent vectors are always parallel along the straight line. To generalize this notion, we need the following concept:

3.3.2 Definition. *Let $M \subseteq \mathbb{R}^n$ be a hypersurface and $c : I \to M$. A smooth map $X : I \to \mathbb{R}^n$ is called a vector field along c if $X(t) \in T_{c(t)}M$ for all $t \in I$. The space of all vector fields along c is denoted by $\mathfrak{X}(c)$.*

3.3.3 Example. Let $c : I \to M$ $\mathcal{C}^\infty$. Then $\dot{c} : I \to \mathbb{R}^n \in \mathfrak{X}(c)$ ($\dot{c}(t) \in T_{c(t)}M$ $\forall t$).

Let $Y \in \mathfrak{X}(M)$, $p \in M$ and $\tilde{Y}$ a smooth extension of Y to some neighborhood of p in $\mathbb{R}^n$. Let $v \in T_pM$. Then by (3.2.2), $D_vY(p) = D\tilde{Y}(p) \cdot v$. If $c : I \to M$ is a curve with $c(0) = p$ and $c'(0) = v$, then

$$\left.\frac{d}{dt}\right|_0 Y(c(t)) = \left.\frac{d}{dt}\right|_0 \tilde{Y}(c(t)) = D\tilde{Y}(\underbrace{c(0)}_{=p})\underbrace{\dot{c}(0)}_{=v} = D_vY(p).$$

To determine $D_vY(p)$ it therefore suffices to know Y along any such curve c. Hence the same is true for $\nabla_v Y(p)$. If $Y \in \mathfrak{X}(M)$ and $c : I \to M$ is smooth, then by the above, $D_{\dot{c}(t)}Y = \frac{d}{dt}(Y \circ c)$. If $Y \in \mathfrak{X}(c)$, we analogously define:

$$D_{\dot{c}(t)}Y(t) := \frac{d}{dt}Y(t) \tag{3.3.1}$$

and

$$\nabla_{\dot{c}(t)}Y(t) := (D_{\dot{c}(t)}Y(t))^{\text{tang}} = D_{\dot{c}(t)}Y(t) - \langle D_{\dot{c}(t)}Y(t), \nu(c(t))\rangle\nu(c(t)). \tag{3.3.2}$$

3.3.4 Lemma. *Let φ be a local parametrization, $\varphi^{-1} = (x^1, \dots, x^{n-1})$, and $c = \varphi \circ u$ a smooth curve in M with local representation $t \mapsto u(t)$. Let $Y \in \mathfrak{X}(c)$ with $Y(t) = \sum_{i=1}^{n-1} Y^i(t) \left.\frac{\partial}{\partial x^i}\right|_{c(t)}$. Then:*

$$\nabla_{\dot{c}(t)}Y(t) = \sum_{k=1}^{n-1}\left(\frac{dY^k}{dt} + \sum_{i,j=1}^{n-1} Y^i(t)\frac{du^j}{dt}\Gamma_{ij}^k(c(t))\right)\left.\frac{\partial}{\partial x^k}\right|_{c(t)}.$$

Proof. Since $Y(t) = \sum_{i=1}^{n-1} Y^i(t)D_i\varphi(\underbrace{\varphi^{-1}(c(t))}_{=u(t)})$, by (3.3.1) we have

$$\begin{aligned} D_{\dot{c}(t)}Y(t) &= \sum_{i=1}^{n-1}\frac{dY^i(t)}{dt}D_i\varphi(u(t)) + \sum_{i,j=1}^{n-1}Y^i(t)D_{ij}\varphi(u(t))\frac{du^j}{dt} \\ &\overset{(3.2.8)}{=} \sum_{k=1}^{n-1}\left(\frac{dY^k(t)}{dt} + \sum_{i,j=1}^{n-1}Y^i(t)\frac{du^j}{dt}\Gamma_{ij}^k(\underbrace{\varphi\circ u(t)}_{=c(t)})\right)\left.\frac{\partial}{\partial x^k}\right|_{c(t)} + (\dots)\cdot\nu. \end{aligned}$$

The claim therefore follows from (3.3.2). □

3.3.5 Definition. *A non-constant curve $c : I \to M$ is called geodesic, if $\nabla_{\dot c(t)}\dot c(t) = 0$ for all t.*

Heuristically this means that $\dot c$ is parallelly transported along c, i.e., the curve goes as straight as the manifold allows. From the point of view of physics, note that $D_{\dot c(t)}\dot c(t) \overset{(3.3.1)}{=} \frac{d}{dt}\dot c(t) = \ddot c(t)$ is the acceleration of a point particle moving along c. $\nabla_{\dot c}\dot c$ is the tangential component of this acceleration. In this picture, a geodesic is a curve in M which feels no acceleration in (any direction tangent to) the hypersurface. The normal component of $\ddot c$ corresponds to the force ($F = m\ddot c$) which is needed to hold the particle within M. Hence c is a geodesic iff $\ddot c(t) \perp T_{c(t)}M$ for all t.

3.3.6 Proposition. *Let φ be a local parametrization, $\varphi^{-1} = (x^1, \dots, x^{n-1})$, and $c = \varphi \circ u$ a smooth curve. c is a geodesic if and only if it satisfies*

$$\ddot u^k(t) + \sum_{i,j=1}^{n-1} \dot u^i(t)\dot u^j(t)\Gamma^k_{ij}(\varphi(u(t))) = 0 \qquad \forall k. \tag{3.3.3}$$

Proof. This follows by applying 3.3.4 and 3.3.5 to $Y = \dot c$: $c = \varphi \circ u$, so

$$\dot c(t) = \sum_i D_i\varphi(u(t)) \cdot \dot u^i(t) = \sum_i \frac{\partial}{\partial x^i}\Big|_{c(t)} \dot u^i(t),$$

i.e., $Y^i(t) = \dot u^i(t)\ \forall i$. □

(3.3.3) is called the system of *geodesic equations*. It is a second order system of nonlinear ODEs. It always has local solutions, but not necessarily global ones.
The following result shows that geodesics are precisely the extremals of the arclength functional. Its proof uses standard methods of the calculus of variations (in fact it shows that the geodesic equations are the Euler-Lagrange equations of the problem of minimizing the arclength of curves connecting two given curves).

3.3.7 Theorem. *Let $p, q \in M$. A curve c connecting p and q is a geodesic if and only if its arclength is an extremal among all lengths of curves connecting p and q.*

Proof. Let $c : [a, b] \to M$, $c(a) = p$, $c(b) = q$. The length of c is extremal if $\frac{d}{ds}\big|_0 L(c^s) = 0$ for every family of curves $(t, s) \mapsto c^s(t) \equiv c(t, s)$ with $c^s(a) = p$, $c^s(b) = q\ \forall s$ and $c^0 = c$. Let c be parametrized by arclength. Then

$$\begin{aligned}
\frac{d}{ds}\Big|_0 L(c^s) &= \frac{d}{ds}\Big|_0 \int_a^b \underbrace{\|\frac{\partial}{\partial t}c(t,s)\|}_{=\langle \dot c(t,s), \dot c(t,s)\rangle^{\frac12}} = \\
&= \int_a^b \frac12 \frac{\frac{\partial}{\partial s}\big|_0 \langle \dot c(t,s), \dot c(t,s)\rangle}{\underbrace{\|\partial_t c(t,0)\|}_{=1}} dt = \\
&= \int_a^b \langle \frac{\partial}{\partial t}\frac{\partial}{\partial s}\Big|_0 c(t,s), \frac{\partial}{\partial t}c(t,0)\rangle dt = \\
&\overset{\text{p.I.}}{=} \underbrace{[\langle \frac{\partial}{\partial s}\Big|_0 c(t,s), \frac{\partial}{\partial t}c(t,0)\rangle]_a^b}_{=0,\text{ since } \begin{smallmatrix} c(b,s)\equiv q \\ c(a,s)\equiv p\end{smallmatrix}} - \int_a^b \langle \underbrace{\frac{\partial}{\partial s}\Big|_0 c(t,s)}_{=:\eta(t)\in T_{c(t,0)}M}, \frac{\partial^2}{\partial t^2}c(t,0)\rangle dt = \\
&= -\int_a^b \langle \eta(t), \ddot c(t)\rangle dt = (*)
\end{aligned}$$

If $L(c)$ is extremal, we may freely choose $c(t,s)$. In particular we may suppose that $\eta(t) = h(t)(\ddot{c}(t) - \langle \ddot{c}(t), \nu_{c(t)} \rangle \nu_{c(t)})$ with $h : [a,b] \to \mathbb{R}^+$ smooth, $h(a) = h(b) = 0$. Then

$$(*) = \int_a^b \underbrace{h(t)}_{\geq 0} (\underbrace{(\langle \ddot{c}, \nu \circ c \rangle)^2 - \langle \ddot{c}, \ddot{c} \rangle}_{\leq 0 \text{ by CSI, since } \|\nu\|=1})dt.$$

Since $\frac{d}{ds}\big|_0 L(c^s) = 0$, we must in fact have equality in the Cauchy Schwarz inequality (CSI) for all t. Therefore, $\ddot{c}(t)$ must be proportional to $\nu(c(t))$ for all t, so $\ddot{c}(t) \in T_{c(t)}M^\perp$, and c is a geodesic.

Conversely, if c is a geodesic then $\frac{d}{dt}(\|\dot{c}\|^2) = 2\langle \dot{c}, \ddot{c} \rangle \overset{\ddot{c} \in T_{c(t)}M^\perp}{=} 0$, so $\|\dot{c}\|$ is constant, implying that c is parametrized proportional to arclength. Hence the above calculation is applicable. Then in $(*)$ we have $\eta(t) \in T_{c(t)}M$, so $\langle \eta(t), \ddot{c}(t) \rangle = 0$ for all t. Therefore, $\frac{d}{ds}\big|_0 L(c^s) = 0$, i.e., $L(c)$ is extremal. $\square$

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