1.1 Submanifolds

In [3, Sec. 2.1] we introduced submanifolds of \( \mathbb{R}^n \); \( M \subseteq \mathbb{R}^n \) is called a submanifold of dimension \( k \) if for every \( p \in M \) there exists an open neighborhood \( W \) of \( p \) in \( \mathbb{R}^n \), an open subset \( U \) of \( \mathbb{R}^k \) and an immersion \( \phi : U \to \mathbb{R}^n \) such that \( \phi : U \to \phi(U) \) is a homeomorphism and \( \phi(U) = W \cap M \). Then \( \phi \) is called a local parametrisation of \( M \). By [3, 2.2.8], any such \( M \) is an abstract manifold whose natural manifold topology is precisely the trace topology of \( \mathbb{R}^n \) on \( M \).

We now want to introduce appropriate notions of submanifolds for abstract manifolds in general. To this end we first need a few results on maps between manifolds.

1.1.1 Definition. Let \( M, N \) be manifolds and let \( f : M \to N \) be smooth. The rank \( \text{rk}_p(f) \) of \( f \) at \( p \in M \) is the rank of the linear map \( T_pf : T_pM \to T_{f(p)}N \).

If \( \phi = (x^1, \ldots, x^m) \) is a chart of \( M \) at \( p \) and \( (y^1, \ldots, y^n) \) a chart of \( N \) at \( f(p) \) then the matrix of \( T_pf : T_pM \to T_{f(p)}N \) with respect to the bases \( (\frac{\partial}{\partial x^i}|_p, \ldots, \frac{\partial}{\partial x^m}|_p) \) of \( T_pM \) and \( (\frac{\partial}{\partial y^i}|_{f(p)}, \ldots, \frac{\partial}{\partial y^n}|_{f(p)}) \) is the Jacobi matrix of \( \phi \circ f \circ \phi^{-1} \) at \( f(p) \) (see [3, 2.4]). Thus \( \text{rk}_p(f) = \text{rk}_{\phi(p)}(\phi \circ f \circ \phi^{-1}) \).

1.1.2 Definition. Let \( f : M \to N \) be smooth. \( f \) is called immersion (submersion) if \( T_pf \) is injective (surjective) for every \( p \in M \).

If \( \text{dim}(M) = m \) and \( \text{dim}(N) = n \) (which henceforth we will indicate by writing \( M^m \) and \( N^n \), respectively) then \( f \) is an immersion (resp. submersion) if and only if \( \text{rk}_p(f) = m \) (resp. = \( n \)) for all \( p \in M \). The following result shows that maps of constant rank locally always are of a particularly simple form.

1.1.3 Theorem. (Rank Theorem) Let \( M^m, N^n \) be manifolds and let \( f : M \to N \) be smooth. Let \( p \in M \) and suppose that \( \text{rk}_p(f) = k \) in a neighborhood of \( p \). Then there exist charts \( (\phi, U) \) of \( M \) at \( p \) and \( (\psi, V) \) of \( N \) at \( f(p) \) such that \( \phi(p) = 0 \in \mathbb{R}^m \), \( \psi(f(p)) = 0 \in \mathbb{R}^n \)

\[ \psi \circ f \circ \phi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^k, 0, \ldots, 0). \]

Proof. By the above, the rank of \( f \) is independent of the chosen charts, so without loss of generality we may assume that \( f : W \to W' \), where \( W \) is open in \( \mathbb{R}^m \) and \( W' \) is open in \( \mathbb{R}^n \), \( p = 0 \), \( f(p) = 0 \) and \( \text{rk}(f) \equiv k \) on \( W \). Since \( \text{rk}(Df(0)) = k \) there exists an invertible \( k \times k \) submatrix of \( Df(0) \) and without loss we may assume that this matrix is given by \( \left( \frac{\partial f^j}{\partial x^i} \right)_{i,j=1}^k \). Now consider the smooth map \( \varphi : W \to \mathbb{R}^m \),

\[ \varphi(x^1, \ldots, x^m) = (f^1(x^1, \ldots, x^m), \ldots, f^k(x^1, \ldots, x^m), x^{k+1}, \ldots, x^m). \]

Then \( \varphi(0) = 0 \) and

\[ D\varphi(0) = \begin{pmatrix} \left( \frac{\partial f^j}{\partial x^i} \right)_{i,j=1}^k & * \\ 0 & I_{m-k} \end{pmatrix} \]

is invertible. By the inverse function theorem \( \varphi \) thereby is a diffeomorphism from some open neighborhood \( W_1 \subseteq W \) of \( 0 \) onto some open neighborhood \( U_1 \) of \( 0 \) in \( \mathbb{R}^m \). Then on \( U_1 \) we have

\[ f \circ \varphi^{-1}(x) = f \circ \varphi^{-1}(x^1, \ldots, x^k, x^{k+1}, \ldots, x^m) = (x^1, \ldots, x^k, f^{k+1}(x), \ldots, f^n(x)). \]
for suitable smooth functions \( f^{k+1}, \ldots, f^n \). Consequently,

\[
D(f \circ \varphi^{-1})(0) = \begin{pmatrix} I_k & 0 \\ \left( \frac{\partial f_r}{\partial x^s} \right)_{r=k+1, \ldots, m} \end{pmatrix}.
\]

Since \( D(f \circ \varphi^{-1}) = Df \circ D\varphi^{-1} \) and \( D\varphi^{-1} \) is bijective it follows that \( \text{rk}(D(f \circ \varphi^{-1})) = \text{rk}(Df) \equiv k \) on \( U_1 \). Then necessarily \( \frac{\partial f_r}{\partial x^s} = 0 \) for \( r = k+1, \ldots, n \) and \( s = k+1, \ldots, m \), i.e., \( f^{k+1}, \ldots, f^n \) depend only on \( x^1, \ldots, x^k \). Now set

\[
T(y^1, \ldots, y^k, y^{k+1}, \ldots, y^m) := (y^1, \ldots, y^k, y^{k+1}, \ldots, y^m + f^{k+1}(y^1, \ldots, y^k), \ldots, y^n + f^n(y^1, \ldots, y^k)).
\]

Then \( T(0) = 0 \) and

\[
DT(y) = \begin{pmatrix} I_k & 0 \\ 0 & I_{n-k} \end{pmatrix},
\]

so \( T \) is a diffeomorphism from some open neighborhood \( \tilde{V} \) of \( 0 \) in \( \mathbb{R}^n \) onto some open \( 0 \in V \subseteq W' \). Choose \( \tilde{U} \subseteq U_1 \) open such that \( f \circ \varphi^{-1}(\tilde{U}) \subseteq V \) and let \( U := \varphi^{-1}(\tilde{U}) \).

Let \( \psi := T^{-1} \), then

\[
\tilde{U} \xrightarrow{\varphi^{-1}} U \xrightarrow{f} V \xrightarrow{\psi} \tilde{V}
\]

and

\[
\psi \circ f \circ \varphi^{-1}(x^1, \ldots, x^k, x^{k+1}, \ldots, x^m) = \psi(x^1, \ldots, x^k, f^{k+1}(x^1, \ldots, x^k), \ldots, f^n(x^1, \ldots, x^k)) = (x^1, \ldots, x^k, 0, \ldots, 0)
\]
on \( \tilde{U} \).

\[
\square
\]

1.1.4 Lemma. Let \( f : M^m \to N^n \) be smooth, let \( p \in M \) and suppose that \( \text{rk}_q(f) = k \). Then there exists a neighborhood \( U \) of \( p \) in \( M \) such that \( \text{rk}_q(f) \geq k \) for all \( q \in U \). In particular, if \( k = \min(m, n) \) then \( \text{rk}_q(f) = k \) for all \( q \in U \).

Proof. Picking charts \( \varphi \) around \( p \) and \( \psi \) around \( f(p) \), \( \text{rk}_p(f) = k \) if and only if there exists a \( k \times k \)-submatrix of \( (D(\psi \circ f \circ \varphi^{-1})) \) with nonzero determinant. By continuity, the same is then true on an entire neighborhood of \( p \). This means that the rank cannot drop locally. If \( k = \min(m, n) \) then it also cannot increase. \( \square \)

1.1.5 Theorem. (Inverse function theorem) Let \( f : M^m \to N^n \) be smooth, let \( p \in M \) and suppose that \( T_p f : T_p M \to T_{f(p)} N \) is bijective. Then there exist open neighborhoods \( U \) of \( p \) in \( M \) and \( V \) of \( f(p) \) in \( N \) such that \( f : U \to V \) is a diffeomorphism.

Proof. For charts \( \varphi \) of \( M \) at \( p \), and \( \psi \) at \( f(p) \) in \( N \) the map \( D(\psi \circ f \circ \varphi^{-1})(\varphi(p)) = T_{f(p)} \psi \circ T_p f \circ T_{\varphi(p)} \varphi^{-1} \) is invertible. Hence by the classical inverse function theory, \( \psi \circ f \circ \varphi^{-1} \) is a diffeomorphism around \( \varphi(p) \) and the claim follows. \( \square \)

1.1.6 Proposition. (Local characterization of immersions) Let \( f : M^m \to N^n \) be smooth and let \( p \in M \). TFAE:

(i) \( T_p f \) is injective.
(ii) \( \text{rk}_p(f) = m \).

(iii) If \( \psi = (\psi^1, \ldots, \psi^n) \) is a chart at \( f(p) \) in \( N \) then there exist \( 1 \leq i_1 < \cdots < i_m \leq n \) such that \( (\psi^{i_1}, \ldots, \psi^{i_m}) \) is a chart at \( p \) in \( M \).

**Proof.** Clearly, (i)\(\Rightarrow\) (ii).

(ii)\(\Rightarrow\) (iii): Let \( \varphi \) be a chart at \( p \) in \( M \). Then \( \text{rk}(D(\psi \circ f \circ \varphi^{-1})(\varphi(p))) = m \), hence there exist \( 1 \leq i_1 < \cdots < i_m \leq n \) with \( \det(D((\psi^{i_1}, \ldots, \psi^{i_m}) \circ f \circ \varphi^{-1})(\varphi(p))) \neq 0 \). By 1.1.5, then, \( (\psi^{i_1}, \ldots, \psi^{i_m}) \) is a diffeomorphism locally around \( p \), hence a chart.

(iii)\(\Rightarrow\) (ii): The linear map \( D((\psi^{i_1}, \ldots, \psi^{i_m}) \circ f \circ \varphi^{-1})(\varphi(p)) \) is bijective, so \( \text{rk}(D(\psi \circ f \circ \varphi^{-1})(\varphi(p))) = m \). \( \square \)

1.1.7 Proposition. (Local characterization of submersions) Let \( f : M^m \to N^n \) be smooth and let \( p \in M \). TFAE:

(i) \( T_pf \) is surjective.

(ii) \( \text{rk}_p(f) = n \).

(iii) If \( \psi = (\psi^1, \ldots, \psi^n) \) is any chart at \( f(p) \) in \( N \) then there exists a chart \( \varphi \) of \( M \) at \( p \) such that \( (\psi^1 \circ f, \ldots, \psi^n \circ f, \varphi^{n+1}, \ldots, \varphi^m) \) is a chart at \( p \) in \( M \).

**Proof.** Again, (i)\(\Rightarrow\) (ii) is obvious.

(ii)\(\Rightarrow\) (iii): Let \( \varphi \) and \( \psi \) be charts at \( p \) and \( f(p) \), respectively. Since \( \text{rk}(D(\psi \circ f \circ \varphi^{-1})(\varphi(p))) = n \), the Jacobi matrix \( D(\psi \circ f \circ \varphi^{-1})(\varphi(p)) \) possesses \( n \) linearly independent columns. By permuting the coordinates of \( \varphi \) we obtain a chart \( \varphi \) such that the first \( n \) columns of \( D(\psi \circ f \circ \varphi^{-1})(\varphi(p)) \) are linearly independent. Now set \( \chi := (\psi^1 \circ f, \ldots, \psi^n \circ f, \varphi^{n+1}, \ldots, \varphi^m) \). Then

\[
D(\chi \circ \varphi^{-1})(\varphi(p)) = \begin{pmatrix} (\frac{\partial \psi^i \circ f \circ \varphi^{-1}}{\partial x^j}(\varphi(p)))_{i,j=1}^n & * \\ 0 & I_{m-n} \end{pmatrix}
\]

(1.1.1)

Hence, by 1.1.5, \( \chi \circ \varphi^{-1} \) is a diffeomorphism around \( \varphi(p) \), and so \( \chi \) is a chart at \( p \).

(iii)\(\Rightarrow\) (ii): Since \( \text{rk}(D(\chi \circ \varphi^{-1})(\varphi(p))) = m \), (1.1.1) implies that \( \text{rk}(D(\psi \circ f \circ \varphi^{-1})(\varphi(p))) = m \). \( \square \)

1.1.8 Proposition. Let \( M^m, N^n, R^r \) be manifolds, \( f : M \to N \) continuous and \( g : N \to R \) an immersion. If \( g \circ f \) is smooth then so is \( f \).

**Proof.** Given \( p \in M \), by 1.1.3 we may choose charts \( (\varphi, U) \) around \( f(p) \) in \( N \), and \( (\psi, V) \) around \( g(f(p)) \) in \( R \) such that

\[
g_{\psi \varphi} := \psi \circ g \circ \varphi^{-1}(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0).
\]

(1.1.2)

Let and \( (\chi, W) \) be a chart in \( M \) around \( p \) and set \( f_{\varphi \chi} := \varphi \circ f \circ \chi^{-1} \).

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\chi \downarrow & & \varphi \downarrow & \psi \downarrow \\
\mathbb{R}^m & \xrightarrow{f_{\varphi \chi}} & \mathbb{R}^n & \xrightarrow{g_{\psi \varphi}} & \mathbb{R}^r
\end{array}
\]
1.1.9 Proposition. Let $M^m$, $N^n$, $R^r$ be manifolds, $f : M \to N$ a surjective submersion and $g : N \to R$ arbitrary. If $g \circ f$ is smooth then so is $g$.

Proof. Using the same notations as in the proof of 1.1.8, by 1.1.3 we may choose the charts $(\chi, W)$ around $p$ and $(\varphi, U)$ around $f(p)$ in such a way that $f_{\varphi \chi} = \varphi \circ f \circ \chi^{-1} = (x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^n)$. As in the proof of 1.1.8, $g_{\varphi \chi} \circ f_{\varphi \chi}$ is a restriction of $\psi \circ (g \circ f) \circ \chi^{-1}$ to an open set, hence is smooth. Thus $(x^1, \ldots, x^n) \mapsto g_{\varphi \chi}(x^1, \ldots, x^n)$ and thereby $g_{\varphi \chi}$ itself is smooth, which implies smoothness of $g$.

After these preparations we are now ready to introduce the notion of submanifold of an abstract manifold.

1.1.10 Definition. Let $M^m$ and $N^n$ be manifolds with $N \subseteq M$ and denote by $j : N \hookrightarrow M$ the inclusion map. $N$ is called an immersive submanifold of $M$ if $j$ is an immersion. $N$ is called a submanifold (or sometimes a regular submanifold), if it is an immersive submanifold and in addition $N$ is a topological subspace of $M$, i.e., if the natural manifold topology of $N$ is the trace topology of the natural manifold topology on $M$.

This definition is a natural generalization of the notion of submanifold of $\mathbb{R}^n$, cf. [3, 2.1.5]. The figure-eight manifold from [3, 2.1.5] (with atlas $\{N, j^{-1}\}$) is an example of an immersive submanifold that is not a regular submanifold.

1.1.11 Remark. If $N$ is a submanifold of $M$ then for each $p \in N$, the map $T_p j : T_p N \to T_p M$ is injective. Hence $T_p j(T_p N)$ is a subspace of $T_p M$ that is isomorphic to $T_p N$. We will therefore henceforth identify $T_p j(T_p N)$ with $T_p N$ and notionally suppress the map $T_p j$, i.e., we will consider $T_p N$ directly as a subspace of $T_p M$.

1.1.12 Theorem. Let $N^n$ be an immersive submanifold of $M^m$. TFAE:

(i) $N$ is a submanifold of $M$ (i.e., $N$ carries the trace topology of $M$).

(ii) Around any $p \in N$ there exists an adapted coordinate system, i.e., for every $p \in N$ there exists a chart $(\varphi, U)$ around $p$ in $M$ such that $\varphi(p) = 0$, $\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^n \times \{0\})$ (with $0 \in \mathbb{R}^{m-n}$) and such that $\varphi|_{U \cap N}$ is a chart of $N$ around $p$.

(iii) Every $p \in N$ possesses a neighborhood basis $U$ in $M$ such that $U \cap N$ is connected in $N$ for every $U \in U$.

Proof. (i)$\Rightarrow$(ii): Let $p \in N$. By assumption, $j : N \hookrightarrow M$ is an immersion. Thus by 1.1.3 there exist charts $(\psi, V)$ around $p$ in $N$ and $(\varphi, \hat{U})$ around $j(p) = p$ in $M$, with $\varphi(p) = 0$, such that $\varphi \circ j \circ \psi^{-1} = (x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^n, 0, \ldots, 0)$. Then $\psi \circ (g \circ f) \circ \chi^{-1}$ is defined on $\chi((g \circ f)^{-1}(V) \cap W)$, $f_{\varphi \chi}$ is defined on $\chi(f^{-1}(U) \cap W)$, and $g_{\varphi \chi}$ is defined on $\varphi(g^{-1}(V) \cap U)$. It follows that $g_{\varphi \chi} \circ f_{\varphi \chi}$ is defined on

\[ \chi(f^{-1}(U) \cap W) \cap f_{\varphi \chi}^{-1}(\varphi(g^{-1}(V) \cap U)) = \chi(f^{-1}(U) \cap W) \cap \chi(f^{-1}(g^{-1}(V) \cap U)) = \chi(f^{-1}(g^{-1}(V)) \cap f^{-1}(U) \cap W) \]

Since $f$ is continuous, this shows that $g_{\varphi \chi} \circ f_{\varphi \chi}$ is a restriction of $\psi \circ (g \circ f) \circ \chi^{-1}$ to an open set, hence is smooth. By (1.1.2), $(g_{\varphi \chi} \circ f_{\varphi \chi})^i = f^i_{\varphi \chi}$ for $1 \leq i \leq n$, hence $f_{\varphi \chi}$ is smooth. Thus, finally, $f$ is smooth. □
The domain of $\varphi \circ j \circ \psi^{-1}$ is $\psi(V \cap j^{-1}(U))$. Since $j$ is continuous, $j^{-1}(U)$ is open in $N$. Shrinking $V$ to $V \cap j^{-1}(U)$ if necessary, we can assume w.l.o.g. that $V \subseteq j^{-1}(U)$. The domain of definition of $\varphi \circ j \circ \psi^{-1}$ then is $\psi(V)$. By (i) there exists some open subset $W$ of $M$ such that $V = W \cap N$ and without loss we may assume that $W = \tilde{U}$ (otherwise replace both $\tilde{U}$ and $W$ by $\tilde{U} \cap W$). Then $V = \tilde{U} \cap N$.

Denote by $pr_1 : \mathbb{R}^n \to \mathbb{R}^n$ the projection map. We have

$$\varphi(V) = \varphi(j(V)) = \varphi \circ j \circ \psi^{-1}(\psi(V)) = \psi(V) \times \{0\},$$

so $pr_1(\varphi(V)) = \psi(V)$, which is open in $\mathbb{R}^n$. Hence the set

$$U := \varphi^{-1}((pr_1(\varphi(V)) \times \mathbb{R}^{m-n}) \cap \varphi(\tilde{U}))$$

is open in $M$ and contains $p$. It follows that $(\varphi, U)$ is a chart of $M$ around $p$ and we claim that $\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^n \times \{0\})$.

To see $\subseteq$, note that obviously $\varphi(U \cap N) \subseteq \varphi(U)$ and $U \cap N \subseteq \tilde{U} \cap N = V$, so $\varphi(U \cap N) \subseteq \varphi(V) \subseteq \mathbb{R}^n \times \{0\}$. Conversely,

$$\varphi(U) \cap (\mathbb{R}^n \times \{0\}) = (pr_1(\varphi(V)) \times \{0\}) \cap \varphi(\tilde{U}) = (\psi(V) \times \{0\}) \cap \varphi(\tilde{U})$$

Now let $\varphi(u) \in \varphi(U) \cap (\mathbb{R}^n \times \{0\})$. Then for some $v \in V$ we have

$$\varphi(u) = (\psi(v), 0) = \varphi \circ j \circ \psi^{-1}(\psi(v)) = \varphi(j(v)) = \varphi(v),$$

so $u = v \in V \subseteq N$ and thereby $\varphi(u) \in \varphi(U \cap N)$.

Finally, $\varphi|_{U \cap N}$ is a chart of $N$ around $p$ since $U \cap N = j^{-1}(U)$ is an open neighborhood of $p$ in $N$ and

$$\varphi|_{U \cap N} \circ \psi^{-1} = \varphi|_{U \cap N} \circ j \circ \psi^{-1} = \varphi \circ j \circ \psi^{-1}|_{U \cap N}$$

$$= (x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^n, 0, \ldots, 0).$$

Identifying $\mathbb{R}^n \times \{0\}$ with $\mathbb{R}^n$, this latter map is the identity on $\mathbb{R}^n$, so $\varphi|_{U \cap N} = \psi|_{U \cap N}$, hence it is a chart.

(ii)$\Rightarrow$(iii): Let $(\varphi, U)$ be a chart as in (ii). Pick $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(0) \subseteq \varphi(U)$ and let $U_\varepsilon := \varphi^{-1}(B_{\varepsilon}(0))$ for $\varepsilon < \varepsilon_0$. Then $U := \{U_\varepsilon \mid \varepsilon < \varepsilon_0\}$ is a neighborhood basis of $p$ in $M$ and

$$\varphi(U_\varepsilon \cap N) = \varphi(U_\varepsilon \cap U \cap N) = B_\varepsilon(0) \cap \varphi(U \cap N) = B_\varepsilon(0) \cap (\mathbb{R}^n \times \{0\})$$

is connected in $\mathbb{R}^n$. Thus $U$ serves the desired purpose.

(iii)$\Rightarrow$(i): Denote by $T_M$ and $T_N$ the topologies on $M$ and $N$, respectively. Since $j : N \hookrightarrow M$ is continuous, for every $W \in T_M$ we get $j^{-1}(W) = W \cap N \in T_N$, so $T_M|_N \leq T_N$. Conversely we will show that any $T_N$-neighborhood of any $p \in N$ is also a $T_M|_N$-neighborhood of $p$. To this end let $p \in N$ and let $U$ be a neighborhood of $p$ in $N$ such that is homeomorphic to a ball in $\mathbb{R}^n$ (e.g. the inverse image of such a ball under a chart). Then $\partial U$ is compact in $N$, so also $j(\partial U) \subseteq \partial U$ is compact in $M$ (since $j$ is continuous). Since $p \in U^c$, $p \notin \partial U$ and so by (iii) there exists some $V \in U$ with $V \cap \partial U = \emptyset$. If we can show that $V \cap N \subseteq U$ then we are done since $V \cap N$ is a neighborhood of $p$ in $T_M|_N$. Assume, therefore, that $V \cap N \subseteq U$. This means that $(V \cap N)\cap (N \setminus U) \neq \emptyset$. Thus $V \cap N$ is connected and $(p \in (V \cap N)\cap (N \setminus U) \neq \emptyset$ as well as $(V \cap N)\cap (N \setminus U) \neq \emptyset$. But this implies $(V \cap N) \cap \partial U \neq \emptyset$ and thereby $V \cap \partial U \neq \emptyset$, a contradiction. \hfill $\square$
1.1.13 \textbf{Remark.} (i) For $M = \mathbb{R}^m$, condition (ii) from 1.1.12 is precisely (T) from [3, 2.1.8] (local trivialization). Therefore, submanifolds of $\mathbb{R}^m$ in the sense of [3] are exactly submanifolds of $\mathbb{R}^m$ in the sense of 1.1.10.

(ii) Consider the subset $N$ of $\mathbb{R}^2$ that consists of the interval $[-1, 1]$ on the $y$-axis, plus the graph of $\sin(1/x)$ between $x = 0$ and $x = 1$. Then $N$ is an immersive submanifold of $\mathbb{R}^2$ that is not a submanifold due to 1.1.12 (iii): in fact, any ball around $(0,0)$ of radius less than 1 intersects $N$ in a non-connected set.

1.1.14 \textbf{Proposition.} Let $N$ be a submanifold of $M$ and let $f : P \to M$ be smooth and such that $f(P) \subseteq N$. Then also $f : P \to N$ is smooth.

\textbf{Proof.} Since $N$ carries the trace topology of $M$ and $f : P \to M$ is continuous, also $f : P \to N$ is continuous. Also, $j : N \hookrightarrow M$ is an immersion and by assumption $j \circ f$ is smooth. The claim therefore follows from 1.1.8. \hfill \Box

1.1.15 \textbf{Corollary.} Let $M$ be a manifold and let $N$ be a subset of $M$. Then $N$ can be endowed with the structure of a submanifold of $M$ in at most one way.

\textbf{Proof.} By definition, $N$ has to carry the trace topology of $M$. Suppose that there are two differentiable structures that make $N$ a submanifold of $M$ and denote $N$ with these structures by $N_1$, $N_2$. Since $j : N_i \to M$ is smooth for $i = 1, 2$, 1.1.14 shows that both $\text{id} : N_1 \to N_2$ and $\text{id} : N_2 \to N_1$ are smooth. Hence $\text{id} : N_1 \to N_2$ is a diffeomorphism and so the differentiable structures on $N$ coincide. \hfill \Box

1.1.16 \textbf{Definition.} Let $M$, $N$ be manifolds. A smooth map $i : N \to M$ is called an \textit{embedding} if $i$ is an injective immersion and if $i$ is a \textit{homeomorphism} from $N$ onto $(i(N), T_{M|_{i(N)})}$.

1.1.17 \textbf{Remark.} (i) If $i : N \to M$ is an embedding then $i(N)$ can be turned into a submanifold of $M$ by declaring $i$ to be a diffeomorphism. The charts of $i(N)$ then are the $\psi \circ i^{-1}$, where $\psi$ is any chart of $N$. This manifold $i(N)$ then is a submanifold of $M$: Let $j : i(N) \hookrightarrow M$ be the inclusion map. Then $i = j \circ i$ is an immersion and $i$ is a diffeomorphism by definition, so $j$ is an immersion. Also, $i(M)$ carries the trace topology by assumption. By 1.1.15 this manifold structure on $i(N)$ is the only one possible.

Next we want to check how to tell whether a given subset $N$ of $M$ can be made into a submanifold of $M$. We first generalized the condition from 1.1.12 (ii):

1.1.18 \textbf{Definition.} Let $M^n$ be a manifold and let $N$ be a subset of $M$. We say that $N$ possesses the \textit{submanifold-property of dimension} $n$ if for every $p \in N$ there exists a chart $(\varphi, U)$ of $p$ in $M$ such that $\varphi(p) = 0$ and $\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^n \times \{0\})$. $(\varphi, U)$ then is called an adapted coordinate system.

1.1.19 \textbf{Theorem.} Let $M^n$ be a manifold and let $N$ be a subset of $M$ possessing the submanifold-property of dimension $n$. Then $N$ can be equipped in a unique way with a differentiable structure such that it becomes an $n$-dimensional submanifold of $M$. If $\text{pr}_1 : \mathbb{R}^m \to \mathbb{R}^n$ denotes the projection then $\mathcal{A} := \{ (\tilde{\varphi} := \text{pr}_1 \circ \varphi, U \cap N) \mid \varphi \text{ is an adapted coordinate system} \}$ is a $C^\infty$-atlas for $N$. In addition, $j : N \hookrightarrow M$ is an embedding.
1.1.22 Theorem. Let \((\varphi_1, U_1), (\varphi_2, U_2)\) be adapted coordinate systems with \((U_1 \cap N) \cap (U_2 \cap N) \neq \emptyset\). We have to show that \(\varphi_1\) and \(\varphi_2\) are \(C^\infty\)-compatible. We first note that since the \(\varphi_i\) are homeomorphisms, so are the \(\tilde{\varphi}_i\), as maps from \(U_i \cap N\) with the trace topology onto \(\text{pr}_1(\varphi_i(U_i)) \cap (\mathbb{R}^n \times \{0\})\).

Let \(\theta : \mathbb{R}^n \to \mathbb{R}^m, \theta(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0)\). Then \(\tilde{\varphi}_1^{-1} = \varphi_1^{-1} \circ \theta\). It follows that \(\tilde{\varphi}_1 \circ \tilde{\varphi}_2^{-1}\) is defined on \(\tilde{\varphi}_2(U_1 \cap U_2 \cap N) \subseteq \text{pr}_1(\varphi_2(U_1 \cap U_2)) \cap (\mathbb{R}^n \times \{0\})\), hence open in \(\mathbb{R}^n\), and

\[
\tilde{\varphi}_1 \circ \tilde{\varphi}_2^{-1} = (\text{pr}_1 \circ \varphi_1) \circ (\text{pr}_1 \circ \varphi_2)^{-1} = \text{pr}_1 \circ \varphi_1 \circ \varphi_2^{-1} \circ \theta
\]

is smooth. Consequently, \(A\) is an atlas for \(N\) and by \([3, 2.2.7]\) the natural manifold topology of \(N\) is precisely the trace topology of \(M\) on \(N\). If \((\varphi, U)\) is an adapted chart then \(\varphi \circ j \circ \tilde{\varphi}_1^{-1} = \theta\), so \(j\) is an immersion. Since \(N\) carries the trace topology, \(j : N \to (j(N), T_{M|j(N)})\) is a homeomorphism, so \(j\) is an embedding. \(\square\)

1.1.20 Proposition. Let \(M^m, N^n\) be manifolds, \(N\) compact and \(i : N \to M\) an injective immersion. Then \(i\) is even an embedding and \(i(N)\) is a submanifold of \(M\) that is diffeomorphic to \(N\).

Proof. We have to show that \(i : (N, T_{M|j(N)})\) is a homeomorphism. We already know that this map is continuous and bijective. But also \(i^{-1}\) is continuous: Let \(A \subseteq N\) be closed, hence compact. Then \((i^{-1})^{-1}(A) = i(A)\) is compact and therefore closed. The final claim follows from 1.1.17 (i). \(\square\)

1.1.21 Corollary. Let \(f : N^n \to M^m\) be an immersion. Then every \(p \in N\) has an open neighborhood \(U\) such that \(f|U : U \to M\) is an embedding. Thus the difference between an immersion and an embedding is of a global nature.

Proof. By 1.1.3 there exist charts \(\varphi\) at \(p\) and \(\psi\) at \(f(p)\) such that \(\psi \circ f \circ \varphi^{-1} = (x^1, \ldots, x^n) \to (x^1, \ldots, x^n, 0, \ldots, 0)\). Thus there exists a compact neighborhood \(V\) of \(p\) such that \(f|V\) is injective. As in the proof of 1.1.20 it follows that \(f|V : V \to (f(V), T_{M|f(V)})\) is a homeomorphism. Let \(U \subseteq V\) be an open neighborhood of \(p\). Then \(f|U\) is an injective immersion and \(f : U \to (f(U), T_{M|f(U)})\) is a homeomorphism, so \(f : U \to M\) is an embedding. \(\square\)

1.1.22 Theorem. Let \(M^m, N^n\) be manifolds and \(f : N \to M\) smooth with \(\text{rk}(f) \equiv k\) on \(N\) \((k < n)\). Let \(q \in f(N)\). Then \(f^{-1}(q)\) is a closed submanifold of \(N\) of dimension \(n - k\).

Proof. Since \(f\) is continuous, \(f^{-1}(q)\) is closed in \(N\). We show that \(f^{-1}(q)\) possesses the submanifold property of dimension \(n - k\). The claim then follows from 1.1.19. Let \(p \in f^{-1}(q)\). Then by 1.1.3 there exist charts \((\varphi, U)\) at \(p\) and \((\psi, V)\) at \(f(p) = q\) such that \(\varphi(p) = 0, \psi(q) = 0\) and

\[
f_{\varphi \psi}(x) = \psi \circ f \circ \varphi^{-1}(x^1, \ldots, x^n) = (x^1, \ldots, x^k, 0, \ldots, 0).
\]

Here, \(f_{\varphi \psi}\) is defined on \(\varphi(U \cap f^{-1}(V)) =: \varphi(W)\). Then \((\varphi, W)\) is a chart of \(N\) at \(p\) and \(\varphi(f^{-1}(q) \cap W) = \varphi(f^{-1}(q)) \cap \varphi(W) = \varphi(f^{-1}(\psi^{-1}(\psi(q)))) \cap \varphi(W) = f_{\varphi \psi}^{-1}(0) \cap \varphi(W) = (\{0\} \times \mathbb{R}^{n-k}) \cap \varphi(W)\). \(\square\)
1.1.23 Corollary. Let \( f : N^n \to M^m \) be smooth with \( m < n \) and let \( q \in N \). If 
\[ \text{rk}_p(f) = m \text{ for all } p \in f^{-1}(q) \]
then \( f^{-1}(q) \) is a closed submanifold of \( N \) of dimension \( n - m \).

Proof. Let \( p \in f^{-1}(q) \). Then \( f \) has maximal rank (= \( m \)) at \( p \), hence by 1.1.4 even in an open neighborhood \( U \) of \( p \) in \( N \). Therefore the rank of \( f \) equals \( m \) on an open neighborhood \( \tilde{N} \) of \( f^{-1}(q) \) in \( N \). The claim now follows by applying 1.1.22 to \( f : \tilde{N} \to M \).

1.1.24 Remark. For \( N = \mathbb{R}^n \) and \( M = \mathbb{R}^m \) this result reduces to the description of submanifolds as zero-sets of regular maps, cf. [3, 2.1.8].

1.1.25 Proposition. Under the assumptions of 1.1.22, let 
\[ L := f^{-1}(q) \] and let \( p \in L \). Then 
\[ T_pL = \ker(T_p f). \]

Proof. For any smooth curve \( c \) in \( L \) with \( c(0) = p \), \( f \circ c \equiv q \), so \( 0 = \frac{d}{dt} |_{t=0} (f \circ c) = T_p f(c'(0)) \). Hence \( T_p L \subseteq \ker(T_p M) \). Since 
\[ \dim(\ker(T_p f)) + \dim(\text{im}(T_p f)) = \dim(T_p N) = n, \] 
\[ \dim(\ker(T_p f)) = n - k = \dim(T_p L), \] 
and equality follows.

1.1.26 Example. Let \( \pi : TM \to M^m \) be the canonical projection and let \( p \in M \). Then \( \pi \) is smooth and \( \text{rk}(\pi) = m \) since with respect to a chart \( \psi \) of \( M \) we have \( \psi \circ \pi \circ T\psi^{-1} = pr : \mathbb{R}^{2m} \to \mathbb{R}^m \) (cf. [3, 2.5.6]). By 1.1.23 it follows that \( \pi^{-1}(p) = T_p M \) is an \( m \)-dimensional submanifold of \( TM \). Moreover, by 1.1.25, for \( v \in T_p M \) we have \( T_{\psi v} T_p M = \ker(T_{\psi v} \pi) \). By the proof of 1.1.22, the submanifold charts of \( T_p M \) are given by \( T\psi |_{T_p M} = T_p \psi. \) As these are linear isomorphisms, the trace topology of \( TM \) on \( T_p M \) is precisely the usual topology of \( T_p M \) as a finite-dimensional vector space. Also, \( T_p \psi \) is a diffeomorphism, so the manifold structure of \( T_p M \) as well is its usual differentiable structure as a finite-dimensional vector space.
Bibliography

