Gauge Theory, Lagrangians, and Symmetries

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Preface

Gauge theory is one of the fundamental underlying structures of modern field theories of physics. Mathematically, the language of principal fiber bundles and connections not only provides a unifying perspective on much of differential geometry but also can be used to formalize gauge theory. In these lecture notes we mostly follow Hamilton's book [**19**] in introducing the gauge-theoretic building blocks of the standard model of particle physics. Concerning prerequisites, we suppose familiarity with the theory of Lie groups and principal fiber bundles to the extent laid out in [**24**, **25**]. The aim of this course is to provide a *mathematical* understanding of the fundamentals of gauge theory as employed in particle physics, but naturally we are aware that in the best case we can only provide a bridge to physics, where these tools are actually put to use. Nonetheless, there is an appendix giving an outlook on how quantum field theory builds on the foundations developed in the main part of the text.

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CHAPTER 1

Clifford algebras and Spin groups

Throughout this chapter, $\mathbb{K}$ will denote any of the fields $\mathbb{R}$ or $\mathbb{C}$, V is a *finite dimensional* vector space over $\mathbb{K}$ with $\mathrm{GL}(V)$ denoting the *general linear group* of invertible linear maps $V \to V$, and $Q\colon V \times V \to \mathbb{K}$ is a *symmetric bilinear* form on V. We will always assume that $\dim V \geq 1$.

1.1. Review of pseudo-orthogonal groups

Recall that the symmetric bilinear form Q on V is *non-degenerate*, if for each $v \in V$ with $v \neq 0$ we can find some $w \in V$ such that $Q(v,w) \neq 0$, and in the real case, Q is *positive (or negative) definite*, if $Q(v,v) > 0$ (or $Q(v,v) < 0$) for all $v \in V \setminus \{0\}$.

1.1.1. Definition. Let Q be non-degenerate, then the *pseudo-orthogonal group* $\mathrm{O}(V,Q)$ is defined as the group of Q-automorphisms on V, i.e.,

$$\mathrm{O}(V,Q) := \{f \in \mathrm{GL}(V) \mid \forall v, w, \in V\colon Q(f(v), f(w)) = Q(v,w)\}.$$

In case $\mathbb{K} = \mathbb{R}$ and Q being positive definite, $\mathrm{O}(V,Q)$ is the *orthogonal group* of the Euclidean vector space (V,Q) (for negative definite Q, we may consider the Euclidean space $(V,-Q)$ and the relation $\mathrm{O}(V,Q) = \mathrm{O}(V,-Q)$ holds).

1.1.2. Example (The standard non-degenerate symmetric bilinear forms on $\mathbb{R}^n$). Let $n \in \mathbb{N}$, $s,t \in \mathbb{N}_0$ such that $n = s+t$ and denote by $e_1, \dots, e_{s+t}$ the standard basis of $\mathbb{R}^n$. We specify a symmetric bilinear form η with signature (s,t) by assigning values to the basis vectors:

- If $s = 0$ (hence $t = n \geq 1$), $\eta(e_j, e_j) := -1$ $(j = 1, \dots, n)$ and $\eta(e_j, e_l) := 0$ $(j \neq l)$.
- If $t = 0$ (hence $s = n \geq 1$), $\eta(e_j, e_j) := 1$ $(j = 1, \dots, n)$ and $\eta(e_j, e_l) := 0$ $(j \neq l)$.
- In case $1 \leq s, t, \leq n$ we define

$$\begin{aligned} \eta(e_j, e_j) &:= 1 \quad (j = 1, \dots, s), \\ \eta(e_j, e_j) &:= -1 \quad (j = s+1, \dots, s+t), \\ \eta(e_j, e_l) &:= 0 \quad (j \neq l). \end{aligned}$$

Upon bilinear extension we obtain a symmetric bilinear form η that is also non-degenerate (since its matrix with respect to the standard basis is non-singular). We will henceforth denote $(\mathbb{R}^{s+t}, \eta)$ simply by $\mathbb{R}^{s,t}$, so that $\mathbb{R}^{s,0}$ is the standard Euclidean space, while $\mathbb{R}^{1,n-1}$ or $\mathbb{R}^{n-1,1}$ are two versions of the *Minkowski spacetime* known from special relativity. For general s and t, the form η defining $\mathbb{R}^{s,t}$ provides an example of a so-called *semi-Riemannian metric* (cf. [**29**]).

Any real n-dimensional vector space V with a non-degenerate symmetric bilinear form Q of signature (s,t) possesses a basis $v_1, \dots, v_s, w_1, \dots, w_t$ with the property

$$Q(v_j, v_l) = \delta_{jl}, \quad Q(w_j, w_l) = -\delta_{jl}, \quad Q(v_j, w_l) = 0,$$

which is called an *orthonormal basis* with respect to Q (see, e.g., [**29**, Chapter 2, Lemma 24]). Considering then the unique linear map with $v_j \mapsto e_j$ and $w_l \mapsto e_{s+l}$, we obtain an isomorphism of (V,Q) with $\mathbb{R}^{s,t}$.

1.1.3. Definition. Let $s, t \in \mathbb{N}_0$ with $n := s + t > 0$ and denote by η the standard form defining $\mathbb{R}^{s,t}$. Then we write $\mathrm{O}(s,t)$ in place of $\mathrm{O}(\mathbb{R}^n, \eta)$. For $s, t \geq 1$ the pseudo-orthogonal groups $\mathrm{O}(1,t)$ and $\mathrm{O}(s,1)$ are called *Lorentz groups*. In the cases $s = 0$ or $t = 0$ we obtain the orthogonal group $\mathrm{O}(n,\mathbb{R})$ and write $\mathrm{O}(n)$ in place of $\mathrm{O}(0,t)$ and $\mathrm{O}(s,0)$.

1.1.4. Example (The standard non-degenerate symmetric bilinear form on $\mathbb{C}^n$). Let the symmetric non-degenerate $\mathbb{C}$-bilinear form $q \colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be defined by $q(y,z) := \sum_{l=1}^n y_l z_l$. (Note the difference to the standard Hermitian form defining a unitary space; instead, here we are considering $\mathbb{C}$-bilinear rather than sesquilinear forms.) It can be shown (see [**20**, Theorem 2.46] or [**33**, Theorem 11.23]) that any n-dimensional complex vector space V with a non-degenerate symmetric $\mathbb{C}$-bilinear form Q possesses a basis $u_1, \ldots, u_n$ which is *Q-orthonormal*, i.e., $Q(u_j, u_l) = \delta_{jl} \quad (1 \leq j, l \leq n)$. Therefore, (V, Q) is isomorphic to $(\mathbb{C}^n, q)$. (Recall that the notion of signature is not meaningful for complex symmetric bilinear forms, since any basis vector v with $Q(v,v) = 1$ could be replaced by $w := iv$ giving $Q(w,w) = -1$ instead.) Clearly, the standard basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$ is q-orthonormal:

$$q(e_j, e_l) = \delta_{jl} \quad (1 \leq j, l \leq n).$$

1.1.5. Useful Facts (Basic properties of $\mathrm{O}(s,t)$). Let $n \in \mathbb{N}$, $s, t \in \mathbb{N}_0$ be such that $s + t = n$ and η be the standard form defining $\mathbb{R}^{s,t}$. Let M denote the symmetric matrix giving $\eta(x,y) = x^T M y$ for all $x, y \in \mathbb{R}^n$, then M is diagonal with s entries of 1 followed by t entries equal to -1.

(a) For any $A \in \mathrm{O}(s,t)$, $\det(A) \in \{-1, 1\}$.

proof. We know that $A \in \mathrm{O}(s,t)$ means $x^T(A^T M A)y = (Ax)^T M (Ay) = \eta(Ax, Ay) = \eta(x,y) = x^T M y$ for all $x, y \in \mathbb{R}^n$, so that $A^T M A = M$. Therefore, $(\det A)^2 \det M = \det M$, which implies $(\det A)^2 = 1$, since M is invertible (by the non-degeneracy of M). □

(b) A real $n \times n$-matrix A belongs to $\mathrm{O}(s,t)$, if and only if the column vectors of A constitute an η-orthonormal basis.

proof. Let the column vectors of A be $v_1, \ldots, v_s, w_1, \ldots, w_t \in \mathbb{R}^n$, so that $v_j = Ae_j$ and $w_l = Ae_{l+s}$. The equation $A^T M A = M$ mentioned in the previous proof is a characterization of pseudo-orthogonality of A. In terms of matrix components the equation reads

$$\eta(Ae_j, Ae_l) = v_j^T M v_l = \delta_{jl}, \quad \eta(Ae_{j+s}, Ae_{l+s}) = w_j^T M w_l = -\delta_{jl}, \quad \eta(Ae_j, Ae_{l+s}) = v_j^T M w_l = 0.$$

□

(c) $\mathrm{O}(s,t)$ is a (matrix) Lie group.

proof. Clearly, $\mathrm{O}(s,t) \subseteq \mathrm{GL}(n,\mathbb{R})$ is a subgroup. Thanks to Cartan's theorem ([**24**, Theorem 21.7]), it suffices to show that $\mathrm{O}(s,t)$ is closed in $\mathrm{GL}(n,\mathbb{R})$. The latter follows from the fact that $\phi(A) := A^T M A$ defines a continuous map from $\mathrm{GL}(n,\mathbb{R})$ into the set of all $n \times n$-matrices and $\mathrm{O}(s,t) = \phi^{-1}(\{M\})$. □

(d) The Lie groups $\mathrm{O}(s,t)$ and $\mathrm{O}(t,s)$ are isomorphic.

proof. Let $\tilde{\eta}$ denote the standard form defining $\mathbb{R}^{t,s}$ and define the linear map $R \colon \mathbb{R}^n \to \mathbb{R}^n$ by linear extension from $e_j \mapsto e_{t+j}$ $(1 \leq j \leq s)$, $e_{s+l} \mapsto e_l$ $(1 \leq l \leq t)$. We obtain $R \in \mathrm{GL}(n,\mathbb{R})$ and $\tilde{\eta}(Rx, Ry) = -\eta(x,y)$ for all $x, y \in \mathbb{R}^n$. Therefore, as subsets of $\mathrm{GL}(n,\mathbb{R})$, we have $R\mathrm{O}(s,t)R^{-1} = \mathrm{O}(t,s)$ and $A \mapsto RAR^{-1}$ is an isomorphism of Lie groups. □

(e) If $s \neq 0$ and $t \neq 0$, then $\mathrm{O}(s,t)$ is not compact.

proof. In a first step, we show that $\mathrm{O}(1,1)$ is not compact: Observe that the (symmetric) matrix $L(\tau) := \begin{pmatrix} \cosh(\tau) & \sinh(\tau) \\ \sinh(\tau) & \cosh(\tau) \end{pmatrix}$ belongs to $\mathrm{O}(1,1)$ for every $\tau \in \mathbb{R}$, since $L(\tau)^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} L(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The set $\{L(\tau) \mid \tau \in \mathbb{R}\} \subseteq \mathrm{O}(1,1)$ cannot be bounded as a subset of the 4-dimensional vector space of 2×2-matrices with respect to any of the equivalent norms, since it has matrix entries that are unbounded as $|\tau| \to \infty$.

In the second step, let $s \geq 1$ and $t \geq 1$. We can map $\mathrm{O}(1,1)$ into $\mathrm{O}(s,t)$ by assigning to any $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{O}(1,1)$ the $n \times n$-matrix

$$\phi(A) := \begin{pmatrix} a & 0 & \cdots & 0 & b \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ c & 0 & \cdots & 0 & d \end{pmatrix}.$$

In fact, $\phi(A)^T M \Phi(A) = M$ follows from $A^T \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, thus $\phi(A) \in \mathrm{O}(s,t)$. Now we obtain an unbounded subset of $\mathrm{O}(s,t)$ by considering $\{\phi(L(\tau)) \mid \tau \in \mathbb{R}\}$. □

(f) We have the following isomorphism of complex Lie algebras

$$\mathfrak{o}(s,t) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{o}(s+t) \otimes_{\mathbb{R}} \mathbb{C}.$$

(The Lie bracket $[.,.]_{\mathbb{C}}$ on the complexification $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ of a real Lie algebra $\mathfrak{g}$ is defined in such a way that $[X \otimes \lambda, Y \otimes \mu]_{\mathbb{C}} = [X,Y] \otimes \lambda\mu$ holds on splitting tensors. If $\mathfrak{g}$ is a Lie subalgebra of the real $n \times n$-matrices with the commutator as Lie bracket, then the Lie bracket $[.,.]_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$ corresponds to the commutator of complex $n \times n$-matrices, as can be shown by a boring calculation using the identification of a complex $n \times n$-matrix B with $(\operatorname{Re} B) \otimes 1 + (\operatorname{Im} B) \otimes i$.)

Sketch of a proof: The characterizing property for a real $n \times n$-matrix C to belong to $\mathfrak{o}(s,t)$ is η-skew-adjointness in the sense that $\eta(Cx,y) + \eta(x,Cy) = 0$ for all $x, y \in \mathbb{R}^n$. (This follows upon evaluating the τ-derivative of the relation $\eta(\exp(\tau C)x, \exp(\tau C)y)) = \eta(x,y)$ at $\tau = 0$.) The extension $\eta_{\mathbb{C}}$ of η to a symmetric non-degenerate $\mathbb{C}$-bilinear form on $\mathbb{C}^n \cong \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C}$ satisfies $\eta_{\mathbb{C}}(x \otimes \lambda, y \otimes \mu) = \lambda\mu\, \eta(x,y)$. It turns out that the elements of $\mathfrak{o}(s,t) \otimes_{\mathbb{R}} \mathbb{C}$ act on $\mathbb{C}^n \cong \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C}$ as $\eta_{\mathbb{C}}$-skew-adjoint linear maps: Any element of $\mathfrak{o}(s,t) \otimes_{\mathbb{R}} \mathbb{C}$ is of the form $A \otimes 1 + B \otimes i$ with $A, B \in \mathfrak{o}(s,t)$ and testing on splitting tensors gives

$$\begin{aligned} \eta_{\mathbb{C}}((A \otimes 1 + B \otimes i)(x \otimes \lambda), y \otimes \mu) &= \eta_{\mathbb{C}}((A \otimes 1)(x \otimes \lambda), y \otimes \mu) + \eta_{\mathbb{C}}(((B \otimes i)(x \otimes \lambda), y \otimes \mu) \\ &= \eta_{\mathbb{C}}(Ax \otimes \lambda, y \otimes \mu) + \eta_{\mathbb{C}}(Bx \otimes \lambda i, y \otimes \mu) = \lambda\mu\, \eta(Ax,y) + \lambda i\mu\, \eta(Bx,y) \\ &= -\lambda\mu\, \eta(x,Ay) - \lambda i\mu\, \eta(x,By) = \ldots = -\eta_{\mathbb{C}}(x \otimes \lambda, (A \otimes 1 + B \otimes i)(y \otimes \mu)). \end{aligned}$$

On the other hand, every $\eta_{\mathbb{C}}$-skew-adjoint linear map L belongs to $\mathfrak{o}(s,t) \otimes_{\mathbb{R}} \mathbb{C}$: From the assumption, $\eta_{\mathbb{C}}(L(x \otimes 1), y \otimes 1) = -\eta_{\mathbb{C}}(x \otimes 1, L(y \otimes 1))$ holds for all $x, y \in \mathbb{R}^n$. With real matrices R and S such that $L = R \otimes 1 + S \otimes i$ (corresponding to the real and imaginary parts of L), we obtain (with a calculation similarly as above)

$$\begin{aligned} \eta(Rx,y) + i\eta(Sx,y) &= \ldots = \eta_{\mathbb{C}}((R \otimes 1 + S \otimes i)(x \otimes 1), y \otimes 1) \\ &= -\eta_{\mathbb{C}}(x \otimes 1, (R \otimes 1 + S \otimes i)(y \otimes 1) = \ldots = -\eta(x,Ry) - i\eta(x,Sy), \end{aligned}$$

therefore, $R, S \in \mathfrak{o}(s,t)$.

As mentioned in Example 1.1.4, $(\mathbb{C}^n, \eta_{\mathbb{C}})$ is isomorphic to the standard space $(\mathbb{C}^n, q)$, which induces an isomorphism (as vector spaces) between the $\eta_{\mathbb{C}}$-skew-adjoint matrices, i.e., $\mathfrak{o}(s,t) \otimes_{\mathbb{R}} \mathbb{C}$ and the q-skew-adjoint matrices $\mathfrak{o}(s+t) \otimes_{\mathbb{R}} \mathbb{C}$. Since the Lie bracket is the commutator in all the cases considered here, the sketch of the proof is complete. □

(g) $\dim \mathrm{O}(s,t) = \dim \mathrm{O}(s+t) = \frac{(s+t)(s+t-1)}{2} = \frac{n(n-1)}{2}$.

proof. Recalling $\dim \mathrm{O}(n) = n(n-1)/2$ ([**24**, Example 18.6(ii)]), only the first equality needs to be justified. An application of (f) gives $2 \dim O(s,t) = 2 \dim \mathfrak{o}(s,t) = \dim \mathfrak{o}(s,t) \otimes_{\mathbb{R}} \mathbb{C} = \dim \mathfrak{o}(s+t) \otimes_{\mathbb{R}} \mathbb{C} = 2 \dim \mathfrak{o}(s+t) = 2 \dim O(s+t)$. □

Definition and description of $\mathbf{SO}^+(s,t)$. Recall (e.g., from [**24**, Proposition 2.4]) that the connected component of the identity G_e in a Lie group G is an open, closed, and normal subgroup and that every connected component is a coset of G_e. For a description of the connected component of the idenity in $\mathrm{O}(s,t)$, properties of the corresponding transformations of $\mathbb{R}^{s,t}$ on subspaces where η is positive or negative definite turn out to be crucial. In the notation of 1.1.5, we have $\eta(x,y) = x^T M y$ with

$$M = \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix},$$

where I_d denotes the $d \times d$-identity matrix and the indicated zero blocks have size $s \times t$ in the upper right and size $t \times s$ in the lower left. This block decomposition of M corresponds to the subspaces $V_+ := \mathrm{span}\{e_1, \ldots, e_s\}$ and $V_- := \mathrm{span}\{e_{s+1}, \ldots e_{s+t}\}$, so that $\mathbb{R}^{s,t} = V_+ \oplus V_-$ and η is positive definite on V_+ and negative definite on V_-. A linear tranformation of $\mathbb{R}^{s,t}$ corresponding (with respect to the standard basis) to an $(s+t) \times (s+t)$-matrix A belongs to $O(s,t)$, if and only if $A^T M A = M$, and we may write A with four submatrices S of size $s \times s$, B of size $s \times t$, C of size $t \times s$, and T of size $t \times t$ according to the above direct sum decomposition in the form

$$A = \begin{pmatrix} S & B \\ C & T \end{pmatrix}. \tag{1.1}$$

1.1.6. Definition. We call $\mathrm{SO}(s,t) := \{A \in \mathrm{O}(s,t) \mid \det A = 1\}$ the *special pseudo-orthogonal group*. Furthermore, based on the representation (1.1) of elements in $\mathrm{O}(s,t)$ in terms of submatrices, we consider also the following two subgroups: The *orthochronous pseudo-orthogonal group* $\mathrm{O}^+(s,t) := \{A \in \mathrm{O}(s,t) \mid \det S > 0\}$ and the *proper orthochronous pseudo-orthogonal group* $\mathrm{SO}^+(s,t) := \mathrm{SO}(s,t) \cap \mathrm{O}^+(s,t) = \{A \in \mathrm{O}(s,t) \mid \det A = 1 \text{ and } \det S > 0\}$.

For the signatures $(s,1)$ and $(1,t)$ with $s,t, \geq 1$ the above names for the subgroups are often used with 'pseudo-orthogonal' replaced by 'Lorentz'.

In case $t = 0$ (positive definite η) we clearly have $\mathrm{O}^+(s,0) = \mathrm{SO}(s,0) = \mathrm{SO}^+(s,0)$ and analogously for $s = 0$ (negative definite η), so that we summarize both with a slight abuse of notation in $\mathrm{O}^+(n) = \mathrm{SO}(n) = \mathrm{SO}^+(n)$.

1.1.7. Remark. The property $\det S > 0$ for a matrix A as in (1.1) can be interpreted in terms of preservation of orientation as a map $V_+ \to A(V_+)$. More generally (cf. [**19**, Lemma 6.1.13]), the latter implies that (i) A preserves orientation on any maximally positive definite subspace of $\mathbb{R}^{s,t}$. Under the additional condition (ii) $\det A = 1$, i.e., A preserves orientation overall on $\mathbb{R}^{s,t}$, (i) in turn can be shown to be equivalent to property (iii), namely, A preserves orientation on any maximally negative definite subspace. Moreover, any two of the properties (i), (ii), (iii) imply the third. In particular, referring to the represention (1.1) for any $A \in \mathrm{O}(s,t)$, we have $\mathrm{SO}^+(s,t) = \{A \in \mathrm{O}(s,t) \mid \det A = 1 \text{ and } \det T > 0\} = \{A \in \mathrm{O}(s,t) \mid \det S > 0 \text{ and } \det T > 0\}$. (Cf. [**19**, Section 6.1] or [**29**, Pages 237-238]; beware of an opposite sign convention for the signature and different notation in [**29**].) In the context of physics in Minkowski space, elements of $\mathrm{SO}^+(1,3)$ are exactly those Lorentz transforms that respect time-orientation and space-orientation.

The following statement gives a full description of the connected components of $\mathrm{O}(s,t)$ and is proved in detail in [**29**, Lemma 6 and Corollary 7 in Chapter 9, pages 237-238].

1.1.8. Proposition. *The subgroup* $\mathrm{SO}^+(s,t)$ *is the connected component of the identity in* $\mathrm{O}(s,t)$. *If* $s,t \geq 1$, *then* $\mathrm{O}(s,t)$ *has four connected components and these can be characterized in terms of the representation* (1.1) *for any* $A \in \mathrm{O}(s,t)$ *by the following cases:* $(++)$ $\det A = 1$ *and* $\det S > 0$, $(+-)$ $\det A = 1$ *and* $\det S < 0$, $(-+)$ $\det A = -1$ *and* $\det S > 0$, $(--)$ $\det A = -1$ *and* $\det S < 0$. *If* $s = 0$ *or* $t = 0$, *then* $\mathrm{O}(n)$ *has two connected components, namely* $\mathrm{SO}^+(n)$ *and the coset* $\mathrm{SO}^-(n) := \{A \in \mathrm{O}(n) \mid \det A = -1\}$.

Reflections as generators of $\mathrm{O}(s,t)$. Recall from linear algebra that a reflection R at a hyperplane H in the standard n-dimensional Euclidean space $(\mathbb{R}^n, \langle . | . \rangle)$ is given by a unit vector $v \in \mathbb{R}^n$ with $H = \{v\}^\perp$ in the form $Rx = x - 2\langle x|v\rangle v$. Moreover, if B is an extension of $\{v\}$ to an orthonormal basis, then the matrix of R with respect to B is simply $\mathrm{diag}(-1, 1, \ldots, 1)$, since $Rv = -v$ while $Rx = x$ for any $x \in H$.

To study the analog for the more general case of $(\mathbb{R}^{s,t}, \eta)$ let v_0 be a vector such that $\eta(v_0, v_0) \neq 0$. Then $v := v_0/|\eta(v_0, v_0)|^{1/2}$ satisfies $\eta(v, v) = \eta(v_0, v_0)/|\eta(v_0, v_0)| = \pm 1$ and the formula

$$Rx := x - 2\frac{\eta(x,v)}{\eta(v,v)}v \quad (x \in \mathbb{R}^{s,t}) \tag{1.2}$$

defines a linear map $\mathbb{R}^{s,t} \to \mathbb{R}^{s,t}$ with the following properties:

(i) $\eta(Rx, Ry) = \eta(x,y) - 2\frac{\eta(x,v)}{\eta(v,v)}\eta(v,y) - 2\frac{\eta(y,v)}{\eta(v,v)}\eta(x,v) + 4\frac{\eta(x,v)\eta(y,v)}{\eta(v,v)^2}\eta(v,v) = \eta(x,y)$,
(ii) $Rv = -v$,
(iii) $Rx = x$ for every $x \in \{v\}^\perp := \{y \in \mathbb{R}^{s,t} \mid \eta(y,v) = 0\}$.

Note that by non-degeneracy of η, the map $\mathbb{R}^n \to \mathbb{R}$, $x \mapsto \eta(x,v)$ has rank one, which implies that the subspace $\{v\}^\perp$ has dimension $n-1$, thus is a hyperplane. We therefore obtain that $R \in \mathrm{O}(s,t)$ and due to (ii), (iii) call it a *reflection at the hyperplane* $H := \{v\}^\perp$, where $v \in \mathbb{R}^{s,t}$ satisfies $\eta(v,v) \in \{-1, 1\}$. Note that we also have (as it should be with a reflection)

(iv) $\det R = -1$,

because it has a diagonal matrix representation with respect to any η-orthonormal basis including v with exactly one time -1, otherwise $+1$, along the diagonal.

We have seen above how to obtain reflections at hyperplanes defined by vectors belonging to one of the following special hyperquadrics (cf. [**29**, Chapter 4, Definition 23]), namely the *pseudosphere*

$$S_+ := \{v \in \mathbb{R}^{s,t} \mid \eta(v,v) = 1\} \tag{1.3}$$

and the *pseudohyperbolic space*

$$S_- := \{v \in \mathbb{R}^{s,t} \mid \eta(v,v) = -1\}. \tag{1.4}$$

Note that $S_- \cap S_+ = \emptyset$, so that a statement of the form $v \in S_- \cup S_+$ means that v is either in S_- or in S_+ and not in both.

We will now prove that all of $\mathrm{O}(s,t)$ is generated from products of reflections of the form (1.2). In our statement and proof of this fact we follow [**16**, Section 11.7, Lemma I]. As a preparation, observe that $\{v\}^\perp$ is isomorphic to $\mathbb{R}^{s-1,t}$, if $v \in S_+$, and to $\mathbb{R}^{s,t-1}$, if $v \in S_-$, which follows directly upon extending v to an η-orthonormal basis of $\mathbb{R}^{s,t}$. Any reflection R_1 defined in $\{v\}^\perp$ (in the sense of the previous isomorphisms) can be extended uniquely to a reflection $\tilde{R}_1$ in $\mathbb{R}^{s,t}$ by setting $\tilde{R}_1 v := v$, $\tilde{R}_1 x := R_1 x$ for $x \in \{v\}^\perp$ and using linear extension.

1.1.9. Theorem (Cartan–Dieudonné). *Every element of* $\mathrm{O}(s,t)$ *can be written as the product of at most* $n+1$ *reflections at hyperplanes of the form* $\{v\}^\perp$ *with* $v \in S_-$ *or* $v \in S_+$.

proof. We will show below that if $a, b \in \mathbb{R}^{s,t}$ satisfy $\eta(a,a) = \eta(b,b) \neq 0$, then one can find a reflection R such that $Ra = \pm b$. Assuming this to be true for the moment, we obtain a proof by induction on the dimension n of $\mathbb{R}^{s,t}$.

In the basic case $n = 1$ the only orthogonal linear maps are $x \mapsto \pm x$, which either is the reflection $x \mapsto -x$ or the square of these.

Suppose now that the statement is true for dimension $n-1$ and let $A \in \mathrm{O}(s,t)$. Choose some $a \in \mathbb{R}^{s,t}$ with $\eta(a,a) \neq 0$ and set $b := Aa$; we may suppose that $\eta(a,a) = \pm 1$. From $\eta(b,b) = \eta(a,a)$ and by the claim made at the beginning of the proof, there is some reflection R such that $Ra = \pm b$. The orthogonal transformation $R_0 := R^{-1}A$ has a as an eigenvector, since $R_0 a = R^{-1} b = \pm a$. Thus, R_0 leaves $\{a\}^\perp$ invariant and defines an orthogonal transformation R_1 of $\mathbb{R}^{s-1,t}$ or of $\mathbb{R}^{s,t-1}$. By induction hypothesis, R_1 is the product of at most n reflections. Each of these reflections extends uniquely to a reflection in $\mathbb{R}^{s,t}$, hence R_0 is the product of at most n reflections, which gives at most $n+1$ reflections as factors for $A = RR_0$.

It remains to prove the following
claim: If $a, b \in \mathbb{R}^{s,t}$ with $\eta(a,a) = \eta(b,b) \neq 0$, then there is a reflection R such that $Ra = \pm b$.
We have

$$\begin{aligned}\eta(a+b,a+b) + \eta(a-b,a-b) &= \eta(a,a) + 2\eta(a,b) + \eta(b,b) + \eta(a,a) - 2\eta(a,b) + \eta(b,b)\\ &= 2\eta(a,a) + 2\eta(b,b) = 4\eta(a,a) \neq 0,\end{aligned}$$

so that $\eta(a+b,a+b) \neq 0$ or $\eta(a-b,a-b) \neq 0$. Consider first the case $\eta(a-b,a-b) \neq 0$: Setting $v := (a-b)/|\eta(a-b,a-b)|^{1/2}$ we have $v \in S_-$ or $v \in S_+$ and define the reflection $Rx := x - 2\frac{\eta(x,v)}{\eta(v,v)}v$ according to (1.2). Observing $\eta(a,v) = -\eta(b,v)$ we obtain

$$\eta(v,v) = \frac{\eta(a,v) - \eta(b,v)}{|\eta(a-b,a-b)|^{1/2}} = \frac{2\eta(a,v)}{|\eta(a-b,a-b)|^{1/2}}$$

and therefore deduce

$$Ra = a - 2\frac{\eta(a,v)}{\eta(v,v)}v = a - (b-a) = b.$$

If $\eta(a-b,a-b) = 0$, then $\eta(a+b,a+b) \neq 0$ and the above applies with $-b$ in place of b. □

A reflection R necessarily has $\det R = -1$, which implies that any two representations of an element $A \in \mathrm{O}(s,t)$ as product of reflections must have both an even or both an odd number of factors. This observation proves already the first part of the following statement.

1.1.10. Corollary. *Let $A \in \mathrm{O}(s,t)$ and for any $v \in S_- \cup S_+$ denote by R_v the corresponding reflection at the hyperplane $\{v\}^\perp$ in $\mathbb{R}^{s,t}$.*

(i) We have $A \in \mathrm{SO}(s,t)$, if and only if A is represented as an even number of reflections.

(ii) We have $A \in \mathrm{SO}^+(s,t)$, if and only if A is represented by a product $R_{v_1} \cdots R_{v_{2m}}$ with an even number of vectors v_j from S_- and an even number of vectors v_j from S_+.

Sketch of a proof: (i) is clear from the discussion prior to the statement of the corollary.

(ii): The image of any maximally positive definite subspace of $\mathbb{R}^{s,t}$ under a pseudo-orthogonal transformation is maximally positive definite and the analogous statement holds with negative definiteness. As indicated in Remark 1.1.7, $A \in \mathrm{SO}(s,t)$ belongs to $\mathrm{SO}^+(s,t)$, if and only if it preserves orientation of both maximally positive and maximally negative definite subspaces.

A vector $v \in S_+$ defines a one-dimensional subspace on which η is positive definite and we have $R_v v = -v$. One can then show that the reflection R_v flips the orientation in every maximally positive definite subspace, while it preserves the orientation in any negative definite subspace. Thus, having an even number of factors R_{v_j} with $v_j \in S_+$ in the product representation of A is equivalent to A respecting orientation on maximally positive definite subspaces. Similarly, $v \in S_-$ defines a reflection R_v flipping orientation on maximally negative definite subspaces and preserving it on positive definite subspaces. Thus, an even number of factors R_{v_j} with $v_j \in S_-$ means that A respects also maximally negative definite subspaces. (For more aspects about the notions used in this reasoning see the references mentioned in 1.1.7 and [**20**, Definition 4.48 and Lemma 4.54].) □

1.2. Clifford algebras

Recall that an *associative* $\mathbb{K}$-*algebra with unit* 1 is a $\mathbb{K}$-vector space A with an additional product $A \times A \to A$, $(a,b) \mapsto a \cdot b$ that is bilinear, associative, and $1 \cdot a = a \cdot 1 = a$ holds for all $a \in A$. In the context of this course, it suffices to consider finite-dimensional algebras. If A and B are associative $\mathbb{K}$-algebras with unit, then we also have natural associative products on the direct sum $A \oplus B$, by setting $(a,b) \cdot (a',b') := (a \cdot a', b \cdot b')$, and on the tensor product $A \otimes B$, namely upon bilinear extension of $(a \otimes b) \cdot (a' \otimes b') := (a \cdot a') \otimes (b \cdot b')$; we then have the unit elements

$(1,1) \in A \oplus B$ and $1 \otimes 1 \in A \otimes B$. For a *homomorphism* $\phi\colon A \to B$ we require $\mathbb{K}$-linearity, $\phi(1) = 1$, and $\phi(a_1 \cdot a_2) = \phi(a_1) \cdot \phi(a_2)$ for all $a_1, a_2 \in A$; if ϕ is bijective it is said to be an *isomorphism*, or also *algebra automorphism* in case $A = B$.

For any $\mathbb{K}$-vector space W, the space $\mathrm{L}(W)$ of all linear maps $W \to W$ is an associative $\mathbb{K}$-algebra with unit element id_W, which we often denote simply by 1, and the product given by composition of maps. It is not commutative, if $\dim W > 1$.

1.2.1. Definition. A *representation* of an associative $\mathbb{K}$-algebra A with unit on a $\mathbb{K}$-vector space W is a homomorphism $\rho\colon A \to \mathrm{L}(W)$. It is called *faithful* if ρ is injective.

For the purpose of this course, we will need only finite-dimensional representations, i.e., homomorphisms $\rho\colon A \to \mathrm{L}(W)$ with W finite-dimensional. We will often consider *complex representations of a real algbebra* A, in which case we mean an $\mathbb{R}$-algebra homomorphism $\rho\colon A \to \mathrm{L}(W)$, where W is a complex vector space and $\mathrm{L}(W)$ is the space of $\mathbb{C}$-linear maps $W \to W$. In this case, we have in addition the representation $\rho_{\mathbb{C}}\colon A \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{L}(W)$ of the complexified algebra satisfying $\rho_{\mathbb{C}}(a \otimes \lambda) = \lambda \rho(a)$.

As known from pairs of operators in $\mathrm{L}(W)$, we consider the notions of *commutator* $[a, b]$ and of *anticommutator* $\{a, b\}$ in the abstract setting for any a, b in an associative $\mathbb{K}$-algebra A:

$$[a, b] := a \cdot b - b \cdot a, \quad \{a, b\} := a \cdot b + b \cdot a. \tag{1.5}$$

As is common with many product notations, we might occasionally get tired of the '$\cdot$' and often write the algebra products simply in the form ab in place of $a \cdot b$.

We now come back to considering (V, Q), where V is a finite-dimensional $\mathbb{K}$-vector space and Q is a symmetric bilinear form on V (here not required to be non-degenerate). Let A be a (finite-dimensional) $\mathbb{K}$-algebra with unit 1.

1.2.2. Definition. A *Clifford map* or *Clifford relation* over (V, Q) (in the algebra A) is a $\mathbb{K}$-linear map $\gamma\colon V \to A$ which satisfies

$$\forall v, w \in V\colon \quad \{\gamma(v), \gamma(w)\} = -2Q(v, w)\,1. \tag{1.6}$$

In the special case $v = w$ we have $\{\gamma(v), \gamma(v)\} = 2\gamma(v)\gamma(v) = 2\gamma(v)^2$ and obtain

$$\forall v \in V\colon \quad \gamma(v)^2 = -Q(v, v)\,1. \tag{1.7}$$

In turn, the relation (1.6) follows from (1.7), since symmetry of $\{\gamma(v), \gamma(w)\}$ allows for the standard polarization formula $\{\gamma(v), \gamma(w)\} = (\{\gamma(v+w), \gamma(v+w)\} - \{\gamma(v), \gamma(v)\} - \{\gamma(w), \gamma(w)\})/2$, which then implies $\{\gamma(v), \gamma(w)\} = -(Q(v+w, v+w) - Q(v, v) - Q(w, w))\,1 = -2Q(v, w)\,1$.

1.2.3. Remark. A first version of the Clifford relations (1.6) or (1.7) appeared in a 1928 paper by Dirac on "The quantum theory of the electron" (see [**22**, Page 59] and [**36**, Pages 56-59] for the historical context with physics). They arose in an attempt to find a first-order Schrödinger-type equation replacing the Klein-Gordon equation on Minkowski space $\mathbb{R}^{1,3}$. The task amounted to finding a sort of "square root" of the second-order partial differential operator $\Box = -\partial_1^2 + \partial_2^2 + \partial_3^2 + \partial_4^2$, the d'Alembertian, known from the principal part in the standard wave equation. We may consider $\Box$ to be the "η-Laplacian" for Minkowski space, defining Δ more generally on $(\mathbb{R}^{s,t}, \eta)$ by

$$\Delta := -\sum_{j,k=1}^{n} \eta(e_j, e_k)\partial_j \partial_k = -\sum_{j=1}^{n} \eta_j \partial_j^2, \quad \text{where } \eta_j := \eta(e_j, e_j)\ (1 \le j \le n).$$

We formally try to find a first-order differential operator D such that $D^2 = D \circ D = \Delta$ and make the ansatz

$$D = \sum_{j=1}^{n} \gamma_j \partial_j, \quad \text{where the } \gamma_j\ (1 \le j \le n) \text{ are "coefficients of an appropriate nature".}$$

Using the symmetry $\partial_j\partial_k = \partial_k\partial_j$, which is valid at least when acting on C^2 functions, we obtain

$$-\sum_{j=1}^{n} \eta_j \partial_j^2 = \Delta \overset{!}{=} D^2 = \sum_{j,k=1}^{n} \gamma_j\gamma_k\partial_j\partial_k = \frac{1}{2}\sum_{j,k=1}^{n} (\gamma_j\gamma_k + \gamma_k\gamma_j)\partial_j\partial_k$$

and conclude that

$$\gamma_j\gamma_k + \gamma_k\gamma_j = 0 = -2\eta(e_j, e_k), \text{ if } j \neq k \quad \text{and} \quad \gamma_j^2 = -\eta_j = -\eta(e_j, e_j).$$

Upon interpreting $\gamma_j := \gamma(e_j)$ $(j = 1, \ldots, n)$ these are just Clifford relations over $(\mathbb{R}^{s,t}, \eta)$.

1.2.4. Definition. A *Clifford algebra* over (V, Q) is an associative $\mathbb{K}$-algebra $\mathrm{Cl}(V, Q)$ with unit 1 together with a Clifford map $\gamma\colon V \to \mathrm{Cl}(V, Q)$ such that the following *universal property* holds: For every associative $\mathbb{K}$-algebra A with Clifford map $\delta\colon V \to A$ there exists a unique algebra homomorphism $\phi\colon \mathrm{Cl}(V, Q) \to A$ such that $\phi(\gamma(v)) = \delta(v)$ holds for all $v \in V$, i.e, the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\gamma} & \mathrm{Cl}(V,Q) \\ & {}_{\delta}\searrow & \downarrow{\scriptstyle \phi} \\ & & A \end{array}$$

The universal property in the definition of a Clifford algebra ensures its uniqueness in the sense of the following statement. The existence of Clifford algebras will be proven below.

1.2.5. Lemma. *Let* $(\mathrm{Cl}(V, Q), \gamma)$ *be a Clifford algebra over* (V, Q).

(i) As an algebra, $\mathrm{Cl}(V, Q)$ *is generated by the subspace* $\gamma(V)$.

(ii) If $(\mathrm{Cl}'(V, Q), \gamma')$ *is also a Clifford algebra over* (V, Q)*, then there is a unique isomorphism* $\phi\colon \mathrm{Cl}(V, Q) \to \mathrm{Cl}'(V, Q))$ *such that* $\phi \circ \gamma = \gamma'$.

proof. (i): Let A be the subalgebra of $\mathrm{Cl}(V, Q)$ generated from $\gamma(V)$ and denote by $\tilde{\gamma}$ the map γ considered with target space A. Thanks to the universal property, there is a unique homomorphism $\tilde{\phi}\colon \mathrm{Cl}(V, Q) \to A$ such that $\tilde{\phi} \circ \gamma = \tilde{\gamma}$. If ι is the inclusion map $A \to \mathrm{Cl}(V, Q)$, then

$$\gamma = \iota \circ \tilde{\gamma} = \iota \circ (\tilde{\phi} \circ \gamma) = (\iota \circ \tilde{\phi}) \circ \gamma.$$

Note that $\iota \circ \tilde{\phi}$ is an algebra homomorphism $\mathrm{Cl}(V, Q) \to \mathrm{Cl}(V, Q)$ and the above relation means that it provides us with the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\gamma} & \mathrm{Cl}(V,Q) \\ & {}_{\gamma}\searrow & \downarrow{\scriptstyle \iota\circ\tilde{\phi}} \\ & & \mathrm{Cl}(V,Q) \end{array}$$

But $\mathrm{id}\colon \mathrm{Cl}(V, Q) \to \mathrm{Cl}(V, Q)$ is another such algebra homomorphism, hence the universal property requires that $\iota \circ \tilde{\phi} = \mathrm{id}$, which implies that ι is surjective.

(ii): Employing the universal property for $(\mathrm{Cl}(V, Q), \gamma)$ with $A = \mathrm{Cl}'(V, Q)$ and $\delta = \gamma'$ defines a unique homomorphism $\phi\colon \mathrm{Cl}(V, Q) \to \mathrm{Cl}'(V, Q))$ such that $\phi \circ \gamma = \gamma'$. Reversing the roles of $(\mathrm{Cl}(V, Q), \gamma)$ and $(\mathrm{Cl}'(V, Q), \gamma')$, we also obtain a unique homomorphism $\phi'\colon \mathrm{Cl}'(V, Q) \to \mathrm{Cl}(V, Q))$ such that $\phi' \circ \gamma' = \gamma$. Combining these properties, we have

$$(\phi' \circ \phi) \circ \gamma = \phi' \circ \gamma' = \gamma \quad \text{and} \quad (\phi \circ \phi') \circ \gamma' = \phi \circ \gamma = \gamma'.$$

Thus, the homomorphisms $\phi' \circ \phi$ and $\phi \circ \phi'$ coincide with the identity on $\gamma(V)$ and on $\gamma'(V)$, respectively, and the claim follows by (i). □

1.2.6. Remark (Clifford relations and reflections). An immediate consequence of the Clifford relations is that $\gamma(v)$ is invertible in $\mathrm{Cl}(V,Q)$ for any $v \in V$ with $Q(v,v) \neq 0$, since it follows from (1.7) that $\gamma(v) \cdot (-\gamma(v)/Q(v,v)) = 1 = (-\gamma(v)/Q(v,v)) \cdot \gamma(v)$. By (1.6) we have then for every $x \in V$,

$$\begin{aligned}\gamma(v)\gamma(x)\gamma(v)^{-1} &= \frac{-1}{Q(v,v)}\gamma(v)\gamma(x)\gamma(v) = \frac{-1}{Q(v,v)}\Big(\{\gamma(v),\gamma(x)\}\,\gamma(v) - \gamma(x)\gamma(v)\gamma(v)\Big)\\ &= \frac{2Q(v,x)}{Q(v,v)}\gamma(v) - \gamma(x).\end{aligned}$$

Let $Q(v,v) = \pm 1$, then $\gamma(v)^{-1} = \mp\gamma(v)$ and we obtain $\mp\gamma(v)\gamma(x)\gamma(v) = \pm 2Q(v,x)\gamma(v) - \gamma(x)$, or

$$\pm\gamma(v)\gamma(x)\gamma(v) = \gamma(x) \mp 2Q(v,x)\gamma(v).$$

If $x \perp v$, then the right-hand side gives $\gamma(x)$, while for any $x = \lambda v$ with $\lambda \in \mathbb{K}$ we obtain $\gamma(x) \mp 2Q(v,x)\gamma(v) = \lambda\gamma(v) \mp 2\lambda Q(v,v)\gamma(v) = \lambda\gamma(v) - 2\lambda\gamma(v) = -\lambda\gamma(v) = -\gamma(x)$, in summary,

$$\pm\gamma(v)\gamma(x)\gamma(v) = \begin{cases}\gamma(x), & \text{if } x \perp v,\\ -\gamma(x), & \text{if } x \in \operatorname{span}\{v\}.\end{cases}$$

Note that $x \mapsto \pm\gamma(v)\gamma(x)\gamma(v)$ is $\mathbb{K}$-linear and, *in case Q is non-degenerate*, we can write any $x \in V$ uniquely as $x = x_\perp + x_\parallel$ with $x_\perp \in \{v\}^\perp$ and $x_\parallel \in \operatorname{span}\{v\}$. Therefore, we may express the effect of $\pm\gamma(v)\gamma(x)\gamma(v)$ by the reflection R_v at the hyperplane $\{v\}^\perp$ on x before applying the Clifford map γ, i.e., for any $v \in V$ with $Q(v,v) = \pm 1$, we have

$$\forall x \in V\colon \quad \pm\,\gamma(v)\gamma(x)\gamma(v) = \gamma(R_v x). \tag{1.8}$$

(Recall that the $\pm$ appearing on the left-hand side is simply representing the sign of $Q(v,v)$.)

1.2.7. Lemma. *A Clifford algebra $(\mathrm{Cl}(V,Q),\gamma)$ exists.*

proof. For $k \in \mathbb{N}$ denote by $T^k(V)$ the k-fold tensor product $V \otimes \cdots \otimes V$ and set $T^0(V) := \mathbb{K}$. We consider the tensor algebra $T(V) := \bigoplus_{k=0}^{\infty} T^k(V)$, the two-sided ideal $I(Q)$ in $T(V)$ generated by the subset

$$\{v \otimes v + Q(v,v)\,1 \mid v \in V\} \subseteq T(V),$$

and define

$$\mathrm{Cl}(V,Q) := T(V)/I(Q).$$

Thus, the product of classes $[a],[b] \in \mathrm{Cl}(V,Q)$ with $a,b \in T(V)$ is given by $[a]\cdot[b] = [a \otimes b]$ and the class $[1]$ is the unit element, which we will also denote by 1.

Let $\pi\colon T(V) \to T(V)/I(Q) = \mathrm{Cl}(V,Q)$ be the canonical surjection, $\iota\colon V \to T(V)$ be the embedding $v \mapsto (0,v,0,0,\ldots)$, and define the linear map $\gamma\colon V \to \mathrm{Cl}(V,Q)$ by $\gamma := \pi \circ \iota$. We have for every $v \in V$,

$$\gamma(v)^2 = [v \otimes v] = [-Q(v,v)\,1] = -Q(v,v)[1] = -Q(v,v)\,1,$$

hence γ satisfies (1.7). As we have already observed (via polarization), this relation suffices to show that γ is a Clifford map.

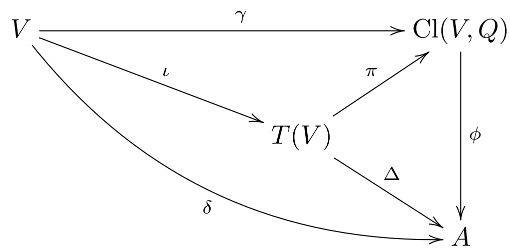
Since $T(V)$ is generated as an algebra by $\iota(V)$, we know that $\gamma(V) = \pi(\iota(V))$ generates $\mathrm{Cl}(V,Q)$, which is a crucial detail for the uniqueness aspect in the following construction showing the universal property for $(\mathrm{Cl}(V,Q),\gamma)$.

Let A be an associative $\mathbb{K}$-algebra with unit and let $\delta\colon V \to A$ be a Clifford map. By the universal property of the tensor algebra ([**16**, Section 3.3]), there is a (unique) homomorphism $\Delta\colon T(V) \to A$ such that $\Delta \circ \iota = \delta$.

We claim that $I(Q) \subseteq \ker\Delta$: It suffices to show that Δ vanishes on the elements $v \otimes v + Q(v,v)\,1$ generating $I(Q)$. Based on the property $\Delta \circ \iota = \delta$ and using that δ is a Clifford map, we obtain

$$\begin{aligned}\Delta(v \otimes v + Q(v,v)\,1) &= \Delta(v \otimes v) + Q(v,v)\Delta(1) = \Delta\left(\iota(v)^2\right) + Q(v,v)\,1\\ &= \Delta(\iota(v))^2 + Q(v,v)\,1 = \delta(v)^2 + Q(v,v)\,1 = 0.\end{aligned}$$

Since $I(Q) \subseteq \ker \Delta$, the homomorphism Δ factors over $I(Q)$ to a homomorphism $\phi\colon \mathrm{Cl}(V,Q) = T(V)/I(Q) \to A$ such that $\phi \circ \pi = \Delta$. We may illustrate the whole construction in the following commutative diagram



where the commutativity for the outermost arrows is justified by $\phi\circ\gamma = \phi\circ\pi\circ\iota = \Delta\circ\iota = \delta$. Note that ϕ is uniquely determined on $\gamma(V)$ by the relation $\phi\circ\gamma = \delta$. Since $\gamma(V)$ generates $\mathrm{Cl}(V,Q)$, we obtain uniqueness of ϕ as algebra homomorphism. □

We collect the results from Lemma 1.2.5 and Lemma 1.2.7 directly into the following statement.

1.2.8. Theorem. *Let (V,Q) be a finite-dimensional vector space with symmetric bilinear form. A Clifford algebra $(\mathrm{Cl}(V,Q),\gamma)$ with Clifford map $\gamma\colon V \to \mathrm{Cl}(V,Q)$ exists and is unique in the sense of Lemma 1.2.5(ii). Moreover, the image $\gamma(V)$ generates the algebra $\mathrm{Cl}(V,Q)$.*

1.2.9. Example. (i) Let $Q = 0$ and consider the construction of $\mathrm{Cl}(V,0)$ as quotient of the tensor algebra $T(V)$. We have to factor out the ideal $I(0)$ generated from $\{v\otimes v \mid v\in V\}$, which means that we obtain as quotient the space of antisymmetric tensors $\Lambda(V)$ and the Clifford product agrees with the exterior product $(a,b)\mapsto a\wedge b$.

(ii) Let $V = \mathbb{R}$ with $Q(x,y) = xy$, i.e., we consider here $\mathbb{R}^{1,0}$. We claim that $\mathrm{Cl}(\mathbb{R},Q) = \mathbb{C}$ with the usual multiplication of complex numbers. Define $\gamma\colon \mathbb{R}\to\mathbb{C}$ by $\gamma(x) := ix$, then $\gamma(x)^2 = -x^2 = -Q(x,x)$ and we see that γ is a Clifford map. We have to show that $(\mathbb{C},\gamma)$ satisfies the universal property: Let $\delta\colon \mathbb{R}\to A$ be another Clifford map and denote the unit in the associative algebra A by e. We need to find a homomorphism $\phi\colon \mathbb{C}\to A$ with $\phi\circ\gamma = \delta$. We obtain the condition $\delta(x) = \phi(\gamma(x)) = \phi(ix)$ for every $x\in\mathbb{R}$, which already forces $\phi(i) := \delta(1)$. The value $\phi(1) := e$ is determined by the requirement that ϕ maps the unit of $\mathbb{C}$ into the unit of A. Using $\mathbb{R}$-linearity, ϕ is thus necessarily given by $\phi(x+iy) = xe + y\delta(1)$ for every $x+iy\in\mathbb{C}$ and it is easy to check that this does indeed define an algebra homomorphism $\mathbb{C}\to A$.

Recall that for any *finite-dimensional* vector space V, we may identify V canonically with $V^{**} := (V^*)^*$. Setting $W := V^*$, this leads to a further identification of the exterior algebra $\Lambda(V)$ with $\Lambda(W^*)$. The latter interpretation has the advantage of being recognizable as antisymmetric multilinear forms on W (or differential forms over some point of W). In this context, for any $X\in W$, we have the *insertion operator* or *contraction* i_X which maps a $(k+1)$-form ω into the k-form $i_X\omega$, given by $i_X\omega(X_1,\dots,X_k) := \omega(X,X_1,\dots,X_k)$ for all $X,X_1,\dots,X_k\in W$ (see, e.g., [**1**, Definition 6.4.7]). The symmetric bilinear form Q on V provides us with the canonical linear map $Q^\flat\colon V\to V^* = W$, where $Q^\flat(v)(u) := Q(v,u)$ for all $v,u\in V$. We can now define the *contraction* for any $v\in V$ and $\omega\in\Lambda^{k+1}(V)$ by

$$v \lrcorner\, \omega := i_{Q^\flat(v)}\omega \in \Lambda^k(V)$$

If $u\in V = \Lambda^1(V)$, we obtain $v\lrcorner\, u = Q(v,u)$. The following antiderivation property then follows immediately from a corresponding one in [**1**, Theorem 6.4.8(i)]:

$$v\lrcorner\,(\mu\wedge\omega) = (v\lrcorner\,\mu)\wedge\omega + (-1)^k\mu\wedge(v\lrcorner\,\omega) \tag{1.9}$$

for $v\in V$, $\mu\in\Lambda^k(V)$, and $\omega\in\Lambda^l(V)$.

We have seen above that in case $Q = 0$ the Clifford algebra is isomorphic to the exterior algebra. As the following result shows, in the general case we still have an isomorphism of the vector space structures. Moreover, the Clifford multiplication can be described as a kind of distortion of the exterior product by the contraction that is defined via Q as above.

1.2.10. THEOREM. *There is a canonical vector space isomorphism $\sigma\colon \mathrm{Cl}(V,Q) \to \Lambda(V)$, called the* symbol map*, with the property that for all $v \in V$, $u \in \mathrm{Cl}(V,Q)$,*

$$\sigma(\gamma(v)\cdot u) = v \wedge \sigma(u) - v \lrcorner \sigma(u).$$

In particular, $\dim \mathrm{Cl}(V,Q) = \dim \Lambda(V) = 2^n$*, if* $\dim V = n$*.*

PROOF. We consider the map $\delta\colon V \to \mathrm{L}(\Lambda(V))$, where $\delta(v)$ for any $v \in V$ is defined by

$$\delta(v)\omega := v \wedge \omega - v \lrcorner \omega \quad (\omega \in \Lambda(V)).$$

Linearity of δ is clear and we will show that it is a Clifford map. For any $v, w \in V$ and $\alpha \in \Lambda^{k+2}(V)$, we have $v \lrcorner (w \lrcorner \alpha)(X_1,\ldots,X_k) = \alpha(Q^\flat(w), Q^\flat(v), X_1,\ldots,X_k) = -\alpha(Q^\flat(v), Q^\flat(w), X_1,\ldots,X_k) = -w \lrcorner (v \lrcorner \alpha)(X_1,\ldots,X_k)$ for all $X_1,\ldots,X_k \in V^*$, i.e.,

$$v \lrcorner (w \lrcorner \alpha) = -w \lrcorner (v \lrcorner \alpha) \tag{1.10}$$

(and for $\alpha \in \mathbb{K} = \Lambda^0(V)$ or $\alpha \in V = \Lambda^1(V)$ both sides are 0). Employing the antisymmetry of the exterior product, (1.9), and (1.10), we obtain

$$\begin{aligned}
\{\delta(v),\delta(w)\}\alpha &= \delta(v)\delta(w)\alpha + \delta(w)\delta(v)\alpha = \delta(v)(w\wedge\alpha - w\lrcorner\alpha) + \delta(w)(v\wedge\alpha - v\lrcorner\alpha)\\
&= v\wedge(w\wedge\alpha - w\lrcorner\alpha) - v\lrcorner(w\wedge\alpha - w\lrcorner\alpha) + w\wedge(v\wedge\alpha - v\lrcorner\alpha) - w\lrcorner(v\wedge\alpha - v\lrcorner\alpha)\\
&= v\wedge w\wedge\alpha - v\wedge(w\lrcorner\alpha) - v\lrcorner(w\wedge\alpha) + v\lrcorner(w\lrcorner\alpha) + w\wedge v\wedge\alpha - w\wedge(v\lrcorner\alpha)\\
&\quad - w\lrcorner(v\wedge\alpha) + w\lrcorner(v\lrcorner\alpha)\\
&= \big(v\wedge w\wedge\alpha + w\wedge v\wedge\alpha\big) - \big(v\lrcorner(w\wedge\alpha) + w\wedge(v\lrcorner\alpha)\big)\\
&\quad - \big(v\wedge(w\lrcorner\alpha) + w\lrcorner(v\wedge\alpha)\big) + \big(v\lrcorner(w\lrcorner\alpha) + w\lrcorner(v\lrcorner\alpha)\big)\\
&= 0 - \big((v\lrcorner w)\wedge\alpha\big) - \big((w\lrcorner v)\wedge\alpha\big) + 0 = -Q(v,w)\alpha - Q(w,v)\alpha = -2Q(v,w)\alpha.
\end{aligned}$$

Varying α, this means

$$\{\delta(v),\delta(w)\} = -2Q(v,w)\,1.$$

By the universal property of $(\mathrm{Cl}(V,Q),\gamma)$, there is a unique homomorphism $\phi\colon \mathrm{Cl}(V,Q) \to \mathrm{L}(\Lambda(V))$ such that $\phi\circ\gamma = \delta$. We define the linear map $\sigma\colon \mathrm{Cl}(V,Q) \to \Lambda(V)$ by $\sigma(u) := \phi(u)(1)$ for every $u \in \mathrm{Cl}(V,Q)$ (note that here $1 \in \Lambda^0(V) \subseteq \Lambda(V)$). By construction, we immediately have for any $v \in V$ and $u \in \mathrm{Cl}(V,Q)$,

$$\sigma(\gamma(v)\cdot u) = \phi(\gamma(v)\cdot u)(1) = \phi(\gamma(v))\phi(u)(1) = \delta(v)\sigma(u) = v\wedge\sigma(u) - v\lrcorner\sigma(u).$$

It remains to show that σ is bijective.

Surjectivity of σ: We will show inductively that for any orthogonal[1] set of vectors $v_1,\ldots,v_m \in V$, we have

$$\sigma(\gamma(v_1)\cdots\gamma(v_m)) = v_1\wedge\ldots\wedge v_m. \tag{1.11}$$

Once this is shown, surjectivity follows upon choosing for v_j vectors from subsets of some fixed orthogonal basis of V, since linear combinations of the right-hand side then generate $\Lambda(V)$. The base case is $\sigma(\gamma(v_1)) = \phi(\gamma(v_1))(1) = \delta(v_1)(1) = v_1\wedge 1 - v_1\lrcorner 1 = v_1 - 0 = v_1$. Suppose the relation has been shown up to m and let $v_1,\ldots,v_{m+1}$ be orthogonal. Then we have

$$\begin{aligned}
&\sigma(\gamma(v_1)\gamma(v_2)\cdots\gamma(v_{m+1})) = \phi(\gamma(v_1)\gamma(v_2)\cdots\gamma(v_{m+1}))(1) = \phi(\gamma(v_1))\,\phi(\gamma(v_2)\cdots\gamma(v_{m+1}))(1)\\
&= \delta(v_1)\sigma(\gamma(v_2)\cdots\gamma(v_{m+1})) = \delta(v_1)(v_2\wedge\ldots\wedge v_{m+1}) = v_1\wedge v_2\wedge\ldots\wedge v_{m+1} - v_1\lrcorner(v_2\wedge\ldots\wedge v_{m+1}),
\end{aligned}$$

where the last term vanishes, because by (an inductive version of) the antiderivation property (1.9), it is a sum of terms with factors $Q(v_1,v_j)$ $(j = 2,\ldots m+1)$ that are all 0.

[1]Note that we do not require them to be orthonormal, because for a degenerate form Q, we can still find a basis of V consisting of orthogonal vectors, but not an orthonormal one.

Injectivity of σ: If $v_1, \dots, v_n$ is a basis of V, then the Clifford algebra $\mathrm{Cl}(V,Q)$ is generated by linear combinations of 1 and all products of the form $\gamma(v_{j_1})\cdots\gamma(v_{j_m})$ with $m \in \mathbb{N}$ and $1 \le j_l \le n$. We may choose the basis to be orthogonal, in which case the Clifford relations show that any two different factors in such a product anticommute. Thus, adjusting signs we need only consider ordered products, e.g., by increasing index from left to right. Then, since $\gamma(v_j)^2 = -Q(v_j, v_j)$ is scalar, any factors appearing more than once can be reduced until there are only different factors in each product. To summarize, $\mathrm{Cl}(V,Q)$ is the linear span of the subset consisting of 1 and all ordered products $\gamma(v_{j_1})\cdots\gamma(v_{j_m})$ with $1 \le m \le n$, $1 \le j_1 < \ldots < j_m \le n$. Since this generating set has 2^n elements, the dimension of $\mathrm{Cl}(V,Q)$ is at most $2^n = \dim \Lambda(V)$. Therefore, the surjective linear map $\sigma\colon \mathrm{Cl}(V,Q) \to \Lambda(V)$ has to be injective as well. □

We extract the information from the injectivity part of the previous proof and combine it with the result about the dimension of $\mathrm{Cl}(V,Q)$, which immediately yields the following statement.

1.2.11. Corollary. *If $v_1, \dots, v_n$ is a basis of V such that $Q(v_j, v_l) = 0$ ($j \ne l$), i.e, an orthogonal basis of V, then*

$$\{1\} \cup \{\gamma(v_{j_1})\cdots\gamma(v_{j_m}) \mid 1 \le m \le n, 1 \le j_1 < \ldots < j_m \le n\}$$

is a basis of $\mathrm{Cl}(V,Q)$ as a vector space. In particular, the Clifford map $\gamma\colon V \to \mathrm{Cl}(V,Q)$ is injective and $\gamma(v_1), \dots, \gamma(v_n)$ is a basis for $\gamma(V)$.

1.2.12. Remark. Based on the results above, we have a practical way to establish isomorphisms of $\mathrm{Cl}(V,Q)$ with associative algebras A in the following way:
Step 1. Find a Clifford map $\delta\colon V \to A$ and apply the universal property to obtain an algebra homomorphism $\phi\colon \mathrm{Cl}(V,Q) \to A$ with $\phi \circ \gamma = \delta$.
Step 2. Choose a Q-orthogonal basis $v_1, \dots, v_n$ of V and show that the linear span of $\phi(1) = 1$ and of all products $\delta(v_{j_1})\cdots\delta(v_{j_m})$ is equal to A. Then we know that ϕ is surjective.
Step 3. If $\dim A = \dim \mathrm{Cl}(V,Q) = 2^n$, then ϕ is an isomorphism.

The technique described in the previous remark will be applied repeatedly in this chapter. Here, we apply it already to establish the following two statements.

1.2.13. Lemma. *If (V', Q') is a finite-dimensional vector space with symmetric bilinear form Q' and R is an orthogonal isomorphism of (V,Q) with (V',Q') in the sense that $R\colon V \to V'$ is linear, bijective, and $Q'(Rv, Rw) = Q(v,w)$ for all $v, w \in V$, then $\mathrm{Cl}(V',Q') \cong \mathrm{Cl}(V,Q)$.*

proof. Denote by γ' the Clifford embedding of V' into $\mathrm{Cl}(V',Q')$ and consider $\delta\colon V \to \mathrm{Cl}(V',Q')$, given by $\delta(v) := \gamma'(Rv)$ for all $v \in V$. The relation $\delta(v)^2 = \gamma'(Rv)^2 = -Q'(Rv,Rv) = -Q(v,v)$ shows that δ is a Clifford map. Let $\phi\colon \mathrm{Cl}(V,Q) \to \mathrm{Cl}(V',Q')$ be the unique homomorphism according to the universal property of $(\mathrm{Cl}(V,Q), \gamma)$. Since R is an isomorphism, $\delta(V) = \gamma'(V')$ and hence $\delta(V)$ is a generating subset for $\mathrm{Cl}(V',Q')$, which shows that ϕ is surjective. Finally, $\dim V = \dim V'$ implies that $\dim \mathrm{Cl}(V,Q) = \dim \mathrm{Cl}(V',Q')$ by Corollary 1.2.11, thus ϕ is an isomorphism. □

1.2.14. Lemma. *There is a unique automorphism $\alpha\colon \mathrm{Cl}(V,Q) \to \mathrm{Cl}(V,Q)$ with $\alpha(\gamma(v)) = \gamma(-v)$ for all $v \in V$. It satisfies $\alpha^2 = \mathrm{id}$ and is called the* parity automorphism.

proof. Consider the Clifford map $\delta\colon V \to \mathrm{Cl}(V,Q)$, $\delta(v) := \gamma(-v)$ for all $v \in V$. By the universal property, there is a unique homomorphism $\alpha\colon \mathrm{Cl}(V,Q) \to \mathrm{Cl}(V,Q)$ such that $\alpha(\gamma(v)) = \delta(v) = \gamma(-v)$ for all $v \in V$. If $v_1, \dots, v_n$ is any Q-orthogonal basis of V, then the linear span of 1 and all products $\delta(v_{j_1})\cdots\delta(v_{j_m})$ coincides with $\mathrm{Cl}(V,Q)$, hence α is surjective and, for dimensional reasons, also injective. For any $v \in V$, $\alpha^2(\gamma(v)) = \alpha(\alpha(\gamma(v))) = \alpha(\gamma(-v)) = \gamma(v)$, which shows that α^2 acts as the identity on $\gamma(V)$. Since this set generates $\mathrm{Cl}(V,Q)$ as an algebra and α is a homomorphism, we have established that $\alpha^2 = \mathrm{id}$. □

As an application of the parity automorphism α, we will use its spectral decomposition to define the so-called *even* and *odd part* of the Clifford algebra $\mathrm{Cl}(V, Q)$.

In a preliminary step, let us show that we cannot have $\alpha = c \cdot \mathrm{id}$ with $c \in \mathbb{K}$: The relation $\alpha^2 = \mathrm{id}$ yields $c^2 = 1$ and thus $\alpha = \pm\mathrm{id}$. The option $\alpha = \mathrm{id}$ is ruled out by $\alpha(\gamma(v)) = -\gamma(v)$ with some $v \neq 0$ (recall that we always assume $\dim V \geq 1$). And $\alpha = -\mathrm{id}$ is impossible, because then $-1 = \alpha(1) = \alpha(1^2) = \alpha(1)^2 = 1$.

We now know that α is a linear operator on the finite-dimensional vector space $\mathrm{Cl}(V, Q)$ satisfying the quadratic polynomial operator identity $\alpha^2 - \mathrm{id} = 0$ and none of degree 1, since $\alpha \neq c \cdot \mathrm{id}$. We deduce that $p(\lambda) = \lambda^2 - 1$ is the minimal polynomial of α and note that $p(\lambda) = (\lambda - 1)(\lambda + 1)$ is the product of two distinct linear factors. Hence α is diagonalizable ([**33**, Theorem 8.11]) and $\mathrm{Cl}(V, Q) = E_1 \oplus E_{-1}$, where E_i denotes the eigenspace for the eigenvalue $j \in \{-1, 1\}$.

Note that $1 \in E_1$, since $\alpha(1) = 1$, and a product $\gamma(v_{j_1}) \cdots \gamma(v_{j_m})$ belongs to E_1, if and only if the number m of factors is even, because $\alpha(\gamma(v_{j_1}) \cdots \gamma(v_{j_m})) = \alpha(\gamma(v_{j_1})) \cdots \alpha(\gamma(v_{j_m})) = (-1)^m \gamma(v_{j_1}) \cdots \gamma(v_{j_m})$. At the same time we see that $\gamma(v_{j_1}) \cdots \gamma(v_{j_m})$ belongs to E_{-1} precisely when m is odd. If the vectors are chosen from a fixed orthogonal basis $v_1, \ldots, v_n$ of V in the way specified in Corollary 1.2.11, then 1 together with the products of an even number of factors constitute a basis for E_1, while the products with an odd number of factors define a basis of E_{-1}. Counting 1 as a product of 0 factors, the identity $0 = (1 + (-1))^n = \sum \binom{n}{l}(-1)^l = \sum \binom{n}{2k} - \sum \binom{n}{2k+1}$ shows that we have equal numbers of basis elements in E_1 and E_{-1}, thus $\dim E_{-1} = \dim E_1 = \dim \mathrm{Cl}(V, Q)/2 = 2^{n-1}$.

1.2.15. Definition. Let E_j denote the eigenspace for the eigenvalue $j \in \{-1, 1\}$ of the parity automorphism $\alpha\colon \mathrm{Cl}(V, Q) \to \mathrm{Cl}(V, Q)$, then we call $\mathrm{Cl}^0(V, Q) := E_1$ the *even part* of the Clifford algebra and $\mathrm{Cl}^1(V, Q) := E_{-1}$ the *odd part*. The even part $\mathrm{Cl}^0(V, Q)$ is a subalgebra of $\mathrm{Cl}(V, Q)$ with unit and is generated by 1 and all products of an even number of factors $\gamma(v_1) \cdots \gamma(v_{2m})$ with $v_j \in V$. We have $\dim \mathrm{Cl}^0(V, Q) = \dim \mathrm{Cl}^1(V, Q) = \dim \mathrm{Cl}(V, Q)/2$.

The Clifford algebras over the standard spaces $(\mathbb{R}^{s,t}, \eta)$ and $(\mathbb{C}^n, q)$. We will henceforth consider only the standard spaces with symmetric bilinear forms and use the notation

$$\mathrm{Cl}(s, t) := \mathrm{Cl}(\mathbb{R}^{s,t}, \eta), \quad \mathrm{Cl}(n) := \mathrm{Cl}(n, 0), \quad \mathbb{C}\mathrm{l}(n) := \mathrm{Cl}(\mathbb{C}^n, q),$$

for the real Clifford algebras over $(\mathbb{R}^{s,t}, \eta)$, $0 \leq s, t \leq n$, $n = s + t$, which is the standard Euclidean space if $t = 0$, and for the complex Clifford algebra over $(\mathbb{C}^n, q)$. Recall that the standard symmetric bilinear forms are all non-degenerate.

In 1.1.5(f) we already made use of the complexification $(\mathbb{C}^n, \eta_{\mathbb{C}})$ of $(\mathbb{R}^{s,t}, \eta)$ with $\eta_{\mathbb{C}}(v \otimes \lambda, w \otimes \mu) = \lambda\mu\, \eta(v, w)$. As noted in Example 1.1.4 we have $(\mathbb{C}^n, \eta_{\mathbb{C}}) \cong (\mathbb{C}^n, q)$, which implies $\mathrm{Cl}(\mathbb{C}^n, \eta_{\mathbb{C}}) \cong \mathbb{C}\mathrm{l}(n)$ by Lemma 1.2.13. We will now show that the Clifford algebra of the complexification is the complexification of the Clifford algebra.

1.2.16. Lemma. *We have the following isomorphism of complex associative algebras with unit:*

$$\mathrm{Cl}(s, t) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}\mathrm{l}(s + t).$$

Any complex representation of $\mathrm{Cl}(s, t)$ *can be obtained as a representation of* $\mathbb{C}\mathrm{l}(s + t)$.

proof. Consider the complexification of $\gamma\colon \mathbb{R}^{s,t} \to \mathrm{Cl}(s, t)$, i.e., the map $\delta\colon \mathbb{C}^n \cong \mathbb{R}^{s,t} \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{Cl}(s, t) \otimes_{\mathbb{R}} \mathbb{C}$, defined on splitting tensors by $\delta(v \otimes \lambda) := \gamma(v) \otimes \lambda$. We have

$$\delta(v \otimes \lambda)^2 = \gamma(v)^2 \otimes \lambda^2 = -\eta(v, v)\, 1 \otimes \lambda^2 = -\lambda^2 \eta(v, v)\, 1 \otimes 1 = -\eta_{\mathbb{C}}(v \otimes \lambda, v \otimes \lambda)\, 1 \otimes 1,$$

which shows that δ is a Clifford map for $(\mathbb{C}^n, \eta_{\mathbb{C}})$. Let $\tilde{\gamma}$ denote the Clifford embedding of $\mathbb{C}^n$ into $\mathrm{Cl}(\mathbb{C}^n, \eta_{\mathbb{C}})$ and let $\phi\colon \mathrm{Cl}(\mathbb{C}^n, \eta_{\mathbb{C}}) \to \mathrm{Cl}(s, t) \otimes_{\mathbb{R}} \mathbb{C}$ be the unique complex algebra homomorphism with $\phi \circ \tilde{\gamma} = \delta$. Surjectivity of ϕ follows from $\delta(\mathbb{C}^n) = \mathrm{span}\{\gamma(v) \otimes \lambda \mid v \in \mathbb{R}^{s,t}, \lambda \in \mathbb{C}\}$, since (the union of $\{1\}$ and) the latter gives a generating subset for the algebra $\mathrm{Cl}(s, t) \otimes_{\mathbb{R}} \mathbb{C}$.

Finally, observing $\dim_{\mathbb{C}} \mathrm{Cl}(s,t) \otimes_{\mathbb{R}} \mathbb{C} = \dim_{\mathbb{R}} \mathrm{Cl}(s,t) = \dim_{\mathbb{C}} \mathrm{Cl}(\mathbb{C}^n, \eta_{\mathbb{C}})$ we may conclude that $\mathrm{Cl}(s,t) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{Cl}(\mathbb{C}^n, \eta_{\mathbb{C}}) \cong \mathbb{C}\mathrm{l}(n)$. □

In the sequel, we will often simplify notation by dropping the reference to the standard embedding $\gamma\colon \mathbb{C}^n \to \mathbb{C}\mathrm{l}(n)$ and consider $x \in \mathbb{C}^n$ directly as the element corresponding to $\gamma(x)$ in $\mathbb{C}\mathrm{l}(n)$, and similarly for the real cases.

We will now show how to obtain a specifically simple model for the even parts of complex standard Clifford algebras. For the following formula, in case $n = 1$, we define $\mathbb{C}\mathrm{l}(n-1) = \mathbb{C}\mathrm{l}(0) := \mathbb{C}$. Note that $\mathbb{C}\mathrm{l}^0(1) \cong \mathbb{C}$, since there are no products with an even number of factors consisting of different basis vectors, which leaves 1 as the only generator for $\mathbb{C}\mathrm{l}^0(1)$ as a $\mathbb{C}$-algebra.

1.2.17. Lemma. *$\mathbb{C}\mathrm{l}^0(n) \cong \mathbb{C}\mathrm{l}(n-1)$.*

PROOF. For any $x = (x_1, \ldots, x_{n-1}) \in \mathbb{C}^{n-1}$ let $x' := (x_1, \ldots, x_{n-1}, 0) \in \mathbb{C}^n$. Let q, q' denote the standard symmetric form on $\mathbb{C}^{n-1}$, $\mathbb{C}^n$, respectively, then we note that $q'(x', e_n) = 0$ and $q'(x', x') = q(x,x)$. Consider $\delta\colon \mathbb{C}^{n-1} \to \mathbb{C}\mathrm{l}^0(n)$, given by $\delta(x) := x' \cdot e_n$. This defines a Clifford map for $(\mathbb{C}^{n-1}, q)$, since

$$\delta(x)^2 = (x' \cdot e_n) \cdot (x' \cdot e_n) = x' \cdot (e_n \cdot x') \cdot e_n = x' \cdot (-x' \cdot e_n - 2q'(x', e_n)) \cdot e_n$$
$$= -x' \cdot x' \cdot e_n \cdot e_n = -(-q'(x', x'))(-q(e_n, e_n)) = -q(x,x).$$

(In case $n = 1$, we have $\delta = 0$ and $q = 0$.) Let $\phi\colon \mathbb{C}\mathrm{l}(n-1) \to \mathbb{C}\mathrm{l}^0(n)$ be the unique homomorphism from the universal property of $\mathbb{C}\mathrm{l}(n-1)$, or $\phi := \mathrm{id}_{\mathbb{C}}$ in case $n = 1$. Note that the image of δ contains all elements $e_j \cdot e_n$ $(1 \leq j < n)$ and therefore $\phi(\mathbb{C}^{n-1}) = \mathbb{C}\mathrm{l}^0(n)$, i.e., ϕ is surjective. Since $\dim \mathbb{C}\mathrm{l}(n-1) = 2^{n-1} = \dim \mathbb{C}\mathrm{l}(n)/2 = \dim \mathbb{C}^0(n)$, we conclude that ϕ is an isomorphism. □

Recall that $\mathrm{Cl}(s,t)$ and $\mathbb{C}\mathrm{l}(n)$ are generated as an algebra by (1 and) the embeddings of the standard orthonormal basis vectors $\gamma(e_a)$ $(a = 1, \ldots, n)$ of $(\mathbb{R}^{s,t}, \eta)$ and $(\mathbb{C}^n, q)$, respectively. The images of these generators in any real ($\mathbb{K} = \mathbb{R}$) or complex ($\mathbb{K} = \mathbb{C}$) representation of $\mathrm{Cl}(s,t)$ on some $\mathbb{K}^N$ determine the representation and deserve a special name. We denote by $\eta_{ab} := \eta(e_a, e_b)$ $(1 \leq a, b, \leq n)$ the matrix elements of η with respect to the standard basis; thus, $(\eta_{ab})_{1 \leq a,b \leq n}$ is the diagonal matrix with s-times 1 followed by t-times -1 along the diagonal.

1.2.18. Definition. Let $\rho\colon \mathrm{Cl}(s,t) \to \mathrm{L}(\mathbb{K}^N)$ be a (real or complex) representation, then we define the *gamma matrices* (of ρ) by

$$\gamma_a := \rho(\gamma(e_a)) \quad (a = 1, \ldots, n).$$

According to the Clifford relations we have the anticommutator relations

$$\{\gamma_a, \gamma_b\} = -2\eta_{ab} I_N,$$

where I_N denotes the unit $N \times N$-matrix. For the commutators of gamma matrices we introduce the notation

$$\gamma_{ab} := \frac{1}{2}[\gamma_a, \gamma_b] = \frac{1}{2}(\gamma_a \gamma_b - \gamma_b \gamma_a).$$

Let us admit here that we will henceforth regularly adopt this abuse of language in the above definition and call elements of $\mathrm{L}(\mathbb{K}^N)$ also matrices, although we mean, in fact, the matrix of the corresponding linear operator on $\mathbb{K}^n$ with respect to the standard basis.

1.2.19. Remark. In the physics literature, the Gamma matrices are often called *Dirac matrices* and the convention about scalar factors typically differ from the above definition (and vary among a spectrum of sources, as does the convention about the signature of the Minkowski metric etc.). To keep track of some alternative in a parallel development, [**19**] introduces what he calls the *physical gamma matrices* by $\Gamma_a := -i\gamma_a$, which then satisfy the anticommutator relations

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab} I_N,$$

which seems to fit nicely, e.g., with Equation (10.66) in [**35**] (where also the signature of the Minkowski metric matches with Hamilton's choice). Furthermore, additional "raising or lowering of indices", with $(\eta^{ab})_{1\le a,b\le n}$ denoting the inverse of $(\eta_{ab})_{1\le a,b\le n}$, as in $\gamma^a = \sum_{b=1}^n \eta^{ab}\gamma_b$, or $\gamma^a = \eta^{ab}\gamma_b$ with the Einstein summation convention, may be involved and has to be taken care of, if one wants to compare the precise forms of basic relations across the scientific communities.

We single out some conventions for the physical gamma matrices that will be used down the line

$$\Gamma^a = \eta^{ac}\Gamma_c, \quad \Gamma^{bc} = \frac{1}{2}[\Gamma^b, \Gamma^c] = \frac{1}{2}(\Gamma^b\Gamma^c - \Gamma^c\Gamma^b).$$

Let us determine the images of the Clifford products (dropping γ) $e_j \cdot e_{j+1}\cdots e_n$ $(1 \le j \le n)$ under the (vector space) isomorphism $\sigma\colon \mathrm{Cl}(s,t) \to \Lambda(\mathbb{R}^n)$, the symbol map, according to Theorem 1.2.10: For any $l \in \{1,\dots,n\}$, we have from the construction in that proof, $\sigma(e_l) = \delta(e_l)(1) = e_l \wedge 1 - e_l \lrcorner 1 = e_l$; the basic relation established in the theorem, namely $\sigma(\gamma(v)\cdot u) = v\wedge\sigma(u) - v\lrcorner\sigma(u)$, then yields $\sigma(e_l\cdot e_k) = e_l \wedge \sigma(e_k) - e_l\lrcorner\sigma(e_k) = e_l\wedge e_k - e_l\lrcorner e_k = e_l \wedge e_k$ for any $l,k\in\{1,\dots,n\}$ with $l \ne k$, and we proceed inductively to obtain

$$\sigma(e_j\cdot e_{j+1}\cdots e_n) = e_j\wedge\sigma(e_{j+1}\cdots e_n) - e_j\lrcorner\sigma(e_{j+1}\cdots e_n) = e_j\wedge\ldots\wedge e_n \quad (1\le j\le n),$$

in particular, $\sigma(e_1\cdots e_n) = e_1\wedge\ldots\wedge e_n$ gives the standard volume form on $\mathbb{R}^n$, which can also be used to choose an orientation on $\mathbb{R}^n$. The above reasoning about the images under σ is still true if we change to another η-orthonormal basis $e'_1,\dots,e'_n$ that is of the same orientation as the standard basis, i.e, $e'_1\wedge\ldots\wedge e'_n = e_1\wedge\ldots\wedge e_n$. We thus obtain also in the Clifford algebra,

$$e'_1\cdots e'_n = \sigma^{-1}(e'_1\wedge\ldots\wedge e'_n) = \sigma^{-1}(e_1\wedge\ldots\wedge e_n) = e_1\cdots e_n.$$

1.2.20. Definition. For any $\lambda\in\mathbb{C}$, an element

$$\omega_\lambda := (e_1\cdots e_n)\otimes\lambda \in \mathrm{Cl}(s,t)\otimes_{\mathbb{R}}\mathbb{C}$$

is called a *chirality element*. We simply write $\omega_\lambda = \lambda e_1\cdots e_n$ and we have seen above that ω_λ does not depend on the choice of oriented η-orthonormal basis of $\mathbb{R}^{s,t}$.

1.2.21. Lemma. *Let n be even, say $n = 2k$ with $k\in\mathbb{N}$, and ω_λ with $\lambda\in\mathbb{C}$ be a chirality element for $\mathrm{Cl}(s,t)$ with $s+t=n$. Then the following hold:*

(i) $\{\omega_\lambda, e_a\} = 0$ and $[\omega_\lambda, e_a\cdot e_b] = 0$ $(1,\le a,b\le n)$,

(ii) if $\lambda^2 = (-1)^{k+t}$, then $\omega_\lambda^2 = 1$.

proof. (i): Applying the Clifford relations repeatedly, we obtain $\omega_\lambda\cdot e_a = \lambda e_1\cdots e_n\cdot e_a = \lambda(-1)^{n-a}e_1\cdots e_a\cdot e_a\cdots\cdot e_n$, while $e_a\cdot\omega_\lambda = \lambda(-1)^{a-1}e_1\cdots e_a\cdot e_a\cdots\cdot e_n$ and, since n is even, $(-1)^{n-a}$ and $(-1)^{a-1}$ have opposite sign, thus the anticommutator relation follows.

We may now apply the anticommutator relations to obtain

$$[\omega_\lambda, e_a\cdot e_b] = \omega_\lambda\cdot e_a\cdot e_b - e_a\cdot e_b\cdot\omega_\lambda = (-e_a\cdot\omega_\lambda)\cdot e_b - e_a\cdot(-\omega_\lambda\cdot e_b) = 0.$$

(ii): In $\omega_\lambda^2 = \lambda^2 e_1\cdots e_n\cdot e_1\cdots e_n$ we repeatedly anticommute some e_j with e_k $(j>k)$, say m times, to arrive at

$$\omega_\lambda^2 = (-1)^m\lambda^2(e_1)^2\cdots(e_n)^2 = (-1)^m\lambda^2(-1)^s = (-1)^{m+k+s+t} = (-1)^{m+k}$$

(because $s+t=n=2k$) and it remains to determine m: First, we have to anticommute e_1 from the middle part $n-1$ times through the subproduct $e_2\cdots e_n$ to the second place from the left, then e_2 is moved from the middle part to the left of the subproduct $e_3\cdots e_n$ by $n-2$ anticommutations; we proceed in this way until the last required anticommutation of e_{n-1} with e_n; thus, $m = (n-1)+(n-2)+\ldots+2+1 = (n-1)n/2 = (2k-1)k$. We obtain $m+k = 2k^2-k+k = 2k^2$ and therefore, $\omega_\lambda^2 = (-1)^{m+k} = (-1)^{2k^2} = 1$. $\square$

1.2.22. Definition. Let $\dim \mathbb{R}^{s,t}$ be even, i.e., $n = s + t$ and $n = 2k$ for some $k \in \mathbb{N}$.

(i) We choose *the chirality element* for $\mathrm{Cl}(s,t)$ to be

$$\omega := -i^{k+t} e_1 \cdots e_n \in \mathrm{Cl}(s,t) \otimes_{\mathbb{R}} \mathbb{C}.$$

(We see that in case $k + t$ even, $\omega \in \mathrm{Cl}(s,t)$.)

(ii) If $\gamma_1, \ldots, \gamma_n$ are the gamma matrices of a complex representation $\rho\colon \mathrm{Cl}(s,t) \to \mathrm{L}(\mathbb{C}^n)$, then

$$\gamma_{n+1} := -i^{k+t} \gamma_1 \cdots \gamma_n \in \mathrm{L}(\mathbb{C}^N)$$

is called the (mathematical) *chirality operator* for the representation ρ. The *physical chirality operator* is defined analogously by $\Gamma_{n+1} := -i^{k+t}\Gamma_1 \cdots \Gamma_n$, and we also set $\Gamma^{n+1} = -i^{k+t}\Gamma^1 \cdots \Gamma^n$. Using Remark 1.2.19 and Lemma 1.2.21 we obtain:

$$\{\Gamma^{n+1}, \Gamma^a\} = 0, \quad [\Gamma^{n+1}, \Gamma^{bc}] = 0, \quad \gamma^{bc} = -\Gamma^{bc}. \tag{1.12}$$

Note that due to Lemma 1.2.21, $\omega^2 = 1$ and ω anticommutes with all elements in $\mathbb{C}^n \cong \gamma(\mathbb{R}^{s,t}) \otimes_{\mathbb{R}} \mathbb{C} \subseteq \mathrm{Cl}(s,t) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}\mathrm{l}(n)$.

1.2.23. Remark. For even n and according to (ii) in the previous lemma, we have $\gamma_{n+1}^2 = I_N$, i.e., $0 = \gamma_{n+1}^2 - I_N = (\gamma_{n+1} - I_N)(\gamma_{n+1} + I_N)$. Note that with $N \geq 1$ we cannot have $\gamma_{n+1} = \pm I_N$, since (i) in the above lemma leads to $0 = \{\gamma_{n+1}, \gamma_a\} = \pm(I_N\gamma_a + \gamma_a I_N) = \pm 2\gamma_a$, hence $\gamma_a = 0$, but the Clifford relation requires $\gamma_a^2 = -\eta_{aa} I_N = \mp I_N \neq 0$. We conclude that the minimal polynomial of γ_{n+1} is $p(\lambda) = \lambda^2 - 1$ (compare with a similar argument for the parity automorphism α earlier), hence γ_{n+1} is diagonalizable with eigenvalues -1 and 1 and we have the eigenspace decomposition $\mathbb{C}^N = E_1 \oplus E_{-1}$, where $E_{\pm 1} = \{\psi \in \mathbb{C}^N \mid \gamma_{n+1}\psi = \pm\psi\}$.

We claim that $\dim E_1 = \dim E_{-1}$, in particular, N has to be even. Pick any standard basis vector e_a and recall that e_a is invertible in $\mathrm{Cl}(s,t)$, hence $\rho(e_a) = \gamma_a \in \mathrm{GL}(n, \mathbb{C})$. We show that $\rho(e_a)$ maps E_1 into E_{-1} and E_{-1} into E_1, which yields $E_1 \cong E_{-1}$. Let $\psi \in E_{\pm 1}$, then applying the anticommutator property in Lemma 1.2.21(i), or rather its proof with $\omega_1 := e_1 \cdots e_n$, we obtain

$$\begin{aligned}\gamma_{n+1}\rho(e_a)\psi &= -i^{k+t}\gamma_1 \cdots \gamma_n \rho(e_a)\psi = -i^{k+t}\rho(e_1 \cdots e_n)\rho(e_a)\psi = -i^{k+t}\rho(\omega_1 \cdot e_a)\psi \\ &= -i^{k+t}\rho(-e_a \cdot \omega_1)\psi = -\rho(e_a)(-i^{k+t}\rho(\omega_1)\psi) = -\rho(e_a)\gamma_{n+1}\psi = -\rho(e_a)(\pm\psi) = \mp\rho(e_a)\psi\end{aligned}$$

and conclude that $\psi \in E_{\pm 1}$ implies $\rho(e_a)\psi \in E_{\mp 1}$.

1.2.24. Example (Clifford algebras for the one-dimensional standard spaces). (i) We have seen already in Example 1.2.9(ii) that $\mathrm{Cl}(1,0) = \mathrm{Cl}(1) = \mathbb{C}$ (as a real 2-dimensional algebra) with $\gamma(x) = ix$ for all $x \in \mathbb{R}^{1,0}$ and the usual multiplication of complex numbers.

(ii) We know that $\dim \mathrm{Cl}(0,1) = 2$ and $\mathrm{Cl}(0,1)$ is the linear span of I and $\gamma(e_1)$, where we temporarily use I to denote the unit in $\mathrm{Cl}(0,1)$. The Clifford relation $\gamma(e_1)^2 = I$ implies $(\gamma(e_1) + I) \cdot (\gamma(e_1) - I) = 0$, hence it will be convenient to use $u_+ := (\gamma(e_1) + I)/2$ and $u_- := -(\gamma(e_1) - I)/2$ as basis and write any element of $\mathrm{Cl}(0,1)$ uniquely in the form $xu_+ + yu_-$ with $x, y \in \mathbb{R}$. Observing $u_+u_- = 0 = u_-u_+$ and $u_+^2 = (\gamma(e_1) + I)(\gamma(e_1) + I)/4 = (2\gamma(e_1) + 2I)/4 = u_+$ and, similarly, $u_-^2 = u_-$, the multiplication then reads

$$(xu_+ + yu_-) \cdot (x'u_+ + y'u_-) = xx'u_+^2 + xy'u_+u_- + yx'u_-u_+ + yy'u_-^2 = xx'u_+ + yy'u_-.$$

Therefore, $u_+ \mapsto (1,0)$ and $u_- \mapsto (0,1)$ and linear extension yields an isomorphism of $\mathrm{Cl}(0,1)$ with the direct sum algebra $\mathbb{R} \oplus \mathbb{R}$.

(iii) From (ii) we may deduce directly that $\mathbb{C}\mathrm{l}(1) \cong \mathrm{Cl}(0,1) \otimes_{\mathbb{R}} \mathbb{C} \cong (\mathbb{R} \oplus \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$ with multiplication in each component separately.

Before proceeding with further examples of Clifford algebras for standard spaces, it is useful to recall the complex Hermitian and traceless (2×2)-matrices, known as *Pauli matrices*,

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.13}$$

They satisfy the following relations for any cyclic permutation (j,k,l) of $(1,2,3)$

$$\sigma_j^2 = I_2 \quad \text{and} \quad \sigma_j\sigma_k = -\sigma_k\sigma_j = i\sigma_l, \tag{1.14}$$

which imply

$$[\sigma_j, \sigma_k] = 2i\sigma_l \quad (\text{if } (j,k,l) \text{ is a cyclic permutation of } (1,2,3)) \tag{1.15}$$

and

$$\{\sigma_j, \sigma_k\} = 2\delta_{j,k}\, I_2 \quad (1 \le j,k \le 3). \tag{1.16}$$

1.2.25. Example (Clifford algebras for the 2-dimensional standard spaces). (i) We know that $\dim \mathrm{Cl}(2,0) = 4$ and observe that the (2×2)-matrices $\gamma_1 := i\sigma_1$ and $\gamma_2 := i\sigma_2$ satisfy the Clifford relations

$$\gamma_1^2 = -I_2 = \gamma_2^2 \quad \text{and} \quad \{\gamma_1, \gamma_2\} = 0.$$

Therefore, we may interpret $\gamma_j = \tilde{\gamma}(e_j)$ $(j = 1,2)$ with the linear map $\tilde{\gamma}\colon \mathbb{R}^{2,0} \to \mathrm{L}(\mathbb{C}^2)$ obtained by linear extension. The Clifford algebra $\mathrm{Cl}(2,0)$ is isomorphic to the 4-dimensional real vector subspace W of $\mathrm{L}(\mathbb{C}^2)$ which is the $\mathbb{R}$-linear span of $\{I_2, \gamma_1, \gamma_2, \gamma_1\gamma_2\}$ and we may consider $\tilde{\gamma}$ with target space $W \supseteq \tilde{\gamma}(\mathbb{R}^{2,0})$ (instead of $\mathrm{L}(\mathbb{C}^2)$) as the corresponding Clifford map. Note that W is isomorphic to the algebra of quaternions $\mathbb{H}$, since for any $a,b,c,d \in \mathbb{R}$,

$$aI_2 + b\gamma_1 + c\gamma_2 + d\gamma_1\gamma_2 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & bi \\ bi & 0 \end{pmatrix} + \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix} + \begin{pmatrix} -di & 0 \\ 0 & di \end{pmatrix} = \begin{pmatrix} a - di & c + bi \\ -c + bi & a + di \end{pmatrix}$$

and $\mathbb{H}$ can be realized as complex (2×2)-matrices of the form $\left(\begin{smallmatrix} \lambda & -\overline{\mu} \\ \mu & \overline{\lambda} \end{smallmatrix}\right)$ with $\lambda, \mu \in \mathbb{C}$ (cf. [**19**, Example 1.1.23]).

(ii) In case of $\mathrm{Cl}(1,1)$ it is convenient to consider $\gamma_1 := i\sigma_1$ and $\gamma_2 := \sigma_2$ to obtain the Clifford relations

$$\gamma_1^2 = -I_2, \quad \gamma_2^2 = I_2, \quad \{\gamma_1, \gamma_2\} = 0.$$

A model of $\mathrm{Cl}(1,1)$ is then given as $W' :=\mathbb{R}\text{-span}\{I_2, \gamma_1, \gamma_2, \gamma_1\gamma_2\} \subseteq \mathrm{L}(\mathbb{C}^2)$. Using instead the basis $(I_2 - \gamma_1\gamma_2)/2, (\gamma_1 - \gamma_2)/2i, (\gamma_1 + \gamma_2)/2i, (I_2 + \gamma_1\gamma_2)/2$ of W', we see that $W' \cong \mathrm{L}(\mathbb{R}^2)$.

(iii) It is easy to see that $\mathrm{Cl}(0,2) \cong \mathrm{L}(\mathbb{R}^2)$ with Clifford map $\gamma\colon \mathbb{R}^{0,2} \to \mathrm{L}(\mathbb{R}^2)$, $(x_1, x_2) \mapsto \left(\begin{smallmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{smallmatrix}\right)$. We verify the Clifford property

$$\gamma(x)^2 = \begin{pmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{pmatrix}^2 = (x_1^2 + x_2^2)I_2 = -\eta(x,x)I_2$$

and note that with $\gamma_1 := \gamma(1,0) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\gamma_2 := \gamma(0,1) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ we have $\gamma_1\gamma_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, so that $\mathrm{span}\{I_2, \gamma_1, \gamma_2, \gamma_1\gamma_2\} = \mathrm{L}(\mathbb{R}^2)$ and $\dim \mathrm{L}(\mathbb{R}^2) = \dim \mathrm{Cl}(0,2)$.

(iv) We claim that $\mathbb{C}\mathrm{l}(2) \cong \mathrm{L}(\mathbb{C}^2)$: The complex dimensions fit and $\gamma(z_1, z_2) := z_1 i\sigma_1 + z_2 i\sigma_2$ defines a Clifford map $\mathbb{C}^2 \to \mathrm{L}(\mathbb{C}^2)$, since (1.14) and (1.16) imply

$$\gamma(z)^2 = (z_1 i\sigma_1 + z_2 i\sigma_2)^2 = -z_1^2\sigma_1^2 - z_1z_2(\sigma_1\sigma_2 + \sigma_2\sigma_1) - z_2^2\sigma_2^2 = -(z_1^2 + z_2^2)I_2.$$

The matrices $I_2, i\sigma_1, i\sigma_2$ together with $(i\sigma_1)(i\sigma_2) = -\sigma_1\sigma_2 = -i\sigma_3$ are linearly independent and generate $\mathrm{L}(\mathbb{C}^2)$.

1.2.26. Remark (Clifford algebras for the 3-dimensional standard spaces). With a bit of theory, e.g., about how to produce Clifford algebras for higher dimensional standard spaces as tensor products of those over lower dimensions (cf. [**26**, Chapter I, Theorems 4.1 and 4.3]), the following isomomorphisms for the 8-dimensional Clifford algebras can be established ([**26**, Chapter I, Section 4, Tables I and II]): $\mathrm{Cl}(3,0) \cong \mathbb{H} \oplus \mathbb{H}$, $\mathrm{Cl}(2,1) \cong \mathrm{L}(\mathbb{C}^2)$ (as real algebra), $\mathrm{Cl}(1,2) \cong \mathrm{L}(\mathbb{R}^2) \oplus \mathrm{L}(\mathbb{R}^2)$, $\mathrm{Cl}(0,3) \cong \mathrm{L}(\mathbb{C}^2)$ (as real algebra), and $\mathbb{C}\mathrm{l}(3) \cong \mathrm{L}(\mathbb{C}^2) \oplus \mathrm{L}(\mathbb{C}^2)$ (as complex algebra).

1.2.27. Example (Clifford algebras for the 4-dimensional standard spaces). Only a few aspects of the 16-dimensional Clifford algebras are illustrated, but we give references for further details.

(i) Let us start with the complex case $\mathbb{C}\mathrm{l}(4)$, where we consider $\gamma\colon \mathbb{C}^4 \to \mathrm{L}(\mathbb{C}^4)$, given by $\mathbb{C}$-linear extension from the assignment $\gamma(e_a) := \gamma_a$ of the following complex (4×4)-matrices γ_a, defined by (2×2)-blocks, to the standard basis vectors e_a $(a = 1,2,3,4)$ of $\mathbb{C}^4$:

$$\gamma_k := \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \ (k = 1,2,3), \quad \gamma_4 := \begin{pmatrix} 0 & iI_2 \\ iI_2 & 0 \end{pmatrix}.$$

We have to verify the Clifford relations

$$\{\gamma_a, \gamma_b\} = -2\delta_{ab} I_4 \quad (1 \le a, b \le 4).$$

For $1 \le j, k \le 3$ we may employ (1.16) to deduce

$$\begin{aligned}\{\gamma_j, \gamma_k\} &= \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\sigma_j\sigma_k & 0 \\ 0 & -\sigma_j\sigma_k \end{pmatrix} + \begin{pmatrix} -\sigma_k\sigma_j & 0 \\ 0 & -\sigma_k\sigma_j \end{pmatrix} = -\begin{pmatrix} \{\sigma_j, \sigma_k\} & 0 \\ 0 & \{\sigma_j, \sigma_k\} \end{pmatrix} = -2\delta_{jk} I_4.\end{aligned}$$

From $\gamma_4\gamma_4 = -I_4$ we get $\{\gamma_4, \gamma_4\} = -2I_4$, and for $1 \le k \le 3$ we calculate $\gamma_k\gamma_4 = \left(\begin{smallmatrix} i\sigma_k & 0 \\ 0 & -i\sigma_k \end{smallmatrix}\right)$ and $\gamma_4\gamma_k = \left(\begin{smallmatrix} -i\sigma_k & 0 \\ 0 & i\sigma_k \end{smallmatrix}\right)$, hence $\{\gamma_k, \gamma_4\} = 0$ and all Clifford relations hold.

By Corollary 1.2.11, the model of $\mathbb{C}\mathrm{l}(4)$ generated from $\gamma(\mathbb{C}^4) \subseteq \mathrm{L}(\mathbb{C}^4)$ possesses the basis

$$B := \{I_4, \gamma_1, \gamma_2, \gamma_3, \gamma_4\} \cup \{\gamma_a\gamma_b \mid 1 \le a < b \le 4\} \cup \{\gamma_a\gamma_b\gamma_c \mid 1 \le a < b < c \le 4\} \cup \{\gamma_1\gamma_2\gamma_3\gamma_4\}, \tag{1.17}$$

which consists of $5+6+4+1 = 16$ elements and thus spans all of $\mathrm{L}(\mathbb{C}^4)$. Therefore, $\mathbb{C}\mathrm{l}(4) \cong \mathrm{L}(\mathbb{C}^4)$.

(ii) To study $\mathrm{Cl}(4,0)$ we may essentially copy the Clifford map from (i), meaning $\gamma\colon \mathbb{R}^4 \to \mathrm{L}(\mathbb{C}^4)$ with $\gamma_a = \gamma(e_a)$ $(a = 1,2,3,4)$ defined in the exact same way, but using $\mathbb{R}$-linear extension. Now the model of $\mathrm{Cl}(4,0)$ is given by $\mathbb{R}\text{-span}\, B$, where B is the basis given in (1.17). It can be shown that $\mathrm{Cl}(4,0)$ is isomorphic to the algebra of quaternionic (2×2)-matrices $\mathrm{L}(\mathbb{H}^2)$, which may also be realized as $\mathbb{H} \otimes_{\mathbb{R}} \mathrm{L}(\mathbb{R}^2)$ (cf. [**26**, Chapter I, Section 4, Table II] and [**16**, Section 10.20]).

(iii) The Clifford algebra $\mathrm{Cl}(1,3)$ over Minkowski space $\mathbb{R}^{1,3}$ can be constructed from the Clifford map $\gamma\colon \mathbb{R}^{1,3} \to \mathrm{L}(\mathbb{C}^4)$ via the generators $\gamma(e_a) := \gamma_{a-1}$ $(a = 1,2,3,4)$, where

$$\gamma_0 := \begin{pmatrix} 0 & iI_2 \\ iI_2 & 0 \end{pmatrix}, \quad \gamma_k := \begin{pmatrix} 0 & i\sigma_k \\ -i\sigma_k & 0 \end{pmatrix} \ (k = 1,2,3).$$

Indeed, we obtain by direct calculations very similar to those in (i) that

$$\gamma_0^2 = -I_4, \quad \gamma_k^2 = I_4 \ (k = 1,2,3), \quad \{\gamma_a, \gamma_b\} = 0 \ (0 \le a, b \le 3, a \ne b)$$

and the real linear hull of the analog of the set B in (1.17) gives $\mathrm{Cl}(1,3)$ as real 16-dimensional subalgebra of $\mathrm{L}(\mathbb{C}^4)$. It can be shown by other means, establishing $\mathrm{Cl}(0,2) \otimes \mathrm{Cl}(1,1) \cong \mathrm{Cl}(1,3)$, that $\mathrm{Cl}(1,3) \cong \mathrm{L}(\mathbb{R}^2) \otimes \mathrm{L}(\mathbb{R}^2) \cong \mathrm{L}(\mathbb{R}^4)$ ([**26**, Chapter I, Section 4, Theorem 4.1 and Table II]).

(iv) With the gamma matrices from (iii), we now set $\gamma_k' := i\gamma_k$ $(k = 1,2,3)$, $\gamma_4' := i\gamma_0$, and obtain the Clifford relations for generators of a model of the algebra $\mathrm{Cl}(3,1)$ as real subalgebra of $\mathrm{L}(\mathbb{C}^4)$,

$${\gamma_k'}^2 = -I_4 \ (k = 1,2,3), \ {\gamma_4'}^2 = I_4, \ \{\gamma_a', \gamma_b'\} = 0 \ (1 \le a, b \le 4, a \ne b).$$

One can prove by other methods that $\mathrm{Cl}(2,0)\otimes\mathrm{Cl}(1,1)\cong\mathrm{Cl}(3,1)$, hence $\mathrm{Cl}(3,1)\cong\mathbb{H}\otimes_{\mathbb{R}}\mathrm{L}(\mathbb{R}^2)\cong\mathrm{L}(\mathbb{H}^2)$ ([**26**, Chapter I, Section 4, Theorem 4.1 and Table II]).

(v) For the remaining cases we do not give explicit constructions of isomorphisms and instead refer to [**26**, Chapter I, Section 4] for the proofs. The results are $\mathrm{Cl}(2,2)\cong\mathrm{L}(\mathbb{R}^4)$ and $\mathrm{Cl}(0,4)\cong\mathrm{L}(\mathbb{H}^2)$.

We will not discuss the classification of the real Clifford algebras $\mathrm{Cl}(s,t)$, which shows a surprising periodicity with respect to $s-t$ modulo 8 in terms of the basic structure as real subalgebra of a real, complex, or quaternionic matrix algebra of growing dimension depending on $s+t$. We refer to [**20**] or [**26**] for more details on this.

In the context of applications in physics, the emphasis is on complex representations of the real Clifford algebras and these can always be naturally extended to become representations of the complexified Clifford algebra. Recall the simple complexification result $\mathrm{Cl}(s,t)\otimes_{\mathbb{R}}\mathbb{C}\cong\mathbb{C}\mathrm{l}(s+t)$ from Lemma 1.2.16. It will therefore be sufficient for our purposes to understand the basic structure of the complex Clifford algebras $\mathbb{C}\mathrm{l}(n)$ $(n\geq 1)$. Furthermore, this will automatically provide us also with a classification of the complex even Clifford subalgebras, since $\mathbb{C}\mathrm{l}^0(n)\cong\mathbb{C}\mathrm{l}(n-1)$ holds thanks to Lemma 1.2.17. So far we know that $\mathbb{C}\mathrm{l}(0):=\mathbb{C}$ by convention, $\mathbb{C}\mathrm{l}(1)\cong\mathbb{C}\oplus\mathbb{C}$ and $\mathbb{C}\mathrm{l}(2)\cong\mathrm{L}(\mathbb{C}^2)$ as seen in examples above, $\mathbb{C}\mathrm{l}(3)\cong\mathrm{L}(\mathbb{C}^2)\oplus\mathrm{L}(\mathbb{C}^2)$ as mentioned in a remark, and $\mathbb{C}\mathrm{l}(4)\cong\mathrm{L}(\mathbb{C}^4)$ as described in the first of the 4-dimensional examples above. We will show that these alternating basic patterns for odd and even dimensions are systematic.

1.2.28. Lemma. *For all* $n\in\mathbb{N}_0$ *we have* $\mathbb{C}\mathrm{l}(n+2)\cong\mathbb{C}\mathrm{l}(n)\otimes\mathbb{C}\mathrm{l}(2)$.

proof. Let ω denote the chirality element for $\mathbb{C}\mathrm{l}(2)\cong\mathrm{Cl}(2,0)\otimes_{\mathbb{R}}\mathbb{C}$ and observe $\mathbb{C}^{n+2}\cong\mathbb{C}^n\oplus\mathbb{C}^2$. We consider the linear map $\delta\colon\mathbb{C}^n\oplus\mathbb{C}^2\to\mathbb{C}\mathrm{l}(n)\otimes\mathbb{C}\mathrm{l}(2)$, $(x,y)\mapsto x\otimes\omega+1\otimes y$ and show that it is a Clifford map: Since ω anticommutes with every element in $\mathbb{C}^2\subseteq\mathbb{C}\mathrm{l}(2)$ and $\omega^2=1$,

$$\begin{aligned}\delta(x,y)^2&=(x\otimes\omega+1\otimes y)\cdot(x\otimes\omega+1\otimes y)\\&=x^2\otimes\omega^2+x\otimes(\omega y)+x\otimes(y\omega)+1\otimes y^2=x^2\otimes 1+x\otimes\{\omega,y\}+1\otimes y^2\\&=x^2\otimes 1+1\otimes y^2=-(q(x,x)+q(y,y))\,1\otimes 1=-q((x,y),(x,y))\,1\otimes 1,\end{aligned}$$

where we abused notation in writing q for the standard bilinear forms on $\mathbb{C}^n$, $\mathbb{C}^2$, and $\mathbb{C}^{n+2}$.

Let us relabel the standard basis in $\mathbb{C}^2$ by e_{n+1},e_{n+2}, then $\omega=-ie_{n+1}e_{n+2}$ and the images of the standard basis vectors are $\delta(e_a,0)=e_a\otimes\omega=-ie_a\otimes e_{n-1}e_{n+2}$ $(a=1,\ldots,n)$ and $\delta(0,e_b)=1\otimes e_b$ $(b=n+1,n+2)$. Taking products of these produces $\delta(e_a,0)^2=-1\otimes 1$, $\delta(e_a,0)\cdot\delta(0,e_{n+1})=-ie_a\otimes(e_{n+1}e_{n+2}e_{n+1})=ie_a\otimes(e_{n+1}^2e_{n+2})=-ie_a\otimes e_{n+2}$, $\delta(e_a,0)\cdot\delta(0,e_{n+2})=-ie_a\otimes(e_{n+1}e_{n+2}^2)=ie_a\otimes e_{n+1}$, and $\delta(0,e_{n+1})\cdot\delta(0,e_{n+2})=1\otimes(e_{n+1}e_{n+2})=i1\otimes\omega$. Now we see that we can produce also $e_a\otimes 1$ from the product $-i\delta(e_a,0)\cdot\delta(0,e_{n+1})\cdot\delta(0,e_{n+2})$. Proceeding in this way, we are able to obtain all elements of a basis for $\mathbb{C}\mathrm{l}(n)\otimes\mathbb{C}\mathrm{l}(2)$ as images under the unique algebra homomorphism $\phi\colon\mathbb{C}\mathrm{l}(n+2)\to\mathbb{C}\mathrm{l}(n)\otimes\mathbb{C}\mathrm{l}(2)$ extending the Clifford map δ according to the universal property, i.e., ϕ is surjective. Since $\dim\mathbb{C}\mathrm{l}(n+2)=2^{n+2}=2^n\cdot 2^2=\dim\mathbb{C}\mathrm{l}(n)\cdot\dim\mathbb{C}\mathrm{l}(2)=\dim\mathbb{C}\mathrm{l}(n)\otimes\mathbb{C}\mathrm{l}(2)$, ϕ is an isomorphism. □

We will apply the previous result combined with the property $\mathrm{L}(\mathbb{C}^l)\otimes\mathrm{L}(\mathbb{C}^m)\cong\mathrm{L}(\mathbb{C}^l\otimes\mathbb{C}^m)$ (see [**16**, Corollary III in Section 1.16] for a more general result with proof), which upon identification of $\mathbb{C}^l\otimes\mathbb{C}^m$ with $\mathbb{C}^{lm}$ results in $\mathrm{L}(\mathbb{C}^l)\otimes\mathrm{L}(\mathbb{C}^m)\cong\mathrm{L}(\mathbb{C}^{lm})$.

1.2.29. Theorem (Structure of complex Clifford algebras). *Let* $k\in\mathbb{N}_0$.

(i) If $n=2k$, *then* $\mathbb{C}\mathrm{l}(n)\cong\mathrm{L}(\mathbb{C}^{2^k})$.

(ii) If $n=2k+1$, *then* $\mathbb{C}\mathrm{l}(n)\cong\mathrm{L}(\mathbb{C}^{2^k})\oplus\mathrm{L}(\mathbb{C}^{2^k})$ *and* $\mathbb{C}\mathrm{l}^0(n)\cong\mathrm{L}(\mathbb{C}^{2^k})$.

(iii) If $n=2k+2$, *then* $\mathbb{C}\mathrm{l}^0(n)\cong\mathrm{L}(\mathbb{C}^{2^k})\oplus\mathrm{L}(\mathbb{C}^{2^k})$.

PROOF. Since $\mathbb{C}l^0(n) \cong \mathbb{C}l(n-1)$, the assertions about $\mathbb{C}l^0(n)$ follow from those about $\mathbb{C}l(n)$.

(i): We use induction on k and start with $\mathbb{C}l(0) = \mathbb{C} \cong \mathrm{L}(\mathbb{C}^1) = \mathrm{L}(\mathbb{C}^{2^0})$. If we already know that $\mathbb{C}l(2k-2) = \mathbb{C}l(2(k-1)) \cong \mathrm{L}(\mathbb{C}^{2^{k-1}})$, then Lemma 1.2.28 and $\mathbb{C}l(2) \cong \mathrm{L}(\mathbb{C}^2)$ imply

$$\mathbb{C}l(2k) \cong \mathbb{C}l(2k-2) \otimes \mathbb{C}l(2) \cong \mathrm{L}(\mathbb{C}^{2^{k-1}}) \otimes \mathrm{L}(\mathbb{C}^2) \cong \mathrm{L}(\mathbb{C}^{2^{k-1}2}) = \mathrm{L}(\mathbb{C}^{2^k}).$$

(ii): We use again induction on k, where the base case now is $\mathbb{C}l(1) \cong \mathbb{C} \oplus \mathbb{C} = \mathrm{L}(\mathbb{C}^{2^0}) \oplus \mathrm{L}(\mathbb{C}^{2^0})$. If we already know that $\mathbb{C}l(2k-1) = \mathbb{C}l(2(k-1)+1) \cong \mathrm{L}(\mathbb{C}^{2^{k-1}}) \oplus \mathrm{L}(\mathbb{C}^{2^{k-1}})$ holds, then we may apply Lemma 1.2.28, $\mathbb{C}l(2) \cong \mathrm{L}(\mathbb{C}^2)$, and the "distributive law" for tensor products to deduce

$$\begin{aligned}\mathbb{C}l(2k+1) &\cong \mathbb{C}l(2k-1) \otimes \mathbb{C}l(2) \cong \Big(\mathrm{L}(\mathbb{C}^{2^{k-1}}) \oplus \mathrm{L}(\mathbb{C}^{2^{k-1}})\Big) \otimes \mathrm{L}(\mathbb{C}^2)\\ &\cong \Big(\mathrm{L}(\mathbb{C}^{2^{k-1}}) \otimes \mathrm{L}(\mathbb{C}^2)\Big) \oplus \Big(\mathrm{L}(\mathbb{C}^{2^{k-1}}) \otimes \mathrm{L}(\mathbb{C}^2)\Big) \cong \mathrm{L}(\mathbb{C}^{2^{k-1}2}) \oplus \mathrm{L}(\mathbb{C}^{2^{k-1}2}) = \mathrm{L}(\mathbb{C}^{2^k}) \oplus \mathrm{L}(\mathbb{C}^{2^k}).\end{aligned}$$

□

1.3. Spinor representations

The structure theorem for the complex Clifford algebras reduces representation theory for these to a classification of representations of matrix algebras of the form $\mathrm{L}(\mathbb{C}^N)$ or $\mathrm{L}(\mathbb{C}^N) \oplus \mathrm{L}(\mathbb{C}^N)$. Recall that a representation $\rho\colon A \to \mathrm{L}(W)$ is said to be *reducible*, if W can be decomposed as a non-trivial direct sum $W = W_1 \oplus W_2$ of ρ-invariant subspaces $W_1, W_2 \subseteq W$. We then obtain the subrepresentations $\rho_j\colon A \to \mathrm{L}(W_j)$, $\rho_j(a) := \rho(a)|_{W_j}$, and may write $\rho = \rho_1 \oplus \rho_2$. If such a non-trivial decomposition is not possible, i.e., if there are no non-trivial ρ-invariant subspaces of W, then ρ is called *irreducible*. An algebra representation is *completely reducible*, if it can be decomposed into a direct sum of irreducible subrepresentations. In case of Clifford algebras, it can be shown[2] that every finite-dimensional representation is completely reducible. Thus, for the complex Clifford algebras $\mathbb{C}l(n)$ it remains to determine the irreducible representations, which needs to be done only up to equivalence in the following sense: Two representations $\rho\colon A \to \mathrm{L}(W)$ and $\pi\colon A \to \mathrm{L}(X)$ of A are called *equivalent*, if there is a vector space isomorphism $f\colon W \to X$, called an *intertwiner*, such that for all $a \in A$, we have $\pi(a) = f \circ \rho(a) \circ f^{-1}$, or in terms of a commutative diagram,

$$\begin{array}{ccc} W & \xrightarrow{\rho(a)} & W \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ X & \xrightarrow[\pi(a)]{} & X \end{array}$$

Note that the kernel of an algebra homomorphism is a two-sided ideal. An algebra A of the form $A = \mathrm{L}(U)$, where U is a finite-dimensional $\mathbb{K}$-vector space, is always *simple*, i.e., has no proper nonzero ideals ([**33**, Theorem 18.3]). Therefore, every representation of $\mathrm{L}(\mathbb{C}^N)$ is faithful.

The so-called *natural or standard representation* ρ_0 of a matrix algebra $\mathrm{L}(\mathbb{K}^N)$ is given by the natural action on $\mathbb{K}^N$, i.e., $\rho_0\colon \mathrm{L}(\mathbb{K}^N) \to \mathrm{L}(\mathbb{K}^N)$, where $\rho_0(M)x := Mx$ for $M \in \mathrm{L}(\mathbb{K}^N)$ and $x \in \mathbb{K}^N$. It turns out that up to equivalence, ρ_0 is the only irreducible $\mathbb{K}$-representation of $\mathrm{L}(\mathbb{K}^N)$ ([**20**, Theorem 8.11]). In view of the structure theorem for complex Clifford algebras, this result settles the matter of representation theory for $\mathbb{C}l(n)$ in case of even dimension $n \geq 0$, and for the even subalgebras $\mathbb{C}l^0(n)$ in case of odd dimension $n \geq 1$.

It remains to clarify the representation theory of the complex algebra $A := \mathrm{L}(\mathbb{C}^N) \oplus \mathrm{L}(\mathbb{C}^N)$. Consider the following two representations $\rho_1, \rho_2\colon A \to \mathrm{L}(\mathbb{C}^N)$, defined by $\rho_1(S,T) := \rho_0(S)$ and $\rho_2(S,T) := \rho_0(T)$ for $S, T \in \mathrm{L}(\mathbb{C}^N)$. Both are irreducible, since any subspace of $\mathbb{C}^N$ invariant under ρ_1 or ρ_2 is also ρ_0-invariant. The representations ρ_1 and ρ_2 cannot be equivalent, because an intertwiner $F \in \mathrm{GL}(N,\mathbb{C})$ would have to satisfy $T = FSF^{-1}$ for all $S, T \in \mathrm{L}(\mathbb{C}^N)$, which

[2] essentially, because they are the group algebras of finite groups (cf. [**26**, Chapter I, Proposition 5.4 and the discussion after Proposition 5.15.])

becomes absurd upon setting $T = 0$ and $S = I_N$. Note that the algebra $\mathrm{L}(\mathbb{C}^N) \oplus \mathrm{L}(\mathbb{C}^N)$ is not simple and that $\ker \rho_1 = \{(0, T) \mid T \in \mathrm{L}(\mathbb{C}^N)\} \cong \mathrm{L}(\mathbb{C}^N)$ is a two-sided ideal, similarly for $\ker \rho_2$.

We claim that ρ_1 and ρ_2 describe the only two equivalence classes of irreducible representations of $A = \mathrm{L}(\mathbb{C}^N) \oplus \mathrm{L}(\mathbb{C}^N)$. Suppose $\rho\colon A \to \mathrm{L}(W)$ is an irreducible representation on a finite-dimensional complex vector space W. Note that we have $A \cong B \otimes \mathrm{L}(\mathbb{C}^N)$ with the commutative $\mathbb{C}$-algebra $B := \mathbb{C} \oplus \mathbb{C}$ and we may apply [**16**, Theorem 11.19.1] to obtain a representation $\delta\colon B \to \mathrm{L}(U)$ and an intertwiner $f\colon U \otimes \mathbb{C}^N \to W$ such that the following holds

$$\forall (\lambda, \mu) \in B, \forall S \in \mathrm{L}(\mathbb{C}^N)\colon \quad f \circ (\delta(\lambda, \mu) \otimes \rho_0(S)) = \rho((\lambda, \mu) \otimes S) \circ f,$$

i.e., ρ is equivalent to $\delta \otimes \rho_0$, where ρ_0 is the standard representation of $\mathrm{L}(\mathbb{C}^N)$. Any δ-invariant subspace $U_1 \subseteq U$ implies that $U_1 \otimes \mathbb{C}^N$ is a ρ-invariant subspace, thus δ has to be irreducible. Therefore, U is one-dimensional, since all operators in the commutative subalgebra $\delta(B) \subseteq \mathrm{L}(U)$ can be diagonalized simultaneously and every joint eigenvector is δ-invariant. This leaves only the option of δ being a multiplicative linear functional $B = \mathbb{C} \oplus \mathbb{C} \to \mathbb{C}$, hence there are $a, b \in \mathbb{C}$ such that $\delta(\lambda, \mu) = a\lambda + b\mu$ for all $\lambda, \mu \in \mathbb{C}$ and the condition of multiplicativity then is easily seen to force either $a = 1$ and $b = 0$, i.e., $\delta(\lambda, \mu) = \delta_1(\lambda, \mu) := \lambda$, or $a = 0$ and $b = 1$, i.e., $\delta(\lambda, \mu) = \delta_2(\lambda, \mu) := \mu$. In summary, we have that ρ is equivalent either to $\delta_1 \otimes \rho_0$, which is in turn equivalent to ρ_1, or to $\delta_2 \otimes \rho_0$, which is in the class of ρ_2.

In view of the structure theorem for complex Clifford algebras, we now know also the representation theory for $\mathbb{C}\mathrm{l}(n)$ in case of odd dimension $n \geq 1$, and for the even subalgebras $\mathbb{C}\mathrm{l}^0(n)$ in case of even dimension $n \geq 2$.

1.3.1. Definition. Let $n \in \mathbb{N}_0$, then the *spinor representation* $\pi_n\colon \mathbb{C}\mathrm{l}(n) \to \mathrm{L}(\Delta_n)$ is defined as follows:

(i) If n is even, then $\Delta_n := \mathbb{C}^{2^{n/2}}$ and π_n corresponds to the unique irreducible representation ρ_0 of $\mathrm{L}(\Delta_n)$ (pre-)composed with the isomorphism $\mathbb{C}\mathrm{l}(n) \cong \mathrm{L}(\Delta_n)$ from the structure theorem.

(ii) If n is odd, then $\Delta_n := \mathbb{C}^{2^{(n-1)/2}}$ and π_n corresponds to the irreducible representation ρ_1 of $\mathrm{L}(\Delta_n) \oplus \mathrm{L}(\Delta_n)$ (pre-)composed with the structure theorem isomorphism $\mathbb{C}\mathrm{l}(n) \cong \mathrm{L}(\Delta_n) \oplus \mathrm{L}(\Delta_n)$.

In both cases, Δ_n is called the space of *Dirac spinors*.

In case of even n, the spinor representation is the unique irreducible representation of $\mathbb{C}\mathrm{l}(n)$ given in (i). In case of odd n, there are two classes of irreducible representations of $\mathbb{C}\mathrm{l}(n)$, one is the spinor representation given in (ii) and the other one corresponds to the choice of ρ_2 there.

The spinor representation induces actions of vectors from $\mathbb{R}^{s,t}$ or of forms from $\Lambda(\mathbb{R}^{s,t})$ as linear operators on spinors, based on applying π_n after a linear map into the Clifford algebra, which was tacitly extended to the complex case by $\mathrm{Cl}(s, t) \hookrightarrow \mathbb{C}\mathrm{l}(n)$: One stems from the embedding $\gamma\colon \mathbb{R}^{s,t} \to \mathrm{Cl}(s, t)$ and the other one can be considered a generalization of the first and uses the inverse of the symbol map $\sigma\colon \mathrm{Cl}(s, t) \to \Lambda(\mathbb{R}^{s,t})$ according to Theorem 1.2.10.

1.3.2. Definition. The *Clifford multiplication* $\mathbb{R}^{s,t} \times \Delta_n \to \Delta_n$ is given by $(X, \psi) \mapsto X \cdot \psi := \pi_n(\gamma(X))\psi$ and the *Clifford multiplication of spinors with forms* $\Lambda(\mathbb{R}^{s,t}) \times \Delta_n \to \Delta_n$ is defined by the assignment $(\omega, \psi) \mapsto \pi_n(\sigma^{-1}(\omega))\psi$.

In particular, for a standard basis vector $e_a \in \mathbb{R}^{s,t}$ the notation $e_a \cdot \psi$ means the spinor $\gamma_a\psi$, if γ_a denotes the corresponding gamma matrix for π_n, i.e., $\gamma_a = \pi_n(e_a)$.

1.3.3. Proposition. *Let $n = 2k$ with $k \in \mathbb{N}$ and γ_{n+1} be the chirality operator for the spinor representation $\pi_n\colon \mathbb{C}\mathrm{l}(n) \to \mathrm{L}(\Delta_n)$.*

(i) We have the decomposition $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$, where $\Delta_n^\pm$ is the eigenspace of γ_{n+1} for the eigenvalue ± 1 and $\dim \Delta_n^\pm = \dim \Delta_n / 2$.

(ii) Both Δ_n^+ and Δ_n^- are invariant under the action of the even Clifford subalgebra $\pi_n(\mathbb{C}\mathrm{l}^0(n))$, while every element of the odd part $\pi_n(\mathbb{C}\mathrm{l}^1(n))$ maps Δ_n^+ into Δ_n^- and Δ_n^- into Δ_n^+. We obtain the isomorphism $\mathbb{C}\mathrm{l}^0(n) \cong \mathrm{L}(\Delta_n^+) \oplus \mathrm{L}(\Delta_n^-)$ as complex algebras and the vector space isomorphism $\mathbb{C}\mathrm{l}^1(n) \cong \mathrm{L}(\Delta_n^-, \Delta_n^+) \oplus \mathrm{L}(\Delta_n^+, \Delta_n^-)$.

PROOF. (i) follows directly from Remark 1.2.23.

(ii): Lemma 1.2.21(i) implies $[\gamma_{n+1}, \gamma_a\gamma_b] = 0$, which shows that the eigenspaces of γ_{n+1} are invariant under the action of $\mathbb{C}\mathrm{l}^0(n)$. By the same lemma, we have the anticommutator relation $\{\gamma_{n+1}, \gamma_a\} = 0$, which shows that every γ_a maps $\Delta_n^\pm$ into $\Delta_n^\mp$ and we deduce that the same is true of $\pi_n(u)$ for every $u \in \mathbb{C}\mathrm{l}^1(n)$. Therefore, the isomorphism $\mathbb{C}\mathrm{l}(n) \cong \mathrm{L}(\Delta_n)$ together with $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$ allows for a block decomposition for any element of $\pi_n(\mathbb{C}\mathrm{l}(n))$ in the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \text{ where } A \in \mathrm{L}(\Delta_n^+), D \in \mathrm{L}(\Delta_n^-), B\colon \Delta_n^- \to \Delta_n^+, C\colon \Delta_n^+ \to \Delta_n^-,$$

where $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right) \in \pi_n(\mathbb{C}\mathrm{l}^0(n))$ and $\left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right) \in \pi_n(\mathbb{C}\mathrm{l}^1(n))$ provide us with the claimed isomorphisms. □

1.3.4. COROLLARY. *Let $n \in \mathbb{N}$ and π_n^0 denote the restriction of the spinor representation π_n to the even subalgebra $\mathbb{C}\mathrm{l}^0(n)$.*

(i) If n is odd, then $\pi_n^0\colon \mathbb{C}\mathrm{l}^0(n) \to \mathrm{L}(\Delta_n)$ is an irreducible algebra representation.

(ii) If n is even, then π_n^0 is the sum of two irreducible subrepresentations $\pi_n^\pm\colon \mathbb{C}\mathrm{l}^0(n) \to \mathrm{L}(\Delta_n^\pm)$, where $\Delta_n^\pm$ is the eigenspace of the chirality operator for the eigenvalue ± 1 and $\dim \Delta_n^\pm = \dim \Delta_n/2$.

PROOF. (i): We have $\dim \mathbb{C}\mathrm{l}^0(n) = \dim \mathbb{C}\mathrm{l}(n)/2 = 2^{n-1} = (\dim \Delta_n)^2 = \dim \mathrm{L}(\Delta_n)$ and claim that π_n^0 is injective. This implies (i), since by dimensionality reasons it is also surjective and thus the image is all of $\mathrm{L}(\Delta_n)$, which acts irreducibly on Δ_n. As for the faithfulness of the representation π_n^0, we only have to note that by the structure theorem, $\mathbb{C}\mathrm{l}^0(n) \cong \mathrm{L}(\mathbb{C}^{2^{(n-1)/2}})$ as an algebra, thus $\mathbb{C}\mathrm{l}^0(n)$ is simple and so the kernel of π_n^0 has to be trivial.

The claim in (ii) follows immediately from Proposition 1.3.3. □

The spaces Δ_n^+ and Δ_n^- are sometimes called the spaces of *left-handed and right-handed Weyl spinors*, but note that the identification with eigenspaces depends on the sign convention used for the chirality operator.

1.3.5. REMARK. For the decomposition into eigenspaces of the chirality operator it does not matter which form of ω is chosen (cf. Definition 1.2.22): γ_{n+1} and Γ_{n+1} only differ by a constant factor, hence have the same eigenspaces. Moreover, $\Gamma^j = \eta^{jj}\Gamma_j$ (no sum), so also Γ^{n+1} is proportional to Γ_{n+1} and therefore has the same eigenspaces.

1.4. Spin groups

As in any ring with unit, the subset of invertible elements in $\mathrm{Cl}(s,t)$ is defined by

$$\mathrm{Cl}^\times(s,t) := \{x \in \mathrm{Cl}(s,t) \mid \exists y \in \mathrm{Cl}(s,t)\colon xy = 1 = yx\}$$

and is a multiplicative group. For $n = s + t$ the analogous statement is true of

$$\mathbb{C}\mathrm{l}^\times(n) := \{z \in \mathbb{C}\mathrm{l}(n) \mid \exists w \in \mathbb{C}\mathrm{l}(n)\colon zw = 1 = wz\}$$

and it is easy to see that

$$\mathrm{Cl}^\times(s,t) = \mathbb{C}\mathrm{l}^\times(n) \cap \mathrm{Cl}(s,t).$$

[In fact, the inclusion $\mathrm{Cl}^\times(s,t) \subseteq \mathbb{C}\mathrm{l}^\times(n) \cap \mathrm{Cl}(s,t)$ is obvious and for the reverse inclusion let $z = x_1 \otimes 1 + x_2 \otimes i$ with $x_j \in \mathrm{Cl}(s,t)$ be a member of the right-hand side; then $x_2 = 0$ and there is some $y_1 \otimes 1 + y_2 \otimes i$ with $y_j \in \mathrm{Cl}(s,t)$ such that $x_1y_1 \otimes 1 + x_1y_2 \otimes i = 1 = y_1x_1 \otimes 1 + y_1x_2 \otimes i$, hence $x_1y_1 = 1 = y_1x_1$, which shows that $x_1 \in \mathrm{Cl}^\times(s,t)$, i.e., $z = x_1 \otimes 1 \in \mathrm{Cl}^\times(s,t) \otimes 1$.]

Since $\mathrm{Cl}(s,t)$ and $\mathbb{C}\mathrm{l}(n)$ are finite-dimensional algebras over the real numbers, there is a natural (real) Lie group structure on $\mathrm{Cl}^\times(s,t)$ and $\mathbb{C}\mathrm{l}^\times(n)$, which follows from a more general statement that we repeat here from [**14**, Page 17].

1.4.1. LEMMA. *Let A be a finite-dimensional real associative algebra with unit element and let $G := A^\times$ be the group of invertible elements in A. Then G is a Lie group with Lie algebra $\mathfrak{g} = A$, where the Lie bracket is given by the commutator in the algebra sense and the exponential map is given by the usual power series.*

PROOF. Most of the statement is covered by [**8**, 16.9.3]. The required arguments are very similar to those for the special case of $\mathrm{GL}(N,\mathbb{R})$ treated as a subset of $\mathrm{L}(\mathbb{R}^N)$, for example, along the lines of [**24**, Examples 4.6.(iii) and 8.4]. ☐

1.4.2. COROLLARY. $\mathrm{Cl}^\times(s,t)$ *and* $\mathbb{C}\mathrm{l}^\times(n)$ *are Lie groups.*

Recall the definition of the pseudosphere S_+ and of the pseudohyperbolic space S_- in $(\mathbb{R}^{s,t},\eta)$ by the equations $\eta(v,v) = 1$ in (1.3) and $\eta(v,v) = -1$ in (1.4), respectively. We have already noticed earlier that any element $v \in V \subseteq \mathrm{Cl}(s,t)$ with $\eta(v,v) \neq 0$ is invertible thanks to the Clifford relation $v^2 = -\eta(v,v)$. In particular, products of elements from $S_- \cup S_+$ are always invertible and these are the basic building blocks for the subgroups of $\mathrm{Cl}^\times(s,t)$, which we define now. By convention, let an "empty product" $v_1 \cdots v_r$ with $r = 0$ be equal to 1.

1.4.3. DEFINITION. The *pin group* is defined as

$$\mathrm{Pin}(s,t) := \{v_1 \cdots v_r \mid r \in \mathbb{N}_0, v_1,\dots,v_r \in S_- \cup S_+\}$$

and the *spin group* is

$$\mathrm{Spin}(s,t) := \mathrm{Pin}(s,t) \cap \mathrm{Cl}^0(s,t) = \{v_1 \cdots v_{2r} \mid r \in \mathbb{N}_0, v_1,\dots,v_{2r} \in S_- \cup S_+\}.$$

The *orthochronous spin group* $\mathrm{Spin}^+(s,t)$ is the subgroup of $\mathrm{Spin}(s,t)$ consisting of all products

$$v_1 \cdots v_{2r}, \text{ where } r \in \mathbb{N}_0 \text{ and the number of factors } v_j \text{ from } S_- \text{ and } S_+ \text{ are both even.}$$

We will also use the notation $\mathrm{Pin}(n)$ for $\mathrm{Pin}(n,0)$ and $\mathrm{Spin}(n)$ for $\mathrm{Spin}(n,0) = \mathrm{Spin}^+(n,0)$.

It is easy to see that the subsets of $\mathrm{Cl}^\times(s,t)$ defined above are subgroups. By endowing them with the trace topology of $\mathrm{Cl}(s,t)$ they become topological groups. We will see later in this section that they are indeed Lie groups.

We observe that any element of $\mathrm{Pin}(s,t)$ belongs either to $\mathrm{Cl}^0(s,t)$ or to $\mathrm{Cl}^1(s,t)$, so that we may define the map

$$\deg\colon \mathrm{Pin}(s,t) \to \mathbb{Z}_2, \quad \deg(u) := \begin{cases} 0, & \text{if } u \in \mathrm{Cl}^0(s,t), \\ 1, & \text{if } u \in \mathrm{Cl}^1(s,t), \end{cases} \tag{1.18}$$

which satisfies $\deg(uw) = \deg(u) + \deg(w)$. Thus, deg is a group homomorphism with kernel equal to $\mathrm{Spin}(s,t)$.

1.4.4. LEMMA. *(i) For $u \in \mathrm{Pin}(s,t)$ and $x \in \mathbb{R}^{s,t} \subseteq \mathrm{Cl}(s,t)$ define the element*

$$R(u,x) := (-1)^{\deg(u)} u x u^{-1}$$

of the Clifford algebra. Then $R(u,x) \in \mathbb{R}^{s,t}$. We obtain the map $R\colon \mathrm{Pin}(s,t) \times \mathbb{R}^{s,t} \to \mathbb{R}^{s,t}$, $(u,x) \mapsto R(u,x) =: R_u(x)$.

(ii) If $v \in S_- \cup S_+$, then the linear map $R_v\colon \mathbb{R}^{s,t} \to \mathbb{R}^{s,t}$, $x \mapsto R(v,x) = R_v(x)$, is the reflection at the hyperplane $\{v\}^\perp$. In particular, $R_v \in \mathrm{O}(s,t)$.

(iii) The map $\lambda\colon \mathrm{Pin}(s,t) \to \mathrm{O}(s,t)$, $u \mapsto R_u$, is a continuous group homomorphism.

PROOF. (i) and (ii): Let $u = v_1 \cdots v_r$ with $\eta(v_j, v_j) = \pm 1$, then $\deg(u)$ equals r modulo 2 and

$$(*)\quad R(u,x) = (-1)^r v_1 \cdots v_r x (v_1 \cdots v_r)^{-1} = (-1)^r v_1 \cdots v_r x v_r^{-1} \cdots v_1^{-1}$$
$$= (-1)^{r-1} v_1 \cdots v_{r-1} R(v_r, x) v_{r-1}^{-1} \cdots v_1^{-1} = \ldots = R(v_1, R(v_2, \ldots R(v_{r-1}, R(v_r, x)) \cdots)),$$

thus it suffices to show that $R(v,x) \in \mathbb{R}^{s,t}$, if $\eta(v,v) = \pm 1$. But this follows already from Equation (1.8) in Remark 1.2.6 upon adaptation to the current notation. The same equation also implies (ii).

(iii): Rereading $(*)$ tells that $R_{v_1 \cdots v_r} = R_{v_1} \cdots R_{v_r}$ for any $v_j \in S_- \cup S_+$ and this implies then $\lambda(uw) = R_{uw} = R_u R_w = \lambda(u)\lambda(w)$ for all $u, w \in \mathrm{Pin}(s,t)$. The topology on $\mathrm{Pin}(s,t)$ is the trace of the topology in the topological group $\mathrm{Cl}^\times(s,t)$, whose topology in turn is inherited from the Euclidean vector space topology of $\mathrm{Cl}(s,t) \supseteq \mathbb{R}^{s,t}$. Thus, the formula for $R(u,x)$ shows continuity of the map R, and in particular also of the map $u \mapsto R_u x = \lambda(u)x$ for every $x \in \mathbb{R}^{s,t}$. Inserting standard basis vectors for x implies that each column of the matrix $R_u \in \mathrm{O}(s,t)$ depends continuously on $u \in \mathrm{Pin}(s,t)$, thus we have continuity of λ. □

Recall, e.g., directly from formula (1.2), that $-v$ and v generate the same reflection, i.e., $R_{-v} = R_v$ for every $v \in \mathbb{R}^{s,t}$ with $\eta(v,v) = \pm 1$, so that λ is not injective. Thanks to the Cartan–Dieudonné Theorem 1.1.9, we obtain surjectivity of λ, since every element $T \in \mathrm{O}(s,t)$ is the product of reflections given in the form R_{v_j} with $v_j \in S_- \cup S_+$, so that the corresponding product of the vectors v_j in $\mathrm{Pin}(s,t)$ defines an element $u \in \mathrm{Pin}(s,t)$ such that $\lambda(u) = T$. Moreover, Corollary 1.1.10 informs us that, for any $u \in \mathrm{Pin}(s,t)$,

$$(1.19)\qquad \lambda(u) \in \mathrm{SO}(s,t) \Leftrightarrow u \in \mathrm{Spin}(s,t) \quad \text{and} \quad \lambda(u) \in \mathrm{SO}^+(s,t) \Leftrightarrow u \in \mathrm{Spin}^+(s,t).$$

These facts provide already proofs for several aspects in the following statement.

1.4.5. THEOREM. *We consider the continuous group homomorphism* $\lambda\colon \mathrm{Pin}(s,t) \to \mathrm{O}(s,t)$ *defined in the previous lemma.*

(i) λ *is surjective, open, and has kernel* $\{-1, 1\}$.

(ii) We have $\mathrm{Spin}(s,t) = \lambda^{-1}(\mathrm{SO}(s,t))$ *and* $\mathrm{Spin}^+(s,t) = \lambda^{-1}(\mathrm{SO}^+(s,t))$ *and both are open subgroups of* $\mathrm{Pin}(s,t)$.

(iii) The restrictions of λ *to* $\mathrm{Spin}(s,t)$ *and to* $\mathrm{Spin}^+(s,t)$ *both have kernel* $\{-1,1\}$ *and map surjectively onto* $\mathrm{SO}(s,t)$ *and* $\mathrm{SO}^+(s,t)$*, respectively.*

(iv) If $s \geq 2$ *or* $t \geq 2$*, then* $\mathrm{Spin}^+(s,t)$ *is connected.*

PROOF. (i): Surjectivity of λ was shown in the discussion preceding the theorem and that λ is an open map follows from a variant of the open mapping principle for topological groups (cf. [**7**, Theorem 4.2.10] or [**9**, 12.16.13 and 12.12.7]).

We determine the kernel of λ. Suppose that $u \in \mathrm{Pin}(s,t)$ and $\lambda(u) = R_u = I$ in $\mathrm{O}(s,t)$. Since $\det R_v = -1$ for all $v \in S_- \cup S_+$, u has to be a product of an even number of factors. Thus, $\deg(u) = 0$ and $x = R_u x = uxu^{-1}$ for every $x \in \mathbb{R}^{s,t}$. In particular, we obtain the following equation, which is linear with respect to u:

$$ue_j = e_j u \quad (j = 1, \ldots, n).$$

Since $u \in \mathrm{Cl}^0(s,t)$ and this subalgebra is the linear span of 1 and all products of an even number of distinct basis vectors, we may[3] write $u = u_0 + e_1 u_1$, where u_0 is even, u_1 is odd, and both u_0 and u_1 are generated from $e_2, \ldots, e_n$. Then e_1 commutes with u_0 and anticommutes with u_1 and we obtain

$$e_1 u_0 + e_1^2 u_1 = e_1 u = u e_1 = u_0 e_1 + e_1 u_1 e_1 = e_1 u_0 - e_1^2 u_1.$$

[3] This idea is from [**34**, Proof of Lemma 5.2.1].

This implies $0 = e_1^2 u_1 = -\eta(e_1, e_1)u_1$ and hence $u_1 = 0$. Proceeding in the same way with e_2 etc, we obtain inductively that $u = u_0$, where u_0 is generated by 1. Thus, $u = c \cdot 1$ with $c \in \mathbb{R}$, but $u \in \mathrm{Pin}(s,t)$ now forces $c = \pm 1$ (The only scalars in $\mathrm{Pin}(s,t)$ are ± 1, cf. (iii) below).

(ii): Follows from (1.19) and the continuity of λ, since $\mathrm{SO}(s,t)$ and $\mathrm{SO}^+(s,t)$ are open subgroups of $\mathrm{O}(s,t)$ due to Proposition 1.1.8. (The proposition said that $\mathrm{SO}^+(s,t)$ is the connected component of the identity in $\mathrm{O}(s,t)$; hence it is open; furthermore, $\mathrm{SO}(s,t)$ is the union of $\mathrm{SO}^+(s,t)$ with the connected component described by the case $(+-)$ in that same proposition.)

(iii): The assertion about the kernel follows from (i), since $\pm 1 \in \mathrm{Spin}^+(s,t)$. (Clearly, $1 \in \mathrm{Spin}^+(s,t)$. If $s > 0$ we have $\mathrm{Spin}^+(s,t) \ni v^2 = -\eta(v,v) = -1$ for any $v \in S_+$; if $s = 0$ we have $\mathrm{Spin}^+(0,n) = \mathrm{Spin}(0,n)$ and $\pm e_1 \in S_-$, hence $\mathrm{Spin}^+(0,n) \ni (-e_1)e_1 = -e_1^2 = \eta(e_1,e_1) = -1$.)

Finally, $\lambda(\mathrm{Spin}(s,t)) = \mathrm{SO}(s,t)$ and $\lambda(\mathrm{Spin}^+(s,t)) = \mathrm{SO}^+(s,t)$ follow from the facts that λ is surjective $\mathrm{Pin}(s,t) \to \mathrm{O}(s,t)$ and that Corollary 1.1.10 characterizes the subsets $\mathrm{SO}(s,t)$ and $\mathrm{SO}^+(s,t)$ within $\mathrm{O}(s,t)$ as images of $\mathrm{Spin}(s,t)$ and $\mathrm{Spin}^+(s,t)$, respectively.

(iv): We have that $\ker\lambda = \{-1,1\}$ is a closed normal subgroup of $\mathrm{Spin}^+(s,t)$ and the corresponding factor map $\tilde{\lambda}\colon \mathrm{Spin}^+(s,t)/\ker\lambda \to \mathrm{SO}^+(s,t)$ is an isomorphism of groups, which is continuous by definition of the quotient topology. Moreover, again appealing to [**9**, 12.16.13] or [**7**, Theorem 4.2.10], $\tilde{\lambda}$ is a homeomorphism. Recalling that $\mathrm{SO}^+(s,t)$ is connected (Proposition 1.1.8), it is even path-wise connected, since it is a manifold ([**23**, Corollary 2.3.5] or [**28**, Proposition 1.11(b)]), and we deduce that $\mathrm{Spin}^+(s,t)/\{-1,1\}$ is path-wise connected as well. Let $q\colon \mathrm{Spin}^+(s,t) \to \mathrm{Spin}^+(s,t)/\{-1,1\}$ denote the canonical surjective homomorphism.

Now we are ready to show that $\mathrm{Spin}^+(s,t)$ is path-wise connected. Let $u, v \in \mathrm{Spin}^+(s,t)$. Their images $q(u)$ and $q(v)$ in the quotient $\mathrm{Spin}^+(s,t)/\{-1,1\}$ can be joined by a continuous curve, say $\alpha\colon [0,1] \to \mathrm{Spin}^+(s,t)/\{-1,1\}$ with $\alpha(0) = q(u)$ and $\alpha(1) = q(v)$. Considering instead the curve α_0 defined by $\alpha_0(\tau) := q(u^{-1})\alpha(\tau)$ we have $\alpha_0(0) = q(u^{-1}u) = 1$ and $\alpha_0(1) = q(u^{-1}v)$. We now note that the quotient map q is in fact a covering (as we shall prove in the following result, Lemma 1.4.6). Consequently (cf., e.g., [**21**, Page 60]), there is a continuous lift $\beta_0\colon [0,1] \to \mathrm{Spin}^+(s,t)$ of α_0 starting at 1, i.e., $q \circ \beta_0 = \alpha_0$ and $\beta_0(0) = 1$. In particular, $q(\beta_0(1)) = \alpha_0(1) = q(u^{-1}v)$. Defining now the path β by $\beta(\tau) := u\beta_0(\tau)$ gives $\beta(0) = u$ and $q(\beta(1)) = q(u)q(\beta_0(1)) = q(v)$, i.e., $\beta(1) = v$ or $\beta(1) = -v$. Therefore, it remains to show that we can always connect $-v$ to v by a continuous path inside $\mathrm{Spin}^+(s,t)$. It suffices to show this for -1 and 1, since then multiplication of the path by v adjusts it to $-v$ and v.

So, finally, we show that -1 and 1 can be joined by a continuous curve in $\mathrm{Spin}^+(s,t)$: By the hypothesis of $s \geq 2$ or $t \geq 2$, we can find two vectors $y_1, y_2 \in \mathbb{R}^{s,t}$ such that $\eta(y_1,y_2) = 0$ and $\pm 1 = \eta(y_1,y_1) = \eta(y_2,y_2)$. Consider the continuous path $\theta\colon [0,\pi/2] \to \mathrm{Spin}^+(s,t)$, where

$$\theta(\tau) := \pm\big(\cos(\tau)y_1 + \sin(\tau)y_2\big)\big(\cos(\tau)y_1 - \sin(\tau)y_2\big).$$

For every τ, both factors belong to $S_\pm$ and thus $\theta(\tau) \in \mathrm{Spin}^+(s,t)$. We have $\theta(0) = \pm y_1^2 = \mp\eta(y_1,y_1) = -1$ and $\theta(\pi/2) = \mp y_2^2 = \pm\eta(y_2,y_2) = 1$. □

We will now state an intermediate topological result that will allow us later to identify a Lie group structure on the pin and spin groups.

1.4.6. Lemma. *The homomorphism λ defines topological double coverings*

$$\mathrm{Pin}(s,t) \to \mathrm{O}(s,t), \quad \mathrm{Spin}(s,t) \to \mathrm{SO}(s,t), \quad \mathrm{Spin}^+(s,t) \to \mathrm{SO}^+(s,t).$$

proof. By Theorem 1.4.5, λ is continuous, open, and surjective with kernel $\{-1,1\}$ in each of the three situations. In each case, the corresponding factor homomorphism $\tilde{\lambda}$, starting from the quotient by $\{-1,1\}$ and mapping into the respective target spaces, is an isomorphism of topological groups (cf. [**9**, 12.16.13] or [**7**, Theorem 4.2.10], as in the beginning of the proof of part (iv) in

Theorem 1.4.5). We illustrate the situation by a commutative diagram in case of $\mathrm{Pin}(s,t)$, for the other two cases this is analogous:

$$\begin{array}{ccc} \mathrm{Pin}(s,t) & \xrightarrow{\lambda} & \mathrm{O}(s,t) \\ & \searrow^{q} & \uparrow \tilde{\lambda} \\ & & \mathrm{Pin}(s,t)/\{-1,1\} \end{array}$$

Thanks to the isomorphism $\tilde{\lambda}$, the map λ is a double covering, if and only if the canonical surjection q is a double covering. The latter is indeed true, as can be seen from the fact that the kernel is discrete, either by using [**27**, Proposition 12.15] (together with the additional investigation that connectedness of the domain of q, built into the definition of covering maps there, is not required) or along the lines of a similar statements for Lie groups in [**24**, Theorem 22.4(iii)], which we will outline here. It is clear that the number of sheets in the above situation is 2, since the pre-image under q of any point consists of two elements. So it suffices to show the following: If G is a topological group with a discrete normal subgroup H, then the canonical surjection $q\colon G \to G/H$ is a (topological) covering map.

Choose a neighborhood U of the neutral element $e \in G$ such that $U \cap H = \{e\}$. Choose an open subset V with $V = V^{-1}$ and $V \cdot V \subseteq U$ (cf. [**24**, Remark 2.3]) and let $V_g := V \cdot g$, the right translation of V by $g \in G$. Note that V_g is a neighborhood of g in G and $q(V_g)$ is a neighborhood of $q(g)$ in the quotient G/H, since q is an open map.
Claim: $v_1 h_1 = v_2 h_2$ with $v_j \in V$, $h_j \in H$ implies $v_1 = v_2$ and $h_1 = h_2$.
This follows, since $v_2^{-1} v_1 = h_2 h_1^{-1}$ belongs to both U and H, thus equals e.

Based on this observation, one can prove the following two facts:
(a) For all $g \in G$ and $h \in H$ the map q induces a homeomorphism $V_g \cdot h \to q(V_g)$,
(b) if $h_1, h_2 \in H$ with $h_1 \neq h_2$, then $(V_g \cdot h_1) \cap (V_g \cdot h_2) = \emptyset$.
Proof of (a): we have surjectivity from $q(V_g \cdot h) = q(V_g)$, injectivity, since $q(v_1 g h) = q(v_2 g h)$ for $v_1, v_2 \in V$ implies $U \supseteq V \cdot V = V \cdot V^{-1} \ni v_1 v_2^{-1} = (v_1 g)(v_2 g)^{-1} \in H$ and then $v_1 v_2^{-1} = e$, and so q is a bijective, continuous, open map, thus, a homeomorphism.
To prove (b), suppose $(V_g \cdot h_1) \cap (V_g \cdot h_2) \neq \emptyset$ and choose $v_1, v_2 \in V$ with $v_1 g h_1 = v_2 g h_2$. Then $V \cdot V \ni v_2^{-1} v_1 = g(h_2 h_1^{-1}) g^{-1} \in gHg^{-1} = H$, which first implies $v_1 v_2^{-1} = e$ and then $h_1 = h_2$.

Note that $q^{-1}(q(V_g)) = q^{-1}(q(V_g \cdot h)) \supseteq V_g \cdot h$ for all $h \in H$ and that $g' \in q^{-1}(q(V_g))$ means $g'H \in V_g H$, i.e., $g' \in V_g h'$ for some $h' \in H$. From (a), (b) we now obtain that for every $g \in G$, $q^{-1}(q(V_g))$ is the disjoint union, with h varying in H, of the sets $V_g \cdot h$, which are all homeomorphic to $q(V_g)$. Therefore, q is a covering map. □

Now we are in a position to identify the above topological covering even as a covering of Lie groups.

1.4.7. Corollary. *On each of the three groups* $\mathrm{Pin}(s,t)$, $\mathrm{Spin}(s,t)$, $\mathrm{Spin}^+(s,t)$, *we can define a unique Lie group structure such that* λ *gives smooth double coverings of Lie groups*

$$\mathrm{Pin}(s,t) \to \mathrm{O}(s,t), \quad \mathrm{Spin}(s,t) \to \mathrm{SO}(s,t), \quad \mathrm{Spin}^+(s,t) \to \mathrm{SO}^+(s,t).$$

proof. In each case we may declare the factor map isomorphism $\tilde{\lambda}$ a diffeomorphism, since $\mathrm{O}(s,t)$, $\mathrm{SO}(s,t)$, $\mathrm{SO}^+(s,t)$ are Lie groups. In this way, we obtain a Lie group structure on the respective quotient groups. The canonical surjection q is a covering map, thus a local homeomorphism, and may be used to induce a smooth structure on the pin and spin groups stemming from the quotients. With these smooth structures, clearly q is a smooth map. Moreover, the pin and spin groups are then Lie groups, which we will check below in some detail for $\mathrm{Pin}(s,t)$ (and the proof below obviously works generally for any covering construction as above). For the moment, let us observe that, as a consequence, the covering map λ then automatically becomes a covering of Lie groups by construction.

Let $g_1, g_2 \in \mathrm{Pin}(s,t)$ and choose first a neighborhood W of $g_1 g_2$ in $\mathrm{Pin}(s,t)$ such that $q|_W$ is a diffeomorphism, then neighborhoods V_j of g_j $(j = 1,2)$ such that $q|_{V_j}$ are diffeomorphisms and that the group multiplication $\mu\colon \mathrm{Pin}(s,t) \times \mathrm{Pin}(s,t) \to \mathrm{Pin}(s,t)$ maps $V_1 \times V_2$ into W. With the notation $N := \ker\lambda = \{-1,1\}$ we have for all $(g', g'') \in V_1 \times V_2$ that $q(\mu(g',g'')) = g'g''N = (g'N)(g''N) = q(g')q(g'') = \tilde{\mu} \circ (q \times q)(g',g'')$, where $\tilde{\mu}$ denotes the *smooth* multiplication in the quotient group $\mathrm{Pin}(s,t)/N$. We have identified $q \circ \mu$ as the composition of smooth maps, hence it is smooth. Since q is a local diffeomorphism, we have shown that μ is smooth. □

1.4.8. Remark. (i) Since the pin group is a Lie group when equipped with the trace topology of $\mathrm{Cl}^\times(s,t)$, [**24**, Theorem 21.10] shows that it is a closed subgroup of $\mathrm{Cl}^\times(s,t)$ and thereby a Lie subgroup according to Cartan's theorem ([**24**, Theorem 21.7]). Indeed there are alternative approaches to the pin group (cf. [**34**, Equations (5.2.10) and (5.2.1)]), which allow one to obtain it directly as a closed subgroup of the Lie group $\mathrm{Cl}^\times(s,t)$. The resulting smooth structure necessarily agrees with the one we constructed due to [**24**, Remark 21.12].

(ii) Based on the classification of coverings in terms of properties of the fundamental group, some of the double coverings with $\mathrm{Spin}^+(s,t)$ established above can be proven to be universal (cf. [**3**, Abschnitt 1.2]). For example, for $n \geq 3$,

$$\mathrm{Spin}(n) \to \mathrm{SO}(n) \quad \text{and} \quad \mathrm{Spin}^+(1,n) \to \mathrm{SO}^+(1,n)$$

are the universal coverings.

1.4.9. Example (Spin groups in dimensions $n \leq 2$). (i) In $\mathbb{R}^{1,0}$ we have $S_- = \emptyset$, $S_+ = \{-e_1, e_1\}$ and obtain $\mathrm{Spin}(1,0) = \{-1,1\} \cong \mathbb{Z}_2$. It covers $\mathrm{SO}(1) = \{1\}$ via the map $k \mapsto k^2$.

(ii) In $\mathbb{R}^{2,0}$ we have $S_- = \emptyset$ and S_+ is the unit circle S^1, so we may parametrize vectors $v \in S_+$ in the form $v = \cos(\alpha)e_1 + \sin(\alpha)e_2$ with $\alpha \in \mathbb{R}$. Note that $\mathrm{Spin}^+(2) \subseteq \mathrm{Cl}^0(2) = \mathbb{R}\text{-span}\{1, e_1e_2\}$ and a quick calculation with $v_j = \cos(\alpha_j)e_1 + \sin(\alpha_j)e_2$ (and basic trigonometric formulae) shows

$$v_1v_2 = (\cos(\alpha_1)e_1 + \sin(\alpha_1)e_2)(\cos(\alpha_2)e_1 + \sin(\alpha_2)e_2) = -\cos(\alpha_1 - \alpha_2)1 - \sin(\alpha_1 - \alpha_2)e_1e_2.$$

Upon setting here $\theta := \pi + (\alpha_1 - \alpha_2)$, we obtain the educated guess that

$$\mathrm{Spin}^+(2) = \{\cos(\theta)1 + \sin(\theta)e_1e_2 \mid \theta \in \mathbb{R}\},$$

where the calculation showed already that $\mathrm{Spin}^+(2)$ contains this subset. Since any element in $\mathrm{Spin}^+(2)$ is generated from products of pairs as above, we only need to check the result of a product of two vectors $\cos(\theta_j)1 + \sin(\theta_j)e_1e_2$ $(j = 1,2)$. Using $e_1e_2e_1e_2 = -e_1^2e_2^2 = -1$ and trigonometric formulae, we have

$$(\cos(\theta_1)1 + \sin(\theta_1)e_1e_2)(\cos(\theta_2)1 + \sin(\theta_2)e_1e_2) = \cos(\theta_1 + \theta_2)1 + \sin(\theta_1 + \theta_2)e_1e_2.$$

By the same observation, we see that $\cos(\theta)1 + \sin(\theta)e_1e_2 \mapsto e^{i\theta}$ induces a group isomorphism with $U(1)$, thus we have shown that

$$\mathrm{Spin}(2) \cong \mathrm{U}(1).$$

The covering map λ onto $\mathrm{SO}(2)$ can be constructed (or guessed) from identifying $v = \cos(\alpha)e_1 + \sin(\alpha)e_2$ with the reflection R_v at $\{v\}^\perp = \mathrm{span}\{-\sin(\alpha)e_1 + \cos(\alpha)e_2\}$ corresponding to the matrix $\begin{pmatrix} -\cos(2\alpha) & -\sin(2\alpha) \\ -\sin(2\alpha) & \cos(2\alpha) \end{pmatrix}$. Calculating $R_{v_1}R_{v_2}$ with v_1 and v_2 as above, then identifying v_1v_2 again as $\cos(\theta)1 + \sin(\theta)e_1e_2$, one can deduce that λ is given by $e^{i\theta} \mapsto \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix}$, or, upon noting that also $\mathrm{U}(1) \cong \mathrm{SO}(2)$, we may interpret λ as the map $\mathrm{U}(1) \to \mathrm{U}(1)$, $z \mapsto z^2$.

(iii) In $\mathbb{R}^{1,1}$ we have $S_- = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_2^2 = 1 + x_1^2\}$ and $S_+ = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1^2 = 1 + x_2^2\}$ as typical hyperbolae, where S_- has its branches above and below the horizontal x_1-axis, S_+ left and right to the vertical x_2-axis. We may parametrize vectors in S_- in the form $\pm(\sinh(\tau)e_1 + \cosh(\tau)e_2)$ and those in S_+ by $\pm(\cosh(\tau)e_1 + \sinh(\tau)e_2)$ with $\tau \in \mathbb{R}$. With a little effort by calculations similar to those in (ii), one can derive that

$$\mathrm{Spin}^+(1,1) = \{\varepsilon(\cosh(\tau)1 + \sinh(\tau)e_1e_2) \mid \tau \in \mathbb{R}, \varepsilon \in \{-1,1\}\}$$

with the multiplication formula

$$\varepsilon_1(\cosh(\tau_1)1 + \sinh(\tau_1)e_1e_2) \cdot \varepsilon_2(\cosh(\tau_2)1 + \sinh(\tau_2)e_1e_2) = \varepsilon_1\varepsilon_2(\cosh(\tau_1+\tau_2)1 + \sinh(\tau_1+\tau_2)e_1e_2).$$

We observe that we may construct an isomorphism with a subgroup of $\mathrm{SL}(2,\mathbb{R})$ via the assignment $\varepsilon(\cosh(\tau)1 + \sinh(\tau)e_1e_2) \mapsto \varepsilon \left(\begin{smallmatrix} e^\tau & 0 \\ 0 & e^{-\tau} \end{smallmatrix}\right)$ and obtain

$$\mathrm{Spin}^+(1,1) \cong \{\pm \left(\begin{smallmatrix} e^\tau & 0 \\ 0 & e^{-\tau} \end{smallmatrix}\right) \mid \tau \in \mathbb{R}\} = \{\left(\begin{smallmatrix} e^\tau & 0 \\ 0 & e^{-\tau} \end{smallmatrix}\right) \mid \tau \in \mathbb{R}\} \cup \{\begin{pmatrix} -e^\tau & 0 \\ 0 & -e^{-\tau} \end{pmatrix} \mid \tau \in \mathbb{R}\} \subseteq \mathrm{SL}(2,\mathbb{R}),$$

which also shows that $\mathrm{Spin}^+(1,1)$ has two connected components.

1.4.10. Remark. Before focussing on Minkowski space in the example to follow, we mention a few other low dimensional examples without going into details. One can show that $\mathrm{Spin}^+(1,2) \cong \mathrm{SL}(2,\mathbb{R})$, $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$, and $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ (cf. [**26**, Chapter I, Section 8]).

1.4.11. Example (The orthochronous spin group for Minkowski space). (i) We will sketch below in (ii) how to prove by means of Lie group covering maps that we have the isomorphism

$$\mathrm{Spin}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C}).$$

Here, we will instead show this by "brave matrix calculations" based on what we learned about $\mathrm{Cl}(1,3)$ in Example 1.2.27(iii). We had the Clifford map $\gamma\colon \mathbb{R}^{1,3} \to \mathrm{L}(\mathbb{C}^4)$ with generators $\gamma(e_a) := \gamma_{a-1}$ $(a = 1,2,3,4)$ given by

$$\gamma_0 := \begin{pmatrix} 0 & iI_2 \\ iI_2 & 0 \end{pmatrix}, \quad \gamma_k := \begin{pmatrix} 0 & i\sigma_k \\ -i\sigma_k & 0 \end{pmatrix} \ (k = 1,2,3)$$

and satisfying

$$\gamma_0^2 = -I_4, \quad \gamma_k^2 = I_4 \ (k=1,2,3), \quad \{\gamma_a, \gamma_b\} = 0 \ (0 \le a, b \le 3, a \neq b).$$

We then obtained $\mathrm{Cl}(1,3)$ as the real 16-dimensional subalgebra of $\mathrm{L}(\mathbb{C}^4)$ which is the real linear hull of the basis set $B := \{I_4, \gamma_0, \gamma_1, \gamma_2, \gamma_3\} \cup \{\gamma_a\gamma_b \mid 0 \le a < b \le 3\} \cup \{\gamma_a\gamma_b\gamma_c \mid 0 \le a < b < c \le 3\} \cup \{\gamma_0\gamma_1\gamma_2\gamma_3\}$. Recall that $\mathrm{Spin}^+(1,3)$ is contained in the even subalgebra $\mathrm{Cl}^0(1,3)$ which is the real linear span of the following subset of B with 8 elements:

$$B_0 := \{I_4\} \cup \{\gamma_a\gamma_b \mid 0 \le a < b \le 3\} \cup \{\gamma_0\gamma_1\gamma_2\gamma_3\}.$$

Elementary calculations using the Pauli matrix relations give all elements of B_0 explicitly by

$$\gamma_0\gamma_k = \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix} \ (k=1,2,3), \qquad \gamma_1\gamma_2 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix},$$

$$\gamma_1\gamma_3 = \begin{pmatrix} \sigma_1\sigma_3 & 0 \\ 0 & \sigma_1\sigma_3 \end{pmatrix} \text{ with } \sigma_1\sigma_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_2\gamma_3 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix},$$

$$\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} iI_2 & 0 \\ 0 & -iI_2 \end{pmatrix} \qquad I_4 = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}.$$

As can be seen from the above, $\mathrm{Cl}^0(1,3) \subseteq L_0 := \mathrm{L}(\mathbb{C}^2) \oplus \mathrm{L}(\mathbb{C}^2)$, since the linear combinations of the elements in B_0 are all block matrices$\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ with $A, B \in \mathrm{L}(\mathbb{C}^2)$. We also see that the basis elements $I_4, \gamma_0\gamma_1, \gamma_0\gamma_3, \gamma_1\gamma_3$ are real matrices, while $\gamma_0\gamma_2, \gamma_1\gamma_2, \gamma_2\gamma_3, \gamma_0\gamma_1\gamma_2\gamma_3$ are all of the form i times a real matrix. Consider linear combinations with coefficients built up bijectively from $a, b, \ldots, g, h \in \mathbb{R}$ in the form

$$U := \frac{a+d}{2} I_4 + \frac{b+c}{2}\gamma_0\gamma_1 + \frac{a-d}{2}\gamma_0\gamma_3 + \frac{c-b}{2}\gamma_1\gamma_3 + \frac{g-f}{2}\gamma_0\gamma_2 + \frac{e-h}{2}\gamma_1\gamma_2$$
$$+ \frac{g+f}{2}\gamma_2\gamma_3 + \frac{e+h}{2}\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & d & -c \\ 0 & 0 & -b & a \end{pmatrix} + i \begin{pmatrix} e & f & 0 & 0 \\ g & h & 0 & 0 \\ 0 & 0 & -h & g \\ 0 & 0 & f & -e \end{pmatrix},$$

which after some playing around can be recognized to have the structure

$$U = \begin{pmatrix} Z & 0 \\ 0 & \tilde{Z} \end{pmatrix} \text{ with } Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \in \mathrm{L}(\mathbb{C}^2) \text{ and } \tilde{Z} = \begin{pmatrix} \overline{z_{22}} & -\overline{z_{21}} \\ -\overline{z_{12}} & \overline{z_{11}} \end{pmatrix}.$$

Observe that $\det \tilde{Z} = \overline{\det Z}$ and hence $\det U = |\det Z|^2$, so that U is invertible if and only if Z is.

Thus far, we have learned that an element $S \in \mathrm{Spin}^+(1,3)$ has to be of the form $S = \left(\begin{smallmatrix} Z & 0 \\ 0 & \tilde{Z} \end{smallmatrix}\right)$ with $Z \in \mathrm{GL}(2,\mathbb{C})$. We also see that in this case, $\tilde{Z} = \overline{\det Z} \cdot (Z^\dagger)^{-1}$, where $Z^\dagger := \overline{Z}^T$. Let us check in more detail the particular case $S = \gamma(x)\gamma(y)$ with x and y both in S_- or both in S_+, i.e., $\eta(x,x) = \eta(y,y) = \pm 1$. A simple calculation shows

$$\gamma(x) = \sum_{a=0}^{3} x_a \gamma_a = \begin{pmatrix} 0 & 0 & i(x_0+x_3) & ix_1+x_2 \\ 0 & 0 & ix_1-x_2 & i(x_0-x_3) \\ i(x_0-x_3) & -ix_1-x_2 & 0 & 0 \\ -ix_1+x_2 & i(x_0+x_3) & 0 & 0 \end{pmatrix} =: \begin{pmatrix} 0 & iM(x) \\ iN(x) & 0 \end{pmatrix}$$

and then

$$S = \gamma(x)\gamma(y) = i^2 \begin{pmatrix} 0 & M(x) \\ N(x) & 0 \end{pmatrix} \begin{pmatrix} 0 & M(y) \\ N(y) & 0 \end{pmatrix} = \begin{pmatrix} -M(x)N(y) & 0 \\ 0 & -N(x)M(y) \end{pmatrix}.$$

Note that

$$\det M(x) = \det \begin{pmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{pmatrix} = (x_0^2 - x_3^2) - (x_1^2 + x_2^2) = \eta(x,x) \tag{1.20}$$

and the same holds for $N(x) = \left(\begin{smallmatrix} x_0-x_3 & -x_1+ix_2 \\ -x_1-ix_2 & x_0+x_3 \end{smallmatrix}\right)$. Therefore, we obtain that

$$\det(-M(x)N(y)) = (-1)^2 \eta(x,x)\eta(y,y) = \eta(x,x)^2 = 1 \quad \text{and} \quad \det(-N(x)M(y)) = 1.$$

We learned that $\gamma(x)\gamma(y)$ with $x, y \in S_-$ or $x, y \in S_+$ is of the form $\left(\begin{smallmatrix} Z & 0 \\ 0 & \tilde{Z} \end{smallmatrix}\right)$ with $\det Z = 1$ and thus $\tilde{Z} = \overline{\det Z} \cdot (Z^\dagger)^{-1} = (Z^\dagger)^{-1}$.

A general element S in $\mathrm{Spin}^+(1,3)$ is a product of pairs $\gamma(x)\gamma(y)$ as above. Since

$$\begin{pmatrix} Z_1 & 0 \\ 0 & (Z_1^\dagger)^{-1} \end{pmatrix} \begin{pmatrix} Z_2 & 0 \\ 0 & (Z_2^\dagger)^{-1} \end{pmatrix} = \begin{pmatrix} Z_1 Z_2 & 0 \\ 0 & ((Z_1 Z_2)^\dagger)^{-1} \end{pmatrix}$$

and the determinant ist multiplicative, we see that such a product is in general of the form

$$S = \begin{pmatrix} Z & 0 \\ 0 & (Z^\dagger)^{-1} \end{pmatrix} \qquad \text{with } Z \in \mathrm{SL}(2,\mathbb{C}).$$

(ii) As a complement to (i), let us give a construction of the double covering map

$$\lambda \colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(1,3),$$

which could also be used for an alternative proof of the ismorphism $\mathrm{Spin}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C})$.

In the course of the calculations in (i) we had introduced a map from $\mathbb{R}^{1,3}$ to the real 4-dimensional vector space of complex Hermitian (2×2)-matrices $H(2,\mathbb{C})$, given by

$$x \mapsto M(x) := \begin{pmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{pmatrix} = x_0 I_2 + \sum_{k=1}^{3} x_k \sigma_k,$$

which has the property $\det M(x) = \eta(x,x)$ as seen in Equation (1.20). Obviously, M is linear and injective, hence a vector space isomorphism $\mathbb{R}^{1,3} \cong H(2,\mathbb{C})$, since the dimensions match.

For any $S \in \mathrm{SL}(2,\mathbb{C})$, the assignment $N \mapsto SNS^\dagger$ is easily seen to give an $\mathbb{R}$-linear bijective map $\rho_0(S) \colon H(2,\mathbb{C}) \to H(2,\mathbb{C})$ with the property $\det(\rho_0(S)N) = \det N$. Furthermore, $S \mapsto \rho_0(S)$ defines a group representation $\rho_0 \colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(H(2,\mathbb{C}))$, since $\rho_0(S_1S_2)N = (S_1S_2)N(S_1S_2)^\dagger = S_1(S_2NS_2^\dagger)S_1^\dagger = \rho_0(S_1)\rho_0(S_2)N$. Intertwining ρ_0 with the ismomorphism $M \colon \mathbb{R}^{1,3} \to H(2,\mathbb{C})$ induces the group representation $\rho \colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(\mathbb{R}^{1,3})$, i.e.,

$$\rho(S)x = M^{-1}\rho_0(S)M(x) \quad \forall S \in \mathrm{SL}(2,\mathbb{C}), x \in \mathbb{R}^{1,3}.$$

We claim that $\rho(S) \in \mathrm{SO}^+(1,3)$ for every $S \in \mathrm{SL}(2,\mathbb{C})$.

To first prove that $\rho(S)$ belongs to $\mathrm{O}(1,3)$, we calculate (recalling $\eta(y,y) = \det M(y)$)

$$\begin{aligned}\eta(\rho(S)x, \rho(S)x) &= \eta(M^{-1}\rho_0(S)M(x), M^{-1}\rho_0(S)M(x)) = \det(\rho_0(S)M(x)) = \det(SM(x)S^\dagger) \\ &= \det S \det M(x) \det S^\dagger = \eta(x,x)\end{aligned}$$

and note that $\eta(\rho(S)x, \rho(S)y) = \eta(x,y)$ follows then by polarization. Moreover, the map $S \mapsto \rho(S)$ could be expressed in terms of coordinates via rational functions, thus is smooth and then connectedness of $\mathrm{SL}(2,\mathbb{C})$ ([**19**, Theorem 1.2.22,2.]) implies that $\rho(\mathrm{SL}(2,\mathbb{C})) \subseteq \mathrm{SO}^+(1,3)$.

We define λ via ρ considered as group homomorphism into $\mathrm{SO}^+(1,3)$ (this preserves smoothness of λ, since $\mathrm{SO}^+(1,3)$ is an embedded submanifold) and finally claim that
(a) $\ker\lambda = \{-I_2, I_2\}$ and
(b) λ is a surjective covering map.

(a): Let $S \in \mathrm{SL}(2,\mathbb{C})$ such that $\lambda(S) = I_4$. This implies that $SNS^\dagger = N$ for all $N \in H(2,\mathbb{C})$. Setting $N = I_2$ we obtain $SS^\dagger = I_2$, thus $S \in \mathrm{SU}(2)$ and we deduce that

$$\forall N \in H(2,\mathbb{C})\colon \quad SN = NS.$$

Since every complex (2×2)-matrix is a complex linear combination of elements $N \in H(2,\mathbb{C})$, we derive that $S = aI_2$ with some $a \in \mathbb{C}$. But then $1 = \det S = a^2$ implies $a = -1$ or $a = 1$.

(b): The result follows from [**24**, Theorem 22.4(iii)] once we have shown that $\lambda_* = T_e\lambda\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{so}^+(1,3)$ is bijective. Recall from 1.1.5(g) that $\dim \mathfrak{so}^+(1,3) = \dim \mathfrak{o}(1,3) = (4\cdot 3)/2 = 6$ and that $\dim \mathfrak{sl}(2,\mathbb{C}) = 2\cdot 4 - 2 = 6$ ([**19**, Theorem 1.2.17,1.]), hence it suffices to check injectivity of λ_*. Let $X \in \mathfrak{sl}(2,\mathbb{C})$ such that $\lambda_*(X) = 0$. Then the curve $c\colon t \mapsto \lambda(\exp(tX))$ is constant in $\mathrm{SO}^+(1,3)$, since $c'(t) = \frac{d}{dt}\lambda(\exp(tX)) = \frac{d}{ds}|_{s=0}\lambda(\exp((t+s)X)) = \frac{d}{ds}|_{s=0}\lambda(\exp(tX))\lambda(\exp(sX)) = T_eL_{\lambda(\exp(tX))}\lambda_*(X) = 0$, where L_g denotes multiplication by g from the left. From $c(0) = I_4$, we deduce that $\exp(tX) \in \ker\lambda = \{-I_2, I_2\}$ for all $t \in \mathbb{R}$. Differentiation of the matrix-valued function $t \mapsto \exp(tX)$ then gives $0 = \frac{d}{dt}|_{t=0}(\exp(tX)) = X$.

We summarize a few insights from the previous extensive example in the following statement.

1.4.12. Corollary. *For Minkowski space $\mathbb{R}^{1,3}$, the orthochronous spin group $\mathrm{Spin}^+(1,3)$ is isomorphic to $\mathrm{SL}(2,\mathbb{C})$. The double covering map $\lambda\colon \mathrm{Spin}^+(1,3) \to \mathrm{SO}^+(1,3)$ is induced by the real vector space isomorphism of $\mathbb{R}^{1,3}$ with the space $H(2,\mathbb{C})$ of Hermitian (2×2)-matrices, given by*

$$x \mapsto M(x) := x_0 I_2 + \sum_{k=1}^{3} x_k\sigma_k, \quad \textit{satisfying } \det M(x) = \eta(x,x),$$

in combination with the action

$$\mathrm{SL}(2,\mathbb{C}) \times H(2,\mathbb{C}) \to H(2,\mathbb{C}), \quad (S,N) \mapsto SNS^\dagger.$$

In the standard representation of $\mathrm{Cl}(1,3)$ on $\mathbb{C}^4$, described in Examples 1.2.27(iii) and 1.4.11(i), we obtain the realization

$$\mathrm{Spin}^+(1,3) = \{\begin{pmatrix} S & 0 \\ 0 & (S^\dagger)^{-1}\end{pmatrix} \mid S \in \mathrm{SL}(2,\mathbb{C})\}.$$

Recall that the spinor representation $\pi_n\colon \mathbb{C}\mathrm{l}(n) \to \mathrm{L}(\Delta_n)$ induces certainly a complex representation of the real subalgebra $\mathrm{Cl}(s,t)$ of $\mathrm{Cl}(s,t) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}\mathrm{l}(n)$, which we may restrict to $\mathrm{Spin}^+(s,t) \subseteq \mathrm{Cl}^0(s,t)$ and clearly have $\pi_n(\mathrm{Spin}^+(s,t)) \subseteq \mathrm{GL}(\Delta_n)$.

1.4.13. Definition. We consider the restriction of the spinor representation π_n to $\mathrm{Spin}^+(s,t)$ with target space $\mathrm{GL}(\Delta_n)$ and obtain the group representation

$$\kappa_n\colon \mathrm{Spin}^+(s,t) \to \mathrm{GL}(\Delta_n),$$

which we call the *spinor representation* of $\mathrm{Spin}^+(s,t)$ (and often drop the index n in κ_n).

The following proposition gives the relation between the Clifford multiplication $\mathbb{R}^{s,t} \times \Delta_n \to \Delta_n$, $x \cdot \psi = \pi_n(x)\psi$, the covering homomorphism $\lambda\colon \mathrm{Spin}^+(s,t) \to \mathrm{SO}^+(s,t)$, and the spinor representation $\kappa = \kappa_n$ of $\mathrm{Spin}^+(s,t)$.

1.4.14. PROPOSITION. *For all $g \in \mathrm{Spin}^+(s,t)$, $x \in \mathbb{R}^{s,t}$, $\psi \in \Delta_n$, we have*

$$\kappa(g)(x \cdot \psi) = (\lambda(g)x) \cdot (\kappa(g)\psi).$$

PROOF. By straightforward application of definitions and basic properties,

$$\kappa(g)(x\cdot\psi) = \pi_n(g)(\pi_n(x)\psi) = \pi_n(gx)\psi = \pi_n(gxg^{-1})\pi_n(g)\psi = \pi_n(\lambda(g)x)\kappa(g)\psi = (\lambda(g)x)\cdot(\kappa(g)\psi).$$

□

Lemma 1.4.1 provided us with the Lie group structure of $\mathrm{Cl}^\times(s,t)$ and the Lie algebra of $\mathrm{Cl}^\times(s,t)$ as isomorphic to $\mathrm{Cl}(s,t)$ with the algebra commutator as Lie bracket. Therefore, we may identify the Lie algebra of $\mathrm{Spin}^+(s,t)$ with a Lie subalgebra of $\mathrm{Cl}(s,t)$.

1.4.15. LEMMA. *Let $n = s+t$ and $M(s,t) := \mathrm{span}\{e_je_k \mid 1 \le j < k \le n\} \subseteq \mathrm{Cl}(s,t)$. Then $M(s,t)$ is a Lie subalgebra of $\mathrm{Cl}(s,t)$ and $\dim M(s,t) = n(n-1)/2$.*

PROOF. We know that the set $\{e_je_k \mid 1 \le j < k \le n\}$ is linearly independent in $\mathrm{Cl}(s,t)$, thus $M(s,t)$ is a vector subspace of dimension $(n-1)+(n-2)+\ldots+2+1 = n(n-1)/2$. It remains to show that $M(s,t)$ is closed under the Lie bracket. Let $j < k$ and $l < m$, then we have (using short-hand notation like $\eta_{jk} = \eta(e_j, e_k)$ etc)

$$\begin{aligned}
[e_je_k, e_le_m] &= e_je_ke_le_m - e_le_me_je_k = e_je_ke_le_m + e_le_je_me_k + 2\eta_{jm}e_le_k \\
&= e_je_ke_le_m + (-e_je_l - 2\eta_{lj})(-e_ke_m - 2\eta_{mk}) + 2\eta_{jm}e_le_k \\
&= e_je_ke_le_m + e_je_le_ke_m + 2(\eta_{lj}e_ke_m + \eta_{mk}e_je_l) + 4\eta_{lj}\eta_{mk} + 2\eta_{jm}e_le_k \\
&= e_je_ke_le_m + e_j(-e_ke_l - 2\eta_{lk})e_m + 2(\eta_{lj}e_ke_m + \eta_{mk}e_je_l + \eta_{jm}e_le_k) + 4\eta_{lj}\eta_{mk} \\
&= 2(-\eta_{lk}e_je_m + \eta_{lj}e_ke_m + \eta_{mk}e_je_l + \eta_{jm}e_le_k + 2\eta_{lj}\eta_{mk}).
\end{aligned}$$

In case $j \ne l$ or $k \ne m$, the last term vanishes and hence $[e_je_k, e_le_m] \in M(s,t)$. In case $j = l$ and $k = m$, we clearly have $[e_je_k, e_je_k] = 0 \in M(s,t)$. □

1.4.16. REMARK. As a vector space, $M(s,t)$ is obviously isomorphic to $\Lambda^2(\mathbb{R}^n)$. As a Lie algebra, $M(s,t)$ is isomorphic to $\mathfrak{o}(s,t)$ as we will show in the following proposition (see also [**34**, Equation (5.2.28)]).

1.4.17. PROPOSITION. *The Lie algebra $\mathfrak{spin}^+(s,t)$ of $\mathrm{Spin}^+(s,t)$ is equal to $M(s,t)$.*

PROOF. We show that every basis element e_je_k $(j < k)$ of $M(s,t)$ occurs as the tangent vector of some curve in $\mathrm{Spin}^+(s,t)$ at the identity. Since we know by local diffeomorphism that $\dim \mathrm{Spin}^+(s,t) = \dim \mathrm{SO}^+(s,t) = n(n-1)/2$ (see also 1.1.5(g)), this will then complete the proof.

Case $\eta_{jj} = \eta_{kk}$: We consider $\alpha(\tau) := \cos(\tau)1 + \sin(\tau)e_je_k$ for $\tau \in \mathbb{R}$. We have $\alpha(\tau) \in \mathrm{Spin}^+(s,t)$ for every $\tau \in \mathbb{R}$, since $\alpha(\tau) = e_j(-\eta_{jj}\cos(\tau)e_j + \sin(\tau)e_k) =: e_jv$ and $\eta(e_j, e_j) = \eta_{jj} = \eta_{jj}^2\cos^2(\tau)\eta_{jj} + \sin^2(\tau)\eta_{kk} = \eta(v,v)$. Clearly, $\alpha(0) = 1$ and $\dot\alpha(0) = e_je_k$.

Case $\eta_{jj} = -\eta_{kk}$: Consider $\alpha(\tau) := \cosh(\tau)1 + \sinh(\tau)e_je_k = e_j(-\eta_{jj}\cosh(\tau)e_j + \sinh(\tau)e_k) =: e_jv$ for $\tau \in \mathbb{R}$. Then $\eta(e_j, e_j) = \eta_{jj} = \eta_{jj}(\cosh^2(\tau) - \sinh^2(\tau)) = \eta_{jj}^2\cosh^2(\tau)\eta_{jj} + \sinh^2(\tau)\eta_{kk} = \eta(v,v)$, which shows that $\alpha(\tau) \in \mathrm{Spin}^+(s,t)$ for every $\tau \in \mathbb{R}$. The values $\alpha(0) = 1$ and $\dot\alpha(0) = e_je_k$ are again obvious. □

Recall from Lemma 1.4.4 and (1.18) that for $u \in \mathrm{Spin}^+(s,t)$, the action of $\lambda(u)$ on a vector $x \in \mathbb{R}^{s,t}$ is given as the unique vector in $\mathbb{R}^{s,t}$ corresponding to the element uxu^{-1} in $\gamma(\mathbb{R}^{s,t}) \subseteq \mathrm{Cl}(s,t)$.

Let $\lambda_* := T_1\lambda$ (the derivative of λ at 1) and $z \in T_1\mathrm{Spin}^+(s,t) = \mathfrak{spin}^+(s,t)$. The formula for the action of λ implies that for every $x \in \mathbb{R}^{s,t}$,

$$\lambda_*(z)x = \frac{d}{d\tau}\left(\lambda(\exp(\tau z))x\right)|_{\tau=0} = \frac{d}{d\tau}\left(\exp(\tau z)x\exp(-\tau z)\right)|_{\tau=0} = zx - xz = [z,x],$$

which a priori looks only like some element in $\mathrm{Cl}(s,t)$, but may be identified with a unique vector in $\mathbb{R}^{s,t}$ in the following way: Writing $z = \sum_{j<k} z_{jk}e_je_k$ and $x = \sum_l x_le_l$, we have $[z,x] = \sum_{j<k,l} z_{jk}x_l[e_je_k, e_l]$ and

$$\begin{aligned}[e_je_k, e_l] &= e_je_ke_l - e_le_je_k = e_je_ke_l + e_je_le_k + 2\eta_{jl}e_k = e_je_ke_l - e_je_ke_l - 2\eta_{kl}e_j + 2\eta_{jl}e_k \\ &= 2\eta_{jl}e_k - 2\eta_{kl}e_j \in \gamma(\mathbb{R}^{s,t}).\end{aligned}$$

In particular, the final expression can be rewritten as $\sum_m(2\eta_{jl}\delta_{km} - 2\eta_{kl}\delta_{jm})e_m =: \sum_m A_{ml}e_m$, from which we may read off the entries of the matrix $A = (A_{ml})_{1\le m,l\le n}$ of $\lambda_*(e_je_k)$ with respect to the standard basis (upon inserting $\eta_{jl} = \eta_{jj}\delta_{jl}$ and $\eta_{kl} = \eta_{kk}\delta_{kl}$) as $A_{ml} = 2\eta_{jj}\delta_{mk}\delta_{jl} - 2\eta_{kk}\delta_{mj}\delta_{kl}$, or $A = 2\eta_{jj}E_{kj} - 2\eta_{kk}E_{jk}$ with E_{pq} denoting the $(n\times n)$-matrix with entry 1 at row p, column q, and 0 elsewhere. We collect the formulae thus obtained:

$$\lambda_*(e_je_k)e_l = 2\eta_{jl}e_k - 2\eta_{kl}e_j \quad \text{and the matrix of } \lambda_*(e_je_k) \text{ is} \quad 2\eta_{jj}E_{kj} - 2\eta_{kk}E_{jk}. \tag{1.21}$$

Since λ is a local diffeomorphism, λ_* is an isomorphism of Lie algebras and we should therefore be able to recover any $z \in \mathfrak{spin}^+(s,t)$ from knowing $\lambda_*(z)$. Indeed, there is an explicit formula for λ_*^{-1}, which we will now derive employing the basis expansion $z = \sum_{j<k} z_{jk}e_je_k$, which gives $\lambda_*(z) = \sum_{j<k} z_{jk}\lambda_*(e_je_k)$. Let $1 \le l < m \le n$ and calculate using (1.21) (and $\eta_{jl} = \eta_{jj}\delta_{lj}$ etc)

$$\begin{aligned}\eta(\lambda_*(z)e_l, e_m) &= \sum_{j<k} z_{jk}\eta(\lambda_*(e_je_k)e_l, e_m) = \sum_{j<k} z_{jk}\eta(2\eta_{jl}e_k - 2\eta_{kl}e_j, e_m) \\ &= 2\sum_{j<k} z_{jk}(\eta_{jl}\eta_{km} - \eta_{kl}\eta_{jm}) = 2\sum_{j<k} z_{jk}\eta_{jj}\eta_{kk}\delta_{lj}\delta_{mk} - 2\sum_{j<k} z_{jk}\eta_{kk}\eta_{jj}\delta_{lk}\delta_{mj},\end{aligned}$$

where the second sum vanishes, since we have $j < k$ and $l < m$, and the first sum collapes to $j = l$ and $k = m$. Therefore, we obtain $\eta(\lambda_*(z)e_l, e_m) = 2z_{lm}\eta_{ll}\eta_{mm}$, which (upon relabeling l as j and m as k for later convenience) yields $z_{jk} = \eta_{jj}\eta_{kk}\eta(\lambda_*(z)e_j, e_k)/2$ and finally gives

$$z = \sum_{j<k} z_{jk}e_je_k = \frac{1}{2}\sum_{j<k}\eta_{jj}\eta_{kk}\eta(\lambda_*(z)e_j, e_k)e_je_k,$$

by which $z \in \mathfrak{spin}^+(s,t)$ is determined from $\lambda_*(z) =: A \in \mathfrak{so}^+(s,t)$. In other words, we have the following explicit formula for $z = \lambda_*^{-1}(A)$:

$$\lambda_*^{-1}(A) = \frac{1}{2}\sum_{j<k}\eta(Ae_j, e_k)\eta_{jj}\eta_{kk}e_je_k. \tag{1.22}$$

To summarize, we have shown the following results.

1.4.18. Corollary. *Let $\lambda_*\colon \mathfrak{spin}^+(s,t) \to \mathfrak{so}^+(s,t)$ be the Lie algebra isomorphism induced by the covering homomorphism $\lambda\colon \mathrm{Spin}^+(s,t) \to \mathrm{SO}^+(s,t)$. For any $z \in \mathfrak{spin}^+(s,t)$ and $x \in \mathbb{R}^{s,t}$, we have $[z,x] \in \gamma(\mathbb{R}^{s,t})$ and the action of $\lambda_*(z)$ can be determined from $\lambda_*(z)x = [z,x]$. For $1 \le j < k \le n$, the matrix corresponding to $\lambda_*(e_je_k) \in \mathfrak{so}^+(s,t)$ is given by* (1.21)*. The inverse $\lambda_*^{-1}\colon \mathfrak{so}^+(s,t) \to \mathfrak{spin}^+(s,t)$ is given by formula* (1.22) *for all $A \in \mathfrak{so}^+(s,t)$.*

1.5. Majorana spinors

1.5.1. Definition. Let V be a complex vector space, G a Lie group and $\rho : G \to \mathrm{GL}(V)$ a representation of G.

(i) A *real structure* on V is a complex antilinear (i.e., conjugate linear) G-equivariant map $\sigma : V \to V$ (i.e., $\sigma(\rho(g)v) = \rho(g)\sigma(v)$) with $\sigma^2 = \mathrm{id}$.

(ii) A *quaternionic structure* on V is a complex antilinear G-equivariant map $J : V \to V$ with $J^2 = -\mathrm{id}$.

1.5.2. PROPOSITION. *Let σ be a real structure on the complex G-representation space V and set*

$$V^\sigma := \{v \in V \mid \sigma(v) = v\}.$$

Then $V = V^\sigma \oplus iV^\sigma$, providing a decomposition of elements of V into real and imaginary parts. Moreover, the representation on V induces real representations of G on V^σ and on iV^σ that are isomorphic.

PROOF. Let $V'^\sigma := \{v \in V \mid \sigma(v) = -v\}$. Then both V^σ and V'^σ are real subspaces of V and for any $v \in V$ we have the decomposition

$$v = \frac{1}{2}(v + \sigma(v)) + \frac{1}{2}(v - \sigma(v)),$$

which shows that $V = V^\sigma \oplus V'^\sigma$. Moreover, by complex antilinearity, $\sigma(iv) = -i\sigma(v)$, so $v \in V^\sigma \Leftrightarrow iv \in V'^\sigma$, i.e., $V'^\sigma = iV^\sigma$.

To see that the representation restricts to V^σ, note that by G-equivariance we have, for $v \in V^\sigma$ and $g \in G$: $\sigma(g \cdot v) = g \cdot \sigma(v) = g \cdot v$, and similarly for V'^σ. Finally, the map $v \mapsto iv$ is G-equivariant because the representation is complex linear, and provides a real isomorphism between V^σ and V'^σ. □

The name 'quaternionic structure' finds its explanation in the following result. In its formulation, we will make use of some basic properties of quaternions $\mathbb{H}$ (cf. [**19**, Section 1.1.1]). By a *quaternionic vector space* we mean a vector space that is a (left or right) $\mathbb{H}$-module.

1.5.3. PROPOSITION. *Let J be a quaternionic structure and ρ a complex representation of G on V. Let*

$$\begin{aligned} I : V &\to V \\ v &\mapsto iv. \end{aligned}$$

Then I, J and $K := IJ$ define the structure of a G-equivariant quaternionic vector space on V.

PROOF. $I(\rho(v)) = i\rho(v) = \rho(iv) = \rho(Iv)$, so I is G-equivariant. The same is true of J by assumption, and consequently also of $K = IJ$. Furthermore, $I^2 = -\mathrm{id} = J^2$, and $JI(v) = J(iv) = -iJ(v) = -IJ(v)$, so I, J, K satisfy the quaternionic identitites. The desired module structure then follows by extending $i \cdot v = I(v)$, $j \cdot v := J(v)$, $k \cdot v := K(v)$ $\mathbb{R}$-linearly to $\mathbb{H} \times V \to V$. □

1.5.4. DEFINITION. As in Definition 1.4.13, let $\kappa\colon \mathrm{Spin}^+(s,t) \to \mathrm{GL}(\Delta)$ denote the complex spinor representation of the orthochronous spin group.

(i) The spinor representation is called *Majorana* if it admits a real structure σ. Then by Proposition 1.5.2 there exists a real subspace

$$\Delta^\sigma = \{s \in \Delta \mid \sigma(s) = s\}$$

of Δ with $\dim_\mathbb{R} \Delta^\sigma = \frac{1}{2} \dim_\mathbb{R} \Delta = \dim_\mathbb{C} \Delta$ and κ induces a real representation of $\mathrm{Spin}^+(s,t)$ on Δ^σ. The elements of Δ^σ are called *Majorana spinors*. Any $\psi \in \Delta$ can be uniquely decomposed as $\psi = \phi_1 + i\phi_2$ with Majorana spinors ϕ_1, ϕ_2. Picking a real basis $\alpha_1, \dots, \alpha_k$ of Δ^σ we have $\sigma(\alpha_i) = \alpha_i$ for $i = 1, \dots, k$

(ii) The spinor representation Δ is called *symplectic Majorana* if it admits a quaternionic structure J. Then Proposition 1.5.3 implies that Δ possesses a $\mathrm{Spin}^+(s,t)$-equivariant structure $I, J, K = IJ$ of a quaternionic vector space. The elements of Δ are called *symplectic Majorana spinors*.

1.5.5. Remark. (i) Given a symplectic Majorana representation Δ, pick any $\chi_1 \neq 0$ in Δ and set $\lambda_1 := J(\chi_1)$. Then $\{\chi_1, \lambda_1\}$ is linearly independent since applying J to $J(\chi_1) = \alpha\chi_1$ would imply $-\chi_1 = \bar{\alpha}J(\chi_1) = |\alpha|^2\chi_1$, a contradiction. If there exists some $\chi_2 \notin \mathrm{span}(\chi_1, \lambda_1)$, we set $\lambda_2 := J(\chi_2)$. Then $\lambda_2 \notin \mathrm{span}(\chi_1, \lambda_1, \chi_2)$, since applying J to $\lambda_2 = \alpha\chi_2 + \beta v$ (with $v \in \mathrm{span}(\chi_1, \lambda_1)$) would imply

$$-\chi_2 = \bar{\alpha}\lambda_2 + w$$

for some $w \in \mathrm{span}(\chi_1, \lambda_1)$, and re-inserting for λ_2 from above then gives $(-1 - |\alpha|^2)\chi_2 \in \mathrm{span}(\chi_1, \lambda_1)$, which is absurd. Continuing in this way, we obtain a basis $\lambda_1, \ldots, \lambda_k, \chi_1, \ldots \chi_k$ of Δ such that

$$J(\chi_i) = \lambda_i$$
$$J(\lambda_i) = -\chi_i.$$

In terms of this basis, J is therefore represented by the standard symplectic matrix $\begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$, which explains the terminology in the previous definition.

(ii) For concrete calculations it is useful to adopt the following conventions from the physics literature: Given a spinor $\psi \in \Delta$, we write $\psi^C = \sigma(\psi)$ or $\psi^C = J(\psi)$, depending on whether Δ is endowed with a real structure σ or a quaternionic structure J. Then the spinor ψ^C is called the *charge conjugate* of ψ. Once a complex basis in Δ has been chosen we may identify Δ with $\mathbb{C}^N$, and in this picture real or quaternionic structures can be expressed via a matrix B:

$$\psi^C = B^{-1}\psi^*, \tag{1.23}$$

with ψ^* denoting the complex conjugate of ψ (which is required due to the anti-linearity of σ resp. J). The conditions $\sigma^2 = \mathrm{id}$ resp. $J^2 = -\mathrm{id}$ translate to $B^*B = I_N$ resp. $B^*B = -I_N$, while $\mathrm{Spin}^+(s,t)$-equivariance ($\sigma(\kappa(g)\psi) = \kappa(g)\sigma(\psi)$) takes the form

$$\kappa(g)^* = B\kappa(g)B^{-1} \qquad \forall g \in \mathrm{Spin}^+(s,t).$$

Recall from Proposition 1.3.3 the direct sum decomosition $\Delta = \Delta_+ \oplus \Delta_-$ of complex Weyl spinors, and let $\pi_+ : \Delta \to \Delta_+$ be the projection along Δ_-.

1.5.6. Definition. Let $\kappa : \mathrm{Spin}^+(s,t) \to \mathrm{GL}(\Delta)$ be the complex spinor representation over an even-dimensional vector space $\Delta = \Delta_n$.

(i) Δ is called *Majorana-Weyl* if it admits a real structure σ that commutes with π_+. Then σ induces real structures on both Weyl spinor spaces $\Delta_\pm$. The elements of

$$\Delta_\pm^\sigma := \{s \in \Delta_\pm \mid \sigma(s) = s\}$$

are called left-handed and right-handed Majorana-Weyl spinors, respectively.

(ii) Δ is called *symplectic Majorana-Weyl* if it possesses a quaternionic structure J that commutes with π_+. Then $I, J, K = IJ$ induce quaternionic vector space structures on both Δ_+ and Δ_-. Elements of $\Delta_\pm$ are called left-handed or right-handed symplectic Majorana-Weyl spinors.

Depending on the dimension n and the signature (s,t) the above types of spinors may or may not exist. E.g., in Minkowski spacetime of dimension 4 there are both Weyl and Majorana spinors, but no Majorana-Weyl spinors (and therefore any Majorana spinor in this case has components in both Δ_+ and Δ_-). There is a complete classification for the existence of these additional structures, cf. [**19**, Section 6.6] and the references given there.

1.6. Spin invariant scalar products

1.6.1. DEFINITION. Let $\Delta = \Delta_n$ be the complex spinor representation of $\mathrm{Cl}(s,t)$ and fix constants $\mu, \nu = \pm 1$. A non-degenerate complex bilinear form $(\,.\,,\,.\,) : \Delta \times \Delta \to \mathbb{C}$ is called a *Majorana form* if

(i) $(X \cdot \psi, \phi) = \mu(\psi, X \cdot \phi)$ for all $X \in \mathbb{R}^{s,t}$ and all $\psi, \phi \in \Delta$.

(ii) $(\psi, \phi) = \nu(\phi, \psi)$ for all $\phi, \psi \in \Delta$.

Concrete calculations are best carried out in coordinates:

1.6.2. LEMMA. *Given a Majorana form $(\,.\,,\,.\,)$ and a complex basis $\{\chi_\alpha\}$ of Δ, let C be the matrix with entries $C_{\alpha\beta} = (\chi_\alpha, \chi_\beta)$. Let $\phi, \psi \in \Delta$ with basis expansion $\psi = \sum_\alpha \psi_\alpha \chi_\alpha$ and $\phi = \sum_\alpha \phi_\alpha \chi_\alpha$, then*

$$(\psi, \phi) = \psi^T C \phi. \tag{1.24}$$

Furthermore, properties (i) and (ii) from Definition 1.6.1 can equivalently be expressed in terms of γ-matrices as

(i) $\gamma_a^T = \mu C \gamma_a C^{-1}$ *for* $a = 1, \ldots, s+t$.

(ii) $C^T = \nu C$.

(The first equation also holds with physical gamma matrices Γ_a instead of γ_a.)

PROOF. Since (1.24) is immediate from the definitions, it remains to show (i) and (ii).

(i) By Definition 1.3.2, (i) from Definition 1.6.1 is equivalent to having, for all ϕ, ψ

$$\begin{aligned}
\forall X : (X \cdot \psi, \phi) = (\pi_n(\gamma(X))\psi, \phi) = \mu(\psi, \pi_n(\gamma(X))\phi) &\Leftrightarrow \forall a : \ (\pi_n(\gamma(e_a))\psi, \phi) = \mu(\psi, \pi_n(\gamma(e_a))\phi) \\
&\Leftrightarrow \forall a : \ (\gamma_a \cdot \psi, \phi) = \mu(\psi, \gamma_a \cdot \phi) \Leftrightarrow \forall a : \ (\gamma_a \psi)^T C \phi = \mu \psi^T C \gamma_a \phi \\
&\Leftrightarrow \forall a : \ \gamma_a^T C = \mu C \gamma_a \Leftrightarrow \forall a : \ \gamma_a^T = \mu C \gamma_a C^{-1}.
\end{aligned}$$

(ii) By (1.24), (ii) in Definition 1.6.1 is equivalent to

$$\forall \phi, \psi : \psi^T C \phi = \nu \phi^T C \psi = \nu \psi^T C^T \phi \Leftrightarrow C = \nu C^T \Leftrightarrow C^T = \nu C.$$

□

In the physics literature, the matrix C is called the *charge conjugation matrix.*

1.6.3. LEMMA. *Every Majorana form is invariant under the action of* $\mathrm{Spin}^+(s,t)$.

PROOF. By 1.6.1 and the definition of Clifford multiplication we have

$$(X \cdot \psi, X \cdot \phi) = \mu(\psi, (X \cdot X) \cdot \phi) = -\mu \eta(X,X)(\psi, \phi) \quad \forall X \in \mathbb{R}^{s,t}. \tag{1.25}$$

By Definition 1.4.3, any $g \in \mathrm{Spin}^+(s,t)$ is of the form $g = Z_1 \cdots Z_{2p+2q}$, with $2p$ of the Z_i from S_-, say $Z_{i_1}, \ldots, Z_{i_{2p}}$, and $2q$ of the Z_i from S_+, say $Z_{j_1}, \ldots, Z_{j_{2p}}$. Applying (1.25) iteratively we obtain

$$(g\psi, g\phi) = (-\mu)^{2p+2q} \prod_{k=1}^{2p} \eta(Z_{i_k}, Z_{i_k}) \prod_{l=1}^{2q} \eta(Z_{j_l}, Z_{j_l})(\psi, \phi) = (\psi, \phi).$$

□

Depending on $n = s + t$, only certain choices of μ and ν are permissible to obtain a Majorana form. Again, a complete list of possible values is available, cf. [**19**, Section 6.7] and the references given there.

1.6.4. Definition. For any spinor $\psi \in \Delta$, its *Majorana conjugate* is the linear functional $\tilde{\psi} \in \Delta^*$ defined by

$$\tilde{\psi} = (\psi, \,.\,).$$

Once a basis $\{\chi_\alpha\}$ for Δ is chosen, then in terms of the charge conjugation matrix C we have $\tilde{\psi}(\phi) = (\psi, \phi) = \psi^T C \phi$, so $\tilde{\psi} = \psi^T C$.

1.7. Dirac forms

1.7.1. Definition. Let $\Delta = \Delta_n$ be the complex spinor representation of $\mathrm{Cl}(s,t)$ and fix a constant $\delta = \pm 1$. A non-degenerate $\mathbb{R}$-bilinear form $\langle \,.\,, \,.\, \rangle : \Delta \times \Delta \to \mathbb{C}$ is called a *Dirac form* if

(i) $\langle X \cdot \psi, \phi \rangle = \delta \langle \psi, X \cdot \phi \rangle$ for all $X \in \mathbb{R}^{s,t}$ and all $\psi, \phi \in \Delta$.

(ii) $\langle \psi, \phi \rangle = \overline{\langle \phi, \psi \rangle}$ for all $\phi, \psi \in \Delta$.

(iii) $\langle \psi, c\phi \rangle = c \langle \psi, \phi \rangle = \langle \bar{c}\psi, \phi \rangle$ for all $\phi, \psi \in \Delta$ and all $c \in \mathbb{C}$.

Note that we do not require a Dirac form to be positive definite. In complete analogy to Lemma 1.6.2 (and with analogous proof), we have the following local characterization of Dirac forms:

1.7.2. Lemma. *Given a Dirac form $\langle \,.\,, \,.\, \rangle$ and a complex basis $\{\chi_\alpha\}$ of Δ, let A be the matrix with entries $A_{\alpha\beta} = \langle \chi_\alpha, \chi_\beta \rangle$. Let $\phi, \psi \in \Delta$ with basis expansion $\psi = \sum_\alpha \psi_\alpha \chi_\alpha$ and $\phi = \sum_\alpha \phi_\alpha \chi_\alpha$, then*

$$\langle \psi, \phi \rangle = \psi^\dagger A \phi. \tag{1.26}$$

Furthermore, properties (i) and (ii) from Definition 1.7.1 can equivalently be expressed in terms of γ-matrices as

(i) $\gamma_a^\dagger = \delta A \gamma_a A^{-1}$ *(equivalently, $\Gamma_a^\dagger = -\delta A \Gamma_a A^{-1}$) for $a = 1, \ldots, s+t$.*

(ii) $A^\dagger = A$.

Moreover, the analogue of Lemma 1.6.3 (again with almost identical proof) reads:

1.7.3. Lemma. *Every Dirac form is invariant under the action of* $\mathrm{Spin}^+(s,t)$.

1.7.4. Definition. A complex representation of the Clifford algebra $\mathrm{Cl}(s,t)$ is called *basis unitary* if all corresponding gamma matrices γ_a are unitary (equivalently, if all Γ_a are unitary).

1.7.5. Lemma. *The complex spinor representation Δ of the Clifford algebra* $\mathrm{Cl}(s,t)$ *can be chosen basis unitary.*

proof. Let $\{e_a\}$ be an orthonormal basis and let G be the multiplicative group in $\mathrm{Cl}(s,t)$ generated by $\{e_a\}$. Using the Clifford relations it immediately follows that G is finite. Pick any Hermitian scalar product $\langle \,.\,, \,.\, \rangle$ on Δ and set, for $\phi, \psi \in \Delta$,

$$(\phi, \psi) := \sum_{g \in G} \langle g\phi, g\psi \rangle.$$

This defines a new Hermitian scalar product on Δ, and for any $a = 1, \ldots, n$ and $\phi, \psi \in \Delta$ we have

$$(e_a \cdot \phi, e_a \cdot \psi) = \sum_{g \in G} \langle g(e_a \cdot \phi), g(e_a \cdot \psi) \rangle = \sum_{g \in G} \langle (ge_a)\phi), (ge_a)\psi) \rangle = \sum_{g \in G} \langle g\phi, g\psi \rangle = (\phi, \psi).$$

Hence with respect to this Hermitian product, all e_a are unitary maps, and so the corresponding gamma matrices are unitary as well. $\square$

Unless otherwise stated we shall henceforth always choose a Hermitian scalar product on the spinor space such that the spinor representation is basis unitary.

Recall from Remark 1.2.19 the identity $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab} I_n$ for the physical gamma matrices. This implies $\Gamma_a^{-1} = \eta(e_a, e_a)\Gamma_a$, and since Γ_a is unitary we get

$$\Gamma_a^\dagger = \Gamma_a^{-1} = \Gamma_a \quad \forall a = 1, \dots, s \tag{1.27}$$

$$\Gamma_a^\dagger = \Gamma_a^{-1} = -\Gamma_a \quad \forall a = s+1, \dots, s+t \tag{1.28}$$

The following result introduces some standard local expressions for Dirac forms:

1.7.6. Proposition. *Given a basis unitary spinor representation of* Cl(s,t)*, suppose that one of the following choices has been made*

(i) $\delta = (-1)^{t+1}$, $\varepsilon \in \{1, i\}$ *such that* $\bar{\varepsilon} = (-1)^{t(t+1)/2}\varepsilon$, $A = \varepsilon\Gamma_{s+1}\cdots\Gamma_{s+t}$, *or*

(ii) $\delta = (-1)^s$, $\varepsilon \in \{1, i\}$ *such that* $\bar{\varepsilon} = (-1)^{s(s-1)/2}\varepsilon$, $A = \varepsilon\Gamma_1\cdots\Gamma_s$.

Then A is unitary and satisfies (i) and (ii) from Lemma 1.7.2, hence defines a Dirac form for the spinor representation.

proof. We show this in the case (i) with $\varepsilon = 1$, the other possibilities are treated analogously. Since $\Gamma_a\Gamma_b = -\Gamma_b\Gamma_a$, for $a = 1, \dots s$ and $b = s+1, \dots, s+t$ we have

$$A\Gamma_a = \Gamma_{s+1}\cdots\Gamma_{s+t}\Gamma_a = (-1)^t\Gamma_a A \underset{(1.27)}{=} -\delta\Gamma_a^\dagger A,$$

if we set $\delta := (-1)^{t+1}$. For $a = s+1, \dots, s+t$ we get

$$A\Gamma_a = (-1)^{a-s-t}\Gamma_{s+1}\cdots\Gamma_a\Gamma_a\cdots\Gamma_{s+t} = (-1)^{t+1}\Gamma_a A \underset{(1.28)}{=} -\delta\Gamma_a^\dagger A,$$

so (i) from Lemma 1.7.2 follows. Concerning property (ii) in that same lemma, we have

$$A^\dagger = \bar{\varepsilon}\Gamma_{s+t}^\dagger\cdots\Gamma_{s+1}^\dagger \underset{(1.28)}{=} \bar{\varepsilon}(-1)^t\Gamma_{s+t}\cdots\Gamma_{s+1} = (-1)^{t(t+1)/2}(-1)^t(-1)^{t(t-1)/2}\Gamma_{s+1}\cdots\Gamma_{s+t} = A.$$

Finally, since each Γ_a is unitary, so is A. □

1.7.7. Definition. The *Dirac conjugate* $\bar{\psi}$ of a spinor $\psi \in \Delta$ with respect to a Dirac form is

$$\bar{\psi} = \langle\psi, .\rangle \in \Delta^*.$$

In terms of a basis $\{\chi_\alpha\}$ for Δ, with $\psi = \sum\psi_\alpha\chi_\alpha$, with respect to the dual basis of Δ^*, we have (cf. (1.26))

$$\bar{\psi} = \psi^\dagger A.$$

Again by (1.26) we can write the Dirac scalar product of spinors ψ, ϕ as

$$\langle\psi, \phi\rangle = \bar{\psi}\phi.$$

Let us now study the relationship between invariant forms and Majorana spinors. To this end, let $\Delta = \Delta_n$ be the complex spinor representation of Cl(s,t) and fix a Majorana form $(. , .)$ and a Dirac form $\langle . , . \rangle$ described by matrices C, A with (cf. Lemma 1.6.2))

$$\Gamma_a^T = \mu C\Gamma_a C^{-1} \quad \forall a = 1, \dots, s+t$$
$$C^T = \nu C$$

and (cf. Lemma 1.7.2)

$$\Gamma_a^\dagger = -\delta A\Gamma_a A^{-1} \quad \forall a = 1, \dots, s+t$$
$$A^\dagger = A.$$

1.7.8. Lemma. *There exists a complex antilinear map $\tau : \Delta \to \Delta$ such that*

$$(\psi, \phi) = \langle\tau(\psi), \phi\rangle \quad \forall\psi, \phi \in \Delta.$$

With respect to a basis $\{\chi_\alpha\}$ as above, $\tau = \psi \mapsto B^{-1}\psi^*$, with $B = (C^\dagger)^{-1}A$. Here, ψ^* denotes the component-wise complex conjugate (to avoid confusion with the Dirac conjugate).

PROOF. Any non-degenerate bilinear form induces an isomorphism between the vector space on which it is defined and its dual. Since the Dirac form is non-degenerate and $\phi \mapsto (\psi, \phi)$ lies in Δ^*, the existence of some $\tau(\psi)$ as above follows, and anti-linearity of τ is a consequence of the anti-linearity of $\langle\, .\, ,\, .\,\rangle$ in the first factor. The remaining claim follows because setting $B = (C^\dagger)^{-1}A$ the map $\tau = \psi \mapsto B^{-1}\psi^*$ indeed satisfies

$$\langle \tau(\psi), \phi\rangle = \tau(\psi)^\dagger A\phi = (B^{-1}\psi^*)^\dagger A\phi = (A^{-1}C^\dagger\psi^*)^\dagger A\phi = \psi^T C(A^{-1})^\dagger A\phi = \psi^T C\phi = (\psi, \phi).$$

□

As an immediate consequence we obtain:

1.7.9. COROLLARY. *For a spinor $\psi \in \Delta$ the Majorana conjugate equals its Dirac conjugate, i.e., $\tilde\psi = \bar\psi$ if and only if $\tau(\psi) = \psi$ or, equivalently, if $B\psi = \psi^*$.*

It can be shown (cf. [**19**, Sec. 6.7]) that this happens for $\psi \neq 0$ if an only if B defines a real structure and ψ is a Majorana spinor.

1.7.10. LEMMA. *The map τ satisfies*

$$\tau(X\cdot\psi) = \mu\delta X\cdot\tau(\psi) \quad \forall \psi \in \Delta, X \in \mathbb{R}^{s,t},$$

or, in terms of the matrix B,

$$\Gamma_a^* = -\mu\delta B\Gamma_a B^{-1} \quad \forall a = 1\dots, s+t.$$

PROOF. Using the definitions of τ, $\langle\, .\, ,\, .\,\rangle$ and $(\, .\, ,\, .\,)$ we calculate:

$$\langle\tau(X\cdot\psi), \phi\rangle = (X\cdot\psi, \phi) = \mu(\psi, X\cdot\phi) = \mu\langle\tau(\psi), X\cdot\phi\rangle = \mu\delta\langle X\cdot\tau(\psi), \phi\rangle.$$

Since the Dirac form is non-degenerate, this implies the first claim. Furthermore, setting $X = e_a$ we have

$$\tau(e_a\cdot\psi) = \mu\delta e_a\tau(\psi) \underset{1.7.8}{\Leftrightarrow} B^{-1}(e_a\cdot\psi)^* = \mu\delta e_a\cdot(B^{-1}\psi^*) \underset{1.3.2}{\Leftrightarrow} B^{-1}\gamma_a^*\psi^* = \mu\delta\gamma_a B^{-1}\psi^*$$
$$\Leftrightarrow B^{-1}\gamma_a^* = \mu\delta\gamma_a B^{-1} \Leftrightarrow B^{-1}(i\Gamma_a)^* = \mu\delta i\Gamma_a B^{-1} \Leftrightarrow \Gamma_a^* = -\mu\delta B\Gamma_a B^{-1}.$$

□

1.7.11. EXAMPLE. Let us flesh out some of these constructions in the case of four-dimensional Minkowski space $\mathbb{R}^{1,3}$. In this case, there exist both Weyl and Majorana spinors, but no Majorana-Weyl spinors. Recall from Example 1.2.27 that the physical gamma matrices for $\mathrm{Cl}(1,3)$ are given by

$$\Gamma_0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \qquad \Gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \quad k = 1,2,3,$$

with σ_k the Pauli matrices (see (1.13)). This is a basis unitary representation of the Clifford algebra, called the *Weyl representation*. By Definition 1.2.22 with $k = 2$ and $t = 3$, the physical chirality operator is given by

$$\Gamma_5 = -i\Gamma_0\Gamma_1\Gamma_2\Gamma_3 = i\Gamma^0\Gamma^1\Gamma^2\Gamma^3 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.$$

Consequently, the eigenspaces of the chirality operator corresponding to the eigenvalue $+1$ and -1 are $\mathbb{C}^2 \times \{0\}$ and $\{0\} \times \mathbb{C}^2$, respectively. In other words (cf. Corollary 1.3.4), the decomposition of any given Weyl spinor $\psi = (\chi, \xi) \in \mathbb{C}^2 \oplus \mathbb{C}^2$ into left- and right-handed Weyl spinors is given simply by $(\chi, 0) + (0, \xi)$. Fixing the choice (ii) in Proposition 1.7.6 (and noting that here we start counting at 0 instead of 1), we have

$$A = \Gamma_0,$$

with $\delta = -1$ and $\varepsilon = 1$. Then Lemma 1.7.2 (i) reads

$$\Gamma_a^\dagger = \Gamma_0\Gamma_a\Gamma_0 \quad (a = 0,1,2,3),$$

and according to Definition 1.7.7 the Dirac conjugate of a spinor ψ is $\bar\psi = \psi^\dagger\Gamma_0$, with Hermitian scalar product

$$\langle\psi, \phi\rangle = \bar\psi\phi = \psi^\dagger\Gamma_0\phi.$$

Here, by Definition 1.7.1 we have

$$\langle X\cdot\psi,\phi\rangle = -\langle\psi, X\cdot\phi\rangle \quad \forall X\in\mathbb{R}^{1,3}.$$

Again writing $\psi = (\chi,\xi)^T$ and inserting Γ_0 from above, we arrive at

$$\langle\psi,\psi\rangle = \bar{\psi}\psi = \chi^\dagger\xi + \xi^\dagger\chi.$$

This Hermitian scalar product is not positive definite. Indeed, both the subspaces of left-handed and right-handed Weyl spinors are null.

For the charge conjugation matrix C defining the Majorana form we make the choice

$$C := i\Gamma_0\Gamma_2 = i\Gamma^2\Gamma^0 = \left(\begin{smallmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2\end{smallmatrix}\right).$$

One then checks that

$$\begin{aligned}\Gamma_a^T &= -C\Gamma_a C^{-1} \quad \forall a = 0,1,2,3\\ C^T &= -C,\end{aligned}$$

so the defining properties from Lemma 1.6.2 are satisfied with $\mu = \nu = -1$. The matrix C is unitary and $\tau = \psi \mapsto B^{-1}\psi^*$, where

$$B = CA = (C^\dagger)^{-1}A = -i\Gamma_2,$$

verifying the condition from Lemma 1.7.8. Also, $B^{-1} = B = B^* = -i\Gamma_2 = i\Gamma^2$, and so the charge conjugate of a Dirac spinor ψ is given by (cf. (1.23))

$$\psi^C = B^{-1}\psi^* = i\Gamma^2\psi^*.$$

In accordance with the remark following Corollary 1.7.9 we have $B^*B = B^2 = I$, so the map $\sigma : \psi \mapsto B\psi^*$ defines a real structure on $\Delta = \mathbb{C}^4$. By Definition 1.5.4, ψ is a Majorana spinor if and only if $\sigma(\psi) = \psi$, i.e., if and only if $\psi^* = B\psi$. In terms of the decomposition $\psi = (\chi,\xi)^T$ this condition means $\xi = i\sigma_2\chi^*$, hence

$$\psi = \left(\begin{smallmatrix} \chi \\ i\sigma_2\chi^*\end{smallmatrix}\right).$$

1.7.12. Remark. The observation that both the subspaces of left-handed and right-handed Weyl spinors are null for the Dirac form is not confined to the particular example of four-dimensional Minkowski space. Indeed, suppose that Δ is of even dimension $n = 2k$ and that η has Lorentzian signature $(n-1,1)$ or $(1,n-1)$. Then $\Delta = \Delta_n^+ \oplus \Delta_n^-$ and choosing a basis for Δ adapted to this splitting it follows that the standard Hermitian product of elements of Δ_n^+ with those of Δ_n^- vanishes. Moreover, each of the choices for A from Proposition 1.7.6 gives A (up to a pre-factor) as a product of an odd number of gamma matrices, and thereby as an element of $\mathrm{Cl}^1(n)$. By Proposition 1.3.3 (ii) therefore, A maps $\Delta_n^\pm$ to $\Delta_n^\mp$. It follows that if $\phi,\psi\in\Delta_n^+$ then inserting them in the Dirac form gives $\langle\psi,\phi\rangle = \psi^\dagger A\phi$, which is a Hermitian scalar product of $\psi\in\Delta_n^+$ with $A\phi\in\Delta_n^-$, hence vanishes (and analogously for $\phi,\psi\in\Delta_n^-$).

CHAPTER 2

Spin structures and Spinor bundles

Having introduced the relevant algebraic structures in the previous chapter, we now turn to a synthesis of these algebraic foundations with the theory of principal fiber bundles (cf., e.g., [**25**]).

2.1. Spin structures

Let (M, g) be a pseudo-Riemannian manifold with metric g of signature (s, t). Then the *orthonormal frame bundle* $(\mathrm{O}(M, g), \pi, M, \mathrm{O}(s, t))$ is an $\mathrm{O}(s, t)$-principal fiber bundle with fiber over $x \in M$ given by

$$\mathrm{O}(M, g)_x = \left\{ \nu_x = (\nu_1, \ldots, \nu_n) \in \mathrm{GL}(M)_x \mid (g_x(\nu_i, \nu_j)) = \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix} \right\}.$$

(Cf. [**25**, Example 2.2.13], and note the different convention for the signature here.)

2.1.1. Remark. (i) By [**25**, Example 4.4.6 (ii)], (M, g) is orientable if and only if its holonomy group is contained in $\mathrm{SO}(s, t)$. By [**25**, Theorem 4.2.4], applied to the connection form A^{LC} induced on $\mathrm{O}(M, g)$ by the Levi-Civita connection on M, $\mathrm{O}(s, t)$ reduces to its holonomy group. Consequently (cf. [**25**, Theorem 2.5.2]), (M, g) is orientable if and only if the orthonormal frame bundle reduces to an $\mathrm{SO}(s, t)$-bundle. Of course, this is just a more sophisticated way of expressing the fact that orientability means the existence of an atlas such that all transition functions have positive determinant of the Jacobian, or equivalently that there exists a covering of M and corresponding local orthonormal frames that transform via matrices with determinant $+1$.

(ii) (M, g) is called *time-orientable* if the frame bundle can be reduced to a principal $\mathrm{O}^+(s, t)$-bundle under the embedding $\mathrm{O}^+(s, t) \hookrightarrow \mathrm{O}(s, t)$. It is called *orientable and time-orientable* if the frame bundle can be reduced to an $\mathrm{SO}^+(s, t)$-bundle. If any of the above reductions exist we call (M, g) itself orientable, time-orientable, or orientable and time-orientable, respectively. A choice of (time-)orientation (i.e., of one covering by frames with the desired transformation behavior) leads to a (time-)oriented manifold.

Suppose now that (M, g) is oriented and time oriented. Then we write

$$\pi_{\mathrm{SO}} : \mathrm{SO}^+(M) \to M$$

for the $\mathrm{SO}^+(s, t)$-frame bundle. Recall from Lemma 1.4.6 the double covering of Lie groups

$$\lambda : \mathrm{Spin}^+(s, t) \to \mathrm{SO}^+(s, t).$$

Using this, we define:

2.1.2. Definition. A *spin structure* on M is a $\mathrm{Spin}^+(s, t)$-principal fiber bundle

$$\pi_{\mathrm{Spin}} : \mathrm{Spin}^+(M) \to M,$$

together with a smooth double covering

$$\Lambda : \mathrm{Spin}^+(M) \to \mathrm{SO}^+(M)$$

such that the following diagram commutes:

$$
\begin{array}{ccccc}
\mathrm{Spin}^+(M) \times \mathrm{Spin}^+(s,t) & \xrightarrow{\ \cdot\ } & \mathrm{Spin}^+(M) & & \\
\Big\downarrow{\scriptstyle \Lambda\times\lambda} & & \Big\downarrow{\scriptstyle \Lambda} & \searrow^{\pi_{\mathrm{Spin}}} & \\
 & & & & M \\
 & & & \nearrow_{\pi_{\mathrm{SO}}} & \\
\mathrm{SO}^+(M) \times \mathrm{SO}^+(s,t) & \xrightarrow{\ \cdot\ } & \mathrm{SO}^+(M) & &
\end{array}
\tag{2.1}
$$

Here, the $\cdot$ in the horizontal lines denotes the standard right action of the structure group on the respective principal bundle.

In the sense of [**25**, Definition 2.2.3], a spin structure therefore is a λ-equivariant bundle morphism $\Lambda : \mathrm{Spin}^+(M) \to \mathrm{SO}^+(M)$, i.e., a λ-reduction of $\mathrm{SO}^+(M)$ (cf. [**25**, Sec. 2.5]).

2.1.3. Definition. Two spin structures $\Lambda : \mathrm{Spin}^+(M) \to \mathrm{SO}^+(M)$ and $\Lambda' : \mathrm{Spin}^+(M)' \to \mathrm{SO}^+(M)$ are called isomorphic if there exists a $\mathrm{Spin}^+(s,t)$-equivariant bundle isomomorphism $F : \mathrm{Spin}^+(M) \to \mathrm{Spin}^+(M)'$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Spin}^+(M) & \xrightarrow{\quad F \quad} & \mathrm{Spin}^+(M)' \\
 & {\scriptstyle \Lambda}\searrow \quad \swarrow{\scriptstyle \Lambda'} & \\
 & \mathrm{SO}^+(M) &
\end{array}
$$

2.1.4. Remark. (i) In the same way as we did above for the tangent bundle TM, one may define spin structures for any principal $\mathrm{SO}^+(s,t)$-bundle.

(ii) Existence of spin structures: While for trivial $\mathrm{SO}^+(s,t)$-bundles there always exist spin structures, in general there are topological obstructions that may prevent such existence results. These obstructions may be characterized in terms of characteristic classes (Stiefel-Whitney classes), but we will not pursue this here (see [**19**, Section 6.9] for precise results and references). A manifold is called *spin* if it possesses a spin structure. This is true, e.g., for all manifolds with trivial tangent bundle, all spheres, or also all orientable two-dimensional manifolds. Moreover, any $\mathbb{R}^{s,t}$ possesses a unique spin structure.

(iii) Any tensor bundle on a manifold M is associated to the frame bundle $\mathrm{GL}(M)$ by [**25**, Example 2.4.6]. If (M,g) is a pseudo-Riemannian manifold of signature (s,t), then $\mathrm{GL}(M)$ reduces to $\mathrm{O}(M)$, hence any tensor bundle is associated to $\mathrm{O}(M)$ in this case by [**25**, Theorem 2.5.8]. If M is, moreover oriented and time-oriented then $\mathrm{O}(M)$ reduces to $\mathrm{SO}^+(M)$. In particular we then have that $TM \cong \mathrm{SO}^+(M) \times_{\rho_{\mathrm{SO}}} \mathbb{R}^{s,t}$ (for ρ_{SO} the standard representation of $\mathrm{SO}^+(s,t)$ on $\mathbb{R}^{s,t}$). By the same reasoning, if M in addition is spin, then all tensor bundles on M are associated to $\mathrm{Spin}^+(M)$. There may also exist further vector bundles associated to $\mathrm{Spin}^+(M)$ on M that do not stem from the reduction of $\mathrm{SO}^+(M)$ to $\mathrm{Spin}^+(M)$.

We now want to look at the relation between sections of $\mathrm{SO}^+(M)$ and $\mathrm{Spin}^+(M)$.

2.1.5. Definition. A local section $e = (e_1, \dots, e_n)$ of $\mathrm{SO}^+(M)$ is called an *n-bein* or *vielbein*. In case $n = 4$ it is also called a *tetrad*.

2.1.6. Lemma. *If M is equipped with a spin structure, then for any vielbein e on a contractible open set $U \subseteq M$ there exist precisely two local sections $\varepsilon_\pm$ of $\mathrm{Spin}^+(M)$ over U with $\Lambda \circ \varepsilon_\pm = e$.*

proof. The image $U' := e(U) \subseteq \mathrm{SO}^+(M)$ is diffeomorphic to U, hence is in particular contractible as well. Consequently,

$$\Lambda|_{\Lambda^{-1}(U')} : \Lambda^{-1}(U') \to U'$$

is a trivial two-sheeted covering (cf., e.g., [**29**, Appendix A, Cor. 14]). It therefore possesses precisely two sections

$$s_\pm : U' \to \Lambda^{-1}(U') \subseteq \mathrm{Spin}^+(M),$$

so we necessarily have $\varepsilon_\pm = s_\pm \circ e$. □

2.2. Spinor bundles

2.2.1. Definition. Given a spin structure $\mathrm{Spin}^+(M) \to M$ on M, let $\kappa : \mathrm{Spin}^+(s,t) \to \mathrm{GL}(\Delta)$ be the spinor representation from Definition 1.4.13. Then the *(Dirac) spinor bundle* is the associated vector bundle

$$S = \mathrm{Spin}^+(M) \times_\kappa \Delta$$

over M (cf. [**25**, Remark 2.4.5]). Sections of S are called *spinor fields* or *spinors*.

2.2.2. Proposition. *Let $S \to M$ be the spinor bundle associated to a spin structure* $\mathrm{Spin}^+(M) \to M$.

(i) *There exists a smooth* Clifford multiplication

$$\begin{aligned} TM \oplus S &\to S \\ (X, \Psi) &\mapsto X \cdot \Psi \end{aligned}$$

(with $\oplus$ denoting the Whitney sum of vector bundles over the same base manifold, cf. [**25**, Rem. 2.4.3 (i)]*) such that for each $p \in M$ its restriction to $T_pM \times S_p \to S_p$ is bilinear. This map also induces a Clifford-multiplication of forms (elements of ΛTM)*[1] *with spinors.*

(ii) *If $n = \dim(M)$ is even then S splits into a direct sum of complex* Weyl spinor bundles *$S = S_+ \oplus S_-$, where*

$$S_\pm := \mathrm{Spin}^+(M) \times_\kappa \Delta^\pm.$$

In this case, Clifford multiplication by a vector maps $S_\pm$ to $S_\mp$.

proof. (i) Recall from Remark 2.1.4 (iii) that $TM \cong \mathrm{SO}^+(M) \times_{\rho_{\mathrm{SO}}} \mathbb{R}^{s,t}$, where ρ_{SO} denotes the standard representation of $\mathrm{SO}^+(s,t)$ on $\mathbb{R}^{s,t}$. Now for any $\varepsilon \in \mathrm{Spin}^+(M)$ we have $\Lambda(\varepsilon) \in \mathrm{SO}^+(M)$ and we define Clifford multiplication $TM \oplus S \to S$ by

$$\begin{aligned} (\mathrm{SO}^+(M) \times_{\rho_{\mathrm{SO}}} \mathbb{R}^{s,t}) \oplus (\mathrm{Spin}^+(M) \times_\kappa \Delta) &\to (\mathrm{Spin}^+(M) \times_\kappa \Delta) \\ ([\Lambda(\varepsilon), x], [\varepsilon, \psi]) &\mapsto [\varepsilon, x \cdot \psi], \end{aligned}$$

where $x \cdot \psi$ is the Clifford multiplication between $x \in \mathbb{R}^{s,t}$ and $\psi \in \Delta$ from Definition 1.3.2. Let us verify that this map is well defined. First, note that any element of $[\nu, y] \in \mathrm{SO}^+(M) \times_{\rho_{\mathrm{SO}}} \mathbb{R}^{s,t}$ with the same footpoint p in M as ε can be written in the form $[\Lambda(\varepsilon), x]$. Indeed, since $\mathrm{SO}^+(s,t)$ acts transitively on the fibers of $\mathrm{SO}^+(M)$ there exists some $A \in \mathrm{SO}^+(s,t)$ with $\nu = \Lambda(\varepsilon) \cdot A$, so $[\nu, y] = [\Lambda(\varepsilon), A \cdot y]$. Now writing $[\varepsilon, \psi] = [\varepsilon \cdot C, \kappa(C^{-1}) \cdot \psi]$ with $C \in \mathrm{Spin}^+(s,t)$ and

$$[\Lambda(\varepsilon), x] = [\Lambda(\varepsilon)\lambda(C), \lambda(C)^{-1}x] = [\Lambda(\varepsilon C), \lambda(C^{-1})x],$$

we indeed obtain

$$\begin{aligned} ([\Lambda(\varepsilon C), \lambda(C^{-1})x], [\varepsilon C, \kappa(C^{-1}) \cdot \psi]) &\mapsto [\varepsilon C, (\lambda(C^{-1})x) \cdot (\kappa(C^{-1})\psi)] \\ &\underset{1.4.14}{=} [\varepsilon C, \kappa(C^{-1})(x \cdot \psi)] = [\varepsilon, x \cdot \psi]. \end{aligned}$$

To show smoothness, pick bundle charts as in [**25**, Theorem 2.3.1]

$$\begin{aligned} \phi_U &: \mathrm{SO}^+(M) \to U \times \mathrm{SO}^+(s,t), \quad \nu \mapsto (\pi(\nu), \varphi_U(\nu)) \\ \tilde{\phi}_U &: \mathrm{Spin}^+(M) \to U \times \mathrm{Spin}^+(s,t), \quad \tilde{\nu} \mapsto (\pi(\tilde{\nu}), \tilde{\varphi}_U(\tilde{\nu})). \end{aligned}$$

Then by [**25**, (2.3.3)] we obtain corresponding charts of $\mathrm{SO}^+(M) \times_{\rho_{\mathrm{SO}}} \mathbb{R}^{s,t}$ and $\mathrm{Spin}^+(M) \times_\kappa \Delta$ of the form $\psi_U^{-1}(p, x) = [\phi_U^{-1}(p, I), x]$ and $\tilde{\psi}_U^{-1}(p, \psi) = [\tilde{\phi}_U^{-1}(p, I), \psi]$, respectively. Since both

[1] These are not differential forms (elements of ΛT^*M), but what is sometimes called *multivectors*.

$p \mapsto \phi_U^{-1}(p, I)$ and $p \mapsto \Lambda(\tilde{\phi}_U^{-1}(p, I))$ are local sections of $\mathrm{SO}^+(M)$, by [**25**, (3.1.7)] there exists a smooth map $p \mapsto A(p) \in \mathrm{SO}^+(s, t)$ such that $\phi_U^{-1}(p, I) = \Lambda(\tilde{\phi}_U^{-1}(p, I)) \cdot A(p)$, and thereby $[\phi_U^{-1}(p, I), x] = [\Lambda(\tilde{\phi}_U^{-1}(p, I)), A(p) \cdot x]$. Smoothness now follows because

$$\begin{aligned}(\psi_U^{-1} \times \tilde{\psi}_U^{-1})((p, x), (p, \psi)) &= ([\phi_U^{-1}(p, I), x], [\tilde{\phi}_U^{-1}(p, I), \psi]) \\ &= ([\Lambda(\tilde{\phi}_U^{-1}(p, I)), A(p) \cdot x], [\tilde{\phi}_U^{-1}(p, I), \psi])\end{aligned}$$

is mapped by the Clifford multiplication defined above to

$$[\tilde{\phi}_U^{-1}(p, I), (A(p) \cdot x) \cdot \psi],$$

and then by further applying $\tilde{\psi}_U$ ends up in $(p, (A(p) \cdot x) \cdot \psi)$, which altogether gives a smooth map.

Turning now to the case of multivectors, recall first that $\Lambda^k TM \cong \mathrm{SO}^+(M) \times_{\rho_k} \Lambda^k \mathbb{R}^{s,t}$, where ρ_k is the k-fold exterior product of ρ_{SO} (cf. [**24**, 23.3 (v)]). Again using Definition 1.3.2, we then define

$$\begin{aligned}(\mathrm{SO}^+(M) \times_{\rho_k} \Lambda^k \mathbb{R}^{s,t}) \oplus (\mathrm{Spin}^+(M) \times_\kappa \Delta) &\to (\mathrm{Spin}^+(M) \times_\kappa \Delta) \\ ([\Lambda(\varepsilon), \omega], [\varepsilon, \psi]) &\mapsto [\varepsilon, \pi_n(\sigma^{-1}(\omega))\psi].\end{aligned}$$

As in the previous case, let us verify that this is well defined. To do this, by linearity it suffices to assume $\omega = v_1 \wedge \cdots \wedge v_k$, where the v_i are orthonormal. Letting $C \in \mathrm{Spin}^+(s, t)$, we have $[\Lambda(\varepsilon), \omega] = [\Lambda(\varepsilon)\lambda(C), \rho_k(\lambda(C^{-1}))\omega] = [\Lambda(\varepsilon C), \rho_k(\lambda(C^{-1}))\omega]$, and by the above definition we get

$$([\Lambda(\varepsilon C), \rho_k(\lambda(C^{-1}))\omega], [\varepsilon C, \kappa(C^{-1}) \cdot \psi]) \mapsto [\varepsilon C, \pi_n(\sigma^{-1}(\rho_k(\lambda(C^{-1}))\omega))(\kappa(C^{-1})\psi)] \tag{2.2}$$

Here, $\rho_k(\lambda(C^{-1}))\omega = (\lambda(C)^{-1}v_1) \wedge \cdots \wedge (\lambda(C)^{-1}v_k)$. By (1.11), therefore,

$$\sigma^{-1}(\rho_k(\lambda(C^{-1}))\omega) = \gamma(\lambda(C^{-1})v_1) \cdots \gamma(\lambda(C^{-1})v_k).$$

Now note that by Proposition 1.4.14 we have

$$\pi_n(\gamma(\lambda(C^{-1})v_i))(\kappa(C^{-1})\psi) \equiv (\lambda(C^{-1})v_i)(\kappa(C^{-1})\psi) = \kappa(C^{-1})(v_i \cdot \psi)$$

for all i. Applying this iteratively, we get

$$\begin{aligned}\pi_n(\sigma^{-1}(\rho_k(\lambda(C^{-1}))\omega)) &= \pi_n(\gamma(\lambda(C^{-1})v_1)) \circ \cdots \circ \pi_n(\gamma(\lambda(C^{-1})v_k))(\kappa(C^{-1})\psi) \\ &= \pi_n(\gamma(\lambda(C^{-1})v_1)) \circ \cdots \circ \pi_n(\gamma(\lambda(C^{-1})v_{k-1}))(\kappa(C^{-1})(v_k \cdot \psi)) \\ &= \cdots = \kappa(C^{-1})(v_1 \cdots v_k \cdot \psi) = \kappa(C^{-1})(\pi_n(\sigma^{-1}(\omega))\psi).\end{aligned}$$

Inserting this in (2.2) we see that

$$([\Lambda(\varepsilon C), \rho_k(\lambda(C^{-1}))\omega], [\varepsilon C, \kappa(C^{-1}) \cdot \psi]) \mapsto [\varepsilon C, \kappa(C^{-1})(\pi_n(\sigma^{-1}(\omega))\psi)] = [\varepsilon, \pi_n(\sigma^{-1}(\omega))\psi],$$

as required. Smoothness follows as in the vector case.

(ii) As was noted in Definitions 1.2.20 and 1.2.22, the chirality element $\omega = -i^{k+t}e_1 \cdots e_n$ does not depend on the choice of vielbein, and from (i) it follows that multiplication of ω with spinors is well defined. The claim therefore follows from a fiber-wise application of Corollary 1.3.4. □

2.2.3. Remark. (i) To obtain a local description of spinors, let e be a local vielbein on an open contractible set $U \subseteq M$ and let $\varepsilon_\pm$ be the local sections of the spin structure bundle associated to e according to Lemma 2.1.6. Let $\Psi : U \to S \equiv \mathrm{Spin}^+(M) \times_\kappa \Delta$ be a local section of the spinor bundle. Then (applying the fiber diffeomorphisms corresponding to $\varepsilon_\pm$, cf. [**25**, (2.3.7)]) there exist two smooth maps $\psi_\pm : U \to \Delta$ such that $\Psi = [\varepsilon_\pm, \psi_\pm]$. Since both ε_+ and ε_- are local sections of $\mathrm{Spin}^+(M)$, there exists a smooth map $C : U \to \mathrm{Spin}^+(s, t)$ such that $\varepsilon_+ = \varepsilon_- \cdot C$ (cf. [**25**, (3.1.7)]). Combining this with (2.1) we obtain

$$e = \Lambda(\varepsilon_+) = \Lambda(\varepsilon_- \cdot C) = \Lambda(\varepsilon_-)\lambda(C) = e \cdot \lambda(C),$$

implying that $C(x) \in \ker(\lambda) = \{\pm I\}$ for each $x \in U$ (cf. Theorem 1.4.5). Since $\varepsilon_+ \neq \varepsilon_-$ it follows that $C \equiv -I$, so $\varepsilon_+ = -\varepsilon_-$ and a fortiori $\psi_\pm = -\psi_\mp$. Fixing a choice of ε and ψ we can write $\Psi = [\varepsilon, \psi]$. Then for example physical Clifford multiplication with a basis vector takes the form $e_a \cdot \Psi = [\varepsilon, \Gamma_a\psi]$ (independently of the choice of ε). Indeed, as an element of $TM \cong \mathrm{SO}^+(M) \times_{\rho_{\mathrm{SO}}} \mathbb{R}^{s,t}$, $e_a = [e, u_a] = [\Lambda(\varepsilon), u_a]$, where u_a is the a-th standard unit vector in $\mathbb{R}^{s,t}$. Thus by Definition 2.2.2,

$$e_a \cdot \Psi = [\Lambda(\varepsilon), u_a] \cdot [\varepsilon, \psi] = [\varepsilon, u_a \cdot \psi] = [\varepsilon, \Gamma_a\psi].$$

Clearly one may proceed analogously for other operations as long as they are linear in ψ.

(ii) Further structures on spinor bundles: In sections 1.5, 1.6 and 1.7 we studied additional structures on the spinor representation space Δ, namely real and quaternionic structures as well as Majorana and Dirac forms. All of these can be transferred to the spinor bundle. In the case of a real structure σ on Δ, given $[\alpha, \psi] \in S = \mathrm{Spin}^+(M) \times_\kappa \Delta$ we set $\sigma_S([\alpha, \psi]) := [\alpha, \sigma(\psi)]$. This gives a well defined operation because for any $C \in \mathrm{Spin}^+(s,t)$ we have

$$\sigma_S([\alpha C, C^{-1}\psi]) = [\alpha C, \sigma(C^{-1}\psi)] \underset{1.5.1}{=} [\alpha C, C^{-1}\sigma(\psi)] = [\alpha, \sigma(\psi)].$$

A similar argument applies to a quaternionic structure J on Δ, and clearly the resulting bundle operations σ_S, J_S are smooth. They can in turn be used to define (symplectic) Majorana or (symplectic) Majorana-Weyl sections of S.

If $(\,.\, , \,.\,)$ is a Majorana form (or $\langle \,.\, , \,.\, \rangle$ is a Dirac form) on Δ, then by Lemma 1.6.3 (or Lemma 1.7.3) it is invariant under the action of $\mathrm{Spin}^+(s,t)$. It therefore induces a bundle metric on S due to [**25**, Theorem 2.4.10], called a Majorana (or Dirac) bundle metric.

2.3. Spin covariant derivative and Dirac operator

In this section we shall see that given a spin structure on a pseudo-Riemannian manifold (M, g), the Levi-Civita connection on TM induces a unique compatible covariant derivative on the spinor bundle S, the so-called *spin covariant derivative*. Throughout, let (M, g) be an oriented and time-oriented pseudo-Riemannian manifold with g of signature (s,t) and Levi-Civita connection ∇ on TM.

Let $e = (e_1, \ldots, e_n)$ be a vielbein on an open set $U \subseteq M$. Then (recalling the definition of the standard bilinear form η from Example 1.1.2 and Remark 1.2.19) there are uniquely determined real-valued one-forms ω_{ab} on U such that

$$\nabla e_a = \omega_{ab}\eta^{bc} \otimes e_c \tag{2.3}$$

and we define

$$\omega_{cab} := \omega_{ab}(e_c). \tag{2.4}$$

2.3.1. Lemma. *The one-forms ω_{ab} are anti-symmetric in a, b:*

$$\omega_{ab} = -\omega_{ba} \qquad (a, b = 1, \ldots, n).$$

Consequently, $\omega_{cab} = -\omega_{cba}$.

Proof. Writing $\langle \,.\, , \,.\, \rangle$ for g we calculate:

$$\langle \nabla_{e_d} e_a, e_s \rangle = \omega_{ab}(e_d)\eta^{bc}\langle e_c, e_s \rangle = \omega_{ab}(e_d)\eta^{bc}\eta_{cs} = \omega_{ab}(e_d)\delta^b{}_s = \omega_{as}(e_d).$$

On the other hand, $\langle e_a, e_s \rangle$ being constant we have $\nabla_{e_d}\langle e_a, e_s \rangle = 0$, so $\langle \nabla_{e_d} e_a, e_s \rangle = -\langle e_a, \nabla_{e_d} e_s \rangle$, which as above is seen to equal $-\omega_{sa}(e_d)$, giving the claim. □

2.3.2. Definition. The *anholonomy coefficients* $\Omega_{ab}{}^c$ of a local vielbein e are defined by

$$[e_a, e_b] = \Omega_{ab}{}^c e_c,$$

where the left hand side denotes the usual Lie bracket (commutator) of smooth vector fields.

2.3.3. Lemma. *Setting $\Omega_{abc} := \Omega_{ab}{}^d\eta_{dc}$ we have*

$$\omega_{cab} = \omega_{ab}(e_c) = \frac{1}{2}(\Omega_{cab} - \Omega_{abc} + \Omega_{bca}).$$

PROOF. Due to $\nabla_{e_d} e_a = \omega_{ab}(e_d)\eta^{bc}e_c$ we have

$$\begin{aligned}\Omega_{abr}\eta^{rs}e_s = \Omega_{ab}{}^s e_s = [e_a, e_b] = \nabla_{e_a}e_b - \nabla_{e_b}e_a \\ = \omega_{br}(e_a)\eta^{rs}e_s - \omega_{ar}(e_b)\eta^{rs}e_s = (\omega_{abr} - \omega_{bar})\eta^{rs}e_s,\end{aligned}$$

so $\Omega_{abr} = \omega_{abr} - \omega_{bar}$. Therefore,

$$\Omega_{cab} - \Omega_{abc} + \Omega_{bca} = \omega_{cab} - \omega_{acb} - (\omega_{abc} - \omega_{bac}) + \omega_{bca} - \omega_{cba} \underset{2.3.1}{=} 2\omega_{cab}.$$

□

Recall from Remark 2.1.4 (iii) that $TM \cong \mathrm{SO}^+(M) \times_{\rho_{\mathrm{SO}}} \mathbb{R}^{s,t}$. Hence by [**25**, Corollary 3.1.13] (more precisely, by its obvious modification replacing $\mathrm{O}(s,t)$ by $\mathrm{SO}^+(\mathrm{s,t})$) the Levi-Civita connection ∇ induces a connection one-form $A_{\mathrm{SO}} \in \Omega^1(\mathrm{SO}^+(M), \mathfrak{so}^+(s,t))$ such that

$$\nabla = \nabla^{A_{\mathrm{SO}}}.$$

Let us recall the explicit relation between A_{SO} and ∇ from [**25**, Example 3.1.10]: e here corresponds to s there and we write A^e_{SO} for $A_e = e^*A_{\mathrm{SO}} = A_{\mathrm{SO}} \circ Te$. Then [**25**, (3.1.15),(3.1.16)] imply that given a local vector field with expansion $Y = y^a e_a$ in terms of the vielbein e,

$$\nabla_X Y = \nabla_X(y^a e_a) = X(y^a)e_a + y^a A^e_{\mathrm{SO}}(X)^c{}_a e_c = (L_X y^c + A^e_{\mathrm{SO}}(X)^c{}_a y^a)e_c \tag{2.5}$$

for any local vector field X. It remains to calculate the local connection coefficients $A^e_{\mathrm{SO}}(\,.\,)^c{}_a$. We have

$$\omega_{rb}(X)\eta^{bc}e_c \underset{(2.3)}{=} \nabla_X e_r \underset{(2.5)}{=} (L_X(\delta^r{}_c) + A^e_{\mathrm{SO}}(X)^c{}_a\delta^a{}_r)e_c = A^e_{\mathrm{SO}}(X)^c{}_r e_c,$$

so that

$$A^e_{\mathrm{SO}}(X)^c{}_a = \omega_{ab}(X)\eta^{bc} \qquad \forall X \in T_pM,\ p \in U. \tag{2.6}$$

We have seen in Lemma 1.4.6 that the Lie group homomorphism $\lambda : \mathrm{Spin}^+(s,t) \to \mathrm{SO}^+(s,t)$ is a covering map. In particular, $\lambda_* : \mathfrak{spin}^+(s,t) \to \mathfrak{so}^+(s,t)$ is a linear isomomorphism. We use this map to define a connection form on $\mathrm{Spin}^+(M)$:

2.3.4. PROPOSITION. *Let $\Lambda : \mathrm{Spin}^+(M) \to \mathrm{SO}^+(M)$ be the covering map from Definition 2.1.2. Then*

$$A_{\mathrm{Spin}} := (\lambda_*)^{-1} \circ (\Lambda^* A_{\mathrm{SO}}) \in \Omega^1(\mathrm{Spin}^+(M), \mathfrak{spin}^+(s,t))$$

is a connection one-form on the principal fiber bundle $\mathrm{Spin}^+(M) \to M$*, called the* spin connection.

PROOF. We have to verify the defining properties of a connection one-form, cf. [**25**, Definition 3.1.3].

(i) Let $g \in \mathrm{Spin}^+(s,t)$ and $Y \in T\mathrm{Spin}^+(M)$. Then

$$\begin{aligned}R^*_g A_{\mathrm{Spin}}(Y) &= (\lambda_*)^{-1} \circ A_{\mathrm{SO}}(T\Lambda \circ TR_g Y) \underset{(2.1)}{=} (\lambda_*)^{-1} \circ A_{\mathrm{SO}}(TR_{\lambda(g)} \circ T\Lambda(Y)) \\ &= (\lambda_*)^{-1} \circ \mathrm{Ad}_{\lambda(g)^{-1}} \circ A_{\mathrm{SO}}(T\Lambda(Y)) = (\lambda_*)^{-1} \circ \mathrm{Ad}_{\lambda(g)^{-1}} \circ \Lambda^* A_{\mathrm{SO}}(Y) \\ &= (\lambda_*)^{-1} \circ \mathrm{Ad}_{\lambda(g)^{-1}} \circ \lambda_* \circ A_{\mathrm{Spin}}(Y),\end{aligned}$$

where we used [**25**, Definition 3.1.3] (i) for A_{SO} and the definition of A_{Spin}. Together with the observation that

$$\mathrm{Ad}_{\lambda(g)^{-1}} \circ \lambda_* = \mathrm{Ad}_{\lambda(g)^{-1}} \circ T_e\lambda = T_e(\mathrm{conj}_{\lambda(g)^{-1}} \circ \lambda) = T_e(\lambda \circ \mathrm{conj}_{g^{-1}}) = \lambda_* \circ \mathrm{Ad}_{g^{-1}}$$

this shows that $R^*_g A_{\mathrm{Spin}}(Y) = \mathrm{Ad}_{g^{-1}} \circ A_{\mathrm{Spin}}(Y)$, as required.

(ii) Let $X \in \mathfrak{spin}^+(s,t)$ and $\varepsilon \in \mathrm{Spin}^+(M)$. For the fundamental vector field $\tilde{X}$ (cf. [**25**, Definition 1.2.1]) corresponding to X we have (with $\exp = \exp^{\mathrm{Spin}^+}$)

$$\begin{aligned} A_{\mathrm{Spin}}(\tilde{X}_\varepsilon) &= A_{\mathrm{Spin}}\Big(\frac{d}{dt}\Big|_0(\varepsilon \cdot \exp(tX))\Big) = (\lambda_*)^{-1} \circ A_{\mathrm{SO}}\Big(T\Lambda\Big(\frac{d}{dt}\Big|_0(\varepsilon \cdot \exp(tX))\Big)\Big) \\ &= (\lambda_*)^{-1} \circ A_{\mathrm{SO}}\Big(\frac{d}{dt}\Big|_0(\Lambda(\varepsilon \cdot \exp(tX)))\Big). \end{aligned}$$

Since $\lambda : \mathrm{Spin}^+(s,t) \to \mathrm{SO}^+(s,t)$ is a Lie group homomorphism, by [**24**, Theorem 8.8] we have

$$\begin{aligned} \Lambda(\varepsilon \cdot \exp^{\mathrm{Spin}^+}(tX)) &\underset{(2.1)}{=} \Lambda(\varepsilon) \cdot \lambda(\exp^{\mathrm{Spin}^+}(tX)) = \Lambda(\varepsilon) \cdot \exp^{\mathrm{SO}^+}(\lambda_*(tX)) \\ &= \Lambda(\varepsilon) \cdot \exp^{\mathrm{SO}^+}(t\lambda_*(X)). \end{aligned}$$

Altogether, we arrive at

$$A_{\mathrm{Spin}}(\tilde{X}_\varepsilon) = (\lambda_*)^{-1} \circ A_{\mathrm{SO}}\Big(\frac{d}{dt}\Big|_0(\Lambda(\varepsilon) \cdot \exp^{\mathrm{SO}^+}(t\lambda_*(X)))\Big) = (\lambda_*)^{-1} \circ A_{\mathrm{SO}}((\lambda_* X)^{\sim}_{\Lambda(\varepsilon)}) = X,$$

where we used [**25**, Definition 3.1.3] (ii) for A_{SO} in the last step. □

Now that we have a spin connection, by [**25**, Definition 3.4.8] there is also a corresponding covariant derivative:

2.3.5. Definition. The covariant derivative $\nabla^{A_{\mathrm{Spin}}}$ on the spinor bundle S is called the *spin covariant derivative*. We will simply denote it by $\nabla \equiv \nabla^{A_{\mathrm{Spin}}}$.

It is immediate from the definition of A_{Spin} in Proposition 2.3.4 that the spin covariant derivative is completely determined by the Levi-Civita connection on M. Our next aim is to derive an explicit local description for it. So suppose that $U \subseteq M$ is open and contractible, let e be a vielbein on U and let $\varepsilon_\pm$ be the associated local sections of $\mathrm{Spin}^+(M)$ defined in Lemma 2.1.6. Pick one of these and call it ε. Then we obtain a corresponding local connection one-form (cf. [**25**, Definition 3.1.5])

$$A^\varepsilon_{\mathrm{Spin}} = \varepsilon^* A_{\mathrm{Spin}} = A_{\mathrm{Spin}} \circ T\varepsilon \in \Omega^1(U, \mathfrak{spin}^+(s,t)).$$

Given a local section $\Psi \in \Gamma(U,S)$ of the spinor bundle, then by [**25**, Theorem 3.4.9] there exists a unique smooth map $\psi : U \to \Delta$ such that $\Psi = [\varepsilon, \psi]$ and for any $X \in \mathfrak{X}(U)$ we have

$$\nabla_X \Psi = [\varepsilon, \nabla_X \psi], \tag{2.7}$$

where

$$\nabla_X \psi = T\psi(X) + \kappa_*(A^\varepsilon_{\mathrm{Spin}}(X))\psi. \tag{2.8}$$

Notationally, we will henceforth suppress the representation κ_* of $\mathfrak{spin}^+(s,t)$ induced by the spinor representation κ on Δ, so we will simply write $\nabla_X \psi = T\psi(X) + A^\varepsilon_{\mathrm{Spin}}(X) \cdot \psi$.

2.3.6. Lemma. *For any $A \in \mathfrak{so}^+(s,t)$, write its components as $A^c{}_a =: w_{ab}\eta^{bc}$. Then the map*

$$\kappa_* \circ (\lambda_*)^{-1} : \mathfrak{so}^+(s,t) \xrightarrow{\cong} \mathfrak{spin}^+(s,t) \longrightarrow \mathrm{End}(\Delta)$$

is given by

$$\kappa_* \circ (\lambda_*)^{-1}(A) = \frac{1}{4} w_{ab}\gamma^{ab}.$$

proof. We first note that

$$w_{ab} = w_{ac}\eta^{cd}\eta_{db} = A^d{}_a \eta_{db} = \eta(A^d{}_a e_d, e_b) = \eta(Ae_a, e_b).$$

Also, $A \in \mathfrak{so}^+(s,t)$, hence is skew-symmetric with respect to η, so

$$w_{ab} = \eta(Ae_a, e_b) = -\eta(e_a, Ae_b) = -w_{ba}.$$

From (1.22) we know that

$$\lambda_*^{-1}(A) = \frac{1}{2}\sum_{a<b} \eta(Ae_a, e_b)\eta_{aa}\eta_{bb}e_a e_b.$$

Note now that κ is the restriction of the algebra representation $\pi_n \colon \mathbb{C}\mathrm{l}(n) \to \mathrm{L}(\Delta_n)$ (cf. Definition 1.3.1). In particular, π_n is linear, so also $\kappa_* = \pi_n$ is an algebra homomorphism, and we have $\kappa_*(e_a e_b) = \kappa_*(e_a)\kappa_*(e_b) = \pi_n(e_a)\pi_n(e_b) = \gamma_a\gamma_b$. Collecting these observations, we obtain

$$\begin{aligned}\kappa_*(\lambda_*^{-1}(A)) &= \frac{1}{2}\sum_{a<b}\eta(Ae_a, e_b)\eta_{aa}\eta_{bb}\gamma_a\gamma_b = \frac{1}{2}\sum_{a<b} w_{ab}\gamma^a\gamma^b = \frac{1}{4}\Big(\sum_{a<b} w_{ab}\gamma^a\gamma^b - \sum_{a>b} w_{ab}\gamma^a\gamma^b\Big)\\ &= \frac{1}{4}\sum_{a<b} w_{ab}(\gamma^a\gamma^b - \gamma^b\gamma^a) = \frac{1}{2}\sum_{a<b} w_{ab}\gamma^{ab}.\end{aligned}$$

Noting that $w_{bb} = 0$ and $w_{ba}\gamma^{ba} = w_{ab}\gamma^{ab}$ now gives the claim. □

Using this we can now establish the desired explict formula:

2.3.7. Proposition. *The spin covariant derivative is locally given by*

$$\nabla_X\psi = T\psi(X) + \frac{1}{4}\omega_{ab}(X)\gamma^{ab}\psi = T\psi(X) - \frac{1}{4}\omega_{ab}(X)\Gamma^{ab}\psi.$$

proof. Because of (2.8) we are left with showing that $\kappa_*(A^{\varepsilon}_{\mathrm{Spin}}(X)) = \frac{1}{4}\omega_{ab}(X)\gamma^{ab}$. Now using Proposition 2.3.4 and the fact that $\Lambda \circ \varepsilon = e$ we obtain

$$A^{\varepsilon}_{\mathrm{Spin}} = A_{\mathrm{Spin}} \circ T\varepsilon = \lambda_*^{-1} \circ A_{\mathrm{SO}} \circ T\Lambda \circ T\varepsilon = \lambda_*^{-1} \circ A_{\mathrm{SO}} \circ Te = \lambda_*^{-1} \circ A^{e}_{\mathrm{SO}}.$$

Since $A^e_{\mathrm{SO}}(X)^c{}_a = \omega_{ab}(X)\eta^{bc}$ by (2.6), the claim then follows from Lemma 2.3.6. □

2.3.8. Lemma. *If $n = \dim(M)$ is even, then the spin covariant derivative preserves the splitting of the spinor bundle into the Weyl spinor bundles S_+ and S_-, i.e.,*

$$\nabla_X\Psi \in \Gamma(S_\pm) \qquad \forall\Psi \in \Gamma(S_\pm),\ X \in \mathfrak{X}(M).$$

proof. The fibers of $S_\pm$ are the eigenspaces of Γ^{n+1} (cf. Remark 1.3.5) and we know from (1.12) that $[\Gamma^{n+1}, \Gamma^{ab}] = 0$. The explicit formula in Proposition 2.3.7 shows that ∇_X maps eigenvectors of Γ^{n+1} into eigenvectors with the same eigenvalue. □

2.3.9. Theorem. *The spin covariant derivative is compatible with the Levi-Civita connection in the sense that, for all $X, Y \in \mathfrak{X}(M)$ and all spinors $\Psi \in \Gamma(S)$ we have*

$$\nabla_X(Y \cdot \Psi) = (\nabla_X Y) \cdot \Psi + Y \cdot \nabla_X\Psi.$$

proof. In order to use the definition of Clifford multiplication given in Proposition 2.2.2 we apply [**25**, Theorem 3.4.9] to write $Y(x) \in TM \cong \mathrm{SO}^+(M) \times_{\rho_{\mathrm{SO}}} \mathbb{R}^{s,t}$ as $Y(x) = [e(x), \hat{Y}(x)]$, where $\hat{Y} : U \to \mathbb{R}^{s,t}$ is smooth (namely, $Y = \hat{Y}^a e_a$), and to obtain

$$\nabla_X Y(x) \equiv \nabla^{A_{\mathrm{SO}}}_X Y = [e(x), T_x\hat{Y} + A^e_{\mathrm{SO}}(X)\hat{Y}].$$

Also, by the discussion following Definition 2.3.5 we have $\Psi = [\varepsilon, \psi]$, and (cf. (2.8)) $\nabla_X\Psi = [\varepsilon, \nabla_X\psi]$, with $\nabla_X\psi = T\psi(X) + \frac{1}{4}\omega_{ab}(X)\gamma^{ab}\psi$ due to Proposition 2.3.7. Inserting these expressions into the Clifford product as defined in Proposition 2.2.2 and keeping in mind that $\lambda \circ \varepsilon = e$, we obtain

$$\begin{aligned}(\nabla_X Y \cdot \Psi)(x) &= [\varepsilon(x), (T_x\hat{Y} + A^e_{\mathrm{SO}}(X)\hat{Y}|_x) \cdot \psi(x)]\\ (Y \cdot \nabla_X\Psi)(x) &= [\varepsilon(x), \hat{Y}|_x \cdot T_x\psi(X) + \hat{Y}|_x \cdot (\frac{1}{4}\omega_{ab}(X)\gamma^{ab}\psi(x))],\end{aligned}$$

while

$$\nabla_X(Y \cdot \Psi)(x) = \nabla_X[\varepsilon, \hat{Y} \cdot \psi]|_x = [\varepsilon(x), T_x(\hat{Y} \cdot \psi) + \frac{1}{4}\omega_{ab}(X)\gamma^{ab}(\hat{Y}|_x \cdot \psi(x))].$$

Now $T_x(\hat{Y} \cdot \psi) = T_x\hat{Y} \cdot \psi(x) + \hat{Y}|_x \cdot T_x\psi(X)$ due to the bilinearity of the algebraic Clifford multiplication, so we are left with showing that (with $Y = \hat{Y}^a e_a$, hence $\hat{Y} = (\hat{Y}^a) = \hat{Y}^a u_a$, u_a the a-th standard unit vector in $\mathbb{R}^{s,t}$):

$$\begin{aligned}\frac{1}{4}\omega_{ab}(X)\gamma^{ab}(\hat{Y}|_x \cdot \psi(x)) &\overset{!}{=} A^e_{\mathrm{SO}}(X)\hat{Y}|_x \cdot \psi(x) + \hat{Y}|_x \cdot (\frac{1}{4}\omega_{ab}(X)\gamma^{ab}\psi(x))\\ &= (A^e_{\mathrm{SO}}(X)^c{}_a\hat{Y}^a(x)u_c) \cdot \psi(x) + \hat{Y}|_x \cdot (\frac{1}{4}\omega_{ab}(X)\gamma^{ab}\psi(x))\\ &\underset{(2.6)}{=} (\omega_{ab}(X)\eta^{bc}\hat{Y}^a(x)u_c) \cdot \psi(x) + \hat{Y}|_x \cdot (\frac{1}{4}\omega_{ab}(X)\gamma^{ab}\psi(x)).\end{aligned}$$

Since this equation is linear in X and in $\hat{Y}$ (separately), it suffices to check it for $X = e_c$ and $Y = e_d$, thus $\omega_{ab}(X) = \omega_{cab}$ and $\hat{Y}^a = \delta^a{}_d$. Inserting this it remains to show that

$$\begin{aligned}\frac{1}{4}\omega_{cab}\gamma^{ab}\gamma_d \cdot \psi(x) &\overset{!}{=} (\omega_{cab}\eta^{bl}\delta^a{}_d u_l) \cdot \psi(x) + u_d \cdot (\frac{1}{4}\omega_{cab}\gamma^{ab}\psi(x))\\ &= \omega_{cdb}\eta^{bl}\gamma_l\psi(x) + \frac{1}{4}\omega_{cab}\gamma_d\gamma^{ab}\psi(x) \qquad (2.9)\\ &= (\omega_{cdb}\gamma^b + \frac{1}{4}\omega_{cab}\gamma_d\gamma^{ab}) \cdot \psi(x).\end{aligned}$$

To verify this we first note that from Definition 1.2.18 it follows that $\{\gamma^a, \gamma_b\} = -2\delta^a{}_b$ and $\{\gamma^a, \gamma^b\} = -2\eta^{ab}$. Using this and the anti-symmetry $\omega_{cba} = -\omega_{cab}$ we have:

$$\begin{aligned}\omega_{cab}(\gamma^{ab}\gamma_d - \gamma_d\gamma^{ab}) &= \frac{1}{2}\omega_{cab}(\gamma^a\gamma^b\gamma_d - \gamma^b\gamma^a\gamma_d - \gamma_d\gamma^a\gamma^b + \gamma_d\gamma^b\gamma^a)\\ &= \omega_{cab}(\gamma^a\gamma^b\gamma_d + \eta^{ab}\gamma_d - \gamma_d\gamma^a\gamma^b - \eta^{ab}\gamma_d) = \omega_{cab}(\gamma^a\gamma^b\gamma_d - \gamma_d\gamma^a\gamma^b)\\ &= \omega_{cab}(-\gamma^a\gamma_d\gamma^b - 2\delta^b{}_d\,\gamma^a + \gamma^a\gamma_d\gamma^b + 2\delta^a{}_d\,\gamma^b) = -2\omega_{cab}\delta^b{}_d\,\gamma^a + 2\omega_{cab}\delta^a{}_d\,\gamma^b\\ &= 2\omega_{cba}\delta^b{}_d\,\gamma^a + 2\omega_{cab}\delta^a{}_d\,\gamma^b = 4\omega_{cab}\delta^a{}_d\,\gamma^b = 4\omega_{cdb}\gamma^b.\end{aligned}$$

Inserting this it follows that (2.9) indeed holds. □

2.3.10. THEOREM. *Let $(\,.\,,\,.\,)_S$ be a Majorana bundle metric and $\langle\,.\,,\,.\,\rangle_S$ be a Dirac bundle metric on the spinor bundle S, defined by Majorana resp. Dirac forms for the spinor representation (cf. Remark 2.2.3 (ii)). Then these metrics are compatible with the spin covariant derivative in the sense that*

$$\begin{aligned}L_X(\Phi, \Psi)_S &= (\nabla_X\Phi, \Psi)_S + (\Phi, \nabla_X\Psi)_S\\ L_X\langle\Phi, \Psi\rangle_S &= \langle\nabla_X\Phi, \Psi\rangle_S + \langle\Phi, \nabla_X\Psi\rangle_S\end{aligned}$$

for all $\Phi, \Psi \in \Gamma(S)$ and all $X \in \mathfrak{X}(M)$.

PROOF. By construction in Remark 2.2.3 (ii), both bundle metrics stem from $\mathrm{Spin}^+(s,t)$-invariant scalar products on the representation space Δ. The claim therefore follows from [**25**, Theorem 3.4.10]. □

2.3.11. DEFINITION. The *Dirac operator* $\not{D} : \Gamma(S) \to \Gamma(S)$ on the spinor bundle S is given by

$$\not{D}\Psi = \eta^{ab}e_a \cdot \nabla_{e_b}\Psi,$$

where ∇ is the spin covariant derivative, and $e = (e_1, \ldots, e_n)$ is a local vielbein, Clifford multiplication is defined as in Proposition 2.2.2, and the summation convention is in force.

As above, let us write $\Psi = [\varepsilon, \psi]$ and note that since $e_a = \delta_a{}^b e_b$ (so that $\delta_a{}^b$ corresponds to the a-th unit vector u_a in $\mathbb{R}^{s,t}$) we have $e_a = [e, u_a] = [\Lambda(\varepsilon), u_a]$. Then

$$\begin{aligned}\not{D}\Psi = \not{D}([\varepsilon, \psi]) = \eta^{ab}e_a \cdot \nabla_{e_b}([\varepsilon, \psi]) &\underset{(2.7)}{=} \eta^{ab}[\Lambda(\varepsilon), u_a] \cdot [\varepsilon, \nabla_{e_b}\psi]\\ &\underset{2.2.2}{=} [\varepsilon, \eta^{ab}\gamma_a\nabla_{e_b}\psi] = [\varepsilon, \gamma^b\nabla_{e_b}\psi]\end{aligned}$$

This means that $\not{D}\Psi = [\varepsilon, \not{D}\psi]$, where

$$
\begin{aligned}
\not{D}\psi = \gamma^a \nabla_{e_a}\psi = i\Gamma^a \nabla_{e_a}\psi \underset{2.3.7}{=} i\Gamma^a(T\psi(e_a) - \frac{1}{4}\omega_{bc}(e_a)\Gamma^{bc}\psi) \\
\underset{(2.4)}{=} i\Gamma^a(T\psi(e_a) - \frac{1}{4}\omega_{abc}\Gamma^{bc}\psi).
\end{aligned}
\tag{2.10}
$$

2.3.12. Remark. To see that $\not{D}$ is well defined, i.e., independent of the chosen local vielbein e it suffices to note that $\not{D}$ can be written as the composition

$$\Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M \otimes S) \xrightarrow{\eta} \Gamma(TM \otimes S) \xrightarrow{\gamma} \Gamma(S),$$

where η denotes the canonical isomomorphism induced by the pseudo-Riemannian metric g and γ is Clifford multiplication: The (musical) isomomorphism between $\mathfrak{X}(M)$ and $\Omega^1(M)$, $X \mapsto X^*$, is uniquely determined by the requirement $X^*(Y) = \langle X, Y\rangle$ for all $Y \in \mathfrak{X}(M)$, with inverse denoted by $\omega \mapsto \omega^\flat$. Then η maps any $\omega \otimes \Phi \in \Gamma(T^*M \otimes S)$ to $\omega^\flat \otimes \Phi$. Any $X \in TM$ can be written as $X = \eta^{ab}e_a^*(X)e_b$, so $\nabla\Psi = \eta^{ab}e_a^* \otimes \nabla_{e_b}\Psi$. Therefore,

$$\gamma(\eta(\nabla\Psi)) = \gamma(\eta^{ab}e_a \otimes \nabla_{e_b}\Psi) = \eta^{ab}e_a \cdot \nabla_{e_b}\Psi,$$

as claimed. Thus the Dirac operator is a well defined first order differential operator on S.

We know from Lemma 2.3.8 that for even n the spin covariant derivative ∇ preserves the splitting $S = S_+ \oplus S_-$, whereas we saw in Proposition 2.2.2 that multiplication by a vector interchanges S_+ and S_-. Therefore we obtain:

2.3.13. Corollary. *If $n = \dim(M)$ is even, then the Dirac operator satisfies $\not{D} : \Gamma(S_\pm) \to \Gamma(S_\mp)$.*

Suppose now that (in arbitrary dimension n), $\langle\, .\,,\, .\,\rangle_S$ is a Dirac bundle metric with $\delta = -1$, i.e.,

$$\langle X \cdot \Phi, \Psi\rangle_S = -\langle \Phi, X \cdot \Psi\rangle_S \qquad \forall X \in TM,\ \forall \Phi, \Psi \in S.$$

Then on the space $\Gamma_0(S)$ of compactly supported sections of S we introduce an L^2-scalar product of spinors:

$$
\begin{aligned}
\langle\, .\,,\, .\,\rangle_{S,L^2} : \Gamma_0(S) \times \Gamma_0(S) \longrightarrow \mathbb{C} \\
\langle \Phi, \Psi\rangle_{S,L^2} := \int_M \langle \Phi, \Psi\rangle_S \, d\mathrm{vol}_g,
\end{aligned}
$$

with vol_g the volume element induced by g.

2.3.14. Theorem. *In the setup described above, the Dirac operator $\not{D} : \Gamma_0(S) \to \Gamma_0(S)$ is formally self-adjoint:*

$$\langle \not{D}\Phi, \Psi\rangle_{S,L^2} = \langle \Phi, \not{D}\Psi\rangle_{S,L^2} \qquad \forall \Phi, \Psi \in \Gamma_0(S).$$

proof. This will follow from the more general Theorem 2.3.21 by applying Remark 2.3.20. □

For applications in the standard model of particle physics it will be important to have available a so-called *twisted* version of spinor bundles. The general setup here is that we are given a principal fiber bundle $P \to M$, a complex representation $\rho : G \to \mathrm{GL}(V)$ and the corresponding associated vector bundle $E = P \times_\rho V$. Also, as above let $S \to M$ be the spinor bundle associated to a given spin structure on M. Using the tensor product of vector bundles (cf. [**25**, Remark 2.4.3 (ii)]), we have:

2.3.15. Definition. The vector bundle $S \otimes E$ is called a *twisted spinor bundle* or *multiplet spinor bundle*.

To obtain a local description, we need an auxiliary construction:

2.3.16. REMARK. (i) Let (P_i, π_{P_i}, M, G_i) $(i = 1, 2)$ be principal fiber bundles over M. The *fiber product* $P_1 \times_M P_2$ of P_1 and P_2 is the principal $G_1 \times G_2$-fiber bundle given as a set by $\{(p_1, p_2) \in P_1 \times P_2 \mid \pi_{P_1}(p_1) = \pi_{P_2}(p_2)\}$, with projection $\pi(p_1, p_2) = \pi_{P_1}(p_1)$. Given bundle charts $\phi_U^{(i)} : P_i|_U \to U \times G_i$, $\phi_U^{(i)}(p_i) = (\pi_{P_i}(p_i), \varphi_U^{(i)}(p_i))$ $(i = 1, 2)$, a bundle chart for $P_1 \times_M P_2$ over U is given by $\phi_U : (p_1, p_2) \mapsto (\pi(p_1, p_2), \varphi_U^{(1)}(p_1) \times \varphi_U^{(2)}(p_2))$, which obviously satisfies the conditions from [**25**, Definition 2.2.1]. An alternative description of the fiber product is to first consider the product bundle $P_1 \times P_2 \to M \times M$ with structure group $G_1 \times G_2$ and denote by $\Delta : M \to M \times M$ the diagonal embedding $x \mapsto (x, x)$. Then $P_1 \times_M P_2$ is the pullback bundle $\Delta^*(P_1 \times P_2)$ in the sense of [**25**, Theorem 2.1.6].

(ii) If $\rho_i : G_i \to \mathrm{GL}(V_i)$ $(i = 1, 2)$ are representations and $E_i = P_i \times_{\rho_i} V_i$ are the corresponding associated vector bundles, then the tensor product $E_1 \otimes E_2$ can be viewed as an associated vector bundle of the principal fiber bundle $P_1 \times_M P_2$ namely as $E_1 \otimes E_2 \cong (P_1 \times_M P_2) \times_\rho (V_1 \otimes V_2)$, where $\rho \equiv \rho_1 \otimes \rho_2$ is the representation

$$\rho : G_1 \times G_2 \longrightarrow \mathrm{GL}(V_1 \otimes V_2)$$
$$\rho(g_1, g_2)(v_1 \otimes v_2) = (\rho_1(g_1)v_1) \otimes (\rho_2(g_2)v_2).$$

Now let $s : U \to P$ be a local gauge (i.e., a local section). Then (cf. [**25**, Theorem 3.4.9]), any local section τ of the associated vector bundle $E = P \times_\rho V$ can be written in the form $\tau = [s, v]$ for some smooth map $v : U \to V$. Identifying V with $\mathbb{C}^r$ by the choice of a basis $(v_1, \dots, v_r)$ we obtain a local frame $\tau_1, \dots, \tau_r$ of E by setting $\tau_i := [s, v_i]$. As in the discussion following Definition 2.3.5, let U be contractible, e a local vielbein, and $\varepsilon : U \to \mathrm{Spin}^+(M)$ a corresponding trivialization (so $e = \Lambda(\varepsilon)$).

Then any section $\Psi \in \Gamma(S \otimes E)$ can locally be written in the form $\Psi = \sum_{i=1}^r \Psi_i \otimes \tau_i$ with $\Psi_i \in \Gamma(U, S)$. Equivalently, $\Psi = [\varepsilon \times s, \psi] \in (\mathrm{Spin}^+(M) \times_M P) \times_{\kappa \otimes \rho} (\Delta \otimes V)$, where ψ is a *multiplet* of the form $\psi = (\psi_1, \dots, \psi_r)^T : U \to \Delta \otimes \mathbb{C}^r$. This decomposition is unique once a choice of ε, s and $(v_1, \dots, v_r)$ has been made.

Next, let A be a connection one-form on P.

2.3.17. DEFINITION. The *twisted spin covariant derivative* ∇^A on the twisted spinor bundle $S \otimes E$ is defined by $\nabla_X^A \Psi := [\varepsilon \times s, \nabla_X^A \hat{\Psi}]$, where $\Psi = [\varepsilon \times s, \hat{\Psi}] \in (\mathrm{Spin}^+(M) \times_M P) \times_{\kappa \otimes \rho} (\Delta \otimes V)$ and

$$\nabla_X^A \hat{\Psi} = T\hat{\Psi}(X) - \frac{1}{4}\omega_{bc}(X)\Gamma^{bc}\hat{\Psi} + (\rho_* A_s(X))\hat{\Psi}.$$

Here, $\hat{\Psi} : U \to \Delta \otimes V$, and the matrices Γ^{bc} act exclusively on the Δ-part of ψ (i.e., separately on each ψ_i), while $\rho_* A_s(X)$ acts exclusively on the V-part of ψ. Concretely, if $\Psi = \sum_{i=1}^r [\varepsilon \times s, \psi_i \otimes w_i]$ with $[\varepsilon, \psi_i] \in \Gamma(U, S)$, and $[s, w_i] \in \Gamma(U, E)$ a local basis for $\Gamma(U, E)$,[2] then we have $\nabla_X^A \Psi = \sum_i [\varepsilon \times s, \nabla_X^A(\psi_i \otimes w_i)]$, with

$$\begin{aligned}
\nabla_X^A(\psi_i \otimes w_i) &= T(\psi_i \otimes w_i)(X) - \Big(\frac{1}{4}\omega_{bc}(X)\Gamma^{bc}\psi_i\Big) \otimes w_i + \psi_i \otimes ((\rho_* A_s(X))w_i) \\
&= T\psi_i(X) \otimes w_i + \psi_i \otimes Tw_i(X) - \dots \\
&= \Big[T\psi_i(X) - \frac{1}{4}\omega_{bc}(X)\Gamma^{bc}\psi_i\Big] \otimes w_i + \psi_i \otimes [Tw_i(X) + (\rho_* A_s(X))w_i] \\
&= (\nabla_X^{\mathrm{Spin}}\psi_i) \otimes w_i + \psi_i \otimes (\nabla_X^A w_i),
\end{aligned} \tag{2.11}$$

where we used Proposition 2.3.7 and [**25**, Theorem 3.4.9] in the last step. Note that this is a slight abuse of notation (albeit one that is analogous to (2.7)): we set $\nabla_X^A w_i := Tw_i(X) + (\rho_* A_s(X))w_i$ to have $\nabla_X^A [s, w_i] = [s, \nabla_X^A w_i]$. From this representation we can read off that $\nabla_X^A \Psi$ is independent of the choice of local sections ε and s (because this is true for $\nabla_X^{\mathrm{Spin}}\psi_i$ and $\nabla_X^A w_i$). Moreover,

[2] As seen above we could even have $w_i = v_i$ constant, but when changing the representative of $[s, v_i]$ using the equivalence relation in E this property is lost.

from the Leibnitz rules for the individual covariant derivatives it follows by a straightforward calculation that the above definition is independent of the choice of local basis $[s, w_i]$.

Clifford multiplication of a twisted spinor $\Psi = \sum_{i=1}^r [\varepsilon \times s, \psi_i \otimes w_i]$ by vectors is defined by natural extension of the construction in Proposition 2.2.2 as

(2.12)
$$(\mathrm{SO}^+(M) \times_{\rho_{\mathrm{SO}}} \mathbb{R}^{s,t}) \oplus ((\mathrm{Spin}^+(M) \times_M P) \times_{\kappa\otimes\rho} (\Delta \otimes V)) \to (\mathrm{Spin}^+(M) \times_M P) \times_{\kappa\otimes\rho} (\Delta \otimes V)$$
$$([\Lambda(\varepsilon), x], \sum_{i=1}^r [\varepsilon \times s, \psi_i \otimes w_i]) \mapsto [\varepsilon \times s, \sum_{i=1}^r (x \cdot \psi_i) \otimes w_i]$$

with $x \cdot \psi_i$ the standard Clifford multiplication $\mathbb{R}^{s,t} \times \Delta \to \Delta$.

We then have the following analogue of Theorem 2.3.9:

2.3.18. Theorem. *The twisted spin covariant derivative is compatible with the Levi-Civita connection in the sense that, for all $X, Y \in \mathfrak{X}(M)$ and all spinors $\Psi \in \Gamma(S \otimes E)$ we have*
$$\nabla_X^A (Y \cdot \Psi) = (\nabla_X Y) \cdot \Psi + Y \cdot \nabla_X^A \Psi.$$

proof. By linearity it suffices to prove this for the case of $\Psi = [\varepsilon \times s, \psi \otimes w]$ with $[\varepsilon, \psi] \in \Gamma(U, S)$, $[s, w] \in \Gamma(U, E)$. Let us write $Y = [e, \hat{Y}]$, $Y \cdot \Psi =: [\varepsilon \times s, \hat{Y} \cdot (\psi \otimes w)]$ for the Clifford multiplication defined above and $\nabla_X Y = [e, \nabla_X \hat{Y}]$, $\nabla_X^A \Psi = [\varepsilon \times s, \nabla_X^A (\psi \otimes w)]$. Then Theorem 2.3.9, applied to $[\varepsilon, \psi]$ says that $\nabla_X^{\mathrm{Spin}} (\hat{Y} \cdot \psi) = \nabla_X \hat{Y} \cdot \psi + \hat{Y} \cdot \nabla_X^{\mathrm{Spin}} \psi$. Therefore,
$$\begin{aligned}\nabla_X^A (\hat{Y} \cdot (\psi \otimes w)) = \nabla_X^A ((\hat{Y} \cdot \psi) \otimes w)) &\underset{(2.11)}{=} (\nabla_X^{\mathrm{Spin}} (\hat{Y} \cdot \psi)) \otimes w + (\hat{Y} \cdot \psi) \otimes (\nabla_X^A w) \\ &= (\nabla_X \hat{Y} \cdot \psi) \otimes w + (\hat{Y} \cdot \nabla_X^{\mathrm{Spin}} \psi) \otimes w + (\hat{Y} \cdot \psi) \otimes (\nabla_X^A w) \\ &= \nabla_X \hat{Y} \cdot (\psi \otimes w) + \hat{Y} \cdot ((\nabla_X^{\mathrm{Spin}} \psi) \otimes w + \psi \otimes (\nabla_X^A w)),\end{aligned}$$
so that by (2.11) we indeed arrive at $\nabla_X^A (Y \cdot \Psi) = \nabla_X Y \cdot \Psi + Y \cdot \nabla_X^A \Psi$. □

Naturally, there is also a generalization of the Dirac operator to twisted spinor bundles:

2.3.19. Definition. The Dirac operator on a twisted spinor bundle $\not{D}_A : \Gamma(S \otimes E) \to \Gamma(S \otimes E)$ is defined by
$$\not{D}_A \Psi := \eta^{ab} e_a \cdot \nabla_{e_b}^A \Psi.$$

For $\Psi = \sum_{i=1}^r [\varepsilon \times s, \psi_i \otimes w_i]$ as above this means that $\not{D}_A \Psi = \sum_{i=1}^r [\varepsilon \times s, \not{D}_A (\psi_i \otimes w_i)]$, where (using again $e_a = [e, u_a]$ as above with u_a the a-th unit vector)

$$\begin{aligned}\not{D}_A (\psi_i \otimes w_i) &= \eta^{ab} u_a \cdot \nabla_{e_b}^A (\psi_i \otimes w_i) \underset{(2.11)}{=} \eta^{ab} u_a \cdot [(\nabla_{e_b}^{\mathrm{Spin}} \psi_i) \otimes w_i + \psi_i \otimes (\nabla_{e_b}^A w_i)] \\ &= (\eta^{ab} \gamma_a \nabla_{e_b}^{\mathrm{Spin}} \psi_i) \otimes w_i + (\eta^{ab} \gamma_a \psi_i) \otimes (\nabla_{e_b}^A w_i) = \not{D} \psi_i \otimes w_i + \eta^{ab} (\gamma_a \cdot \psi_i) \otimes (\nabla_{e_b}^A w_i).\end{aligned} \tag{2.13}$$

Here, $\not{D}$ is the Dirac operator from Definition 2.3.11. The argument from Remark 2.3.12, with $S \otimes E$ instead of S and the Clifford multiplication from above instead of γ shows that $\not{D}_A$ is well defined. Alternatively, we can also check by a direct calculation that Definition 2.3.19 remains unchanged if we switch to a different local frame f_c: Indeed, we can then express e_a as $e_a = \langle e_a, f_c \rangle \eta^{cc} f_c$, so that
$$\eta^{ab} e_a \cdot \nabla_{e_b}^A \Psi = \underbrace{\eta^{ab} \langle e_a, f_c \rangle \langle e_b, f_d \rangle}_{= \langle f_c, f_d \rangle = \eta_{cd}} \eta^{cc} \eta^{dd} f_c \cdot \nabla_{f_d}^A \Psi = \eta^{cd} f_c \cdot \nabla_{f_d}^A \Psi.$$

Using (2.10), (2.13) can also be written locally in terms of the physical gamma matrices as

$$\begin{aligned}\not{D}_A (\psi_j \otimes w_j) &= i \Gamma^a [T \psi_j (e_a) \otimes w_j - \frac{1}{4} \omega_{abc} \Gamma^{bc} \psi_j \otimes w_j + \psi_j \otimes \nabla_{e_a}^A w_j] \\ &= i \Gamma^a \Big(T(\psi_j \otimes w_j)(e_a) - \frac{1}{4} \omega_{abc} \Gamma^{bc} \psi_j \otimes w_j \Big) + i \Gamma^a \psi_j \otimes ((\rho_* A_s(e_a)) w_j)\end{aligned} \tag{2.14}$$

(with the convention agreed upon after (2.11)), or, using Definition 2.3.17,

$$\not{D}_A\Psi = i\Gamma^a\nabla^A_{e_a}\Psi \qquad (\Psi\in\Gamma(S\otimes E)). \tag{2.15}$$

As in the discussion preceding Theorem 2.3.14 let $\langle\,.\,,\,.\,\rangle_S$ be a Dirac bundle metric with $\delta=-1$, and let $\langle\,.\,,\,.\,\rangle_E$ be a Hermitian bundle metric on E. Then there is an induced bundle metric $\langle\,.\,,\,.\,\rangle_{S\otimes E}$ on $S\otimes E$ and we have the following analogue of Theorem 2.3.10:

$$L_X\langle\Phi,\Psi\rangle_{S\otimes E} = \langle\nabla^A_X\Phi,\Psi\rangle_{S\otimes E} + \langle\Phi,\nabla^A_X\Psi\rangle_{S\otimes E} \qquad \forall\Phi,\Psi\in\Gamma(S\otimes E)\ \forall X\in\mathfrak{X}(M). \tag{2.16}$$

To verify this, we may again restrict to the case where both Φ and Ψ are splitting, so suppose that $\Phi=[\varepsilon\times s,\phi\otimes v]$ and $\Psi=[\varepsilon\times s,\psi\otimes w]$. Also, by [**25**, Th. 2.4.10],

$$\langle\Phi,\Psi\rangle_{S\otimes E}(p) = \langle\phi(p)\otimes v(p),\psi(p)\otimes w(p)\rangle_{\Delta\otimes V} = \langle\phi(p),\psi(p)\rangle_\Delta\langle v(p),w(p)\rangle_V = (\langle\phi,\psi\rangle_S\langle v,w\rangle_E)(p).$$

Based on this, Theorem 2.3.10 gives

$$\begin{aligned} L_X\langle\Phi,\Psi\rangle_{S\otimes E} &= L_X(\langle\phi,\psi\rangle_S\langle v,w\rangle_E) = L_X(\langle\phi,\psi\rangle_S)\langle v,w\rangle_E + \langle\phi,\psi\rangle_S L_X\langle v,w\rangle_E \\ &= (\langle\nabla_X\phi,\psi\rangle_S + \langle\phi,\nabla_X\psi\rangle_S)\langle v,w\rangle_E + \langle\phi,\psi\rangle_S(\langle\nabla^A_X v,w\rangle_E + \langle v,\nabla^A_X w\rangle_E) \\ &= \langle\nabla_X\phi\otimes v + \phi\otimes\nabla^A_X v,\psi\otimes w\rangle_{\Delta\otimes V} + \langle\phi\otimes v,\nabla_X\psi\otimes w + \psi\otimes\nabla^A_X w\rangle_{\Delta\otimes V} \\ &\underset{(2.11)}{=} \langle\nabla^A_X(\phi\otimes v),\psi\otimes w\rangle_{\Delta\otimes V} + \langle\phi\otimes v,\nabla^A_X(\psi\otimes w)\rangle_{\Delta\otimes V} \\ &= \langle\nabla^A_X\Phi,\Psi\rangle_{S\otimes E} + \langle\Phi,\nabla^A_X\Psi\rangle_{S\otimes E}. \end{aligned}$$

2.3.20. Remark. In case $E=M\times\mathbb{K}$ is trivial, G and A are arbitrary and $\rho: g\mapsto \mathrm{id}_{\mathbb{K}}$ is the trivial representation (hence $\rho_*=0$), the twisted spinor space $\Gamma(S\otimes E)$ reduces to the spinor space $\Gamma(S)$ (since $S\otimes E\cong S$) and so do all the related notions: Any $\Psi\in\Gamma(S\otimes E)$ can uniquely be written as $\Psi=\psi\otimes 1$ for $\psi\in\Gamma(S)$, $\nabla^A_X\Psi=\nabla^{\mathrm{Spin}}_X\psi$ by (2.11), and $\not{D}_A\Psi=\not{D}\psi$ by (2.13).

Generalizing the pure spinor case, on the space $\Gamma_0(S\otimes E)$ of complex compactly supported sections of $S\otimes E$ we introduce an L^2-scalar product of twisted spinors by

$$\begin{aligned} \langle\,.\,,\,.\,\rangle_{S\otimes E,L^2} &: \Gamma_0(S\otimes E)\times\Gamma_0(S\otimes E)\longrightarrow\mathbb{C} \\ \langle\Phi,\Psi\rangle_{S\otimes E,L^2} &:= \int_M\langle\Phi,\Psi\rangle_{S\otimes E}\,d\mathrm{vol}_g. \end{aligned}$$

Then we have the following generalization of Theorem 2.3.14

2.3.21. Theorem. *The twisted Dirac operator $\not{D}_A:\Gamma_0(S\otimes E)\to\Gamma_0(S\otimes E)$ is formally self-adjoint:*

$$\langle\not{D}_A\Phi,\Psi\rangle_{S\otimes E,L^2} = \langle\Phi,\not{D}_A\Psi\rangle_{S\otimes E,L^2} \qquad \forall\Phi,\Psi\in\Gamma_0(S\otimes E).$$

proof. With the vielbein e as above we denote by $\alpha^j\in\Omega^1(U)$ the dual frame of $(e_1,\dots,e_n)$, i.e., $\alpha^j(e_i)=\delta^j{}_i$. We define $\sigma\in\Omega^1(M)$ by

$$\sigma(X) := \langle X\cdot\Phi,\Psi\rangle_{S\otimes E} \qquad (X\in\mathfrak{X}(M)),$$

and set $\sigma =: \sigma_j\alpha^j$. Then its covariant differential $\nabla\sigma$ is a $(0,2)$-tensor field and its divergence is given by (cf. [**29**, p. 84])

$$\mathrm{div}\sigma = C_{12}(\nabla\sigma) = \sum_m\langle e_m,e_m\rangle\nabla\sigma(e_m,e_m) = \sum_m\langle e_m,e_m\rangle\nabla_{e_m}\sigma(e_m) = \eta^{il}(\nabla_{e_i}\sigma)(e_l). \tag{2.17}$$

By (2.3), (2.4) we have $\nabla_{e_i}e_k=\omega_{ikr}\eta^{rl}e_l$. Also, $0=e_i(\alpha^j(e_k))=(\nabla_{e_i}\alpha^j)(e_k)+\alpha^j(\nabla_{e_i}e_k)$, so

$$\nabla_{e_i}\alpha^j = (\nabla_{e_i}\alpha^j)(e_k)\alpha^k = -\alpha^j(\nabla_{e_i}e_k)\alpha^k = -\omega_{ikr}\eta^{rj}\alpha^k$$

Therefore,

$$\begin{aligned} \nabla_{e_i}\sigma &= \nabla_{e_i}(\sigma_j\alpha^j) = e_i(\sigma_j)\alpha^j + \sigma_j\nabla_{e_i}\alpha^j = e_i(\sigma_j)\alpha^j - \omega_{ikr}\eta^{rj}\sigma_j\alpha^k \\ &= (e_i(\sigma_j) - \omega_{ijr}\eta^{rk}\sigma_k)\alpha^j. \end{aligned} \tag{2.18}$$

Combining (2.17) and (2.18) we obtain:

$$\mathrm{div}\sigma = \eta^{il} \cdot (e_i(\sigma_l) - \omega_{ilr}\eta^{rk}\sigma_k). \tag{2.19}$$

Using this we calculate:

$$\begin{aligned}
\langle \not{D}_A\Phi, \Psi\rangle_{S\otimes E} &= \langle \eta^{ac} e_c \nabla^A_{e_a}\Phi, \Psi\rangle_{S\otimes E} \underset{2.3.18}{=} \langle \eta^{ac}(\nabla^A_{e_a}(e_c\cdot\Phi) - (\nabla_{e_a}e_c)\cdot\Phi), \Psi\rangle_{S\otimes E} \\
&\underset{(2.3)}{=} \langle \eta^{ac}\nabla^A_{e_a}(e_c\cdot\Phi), \Psi\rangle_{S\otimes E} - \langle \eta^{ac}\eta^{bk}\omega_{acb}e_k\cdot\Phi, \Psi\rangle_{S\otimes E} \\
&\underset{(2.16)}{=} \eta^{ac}e_a(\langle e_c\cdot\Phi, \Psi\rangle_{S\otimes E}) - \eta^{ac}\langle e_c\cdot\Phi, \nabla^A_{e_a}\Psi\rangle_{S\otimes E} - \eta^{ac}\eta^{bk}\omega_{acb}\sigma(e_k) \\
&= \eta^{ac}e_a(\sigma_c) - \eta^{ac}\eta^{bk}\omega_{acb}\sigma_k - \eta^{ac}\langle e_c\cdot\Phi, \nabla^A_{e_a}\Psi\rangle_{S\otimes E} \\
&\underset{(2.19)}{=} \mathrm{div}\sigma - \eta^{ac}\langle e_c\cdot\Phi, \nabla^A_{e_a}\Psi\rangle_{S\otimes E} \underset{\delta=-1}{=} \mathrm{div}\sigma + \langle \Phi, \eta^{ac}e_c\cdot\nabla^A_{e_a}\Psi\rangle_{S\otimes E} \\
&= \mathrm{div}\sigma + \langle\Phi, \not{D}_A\Psi\rangle_{S\otimes E}.
\end{aligned}$$

Integrating this equality over M and noting that $\int_M \mathrm{div}\sigma\, d\mathrm{vol}_g = 0$ by Stokes' theorem yields the claim. ☐

To conclude this section we also introduce a notion of chirality into the twisted spinor setting. Let $n = \dim(M)$ be even, so that the spinor bundle splits into Weyl spinor bundles $S = S_+ \oplus S_-$. Further, together with the principal fiber bundle $P \to M$ we consider two representations $\rho_\pm : G \to \mathrm{GL}(V_\pm)$ of its structure group G on complex vector spaces $V_\pm$, and we let $E_\pm := P \times_{\rho_\pm} V_\pm$ be the corresponding associated vector bundles.

2.3.22. Definition. The vector bundle $(S\otimes E)_+ = (S_+\otimes E_+)\oplus(S_-\otimes E_-)$ is called a *twisted chiral spinor bundle*. Also, we set $(S\otimes E)_- = (S_-\otimes E_+)\oplus(S_+\otimes E_-)$.

Sections of $(S\otimes E)_+$ can be written in the form $\psi = \psi_+ + \psi_-$, where $\psi_\pm : M \to S_\pm\otimes E_\pm$.

2.3.23. Definition. Let A be a connection one-form on P. The *twisted chiral spin covariant derivative* on $(S\otimes E)_+$ is defined by $\nabla^A_X\Psi := [\varepsilon\times s, \nabla^A_X\psi]$, where

$$\nabla^A_X\psi := T\psi(X) - \frac{1}{4}\omega_{bc}(X)\Gamma^{bc}\psi + (\rho_{+*}A_s(X))\psi_+ + (\rho_{-*}A_s(X))\psi_-.$$

The discussion following Definition 2.3.17 applies (mutatis mutanis) also in this setup. Further, the corresponding *Dirac operator* $\not{D}_A : \Gamma((S\otimes E)_+) \to \Gamma((S\otimes E)_-)$ is defined locally by

$$\not{D}_A\psi := i\Gamma^a\Big(T\psi(e_a) - \frac{1}{4}\omega_{abc}\Gamma^{bc}\psi + (\rho_{+*}A_{e_a}(X))\psi_+ + (\rho_{-*}A_{e_a}(X))\psi_-\Big),$$

again to be interpreted analogously to (2.14). We can decompose $\not{D}_A$ into

$$\not{D}_{A\pm} : \Gamma(S_\pm\otimes E_\pm) \to \Gamma(S_\mp\otimes E_\pm).$$

A general feature of Dirac operators is that the principal part of their square equals the Laplace-Beltrami operator on (M, g), recall also Remark 1.2.3 for the flat case. This is indeed also true for the operators introduced here, as follows from the *Lichnerowicz-Weitzenböck formula*, cf. [**19**, Section 6.12] for references.

CHAPTER 3

The classical Lagrangians of Gauge Theory

Our modest aim in this chapter is to obtain a formal (i.e., mathematical, as opposed to physical) understanding of each term of the Lagrangian of the standard model of particle physics. In its full glory, this *Yang–Mills-Dirac-Higgs-Yukawa Lagrangian* reads

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_D[\Psi, A] + \mathcal{L}_H[\Phi, A] + \mathcal{L}_Y[\Psi_L, \Phi, \Psi_R] + \mathcal{L}_{YM}[A] \\ &= \operatorname{Re}(\bar{\Psi} \not{D}_A \Psi) + \langle d_A\Phi, d_A\Phi\rangle_E - V(\Phi) - 2g_Y \operatorname{Re}(\bar{\Psi}_L \Phi \Psi_R) - \frac{1}{2}\langle F_M^A, F_M^A\rangle_{\mathrm{Ad}(P)}.\end{aligned} \tag{3.1}$$

We will devote a separate section to each individual term here. Throughout this chapter, (M, g) will denote an n-dimensional oriented pseudo-Riemannian manifold. In physics, the signature of g will typically be Lorentzian (M modelling the spacetime underlying the model), but we will allow arbitrary signature.

Although, as stated above, we cannot provide here a deeper physical understanding of $\mathcal{L}$, we nevertheless will start out with a section devoted to motivating the use of gauge theory and Lagrangians in field theories of physics, following [**4**].

3.1. Gauge theory and variational principles

When describing particles in (semiclassical, or "first-quantized") physics one uses a wave function (or particle field) $\psi : M \to V$, where M is the (Lorentzian) spacetime manifold and V is some (usually complex) vector space. In V one chooses some basis corresponding to certain states of the particle. Moreover, the value of ψ at some spacetime point x implicitly depends on a choice of reference frame at x (e.g., coordinate axes or axes of isospin, etc.). Of course the underlying physics has to be independent of these choices so we need some mathematical machinery to express the transformation behavior of the quantities used in this description. Let P_x denote the set of reference frames at $x \in M$. Then any two elements of P_x will be related by an element of some group G of transformations (e.g., rotations, Lorentz transformations, etc.). For $p \in P_x$ and $g \in G$, let pg denote the transformed frame. Typically, g will also induce a transformation of V, $v \mapsto g \cdot v$, such that, if $\psi(p)$ is the value of ψ relative to $p \in P_x$, then $\psi(pg) = g^{-1}\psi(p)$ is the value relative to pg. From these considerations, it is quite natural to endow the disjoint union P of the P_x with the structure of a principal G-bundle and to observe the automatic occurrence of the associated vector bundle $P \times_G V$. The transformation rule $\psi(pg) = g^{-1}\psi(p)$ shows that $\psi \in C^\infty(P, V)^G$ (cf. [**25**, (2.3.9)]), and by [**25**, Theorem 2.3.4] we have

$$C^\infty(P, V)^G \cong \Gamma(P \times_G V),$$

i.e., ψ can naturally be viewed as a section of the associated vector bundle $P \times_G V$. Examples of such particle fields are the Schrödinger wave function, the Klein-Gordon field, or the Dirac electron field.

A (smooth) choice of reference frame over some subregion U of the spacetime M is a map $s_U : U \to P$ such that $s_U(x) \in P_x$ for each $x \in U$, i.e., $s_U \in \Gamma(U, P)$ is a *section* of P over U, and it is now evident why such a choice is usually termed a *local gauge*. Using s_U we may pull back ψ to $U \subseteq M$, obtaining a local wave function $\psi_U := U \to V$, $\psi_U = \psi \circ s_U$. If $s_W : W \to P$ is another local gauge then there exists a smooth map $g_{UW} : U \cap W \to G$ such that $s_W = s_U \cdot g_{UW}$. Then

$$\psi_W(y) = \psi(s_W(y)) = \psi(s_U(y)g_{UW}(y)) = g_{UW}(y)^{-1} \cdot \psi_U(y) \qquad (y \in U \cap W)$$

gives the transformation behavior of the local wave functions under a change of gauge.

If the principal fiber bundle is endowed with a connection form then this opens the possibility of 'geometrizing' the forces of this potential. Let us motivate this point by looking at the dynamics of point particles in Newtonian physics versus general relativity: The trajectories of two massive particles in three-dimensional space (say, in the vicinity of planet earth) may be tangential without coinciding. For example, if one particle is faster than the other at the point of contact of the curves they describe, it may escape the gravitational pull of the earth, whereas the slower one will fall to the ground. From the Newtonian point of view this behavior is explained by the action of the force of gravity. On the other hand, when considered in the framework of general relativity, i.e., in four-dimensional spacetime, then the worldlines of the particles are *not* tangential (their four-velocities differ). Rather, both of them follow geodesics in a four dimensional Lorentzian manifold, and such geodesics are uniquely determined by initial position and (four-)velocity. In this sense, adding another dimension allows one to 'geometrize' the force of gravity.

Going one step further, assume now that one of the particles is electrically charged while the other is neutral. Then even if their initial four-velocities coincide, their trajectories in spacetime will differ. A geometrization of the electrical force between them can be effected by adding yet another dimension, the "charge dimension". This results in a five-dimensional space (due to Kaluza and Klein) which indeed is a principal fiber bundle with structure group $U(1)$. A suitable connection on this bundle allows one to speak of geodesics in this new space and as in the previous extension of Euclidean space to Lorentzian spacetime it gives an equivalent geometric description of gravity and electromagnetism simultaneously: a unification, but not a grand unification so to say. Indeed there are many more types of charge (isospin, hypercharge, color, weak charge) whose geometrization requires further principal fiber bundles with corresponding (non-abelian) structure groups.

Given a connection A on P and a local section s_U, the pullback of A under s_U is called a *gauge potential*. The pullback F^{s_U} under s_U of the corresponding curvature two-form $F \equiv F^A = D_A A \in \Omega^2(P, \mathfrak{g})$ (cf. [**25**, Definition 3.5.1]) is then interpreted as the field strength of the gauge potential. The fact that F^A always satisfies the Bianchi identity $D_A F^A = 0$ ([**25**, Theorem 3.5.3]) in this context corresponds to the homogeneous field equation for the underlying particle field. If $s_W = s_U \cdot g_{UW}$ is another local gauge then by [**25**, (3.5.3)] we have the transformation rule

$$F^{s_W} = \mathrm{Ad}(g_{UW}^{-1}) \circ F^{s_U}. \tag{3.2}$$

In particular, for G abelian (so that $\mathrm{Ad}(g) = \mathrm{id}_{\mathfrak{g}}$ for all g), the local gauge potentials F^{s_U} patch together to give a well-defined Lie algebra valued two-form on the spacetime M itself.

As in classical mechanics, to describe the dynamics of the particle one relies on a variational principle. Here, a real-valued function (the so-called action density) is assigned to each particle field. The corresponding particle then obeys the Euler-Lagrange equation stemming from the first variation of the action density. In this sense the particle obeys the principle of least action. The action density itself is closely linked to a function on the space of vector valued one-jets on P, the Lagrangian. Let us examine this construction in a bit more detail. The space of one-jets of maps from P to V is defined as

$$J(P, V) := \{(p, v, \theta) \mid p \in P,\ v \in V,\ \theta : T_pP \to V \text{ linear}\}.$$

To endow $J(P, V)$ with a manifold structure, let $\varphi : P \supseteq U \to \mathbb{R}^N$ be a manifold chart of P and let

$$\begin{aligned} \tilde{\varphi} : J(P, V)|_U &\to \varphi(U) \times V \times L(\mathbb{R}^N, V) \\ \tilde{\varphi}(p, v, \theta) &:= (\varphi(p), v, \theta \circ (T_p\varphi)^{-1}). \end{aligned}$$

Then it is easy to check that this induces a smooth atlas for $J(P, V)$.

A *Lagrangian* now is a smooth map $L : J(P,V) \to \mathbb{R}$ that is invariant under the action of the structure group in the sense that for each $(p,v,\theta) \in J(P,V)$ and each $g \in G$ we have

$$L(pg, g^{-1} \cdot v, g^{-1} \cdot \theta \circ TR_{g^{-1}}) = L(p,v,\theta). \tag{3.3}$$

Why precisely this type of invariance? We expect the Lagrangian to induce a well-defined function depending on p, $\psi(p)$ and $T_p\psi$ (cf. the usual Lagrangians of classical physics). Recall from [**25**, Theorem 2.3.4] that an equivalent description of the space of sections of $P \times_G V$, i.e., of the space of particle fields, is

$$\Gamma(P \times_G V) \cong C^\infty(P,V)^G = \{\psi \in C^\infty(P,V) \mid \psi(p \cdot g) = g^{-1} \cdot \psi(p)\}, \tag{3.4}$$

and we shall henceforth use this identification without further notice. The important point to note now is that any Lagrangian $L : J(P,V) \to \mathbb{R}$ induces a well-defined function $\mathcal{L}_0 : C^\infty(P,V)^G \to C^\infty(M)$ given (for $x \in M$, $\psi \in C^\infty(P,V)^G$ and $p \in P_x$) by

$$\mathcal{L}_0(\psi)(x) := L(p, \psi(p), T_p\psi).$$

To see this, note that $\psi \circ R_g = g^{-1} \cdot \psi$ implies $T_{pg}\psi \circ T_pR_g = g^{-1} \cdot T_p\psi$, and so

$$L(pg, \psi(pg), T_{pg}\psi) = L(pg, g^{-1} \cdot \psi(p), g^{-1} \cdot T_p\psi \circ TR_{g^{-1}}) = L(p, \psi(p), T_p\psi).$$

In practical applications it turns out that most Lagrangians are *G-invariant* in the sense that

$$L(p, g \cdot v, g \cdot \theta) = L(p,v,\theta). \tag{3.5}$$

On physical grounds one is interested only in Lagrangians L whose corresponding $\mathcal{L}_0$ is *gauge invariant*. By this we mean that for any element $f \in \mathcal{G}(P)$ of the gauge group of P ([**25**, Definition 3.5.16]) we require that $\mathcal{L}_0(f^*\psi) = \mathcal{L}_0(\psi)$, where $f^*\psi = \psi \circ f$. However, even a G-invariant Lagrangian L in general does not give rise to a gauge invariant $\mathcal{L}_0$: To see this, let $f \in \mathcal{G}(P)$. By [**25**, Lemma 3.5.17], f is of the form $f(p) = p\sigma_f(p)$ for some $\sigma_f \in C^\infty(P,G)^G$. By (3.4) we then have $f^*\psi = \sigma_f^{-1} \cdot \psi$. To check the gauge invariance (or lack thereof) of $\mathcal{L}_0$ we have to calculate $T_p(f^*\psi)$. So let $X \in T_pP$ and let $\gamma : \mathbb{R} \to P$ be a smooth curve with $\gamma'(0) = X$. Then

$$\begin{aligned} T_p(f^*\psi)(X) &= \frac{d}{dt}\Big|_0 \sigma_f(\gamma(t))^{-1} \cdot \psi(\gamma(t)) = \frac{d}{dt}\Big|_0 \sigma_f(p)^{-1} \cdot \psi(\gamma(t)) + \frac{d}{dt}\Big|_0 \sigma_f(\gamma(t))^{-1} \cdot \psi(p) \\ &= \sigma_f(p)^{-1} \cdot T_p\psi(X) + \frac{d}{dt}\Big|_0 \sigma_f(\gamma(t))^{-1} \cdot \sigma_f(p) \cdot \sigma_f(p)^{-1} \cdot \psi(p) \\ &= \sigma_f(p)^{-1} \cdot T_p\psi(X) + TR_{\sigma_f(p)} \circ T_p(\sigma_f^{-1})(X) \cdot \sigma_f(p)^{-1} \cdot \psi(p). \end{aligned}$$

Consequently,

$$\begin{aligned} \mathcal{L}_0(f^*\psi)(x) &= L(p, (f^*\psi)(p), T_p(f^*\psi)_p) \\ &= L(p, \sigma_f(p)^{-1} \cdot \psi(p), \sigma_f(p)^{-1} \cdot T_p\psi + TR_{\sigma_f(p)} \circ T_p(\sigma_f^{-1}) \cdot \sigma_f(p)^{-1} \cdot \psi(p)). \end{aligned}$$

Comparing with (3.5) we see that the obstruction against gauge invariance of $\mathcal{L}_0$ is the term $TR_{\sigma_f(p)} \circ T_p(\sigma_f^{-1}) \cdot \sigma_f(p)^{-1} \cdot \psi(p)$. For the practically minded physicist the task therefore is to find an object which, when incorporated into the definition of $\mathcal{L}_0$ will cancel this term. An elegant solution to this problem is the replacement of the standard derivative $T_p\psi$ in the definition of $\mathcal{L}_0$ by the absolute differential $D_A\psi$ with respect to a connection A (which itself then has to be included as a new variable). In this sense, connections are "forced onto" our hypothetical physicist (a strategy that is related to what is called *minimal coupling* in physics). Indeed, we have:

3.1.1. THEOREM. *Let $L : J(P,V) \to \mathbb{R}$ be a G-invariant Lagrangian and let $\mathcal{C}(P)$ be the space of connections on P. Let*

$$\begin{aligned} \mathcal{L} : C^\infty(P,V)^G \times \mathcal{C}(P) &\to C^\infty(M) \\ \mathcal{L}(\psi, A)(x) &:= L(p, \psi(p), D_A\psi(p)), \end{aligned}$$

where $x \in M$ and $p \in P_x$. Then $\mathcal{L}$ is well-defined and gauge invariant, i.e., $\mathcal{L}(f^*\psi, f^*A) = \mathcal{L}(\psi, A)$ for any $f \in \mathcal{G}(P)$. $\mathcal{L}(\psi, A)$ is called the action density *of the pair (ψ, A).*

PROOF. By [**25**, Theorem 3.4.4], $D_A\psi \in \Omega^1_{\mathrm{hor}}(P,V)^G$, so by [**25**, Definition 3.2.1] we have

$$(D_A\psi)_{pg} \circ TR_g = (R_g^* D_A\psi)(p) = g^{-1} \cdot D_A\psi(p), \tag{3.6}$$

so that

$$L(pg, \psi(pg), (D_A\psi)(pg)) = L(pg, g^{-1}\psi(p), g^{-1} \cdot D_A\psi(p) \circ T_{pg}R_{g^{-1}}) \underset{(3.3)}{=} L(p, \psi(p), D_A\psi(p)),$$

verifying that $\mathcal{L}$ is well-defined. To check gauge invariance we note that due to [**25**, Theorem 3.5.18] we have $D_{f^*A}(f^*\psi) = f^*(D_A\psi)$, so

$$\mathcal{L}(f^*\psi, f^*A)(x) \underset{(3.4)}{=} L(p, \sigma_f(p)^{-1} \cdot \psi(p), f^*(D_A\psi)(p)).$$

For any $X \in T_pP$ we have $f^*(D_A\psi)_p(X) = D_A\psi(T_pf(X))$, and since $f(p) = p \cdot \sigma_f(p)$, the product rule [**25**, Lemma 1.2.3] gives for any smooth curve $\gamma : \mathbb{R} \to P$ with $\gamma'(0) = X$:

$$T_pf(X) = \frac{d}{dt}\Big|_0 \gamma(t) \cdot \sigma_f(\gamma(t)) = TR_{\sigma_f(p)}(X) + (TL_{\sigma_f(p)^{-1}} \circ T\sigma_f(X))^{\sim}(f(p)).$$

The second term here is a fundamental vector field, hence vertical, so it is annihilated by $D_A\psi \in \Omega^1_{\mathrm{hor}}(P,V)^G$, leading to

$$f^*(D_A\psi)_p(X) = D_A\psi(T_pf(X)) = D_A\psi(TR_{\sigma_f(p)}(X)) \underset{(3.6)}{=} \sigma_f(p)^{-1} \cdot D_A\psi(p)(X).$$

Altogether, we arrive at

$$\mathcal{L}(f^*\psi, f^*A)(x) = L(p, \sigma_f(p)^{-1} \cdot \psi(p), \sigma_f(p)^{-1} \cdot D_A\psi(p)) \underset{(3.5)}{=} L(p, \psi(p), D_A\psi(p)) = \mathcal{L}(\psi, A).$$

□

Often what we (following [**4**]) have called action density is simply referred to as Lagrangian, and we will also no longer insist on the distinction we have made so far.

The map $\mathcal{L}$ is the fundamental object of interest for modelling the dynamics of a physical field theory. More precisely, the dynamics of the theory is encoded in the following *principle of least action*: For any $U \Subset M$ (i.e., U open and relatively compact), let $\bar{\mathcal{L}}^A_U(\psi) := \int_U \mathcal{L}(\psi, A)\, d\mathrm{vol}_g$. Then a particle field $\psi \in C^\infty(P,V)^G$ is called *stationary* relative to $\mathcal{L}^A$ if for each $U \Subset M$ and each $\varphi \in C^\infty(P,V)^G$ whose support projects to a subset of U we have

$$\frac{d}{dt}\Big|_0 \bar{\mathcal{L}}^A_U(\psi + t\varphi) = 0.$$

It can be shown (cf. [**4**]) that this condition is equivalent to the corresponding Euler-Lagrange equation, which for the particle field gives the equation of motion (e.g., the Dirac equation for ψ a free electron field, etc.). By defining a suitable notion of current one can also derive inhomogeneous field equations. The actual derivation and analysis of these field equations is a separate topic that mostly[1] goes beyond the aims of this lecture course. Instead, we now start our discussion of the individual terms of the Lagrangian (3.1) of the standard model.

3.2. The Yang–Mills Lagrangian

To begin our tour through the Lagrangian of the standard model we look at *pure Yang–Mills theories* (sometimes called *gluodynamics*). Physically, these correspond to "pure" field theories without additional matter fields. The basic setup is as follows (cf. also [**34**, Section 6.1]):

- The theory is implemented on a principal fiber bundle $P \to M$ with M an n-dimensional oriented pseudo-Riemannian manifold and compact structure group G.
- There is a gauge potential (i.e., connection one-form) A on P which mediates the fundamental interaction. Its field strength is then given by (cf. Section 3.1) the corresponding curvature two-form F^A.

[1] See, however, the derivation of the Maxwell equations on general pseudo-Riemannian manifolds in Section 3.2 below

- The Lie algebra $\mathfrak{g}$ of G is endowed with an Ad-invariant scalar product $\langle\,.\,,\,.\,\rangle_{\mathfrak{g}}$.
- A $\langle\,.\,,\,.\,\rangle_{\mathfrak{g}}$-orthonormal basis $T_1, \ldots, T_r$ for $\mathfrak{g}$ has been chosen.

Recall from [**25**, Theorem 3.2.3] the isomorphism

$$\Omega^2_{\mathrm{hor}}(P, \mathfrak{g})^{(G,\mathrm{Ad})} \cong \Omega^2(M, \mathrm{Ad}(P)),$$

where $\mathrm{Ad}(P) = P \times_G \mathfrak{g}$, according to which we may identify the curvature form F^A with an element F^A_M of $\Omega^2(M, \mathrm{Ad}(P))$ via the relation

$$F^A_M(x)(t_1, t_2) = [p, F^A(p)(X_1, X_2)] \tag{3.7}$$

whenever $t_i \in T_xM$, $p \in P_x$ and $X_i \in T_pP$ with $T\pi(X_i) = t_i$ $(i = 1, 2)$.

By [**25**, Theorem 2.4.10], $\langle\,.\,,\,.\,\rangle_{\mathfrak{g}}$ induces a bundle metric $\langle\,.\,,\,.\,\rangle_{\mathrm{Ad}(P)}$ on the associated vector bundle $\mathrm{Ad}(P)$ by setting

$$\langle [p, v], [p, w] \rangle_{\mathrm{Ad}(P)} := \langle v, w \rangle_{\mathfrak{g}}.$$

3.2.1. Remark. Recall that g induces a scalar product on k-forms, $\langle\,.\,,\,.\,\rangle : \Omega^k(M) \times \Omega^k(M) \to C^\infty(M)$ as follows: if x^μ are local coordinates and $\omega, \tau \in \Omega^k(M)$, with $\omega_{\mu_1,\ldots,\mu_k} := \omega(\partial_{\mu_1}, \ldots, \partial_{\mu_k})$, and raising indices via the metric g (so that $\omega^{\mu_1,\ldots,\mu_k} = \omega_{\nu_1,\ldots,\nu_k} g^{\nu_1\mu_1} \cdots g^{\nu_k\mu_k}$) we have

$$\langle \omega, \tau \rangle_{\Omega^k} = \sum_{\mu_1 < \cdots < \mu_k} \omega_{\mu_1,\ldots,\mu_k} \tau^{\mu_1,\ldots,\mu_k} = \frac{1}{k!} \sum_{\mu_1,\ldots,\mu_k} \omega_{\mu_1,\ldots,\mu_k} \tau^{\mu_1,\ldots,\mu_k} \equiv \frac{1}{k!} \omega_{\mu_1,\ldots,\mu_k} \tau^{\mu_1,\ldots,\mu_k},$$

which is easily seen to be independent of the chosen chart. For vector-valued k-forms $F, G \in \Omega^k(M, E)$, with E a vector bundle equipped with a bundle metric $\langle\,.\,,\,.\,\rangle_E$ (in our case $E = \mathrm{Ad}(P)$ with $\langle\,.\,,\,.\,\rangle_{\mathrm{Ad}}$), one defines a corresponding scalar product by choosing any local basis $e_1, \ldots, e_r$ for $\Gamma(E)$, expanding $F = \sum_i F_i \otimes e_i$ and $G = \sum_j G_j \otimes e_j$ with $F_i, G_j \in \Omega^k(M)$ and then setting

$$\langle F, G \rangle_E := \sum_{i,j=1}^r \langle F_i, G_j \rangle_{\Omega^k} \langle e_i, e_j \rangle_E$$

(cf. [**25**, Section 5.2]).

3.2.2. Definition. The *Yang–Mills Lagrangian* is defined by

$$\mathcal{L}_{YM}[A] = -\frac{1}{2} \langle F^A_M, F^A_M \rangle_{\mathrm{Ad}(P)}.$$

Clearly, for fixed $A \in \mathcal{C}(P)$, $\mathcal{L}_{YM}[A] \in C^\infty(M, \mathbb{R})$. Furthermore, the essential criterion of gauge invariance,

$$\mathcal{L}_{YM}[f^*A] = \mathcal{L}_{YM}[A] \qquad \forall A \in \mathcal{C}(P)\ \forall f \in \mathcal{G}(P) \tag{3.8}$$

is satisfied by [**25**, Theorem 5.2.7]. Let us next derive local coordinate expressions for the Yang–Mills Lagrangian. Given a local gauge $s \in \Gamma(U, P)$, the local field strength is given by[2]

$$F^A_s = s^* F^A \in \Omega^2(U, \mathfrak{g}).$$

Let x^μ be local coordinates on U, then due to (3.7) we have

$$F^A_{M\mu\nu} = F^A_M(\partial_\mu, \partial_\nu) = F^A(Ts(\partial_\mu), Ts(\partial_\nu)) = F^A_s(\partial_\mu, \partial_\nu) =: F^A_{\mu\nu}.$$

Using the orthonormal basis T_a of $\mathfrak{g}$ we obtain local frames $\tilde{T}_a(x) := [s(x), T_a]$ for $\mathrm{Ad}(P)$ and a corresponding expansion $F^A_s = F^{Aa}_s \otimes \tilde{T}_a$, so that

$$F^A_{\mu\nu} = F^{Aa}_{\mu\nu} \tilde{T}_a, \tag{3.9}$$

[2] In [**25**] the notation is F^s, A^s, etc., but since we already have the superscript A we will synonymously attach the s as lower index whenever convenient.

with $F_s^{Aa} \in \Omega^2(U)$ and $F_{\mu\nu}^{Aa} = F_s^{Aa}(\partial_\mu, \partial_\nu) \in C^\infty(U)$. Here and below we always use the summation convention. For the Lagrangian we get using Remark 3.2.1:

$$\begin{aligned}\mathcal{L}_{YM}[A] &= -\frac{1}{2}\langle F_M^A, F_M^A\rangle_{\mathrm{Ad}(P)} = -\frac{1}{2}\langle F_s^{Aa}, F_s^{Ab}\rangle_{\Omega^2(U)}\langle \tilde{T}_a, \tilde{T}_b\rangle_{\mathrm{Ad}(P)} \\ &= -\frac{1}{4}F_{\mu\nu}^{Aa}F^{Ab\mu\nu}\underbrace{\langle T_a, T_b\rangle_{\mathfrak{g}}}_{=\delta_{ab}} = -\frac{1}{4}F_{\mu\nu}^{Aa}F_a^{A\mu\nu}.\end{aligned} \tag{3.10}$$

For the local field strength F_s we have by the structure equation ([**25**, Theorem 3.5.3]):

$$F_s^A = s^*F^A = s^*dA + \frac{1}{2}s^*[A, A] = ds^*A + \frac{1}{2}[s^*A, s^*A] = dA_s + \frac{1}{2}[A_s, A_s],$$

and so

$$F_{\mu\nu}^A = dA_s(\partial_\mu, \partial_\nu) + \frac{1}{2}[A_s, A_s](\partial_\mu, \partial_\nu).$$

Here, $\frac{1}{2}[A_s, A_s](\partial_\mu, \partial_\nu) = [A_s(\partial_\mu), A_s(\partial_\nu)]$ by [**25**, Lemma 3.5.2], so that (using [**25**, (3.4.1)])

$$\begin{aligned}F_{\mu\nu}^A &= \partial_\mu(A_s(\partial_\nu)) - \partial_\nu(A_s(\partial_\mu)) - A_s(\underbrace{[\partial_\mu, \partial_\nu]}_{=0}) + [A_s(\partial_\mu), A_s(\partial_\nu)] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].\end{aligned} \tag{3.11}$$

Denote by f_{abc} the *structure constants* determined by $[T_a, T_b] = \sum_{c=1}^r f_{abc}T_c$. Then we have an expansion $A_\mu = A_\mu^a T_a$. Also,

$$[s(x), F^A(Ts(\partial_\mu), Ts(\partial_\nu))] \underset{(3.7)}{=} F_M^A(x)(\partial_\mu, \partial_\nu) \underset{(3.9)}{=} F_{\mu\nu}^A = F_{\mu\nu}^{Aa}\tilde{T}_a = [s(x), F_{\mu\nu}^{Aa}T_a],$$

so (3.9) becomes $F_{\mu\nu}^A = F_{\mu\nu}^{Aa}T_a$ when F^A is viewed as a $\mathfrak{g}$-valued form on P via (3.7). Consequently,

$$F_{\mu\nu}^{Aa} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{bca}A_\mu^b A_\nu^c. \tag{3.12}$$

3.2.3. Lemma. *The structure constants of $\mathfrak{g}$ with respect to the orthonormal basis $\{T_a\}$ satisfy $f_{abc} + f_{bac} = 0$ and $f_{bca} + f_{bac} = 0$, and thereby also $f_{bca} = f_{abc}$.*

proof. The first equation follows from the antisymmetry of the Lie bracket. For the second, we use the Ad-invariance of $\langle\,.\,,\,.\,\rangle_{\mathfrak{g}}$ to calculate:

$$0 = \frac{d}{dt}\Big|_0 \langle \mathrm{Ad}_{\exp(tT_b)}(T_c), \mathrm{Ad}_{\exp(tT_b)}(T_a)\rangle_{\mathfrak{g}} = \langle [T_b, T_c], T_a\rangle_{\mathfrak{g}} + \langle T_c, [T_b, T_a]\rangle_{\mathfrak{g}}.$$

□

Thus (3.12) is equivalent to

$$F_{\mu\nu}^{Aa} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{abc}A_\mu^b A_\nu^c,$$

so (3.10) becomes

$$\begin{aligned}\mathcal{L}_{YM}[A] = &-\frac{1}{4}F_{\mu\nu}^{Aa}F_a^{A\mu\nu} = \\ &-\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu) \\ &-\frac{1}{2}f_{abc}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)A^{b\mu}A^{c\nu} - \frac{1}{4}f_{abc}f_{ade}A_\mu^b A_\nu^c A^{d\mu}A^{e\nu}.\end{aligned} \tag{3.13}$$

Here, the term in the second line is quadratic in the gauge field and describes free (non-interacting) gauge bosons. If the structure group is abelian, it is the only remaining term. The terms in the third line are cubic and quartic in the gauge field. They describe the interaction between gauge bosons in non-abelian gauge theories. In QCD (quantum chromodynamics) they are called 3-gluon vertex and 4-gluon vertex, respectively.

Next we want to look at the principle of least action in the current setup. For convenience, let us assume that M is closed, i.e., compact without boundary. Then the following definition makes sense:

3.2.4. DEFINITION. The *Yang–Mills action* for the principal G-bundle $P \to M$ is the map

$$S_{YM} : \mathcal{C}(P) \to \mathbb{R}$$
$$S_{YM}[A] = -\frac{1}{2}\langle F_M^A, F_M^A\rangle_{\mathrm{Ad}(P),L^2} = -\frac{1}{2}\int_M \langle F_M^A, F_M^A\rangle_{\mathrm{Ad}(P)}\, d\mathrm{vol}_g.$$

A connection A on P is a *critical point* of the Yang–Mills action if

$$\frac{d}{dt}\Big|_0 S_{YM}[A+t\alpha] = 0 \qquad \forall \alpha \in \Omega^1_{\mathrm{hor}}(P,\mathfrak{g})^{\mathrm{Ad}} \cong \Omega^1(M,\mathrm{Ad}(P)).$$

By [**25**, Theorem 5.2.10] we have:

3.2.5. THEOREM. *A connection form A is a critical point of the Yang–Mills action if and only if it satisfies the* Yang–Mills equation $\delta_A F_M^A = 0$.

Here, the *codifferential* induced by A is the map $\delta_A : \Omega^{k+1}(M,\mathrm{Ad}(P)) \to \Omega^k(M,\mathrm{Ad}(P))$ defined by (see [**25**, Definition 5.2.3])

$$\delta_A := (-1)^{nk+p+1} * d_A * .$$

The Yang–Mills equation is a second order partial differential equation for the connection A.

Let us also derive a local expression for the Yang–Mills equation. Since the Hodge operator is an isomorphism, we need a local representation of the equation $d_A * F^A = 0$. Here, the representation of the structure group is $\rho = \mathrm{Ad}$, so $\rho_* = \mathrm{ad}$. Moreover, [**25**, (3.4.3)] shows that $d_A\omega = d\omega + \mathrm{ad}(A)\wedge\omega$, and since $\mathrm{ad}(X)(Y) = [X,Y]$, [**25**, (3.4.5),(3.5.5)] give

$$d_A(*F^A) = d(*F^A) + [A, *F^A],$$

Finally, applying s^* for a local gauge s gives the local form of the Yang–Mills equation as

$$d(*F_s^A) + [A_s, *F_s^A] = 0.$$

By [**25**, Theorem 3.5.5], the curvature form automatically satisfies the Bianchi identity $d_A F_M^A = 0$. It therefore also solves the equation $(d_A + \delta_A)^2 F^A = 0$, which is a direct generalization of the Laplace–Beltrami equation ($\Box\omega = (d+\delta)^2\omega = 0$). In this sense, curvature forms that are solutions of the Yang–Mills equation can be viewed as *harmonic forms*.

3.2.6. EXAMPLE. (The Maxwell equations) Let $G = U(1)$, which is abelian. By what was said after (3.2), the local curvature forms F_s therefore are independent of the choice of local gauge s and define a global 2-form $F_M \in \Omega^2(M,\mathfrak{u}(1))$ on the base manifold M. In this case, $\rho = \mathrm{Ad}^{U(1)}$, so $\rho_* = \mathrm{ad}^{\mathfrak{u}(1)} \equiv 0$ since the Lie bracket on $\mathfrak{u}(1)$ vanishes. Therefore, $D_A\omega = d\omega + \mathrm{ad}(A)\wedge\omega = d\omega + [A,\omega] = d\omega$ for any connection A and any ω (cf. [**25**, (3.4.3)]), so also $d_A = d$, and the system consisting of the Bianchi identity and the Yang–Mills equation,

$$dF_M = 0$$
$$d * F_M = 0$$

is called the *Maxwell equations for* a source-free electromagnetic field. In particular, for $M = \mathbb{R}^{1,3}$ the Minkowski space we obtain precisely the usual Maxwell equations, cf. [**25**, Section 5.1]. On the other hand, one may also generalize the Maxwell equations to any abelian structure group G, where they are still linear, or even to non-abelian G, where they become nonlinear PDEs for A.

3.2.7. DEFINITION. A connection that satisfies the Yang–Mills equation is called a *Yang–Mills connection*.

It follows from (3.8), combined with Theorem 3.2.5 that if A is a Yang–Mills connection and $f \in \mathcal{G}(P)$ is an element of the gauge group, then also f^*A is a Yang–Mills connection. The quotient of the space of Yang–Mills connections by the gauge group $\mathcal{G}(P)$ is called the *Yang–Mills moduli space*.

3.2.8. Example. (Instantons) Let $P \to M$ be a principal fiber bundle with structure group G and with M an oriented Riemannian 4-manifold. In this case, for the Hodge star operator we have $** = \mathrm{id}$ on $\Omega^2(M)$. Connections A whose curvature form $F^A_M \in \Omega^2(M, \mathrm{Ad}(P))$ satisfy $*F^A_M = F^A_M$ or $*F^A_M = -F^A_M$ are called *self-dual* or *anti-self-dual instantons*, respectively. Since any connection satisfies the Bianchi-identity, it follows that both types of instantons automatically are Yang–Mills connections. The defining equations for instantons are examples of so-called *BPS equations* in the sense that they are first order equations whose solutions necessarily are also solutions of the field equations. Since $F^{f^*A} = f^*F^A = \mathrm{Ad}(\sigma_f^{-1}) \circ F^A$ for any $f \in \mathcal{G}(P)$ (see [**25**, Theorem 3.5.18]), the gauge group maps the instanton spaces into themselves, and so one can define corresponding *instanton moduli spaces*, which play a central role in Donaldson theory.

3.2.9. Remark. In quantum field theory it turns out that the Yang–Mills Lagrangian

$$\mathcal{L}_{YM}[A] = -\frac{1}{2}\langle F^A_M, F^A_M\rangle_{\mathrm{Ad}(P)} = -\frac{1}{4}F^{Aa}_{\mu\nu}F^{A\mu\nu}_a$$

describes *massless* gauge bosons. One then can add (in a local gauge) a mass term

$$\frac{1}{2}m^2 A^\nu_a A^a_\nu \tag{3.14}$$

to $\mathcal{L}_{YM}[A]$ to describe gauge bosons of mass m. This would suggest to add to $\mathcal{L}_{YM}[A]$ an invariant term $\frac{1}{2}m^2\langle A_M, A_M\rangle_{\mathrm{Ad}(P)}$. However, A_M is not an element of $\Omega^1(M, \mathrm{Ad}(P))$ (because A is not horizontal by the very definition of connection forms, only the difference of two connection forms is of this type, cf. [**25**, Remark 3.2.2]). Thus adding such a term does not provide a gauge-invariant Lagrangian. To actually derive a gauge invariant Lagrangian that describes non-zero mass gauge bosons will require the *Higgs mechanism*, see Chapter 4.

To conclude this section, we are going to collect some fundamental relations from gauge theory and compare the formalisms most common in mathematics and physics. As before we assume that $\mathfrak{g}$ is equipped with an Ad-invariant scalar product $\langle\,.\,,\,.\,\rangle_{\mathfrak{g}}$, but now denote the orthonormal basis of $\mathfrak{g}$ by $S_1, \dots, S_r$. We also fix a so-called *coupling constant* $g > 0$.[3]

Beginning with the conventions mainly used in mathematics, define a new scalar product

$$\langle\,.\,,\,.\,\rangle'_{\mathfrak{g}} := \frac{1}{g^2}\langle\,.\,,\,.\,\rangle_{\mathfrak{g}},$$

which then has the orthonormal basis $T_a = gS_a$ $(a = 1, \dots, r)$. We can then expand the connection $A \in \Omega^1(P, \mathfrak{g})$ and its curvature $F^A \in \Omega^2(P, \mathfrak{g})$ as

$$A = \sum_{a=1}^r A^a \otimes T_a \qquad F^A = \sum_{a=1}^r F^a \otimes T_a,$$

with $A^a \in \Omega^1(P)$ and $F^a \in \Omega^2(P)$. For s a local gauge and $\Phi \in \Gamma(\mathrm{Ad}(P))$ we can write Φ over U as $\Phi = [s, \phi]$ with $\phi \in C^\infty(U, \mathfrak{g})$, and by [**25**, Theorem 3.4.9] then $\nabla^A_X\Phi = [s, \nabla^A_X\phi]$, where $\nabla^A_X\phi(x) = T\phi(X_x) + \mathrm{ad}(A_s(X_x))\phi(x)$. Then given local coordinates x^μ on U, $\nabla^A_\mu\phi \equiv \nabla^A_{\partial_\mu}\phi = \partial_\mu\phi + A_\mu\phi$, with $A_\mu := A_s(\partial_\mu)$ and $A_\mu\phi = \mathrm{ad}(A_s(\partial_\mu))\phi \equiv [A_\mu, \phi]$. In this sense,

$$\nabla^A_\mu = \partial_\mu + A_\mu. \tag{3.15}$$

Furthermore, we know from (3.11) that the local curvature is given by

$$F^A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

[3]This is slightly at odds with our notation g for the metric on M, but we follow Hamilton's conventions and believe that no confusion will ensue.

Finally (cf. (3.10)), the Yang–Mills Lagrangian is then defined to be

$$\mathcal{L}_{YM}[A] = -\frac{1}{4}\langle F^{\mu\nu}, F_{\mu\nu}\rangle'_{\mathfrak{g}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a.$$

In physics, on the other hand, it is more common to pick a Hermitian scalar product $\langle\, .\, ,\, .\, \rangle_{i\mathfrak{g}}$ on $i\mathfrak{g}$ associated to $\langle\, .\, ,\, .\, \rangle_{\mathfrak{g}}$ via the orthonormal basis $\frac{1}{i}S_a$ $(a = 1, \dots, r)$ on $i\mathfrak{g}$. Then we expand the gauge field $B \in \Omega^1(P, i\mathfrak{g})$ and its curvature $H \in \Omega^2(P, i\mathfrak{g})$ as

$$B = \frac{1}{i}\sum_{a=1}^{r} B^a \otimes S_a \qquad H = \frac{1}{i}\sum_{a=1}^{r} H^a \otimes S_a.$$

There are two different sign conventions for the covariant derivative, namely

$$\nabla^A_\mu = \partial_\mu \pm igB_\mu,$$

with corresponding curvatures

$$H_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \pm ig[B_\mu, B_\nu]$$

and Lagrangian

$$\mathcal{L}_{YM}[B] = -\frac{1}{4}\langle H^{\mu\nu}, H_{\mu\nu}\rangle_{i\mathfrak{g}} = -\frac{1}{4}H^a_{\mu\nu}H^{\mu\nu}_a.$$

An advantage of the physical convention is that the coupling constant is appearing explicitly in the covariant derivative and the curvature. In the first case, it describes the coupling of the gauge field to other fields, and in the second case the coupling between gauge bosons in non-abelian gauge theories.

The relation between these conventions is given by $A = \pm igB$ and $F = \pm igH$. If the representation of G on a vector space V is unitary, then A_μ acts as a skew-Hermitian operator and B as a Hermitian operator. Moreover, $\nabla^A_\mu = \nabla^B_\mu$, $A^a = \pm B^a$, $F^a = \pm H^a = \mp H_a = F_a$. We will mostly follow the mathematical conventions below.

3.3. Modelling matter fields

Having discussed pure gauge theories in the previous section, let us very briefly consider the general setup for dealing with matter fields in the framework of gauge theory (cf. [**34**, Section 7.1]).

One usually considers a spacetime manifold M that is endowed with certain additional structures, encoded in either the frame bundle $\mathrm{GL}(M)$ (cf. [**25**, Example 2.2.12]) or the spin structure $S(M)$ (cf. Definition 2.1.2). In what follows (in this section) let Q stand for either $\mathrm{GL}(M)$ or $S(M)$, calling Q the spacetime principal bundle and call its structure group S. If we are given in addition another, so-called *gauge principal bundle* P, we form the fiber product $Q \times_M P$ (see Remark 2.3.16), which is a principal $S \times G$ bundle over M. The bundle projections will be denoted by π_Q and π_P, and the right actions by R^Q and R^P, respectively. For the induced right action of $S \times G$ on $Q \times_M P$ we will simply write R, and π for the projection.

Then a classical matter field model consists of the following building blocks:

(A) The primary underlying space is a tensor product $E = E_s \otimes E_i$, where E_s is the bundle of *spacetime degrees of freedom* and E_i is the bundle of *internal degrees of freedom*. For E_s there are two versions:

- Either E_s is a tensor bundle over M that is associated to the frame bundle $\mathrm{GL}(M)$ (*bosonic matter*), or
- E_s is a spinor bundle associated to the spin structure $S(M)$ (*fermionic matter*).

In the unified notation introduced above we therefore have $E_s = Q \times_\mu F_s$, where F_s is a finite-dimensional vector space carrying a representation μ of S.

The bundle E_i is associated to the gauge principal bundle P: $E_i = P \times_\sigma F_i$, where F_i also is a finite-dimensional vector space carrying a representation σ of the gauge group G. In addition to σ, F_i may be endowed with a further Lie group representation implementing further degrees of freedom called flavor.

By Remark 2.3.16, E is associated to $Q \times_M P$, with typical fiber $F = F_s \otimes F_i$ carrying the tensor product representation $\mu \otimes \sigma$ of $S \times G$.

(B) A *matter field* of spacetime type μ and of gauge type σ (of type (μ, σ) for short) is a section $\Psi \in \Gamma(E)$. By [**25**, Theorem 2.3.4], Ψ can equivalenty be viewed as an element $\bar{\Psi} \in C^\infty(Q \times_M P, F)^{S\times G}$, where

$$\Psi(x) = [p, \bar{\Psi}(p)] \qquad (p \in \pi^{-1}(x)).$$

By the principle of minimal coupling (cf. Section 3.1), the coupling of the matter field with the gauge potential is encoded in the corresponding covariant derivative. Exactly as in Definition 2.3.17, connections on Q and P define covariant derivatives on the associated bundles E_s and E_i, which in turn induce a covariant derivative on $E = E_s \otimes E_i$ that realizes the minimal coupling. This covariant derivative distributes over the tensor product representation, see (2.11).

(C) Let s_P and s_Q be local gauges of P and Q, respectively. Then $s := s_P \times s_Q$ is a local gauge for $Q \times P$ which induces a local gauge for $P \times_M Q$ (also denoted s) via

$$\begin{aligned} s &: U \to P \times_M Q \\ s(x) &:= (s_P(x), s_Q(x)). \end{aligned}$$

In this local gauge, the matter field $\bar{\Psi}$ is represented by $s^*\bar{\Psi}$, and this representation is compatible with the tensor product structure of F.

(D) The (infinite dimensional) vector space $\Gamma(E)$ is called the *matter configuration space*. Gauge transformations of $P \times_M Q$ act on $\Gamma(E)$ on the left, see (3.17) below.

In the remaining sections of this chapter we will implement this general program for the matter fields of the standard model.

3.4. The Klein–Gordon and Higgs Lagrangians

We now turn to the case of matter fields that couple to the gauge field, starting out in the present section with scalar fields. As before, (M, g) is an oriented pseudo-Riemannian manifold.

3.4.1. Definition. A *complex scalar field* is a smooth map $\phi : M \to \mathbb{C}$. A *multiplet of complex scalar fields* is a smooth map $\phi : M \to \mathbb{C}^r$ for some $r > 1$.

On $\mathbb{C}^r$ we consider the standard Hermitian scalar product $\langle v, w\rangle = v^\dagger \cdot w$. Given a multiplet of scalar fields ϕ, its differential $T\phi \equiv d\phi$ is an element of $\Omega^1(M, \mathbb{C}^r)$, and as a simple special case of Remark 3.2.1 we obtain an induced scalar product on this space of vector valued one-forms.

3.4.2. Definition. The *free Klein–Gordon Lagrangian* for a multiplet of complex scalar fields $\phi : M \to \mathbb{C}^r$ of mass m is given by

$$\mathcal{L}_{KG}[\phi] = \langle d\phi, d\phi\rangle - m^2\langle \phi, \phi\rangle.$$

Here, $\langle d\phi, d\phi\rangle$ is called the *kinetic term* and $-m^2\langle\phi,\phi\rangle$ is called the *Klein–Gordon term*.

For fixed ϕ, $\mathcal{L}_{KG}[\phi] : M \to \mathbb{R}$ is smooth. Writing $\phi = (\phi^1, \dots, \phi^r)$ we have $d\phi = (d\phi^1, \dots, d\phi^r) = \sum_{i=1}^r d\phi^i u_i$ (u_i the i-th unit vector in $\mathbb{C}^r$). In the notation of Remark 3.2.1 we then obtain $\langle u_i, u_j\rangle_{\mathbb{C}^r} = \delta_{ij}$, and so

$$\langle d\phi, d\phi\rangle = \sum_{i=1}^r \langle d\phi^i, d\phi^i\rangle_{\Omega^1} = \sum_{i=1}^r (d\phi^i)^\mu (d\phi^i)_\mu = \langle \partial^\mu\phi, \partial_\mu\phi\rangle_{\mathbb{C}^r} \equiv \langle \partial^\mu\phi, \partial_\mu\phi\rangle.$$

More generally, we define:

3.4.3. DEFINITION. A *potential* is a smooth function $V : \mathbb{R} \to \mathbb{R}$. Then setting $V(\phi) := V(\langle \phi, \phi \rangle)$, the *Higgs Lagrangian* for a multiplet of complex scalar fields ϕ with potential V is

$$\mathcal{L}_H[\phi] = \langle d\phi, d\phi \rangle - V(\phi).$$

If V contains terms of order higher than two in the field ϕ then it describes a *direct interaction* between particles of the field ϕ. As we shall see later, the potential of the Higgs field is a quadratic polynomial in $\phi^\dagger \phi$, hence is of fourth order in ϕ.

Turning now to a scalar field coupled to a gauge field, we make the following general assumptions:

- (M, g) is an oriented pseudo-Riemannian manifold.
- $P \to M$ is a principal fiber bundle with compact structure group G of dimension r.
- $\rho : G \to \mathrm{GL}(W)$ is a complex representation, and we call $E = P \times_\rho W \to M$ the associated vector bundle.
- W is equipped with a ρ-invariant scalar product $\langle\, .\, , .\, \rangle_W$, inducing (via [**25**, Theorem 2.4.10]) a bundle metric $\langle\, .\, , .\, \rangle_E$ on E.

3.4.4. DEFINITION. Let $\phi \in \Gamma(E)$. If $\dim(W) = 1$, ϕ is called a *complex scalar field*, while if $\dim(W) > 1$ then ϕ is called a *multiplet of complex scalar fields*, and W is called a *multiplet space*.

As in the previous section, in what follows we make use of the covariant differential $d_A : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$ induced by the absolute differential $D_A : \Omega^k_{\mathrm{hor}}(P, W)^{(G,\rho)} \to \Omega^{k+1}_{\mathrm{hor}}(P, W)^{(G,\rho)}$ (cf. [**25**, (3.4.6)]).

3.4.5. DEFINITION. The *Klein–Gordon Lagrangian* for a multiplet $\Phi \in \Gamma(E)$ of complex scalar fields of mass m coupled to a gauge field A is

$$\mathcal{L}_{KG}[\Phi, A] = \langle d_A \Phi, d_A \Phi \rangle_E - m^2 \langle \Phi, \Phi \rangle_E.$$

Again, for fixed Φ and A, $\mathcal{L}_{KG}[\Phi, A] : M \to \mathbb{R}$ is smooth. The corresponding *action functional* (on a closed manifold M) is

$$S_{KG} : \Gamma(E) \times \mathcal{C}(P) \to \mathbb{R}$$

$$S_{KG}[\Phi, A] = \int_M \mathcal{L}_{KG}[\Phi, A] \, \mathrm{dvol}_g.$$

Since $\nabla^A = d_A | \Omega^0(M, E)$, Remark 3.2.1 gives the local representation of the kinetic term:

$$\langle d_A \Phi, d_A \Phi \rangle_E = \langle \nabla^{A\mu} \Phi, \nabla^A_\mu \Phi \rangle_E.$$

In terms of a local gauge $s \in \Gamma(U, P)$ we have $\Phi|_U = [s, \phi]$ for some $\phi \in C^\infty(U, W)$, and exactly as in the discussion preceding (3.15) it follows that

$$\nabla^A_\mu \Phi = [s, \nabla^A_\mu \phi], \quad \nabla^A_\mu \phi = \partial_\mu \phi + A_\mu \phi.$$

The term $A_\mu \phi := \rho_*(A_\mu)\phi$ is called the *minimal coupling*.

Let us now identify W with $\mathbb{C}^N$ (by the choice of a basis) and the scalar product on W with the standard Hermitian product $\langle v, w \rangle = v^\dagger w$. By our general assumption $\langle\, .\, , .\, \rangle_W$ is G-invariant, i.e., ρ is a unitary representation. By [**24**, Theorem 23.9], $\rho_*(\mathfrak{g})$ therefore acts on $W \equiv \mathbb{C}^r$ by skew Hermitian matrices. Since $A_\mu \in \mathfrak{g}$, this implies that the matrices A_μ are skew Hermitian as well:

$$A_\mu^\dagger = -A_\mu.$$

Combining the above observations we obtain the following local representation of the Lagrangian with respect to the local gauge s and the local coordinates x^μ:

$$\mathcal{L}_{KG}[\Phi, A] = (\partial^\mu \phi)^\dagger (\partial_\mu \phi) - m^2 \phi^\dagger \phi + (\partial^\mu \phi)^\dagger (A_\mu \phi) - (\phi^\dagger A_\mu)(\partial_\mu \phi) - \phi^\dagger A^\mu A_\mu \phi. \tag{3.16}$$

Here, the first two terms are the Klein–Gordon Lagrangian for a free multiplet of complex scalar fields of mass m. They are quadratic in ϕ. The remaining terms, which are cubic resp. quartic in (ϕ, A) and are called the *interaction terms*, describe the interaction (or coupling) between the gauge field and the multiplet of scalar fields. In quantum field theory this amounts to an indirect interaction between particles of the scalar field, mediated by gauge bosons.

3.4.6. Remark. Here (and also later for the Dirac Lagrangian) we see the following mixing effects caused by the action ρ of G in case it does not act diagonally on $W = \mathbb{C}^N$:

- ρ mixes components of the multiplet, so the identification of a section of $E = P \times_\rho W$ with a map into W depends on the choice of gauge.
- Via the induced representation ρ_* of $\mathfrak{g}$ on W, the gauge field A mixes different components of the multiplet in the interaction terms of the Lagrangian.

3.4.7. Definition. A section Φ of the associated bundle $E = P \times_\rho W$ with $\rho_* : \mathfrak{g} \to L(W, W)$ non-trivial is called a *charged scalar*.

According to Remark 3.4.6, charged scalars have a non-trivial coupling to the gauge field A.

Our next goal is to prove the gauge invariance of the Klein–Gordon Lagrangian. To do this we first have to gain a better understanding of the action of the gauge group on associated vector bundles. Recall from [**25**, Lemma 3.5.17] that any $f \in \mathcal{G}(P)$ is of the form $f(p) = p\sigma_f(p)$ for some $\sigma_f \in C^\infty(P, G)^G$ (and conversely). The map σ_f can be used to obtain an action of $\mathcal{G}(P)$ on sections of an associated bundle:

3.4.8. Lemma. *Let $E = P \times_\rho W$ be associated to P. Then $\mathcal{G}(P)$ acts on E via vector bundle isomorphisms given by*

$$\begin{aligned} &\mathcal{G}(P) \times E \to E \\ &(f, [p, v]) \mapsto f \cdot [p, v] := [f(p), v] = [p \cdot \sigma_f(p), v]. \end{aligned}$$

proof. The action is obviously fiber linear and f^{-1} induces the inverse transformation, so it only remains to convince ourselves that the map is well-defined: Let $[p, v] = [p', v']$, i.e., $p' = p \cdot g$, $v' = \rho(g)^{-1} v$ for some $g \in G$. Then

$$[f(p'), v'] = [f(p \cdot g), \rho(g)^{-1} v] = [f(p) \cdot g, \rho(g)^{-1} v] = [f(p), v].$$

□

This operation then automatically also induces an action of $\mathcal{G}(P)$ on sections of E: For $\Phi \in \Gamma(E)$ and $f \in \mathcal{G}(P)$ we define $f \cdot \Phi \in \Gamma(E)$ by

$$(f \cdot \Phi)(x) := f \cdot \Phi(x). \tag{3.17}$$

In order to give meaning to the problem of gauge invariance of $\mathcal{L}_{KG}$ we need to give a definition of the pullback of $\Phi \in \Gamma(E)$ under $f \in \mathcal{G}(P)$. Offhand this does not make sense because $\Phi : M \to E$, whereas $f : P \to P$. However,

$$\Gamma(E) = \Omega^0(M, E) \cong \Omega^0_{\mathrm{hor}}(P, W)^{(G,\rho)} = C^\infty(P, W)^G$$

(see [**25**, Theorems 2.3.4, 3.2.3]), and the pullback under f on elements of the right hand side is well defined. As in [**25**], for any k we denote the isomorphism $\Omega^k(M, E) \to \Omega^k_{\mathrm{hor}}(P, W)^{(G,\rho)}$ by $\omega \mapsto \bar{\omega}$ and its inverse by $\tau \mapsto \hat{\tau}$. To transfer the pullback operation from $\Omega^k_{\mathrm{hor}}(P, W)^{(G,\rho)}$ to $\Omega^k(M, E)$ we set, for any $\omega \in \Omega^k(M, E)$ and $f \in \mathcal{G}(P)$:

$$f^*\omega := (f^*\bar{\omega})^\wedge. \tag{3.18}$$

Let us work this out explicitly, beginning with the case $k = 0$, i.e., for a $\Phi \in \Gamma(E)$. By [**25**, Theorem 2.3.4], the relation between Φ and $\bar\Phi$ is $\Phi(x) = [p, \bar\Phi(p)]$ ($p \in P_x$ arbitrary). Since $\bar\Phi \in C^\infty(P, W)^G$ we have (suppressing ρ)

$$(f^*\bar\Phi)(p) = \bar\Phi(f(p)) = \bar\Phi(p \cdot \sigma_f(p)) = \sigma_f(p)^{-1} \cdot \bar\Phi(p).$$

Therefore,

$$\begin{aligned}(f^*\Phi)(x) &= (f^*\bar\Phi)^\wedge(x) = [p, (f^*\bar\Phi)(p)] = [p, \sigma_f(p)^{-1} \cdot \bar\Phi(p)] = [p \cdot \sigma_f(p)^{-1}, \bar\Phi(p)] \\ &= [f(p)^{-1}, \bar\Phi(p)] \underset{(3.17)}{=} (f^{-1} \cdot \Phi)(x),\end{aligned} \tag{3.19}$$

meaning that pullback under f of elements of $\Gamma(E)$ in the sense of (3.18) is precisely multiplication by f^{-1} in the sense of (3.17).

Turning now to general k, let $\omega \in \Omega^k(M, E)$, $f \in \mathcal{G}(P)$, $x \in M$, $p \in P_x$, $t_1, \dots, t_k \in T_xM$ and $X_1, \dots, X_k \in T_pP$ with $T\pi(X_i) = t_i$. Then (cf. the proof of [**25**, Theorem 3.2.3])

$$\begin{aligned}(f^*\omega)_x(t_1, \dots, t_k) &= (f^*\bar\omega)^\wedge_x(t_1, \dots, t_k) = [p, (f^*\bar\omega)_p(X_1, \dots, X_k)] \\ &= [p, \bar\omega_{f(p)}(T_pf(X_1), \dots, T_pf(X_k))] = [p, \bar\omega_{p\cdot\sigma_f(p)}(T_pf(X_1), \dots, T_pf(X_k))] \\ &= [p, (R^*_{\sigma_f(p)}\bar\omega)_p(TR^{-1}_{\sigma_f(p)}(T_pf(X_1)), \dots, TR^{-1}_{\sigma_f(p)}(T_pf(X_k)))].\end{aligned}$$

The last line here was introduced in order to allow us to exploit the fact that $\bar\omega \in \Omega^k_{\mathrm{hor}}(P, W)^{(G,\rho)}$. Indeed, applying [**25**, Definition 3.2.1 (ii)] and noting that $R^{-1}_{\sigma_f} \circ f = \mathrm{id}_P$ we arrive at (again suppressing ρ)

$$\begin{aligned}(f^*\omega)_x(t_1, \dots, t_k) &= [p, \sigma_f(p)^{-1}\bar\omega_p(X_1, \dots, X_k)] = [p \cdot \sigma_f(p)^{-1}, \bar\omega_p(X_1, \dots, X_k)] \\ &= f^{-1} \cdot [p, \bar\omega_p(X_1, \dots, X_k)] = (f^{-1} \cdot \omega)_x(t_1, \dots, t_k).\end{aligned} \tag{3.20}$$

Thus also in the general case, pullback amounts to multiplication by f^{-1} in the sense of (3.17). There is, moreover, a simple relation between the case $k = 0$ and $k > 0$:

$$f^{-1}(\omega(t_1, \dots, t_k)) = (f^{-1}\omega)(t_1, \dots, t_k).$$

Indeed, $f^{-1}(\omega(t_1, \dots, t_k)) = f^{-1}([p, \bar\omega_p(X_1, \dots, X_k)])$, so this follows immediately from (3.20).

After these preparations we can now prove the desired result on the gauge invariance of $\mathcal{L}_{KG}$:

3.4.9. Theorem. *The Klein–Gordon Lagrangian of a multiplet of complex scalar fields, coupled to a gauge field, is gauge invariant:*

$$\mathcal{L}_{KG}[f^{-1} \cdot \Phi, f^*A] = \mathcal{L}_{KG}[\Phi, A] \qquad (f \in \mathcal{G}(P), \Phi \in \Gamma(E), A \in \mathcal{C}(P)).$$

proof. We will indeed show that each term in $\mathcal{L}_{KG}$ is gauge invariant, starting with the kinetic term $\langle d_A\Phi, d_A\Phi\rangle_E$. Recall from [**25**, (3.4.6)] the relation between d_A and the absolute differential D_A:

$$\begin{array}{ccc}\Omega^k(M,E) & \xrightarrow{\ d_A\ } & \Omega^{k+1}(M,E) \\ {\scriptstyle \omega\mapsto\bar\omega}\big\downarrow\big\uparrow{\scriptstyle \tau\mapsto\hat\tau} & & {\scriptstyle \omega\mapsto\bar\omega}\big\downarrow\big\uparrow{\scriptstyle \tau\mapsto\hat\tau} \\ \Omega^k_{\mathrm{hor}}(P,W)^{(G,\rho)} & \xrightarrow{\ D_A\ } & \Omega^{k+1}_{\mathrm{hor}}(P,W)^{(G,\rho)}\end{array}$$

This, together with (3.18) allows us to transfer the gauge invariance of D_A, i.e., $D_{f^*A}(f^*\omega) = f^*(D_A\omega)$ ([**25**, Theorem 3.5.18]) to d_A:

$$\begin{aligned}d_{f^*A}(f^*\Phi) &= (D_{f^*A}\overline{f^*\Phi})^\wedge = (D_{f^*A}(f^*\bar\Phi))^\wedge = (f^*D_A\bar\Phi)^\wedge \\ &= (f^*\overline{d_A\Phi})^\wedge = f^*(d_A\Phi) \underset{(3.20)}{=} f^{-1} \cdot d_A\Phi.\end{aligned} \tag{3.21}$$

Remark 3.2.1, combined with the G-invariance of $\langle\,.\,,\,.\,\rangle_W$ then yields

$$\langle d_{f^*A}(f^*\Phi), d_{f^*A}(f^*\Phi)\rangle_E = \langle f^{-1} \cdot d_A\Phi, f^{-1} \cdot d_A\Phi\rangle_E = \langle d_A\Phi, d_A\Phi\rangle_E,$$

as claimed. Finally, for the mass term we directly have

$$-m^2\langle f^*\Phi, f^*\Phi\rangle_E = -m^2\langle f^{-1}\cdot\Phi, f^{-1}\cdot\Phi\rangle_E = -m^2\langle \Phi,\Phi\rangle_E.$$

□

The Klein–Gordon Lagrangian contains the connection form in non-dynamic form, meaning that it does not involve derivatives of A, which therefore figures merely as a fixed background field. To describe the combined dynamics of the scalar field, the gauge field and their interaction one considers the *Yang–Mills–Klein–Gordon Lagrangian*

$$\mathcal{L}_{KG}[\Phi,A] + \mathcal{L}_{YM}[A] = \langle d_A\Phi, d_A\Phi\rangle_E - m^2\langle\Phi,\Phi\rangle_E - \frac{1}{2}\langle F_M^A, F_M^A\rangle_{\mathrm{Ad}(P)}.$$

To describe the dynamics of a scalar field with a potential coupled to a gauge field, one uses the Higgs Lagrangian:

3.4.10. Definition. The *Higgs–Lagrangian* for a multiplet of complex scalar fields coupled to a gauge field is

$$\mathcal{L}_H[\Phi,A] := \langle d_A\Phi, d_A\Phi\rangle_E - V(\Phi),$$

where V is a gauge invariant potential (i.e., $V(f^{-1}\cdot\Phi) = V(\Phi)$ for all $f\in\mathcal{G}(P)$).

The only type of potential we shall consider is one of the form $V(\Phi) = V(\langle\Phi,\Phi\rangle_E)$ with (the r.h.s.) $V:\mathbb{R}\to\mathbb{R}$. Gauge invariance is then clear (from the proof of Theorem 3.4.9). The Higgs–Lagrangian describes the interaction between particles of the scalar field and particles of the gauge field, and simultaneously the direct interaction between the particles of the scalar field (in case V contains terms of order greater than 2 in Φ).

From the above remarks and the proof of Theorem 3.4.9 we immediately conclude:

3.4.11. Theorem. *The Higgs Lagrangian for a multiplet of complex scalar fields coupled to a gauge field is gauge invariant:*

$$\mathcal{L}_H[f^{-1}\cdot\Phi, f^*A] = \mathcal{L}_H[\Phi,A] \qquad (f\in\mathcal{G}(P), \Phi\in\Gamma(E), A\in\mathcal{C}(P)).$$

It will come as no surprise that the sum of the Higgs- and the Yang–Mills Lagrangian is called the *Yang–Mills–Higgs* Lagrangian:

$$\mathcal{L}_H[\Phi,A] + \mathcal{L}_{YM}[A] = \langle d_A\Phi, d_A\Phi\rangle_E - V(\Phi) - \frac{1}{2}\langle F_M^A, F_M^A\rangle_{\mathrm{Ad}(P)}. \tag{3.22}$$

3.5. The Dirac Lagrangian

Fermions are particles that follow the Fermi-Dirac statistics and have half integer spin. They are described by spinor fields on spacetime. In this section we make the following general assumptions:

- (M,g) is an n-dimensional oriented and time-oriented pseudo-Riemannian manifold of signature (s,t).
- There is a spin structure $\mathrm{Spin}^+(M)$, with corresponding spinor bundle $S\to M$.
- $\langle\,.\,,\,.\,\rangle$ is a (not necessarily positive definite) Dirac form on the Dirac spinor space $\Delta=\Delta_n$ with associated Dirac bundle metric $\langle\,.\,,\,.\,\rangle_S$. We also write $\bar\Psi\Phi$ for $\langle\Psi,\Phi\rangle_S$.

3.5.1. Definition. The *Dirac Lagrangian* for a free spinor field $\Psi\in\Gamma(S)$ of mass m is

$$\mathcal{L}_D[\Psi] = \mathrm{Re}\langle\Psi, D\!\!\!/\Psi\rangle_S - m\langle\Psi,\Psi\rangle_S = \mathrm{Re}(\bar\Psi D\!\!\!/\Psi) - m\bar\Psi\Psi,$$

where $D\!\!\!/:\Gamma(S)\to\Gamma(S)$ is the Dirac operator (see Definition 2.3.11), and $\bar\Psi$ refers to the Dirac conjugate of Ψ, cf. Definition 1.7.7. In the above expression, $\mathrm{Re}(\bar\Psi D\!\!\!/\Psi)$ is called the *kinetic term* and $-m\bar\Psi\Psi$ is called the *Dirac mass term.*

3.5.2. Remark. Taking the real part in the above definition is necessary since Lagrangians have to be real valued. If the Dirac form $\langle\,.\,,\,.\,\rangle_S$ has $\delta = -1$ then the proof of Theorem 2.3.21 (together with Remark 2.3.20) shows that

$$(\langle \Psi, \not{D}\Psi\rangle_S - \langle \not{D}\Psi, \Psi\rangle_S)d\mathrm{vol}_g = \mathrm{div}(\sigma)d\mathrm{vol}_g \tag{3.23}$$

for some one-form σ on M. Therefore,

$$\begin{aligned}\mathrm{Re}(\langle \Psi, \not{D}\Psi\rangle_S)d\mathrm{vol}_g &= \frac{1}{2}(\langle \Psi, \not{D}\Psi\rangle_S + \langle \Psi, \not{D}\Psi\rangle_S^*)d\mathrm{vol}_g \\ &\underset{1.7.1}{=} \frac{1}{2}(\langle \Psi, \not{D}\Psi\rangle_S + \langle \not{D}\Psi, \Psi\rangle_S)d\mathrm{vol}_g \\ &\underset{(3.23)}{=} \langle \Psi, \not{D}\Psi\rangle_S d\mathrm{vol}_g - \frac{1}{2}\mathrm{div}(\sigma)d\mathrm{vol}_g.\end{aligned}$$

Thus Stokes' theorem implies that the action defined by $\langle \Psi, \not{D}\Psi\rangle_S$ and its real part coincide if Ψ has compact support (and M has no boundary).

Let us now look at the coupling of a spinor field to a scalar field, describing the interaction between fermions and gauge bosons (e.g., between electrons and photons in quantum electrodynamics (QED), or between quarks and gluons in quantum chromodynamics (QCD)). Additionally to the conventions noted at the beginning of this section, we fix

- a principal fiber bundle $P \to M$ with compact structure group G of dimension r, and
- a complex representation $\rho : G \to \mathrm{GL}(V)$ and the associated vector bundle $E = P \times_\rho V \to M$, as well as
- a G-invariant Hermitian scalar product $\langle\,.\,,\,.\,\rangle_V$ on V and the corresponding bundle metric $\langle\,.\,,\,.\,\rangle_E$ (according to [**25**, Theorem 2.4.10]). This and the Dirac form on the spinor bundle S induce a Hermitian scalar product $\langle\,.\,,\,.\,\rangle_{S\otimes E}$ on the twisted spinor bundle $S \otimes E$ (cf. the discussion around (2.16)). As before, we abbreviate $\langle \Psi, \Phi\rangle_{S\otimes E}$ by $\bar{\Psi}\Phi$.

Let $s : U \to P$ be a local gauge and fix an orthonormal basis $v_1, \ldots, v_N$ of V. Then the twisted spinors Ψ, Φ (or actually their pullbacks under s) correspond to multiplets $\Psi = (\Psi_1, \ldots, \Psi_N)^T$, $\Phi = (\Phi_1, \ldots, \Phi_N)^T$, where $\Phi_i, \Psi_i \in \Gamma(U, S)$. The scalar product in $S \otimes E$ then takes the form

$$\bar{\Psi}\Phi = \sum_{j=1}^{N} \bar{\Psi}_j \Phi_j.$$

3.5.3. Definition. The *Dirac Lagrangian* for a twisted spinor field $\Psi \in \Gamma(S \otimes E)$ of mass m coupled to a gauge field A on the principal fiber bundle P is

$$\mathcal{L}_D[\Psi, A] = \mathrm{Re}\langle \Psi, \not{D}_A\Psi\rangle_{S\otimes E} - m\langle \Psi, \Psi\rangle_{S\otimes E} = \mathrm{Re}(\bar{\Psi}\not{D}_A\Psi) - m\bar{\Psi}\Psi.$$

Here, $\not{D}_A : \Gamma(S \otimes E) \to \Gamma(S \otimes E)$ is the twisted Dirac operator from Definition 2.3.19. The corresponding action (for M a closed manifold) is

$$\begin{aligned}S_D &: \Gamma(S \otimes E) \times \mathcal{C}(P) \to \mathbb{R} \\ S_D[\Psi, A] &= \int_M \mathcal{L}_D[\Psi, A]\, d\mathrm{vol}_g.\end{aligned}$$

To obtain a local description, let us additionally pick a local vielbein e for TM with corresponding local trivialization ε of $\mathrm{Spin}^+(M)$ (cf. Lemma 2.1.6). Then from (2.14) and the discusion leading up to (3.15) we obtain

$$\mathcal{L}_D[\Psi, A] = \mathrm{Re}\sum_{j=1}^{N} i\bar{\psi}_j\Gamma^p\Big(\partial_p - \frac{1}{4}\omega_{pqr}\Gamma^{qr}\Big)\psi_j - \sum_{j=1}^{N} m\bar{\psi}_j\psi_j + \mathrm{Re}\sum_{j=1}^{N} i\bar{\psi}_j\Gamma^p(A_p\psi)_j. \tag{3.24}$$

Some care has to be taken in interpreting this formula correctly: First, the ψ_j here correspond to what was written as $\psi_j \otimes w_j$ in (2.14), with the understanding (discussed prior to (2.11)) that the gamma-matrices only affect the Δ-part (i.e., the ψ_j) and A_s affects only the V-part (i.e., the second factor in the tensor product). Finally, in contrast to (3.15) (where local coordinates were used instead of a local vielbein), ∂_p stands for the application of the vector field e_p to a scalar function, and $A_p := \rho_* A_s(e_p)$. So in proper longhand notation, (3.24) reads

$$\begin{aligned}\mathcal{L}_D[\Psi, A] = &\operatorname{Re}\sum_{j=1}^{N} i\bar{\psi}_j\Gamma^p\Big(T(\psi_j\otimes w_j)(e_p) - \frac{1}{4}\omega_{pqr}\Gamma^{qr}\psi_j\otimes w_j\Big) - \sum_{j=1}^{N} m\bar{\psi}_j\psi_j \\ &+ \operatorname{Re}\sum_{j=1}^{N} i\bar{\psi}_j\Gamma^p\psi_j\otimes((\rho_* A_s(e_p))w_j).\end{aligned} \tag{3.25}$$

The first two sums in (3.24) constitute the Dirac Lagrangian for a free multiplet of fermions, with *kinetic term* $\operatorname{Re}\sum_{j=1}^{N} i\bar{\psi}_j\Gamma^p\partial_p\psi_j$, followed by a coupling term between the spinor field and the metric encoded in ω_{pqr}, and the Dirac mass term. The third sum in (3.24) is cubic in the fields (i.e., in (Ψ, A)) and is the *interaction term*, describing the interaction between the fermions and the gauge field (and thereby an indirect interaction between the fermions). Note that since the gauge field A acts by skew Hermitian matrices, the interaction term is automatically real, so taking the real part in the final term could be omitted.

3.5.4. Definition. If $\Psi \in \Gamma(S\otimes E)$ is a section of a twisted spinor bundle with $E = P\times_\rho V$ and $\rho_* : \mathfrak{g} \to V$ non-trivial, then Ψ is called a *charged fermion*.

As can be seen from (3.25), charged fermions have a non-trivial coupling to the gauge field A.

3.5.5. Theorem. *The Dirac Lagrangian for a twisted spinor field is gauge invariant:*

$$\mathcal{L}_D[f^{-1}\Psi, f^*A] = \mathcal{L}_D[\Psi, A] \qquad \forall\Psi\in\Gamma(S\otimes E)\ \forall A\in\mathcal{C}(P)\ \forall f\in\mathcal{G}(P).$$

proof. Let us first clarify what we mean by $f^{-1}\cdot\Psi$. Since f is a bundle automorphism of P it only acts on the E-part of Ψ — more precisely, for a local tensor product $\psi\otimes w$ as introduced before (2.11) we set $f^{-1}\cdot(\psi\otimes w) := \psi\otimes(f^{-1}w)$. Thus for any smooth local vector field X on U we have (recalling from [**25**, 3.4.8] that $\nabla^A = d_A|_{\Gamma(E)}$)

$$\begin{aligned}\nabla_X^{f^*A}(f^{-1}\cdot(\psi\otimes w)) &\underset{(2.11)}{=} (\nabla_X^{\mathrm{Spin}}\psi)\otimes(f^{-1}w) + \psi\otimes\nabla_X^{f^*A}(f^{-1}\cdot w) \\ &\underset{(3.21)}{=} (\nabla_X^{\mathrm{Spin}}\psi)\otimes(f^{-1}w) + \psi\otimes f^{-1}(\nabla_X^A w) \\ &= f^{-1}\cdot\big((\nabla_X^{\mathrm{Spin}}\psi)\otimes w + \psi\otimes\nabla_X^A w\big) = f^{-1}\cdot\nabla_X^A(\psi\otimes w).\end{aligned}$$

This shows that $\nabla_X^{f^*A}(f^{-1}\cdot\Psi) = f^{-1}\cdot\nabla_X^A\Psi$ for each $\Psi\in\Gamma(S\otimes E)$.

By Definition 2.3.19 we have $\not{D}_A\Psi = \eta^{ab}e_a\cdot\nabla_{e_b}^A\Psi$. Here, according to (2.12), the Clifford multiplication only acts on the first component of the local tensor product representation of sections of $\Gamma(S\otimes E)$. On the other hand, as we have seen above the action of f^{-1} only involves the second component. Therefore these operations commute and we get

$$\begin{aligned}f^{-1}\cdot(\not{D}_A\Psi) = f^{-1}\cdot(\eta^{ab}e_a\cdot\nabla_{e_b}^A\Psi) &= \eta^{ab}e_a\cdot(f^{-1}\cdot\nabla_{e_b}^A\Psi) \\ &= \eta^{ab}e_a\cdot\nabla_{e_b}^{f^*A}(f^{-1}\cdot\Psi) = \not{D}_{f^*A}(f^{-1}\Psi).\end{aligned}$$

Consequently,

$$\langle f^{-1}\cdot\Psi, \not{D}_{f^*A}(f^{-1}\cdot\Psi)\rangle_{S\otimes E} = \langle f^{-1}\cdot\Psi, f^{-1}\cdot(\not{D}_A\Psi)\rangle_{S\otimes E} = \langle\Psi, \not{D}_A\Psi\rangle_{S\otimes E},$$

where the last equality follows from the definition of $\langle\,.\,,\,.\,\rangle_{S\otimes E}$ on splitting tensors and the G-invariance of $\langle\,.\,,\,.\,\rangle_V$ (as in the proof of Theorem 3.4.9). For the same reason, $\langle f^{-1}\cdot\Psi, f^{-1}\cdot\Psi\rangle_{S\otimes E} = \langle\Psi,\Psi\rangle_{S\otimes E}$, establishing also the gauge invariance of the mass term. □

3.5.6. Remark. It now follows from Remark 2.3.20 that the previous result automatically also implies gauge invariance of the Lagrangian for a free spinor field.

3.5.7. Example. The *strong interaction* in QCD is modelled using $G = \mathrm{SU}(3)$ and $V \cong \mathbb{C}^3$. It has six multiplets Ψ_f, called *quarks*, with the *flavors* u, d, s, c, t, b. The three components of every multiplet are called *colors*. The interaction term of the corresponding Lagrangian contains a gauge field A_μ with values in $\mathfrak{su}(3)$ (modelling eight gluons). It mixes different colors of a quark of a given flavor, but not different flavors (which would require *weak interaction*). So the Lagrangian for QCD has the form

$$\mathcal{L}_D[\Psi, A] = \sum_{f\in\{u,d,s,c,t,b\}} \big(\operatorname{Re}(\bar\Psi_f \not{D}_A \Psi_f) - m_f \bar\Psi_f \Psi_f\big).$$

3.5.8. Remark. (The problem of particle masses) By the very form of the mass term $-m\bar\Psi\Psi$ in the Dirac Lagrangian it is clear that each component of the multiplet Ψ has the same mass m. If one would assign different masses to different components then this would destroy the gauge invariance of the mass term (recall that the gauge group mixes the components of the multiplet). This causes problems for the standard model, which aims to describe particles with very different masses, like electron and electron neutrino as $\mathrm{SU}(2)\times U(1)$-doublets, while at the same time keeping the Lagrangian gauge invariant. In fact, an additional complication is that left-handed and right-handed fermions transform with respect to different representations of $\mathrm{SU}(2)\times U(1)$. This implies that a gauge invariant mass term cannot be defined even if all components of the multiplet had the same mass. We will return to this question below and in the next chapter, where a solution to this problem based on the Higgs field will be described.

As before, to make the spinor multiplet Ψ and the gauge field A dynamic we can combine the Lagrangians into the *Yang–Mills–Dirac Lagrangian*

$$\mathcal{L}_D[\Psi, A] + \mathcal{L}_{YM}[A] = \operatorname{Re}(\bar\Psi \not{D}_A \Psi) - m\bar\Psi\Psi - \frac{1}{2}\langle F_M^A, F_M^A\rangle_{\mathrm{Ad}(P)}.$$

Our next aim is to describe chiral fermions. To do this, we restrict to M being of even dimension (as well as oriented and time-oriented), to have Lorentzian signature ($(1, n-1)$ or $(n-1, 1)$), and to be equipped with a spin structure. In the case of the standard model one simply takes four-dimensional Minkowski space for M.

Recall from Proposition 1.7.6 that for both choices of the matrix A there, A consists of a product of an odd number of gamma matrices. Therefore, if we decompose the spinor bundle into left- and right handed Weyl spinors, $S = S_L \oplus S_R$, then the Dirac bundle metric is null on both subbundles S_L and S_R, see Remark 1.7.12. Thus for $\Psi, \Phi \in \Gamma(S)$ we have

$$\bar\Psi\Phi = \langle\Psi, \Phi\rangle_S = \langle\Psi_L, \Phi_R\rangle_S + \langle\Psi_R, \Phi_L\rangle_S = \bar\Psi_L\Phi_R + \bar\Psi_R\Phi_L. \tag{3.26}$$

Note that this implies that the decomposition $S = S_L \oplus S_R$ is not orthogonal with respect to the Dirac bundle metric.

By the definition of the scalar product on the twisted spinor bundle, (3.26) remains valid also on

$$S\otimes E = S_L\otimes E \oplus S_R \otimes E.$$

In particular, the Dirac Lagrangian for a twisted spinor field $\Psi \in \Gamma(S\otimes E)$ of mass m coupled to a gauge field A on the principal fiber bundle P (see Definition 3.5.3) under the above assumptions takes the form

$$\mathcal{L}_D[\Psi, A] = \operatorname{Re}(\bar\Psi \not{D}_A\Psi) - m\bar\Psi\Psi = \operatorname{Re}(\bar\Psi_L \not{D}_A \Psi_L + \bar\Psi_R \not{D}_A \Psi_R) - 2m\operatorname{Re}(\bar\Psi_L\Psi_R).$$

To conclude this section we want to generalize these observations to the case of twisted chiral spinor bundles (cf. Definition 2.3.22). So let M be an even dimensional Lorentzian spin manifold as above and consider a twisted chiral spinor bundle

$$(S\otimes E)_+ = (S_L\otimes E_L) \oplus (S_R \otimes E_R),$$

where E_L and E_R are complex vector bundles associated to $P \to M$ via representations $\rho_L : G \to \mathrm{GL}(V_L)$ and $\rho_R : G \to \mathrm{GL}(V_R)$. Let us also fix G-invariant Hermitian scalar products on V_L and V_R, which in turn induce Hermitian bundle metrics $\langle\,.\,,\,.\,\rangle_{E_L}$ and $\langle\,.\,,\,.\,\rangle_{E_R}$.

We then introduce a *massless Dirac Lagrangian* by

$$\begin{aligned}\mathcal{L}_D[\Psi, A] :&= \mathrm{Re}\langle\Psi, \not{D}_A\Psi\rangle_{S\otimes E} = \mathrm{Re}\left(\langle\Psi_L, \not{D}_A\Psi_L\rangle_{S\otimes E_L} + \langle\Psi_R, \not{D}_A\Psi_R\rangle_{S\otimes E_R}\right)\\ &= \mathrm{Re}(\bar{\Psi}\not{D}_A\Psi) = \mathrm{Re}(\bar{\Psi}_L\not{D}_A\Psi_L + \bar{\Psi}_R\not{D}_A\Psi_R).\end{aligned}$$

Keep in mind that the Dirac operator here only affects the S-components. Gauge invariance is clear from the proof of Theorem 3.5.5.

If we try to add an appropriate mass term to this Lagrangian, a natural choice seems to be

$$-m\bar{\Psi}\Psi = -2m\,\mathrm{Re}(\bar{\Psi}_L\Psi_R).$$

However, this term is ill-defined as can be seen by trying to write out the scalar product that is to be used here. It can neither be $\langle\Psi_L, \Psi_R\rangle_{S\otimes E_L}$ nor $\langle\Psi_L, \Psi_R\rangle_{S\otimes E_R}$ because Ψ_L is a section of the first bundle whereas Ψ_R is one of the second, and so there is no scalar product in which to insert them both.

Instead, one might try to use something like this:

3.5.9. Definition. Let V_L and V_R be unitary representations of a Lie group G. A *mass pairing* is a G-invariant form

$$\kappa : V_L \times V_R \to \mathbb{C}$$

that is complex anti-linear in the first argument and complex linear in the second.

As usual, by [**25**, Theorem 2.4.10] a mass pairing induces a corresponding bundle form $\kappa : E_L \oplus E_R \to \mathbb{C}$, which then can be used to define a gauge invariant Dirac mass term for chiral twisted spinors. At least in theory, that is: As the following result shows, in many cases there does not exist a non-trivial mass pairing:

3.5.10. Theorem. *(Triviality of mass pairings) If V_L and V_R are irreducible, unitary and non-isomorphic representations of G, then any mass pairing κ vanishes identically.*

proof. The dual $\bar{V}_L^*$ of the complex conjugate of V_L can be identified with the space $\{\alpha : V_L \to \mathbb{C} \mid \alpha \text{ is } \mathbb{C}\text{−antilinear}\}$. The corresponding representation of G is given by $(g\cdot\alpha)(v_L) := \alpha(g^{-1}\cdot v_L)$ for $g \in G$ and $v_L \in V_L$. Then the map

$$\begin{aligned}F : V_L &\to \bar{V}_L^*\\ v_L &\mapsto \langle\,.\,, v_L\rangle_{V_L}\end{aligned}$$

(with $\langle\,.\,,\ \rangle_{V_L}$ the G-invariant Hermitian form on V_L) gives a complex linear G-equivariant isomorphism. Indeed,

$$(g \cdot F(v_L))(w_L) = F(v_L)(g^{-1}w_L) = \langle g^{-1}w_L, v_L\rangle = \langle w, gv_L\rangle = F(gv_L)(w_L).$$

Assuming now that there exists some $\kappa \neq 0$, the map

$$\begin{aligned}G : V_R &\to \bar{V}_L^*\\ v_R &\mapsto \kappa(\,.\,, v_R)\end{aligned}$$

is also complex linear and G-equivariant. Then $F^{-1} \circ G : V_R \to V_L$ is complex linear and G-equivariant. It is also non-zero since $\kappa \neq 0$. By Schur's Lemma ([**24**, Theorem 23.5]) it follows that $V_L \cong V_R$, a contradiction. □

3.5.11. Remark. Experimental particle physics requires the existence of twisted chiral fermions with non-zero mass (since weak interaction is not invariant under parity inversion). Combining Remarks 3.2.9 and 3.5.8 with this observation, we have now seen the following instances where it is unclear how to define gauge invariant mass terms:

- non-zero masses for gauge bosons
- different masses for fermions in the same gauge multiplet
- non-zero masses for twisted chiral fermions

As we shall see in the next chapter, all of these problems can be solved by the introduction of the Higgs field.

3.6. Yukawa Couplings

Yukawa couplings are a tool in the standard model to define a mass for twisted chiral fermions. They are trilinear forms taking two twisted chiral spinors and one scalar field as arguments. The aim is to have the G-representation of the scalar field cancel the difference between the representations of the twisted chiral spinors so as to obtain an overall gauge invariant expression. Let (M, g) be an oriented and time-oriented Lorentzian spin manifold of dimension n and with signature $(1, n-1)$ or $(n-1, 1)$. Also, let $P \to M$ be a principal fiber bundle with structure group G.

3.6.1. Definition. Let V_L, V_R, W be unitary representation spaces of the compact Lie group G. A *Yukawa form* is a map

$$\tau : V_L \times W \times V_R \to \mathbb{C}$$

that is invariant under the action of G, complex anti-linear in V_L, real linear in W and complex linear in V_R.

3.6.2. Definition. Let τ be a Yukawa form. Then given a real constant g_Y, the G-invariant map

$$\begin{aligned} &(\Delta_L \otimes V_L) \times W \times (\Delta_R \otimes V_R) \to \mathbb{R} \\ &(\lambda_L \otimes v_L, \phi, \lambda_R \otimes v_R) \mapsto -2g_Y \operatorname{Re}\left(\bar{\lambda}_L \lambda_R \tau(v_L, \phi, v_R)\right) \end{aligned}$$

is called a Yukawa coupling.[4]

Here, $\bar{\lambda}_L$ is the Dirac conjugate of λ_L (cf. Definition 1.7.7), so $\bar{\lambda}_L \lambda_R = \langle \lambda_L, \lambda_R \rangle$ is the Dirac form evaluated on (λ_L, λ_R). By Lemma 1.7.3, this expression is invariant under the action of the representation κ of Spin^+. Together with the G-invariance of τ it follows that the Yukawa coupling is indeed invariant under the action of G (cf. Remark 2.3.16).

On the bundle level, the Yukawa coupling (written simply as $\bar{\Psi}_L \Phi \Psi_R$) therefore gives rise to a gauge invariant *Yukawa–Lagrangian*

$$\mathcal{L}_Y[\Psi_L, \Phi, \Psi_R] := -2g_Y \operatorname{Re}(\bar{\Psi}_L \Phi \Psi_R) = -g_Y(\bar{\Psi}_L \Phi \Psi_R) - g_Y(\bar{\Psi}_L \Phi \Psi_R)^*,$$

where $\Psi_L \in \Gamma(S_L \otimes E_L)$, $\Phi \in \Gamma(F)$, and $\Psi_R \in \Gamma(S_R \otimes E_R)$, and E_L, F, E_R are complex vector bundles associated to the principal G-bundle P via the representations V_L, W, V_R.

Finally, as announced at the beginning of this chapter, the Lagrangian of the standard model of particle physics is the sum of all the Lagrangians we have discussed so far, i.e., it is the *Yang–Mills-Dirac-Higgs-Yukawa Lagrangian*

$$\boxed{\begin{aligned} \mathcal{L} &= \mathcal{L}_D[\Psi, A] + \mathcal{L}_H[\Phi, A] + \mathcal{L}_Y[\Psi_L, \Phi, \Psi_R] + \mathcal{L}_{YM}[A] \\ &= \operatorname{Re}(\bar{\Psi} \not{D}_A \Psi) + \langle d_A \Phi, d_A \Phi \rangle_E - V(\Phi) - 2g_Y \operatorname{Re}(\bar{\Psi}_L \Phi \Psi_R) - \frac{1}{2}\langle F_M^A, F_M^A \rangle_{\mathrm{Ad}(P)}. \end{aligned}}$$

To conclude this chapter we want to compare two ways of obtaining mass terms for spinor fields. Up to now, we have used a Dirac form $\langle\, .\, ,\, .\, \rangle$ and the corresponding bundle form $\langle\, .\, ,\, .\, \rangle_S$ to assign to a spinor field $\Psi \in \Gamma(S)$ a mass term

$$-m\langle \Psi, \Psi \rangle_S = -m\bar{\Psi}\Psi.$$

[4] In the literature sometimes the constant g_Y is also called a Yukawa coupling

A second way to do this is to employ a Majorana form:

3.6.3. Definition. Let $(. , .)$ be a Majorana form on the spinor space Δ (as in Definition 1.6.1), with $(. , .)_S$ the corresponding bundle form. Then we call

$$-m \operatorname{Re}(\Psi, \Psi)_S$$

a *Majorana mass term.*

By Lemmas 1.7.3 and 1.6.3, both Dirac and Majorana mass terms are invariant under the action of the spin group.

3.6.4. Example. Let us compare the Dirac and Majorana mass terms in the case of M being the four-dimensional Minkowski space, using the results from Example 1.7.11. The Dirac form is given by the matrix

$$A = \Gamma_0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix},$$

while the Majorana form has the matrix representation

$$C = i\Gamma_0\Gamma_2 = i\Gamma^2\Gamma^0 = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}.$$

Decomposing a Dirac spinor Ψ into left- and right-handed Weyl spinors $\Psi = (\Psi_L, \Psi_R)^T$, the Dirac mass term is given by

$$-m\langle\Psi, \Psi\rangle = -m\Psi^\dagger A\Psi = -m(\Psi_L^\dagger \Psi_R + \Psi_R^\dagger \Psi_L),$$

while for the Majorana mass term we obtain

$$-m \operatorname{Re}(\Psi, \Psi) = -m \operatorname{Re}(\tilde{\Psi}\Psi) = -m \operatorname{Re}(\Psi^T C\Psi) = m \operatorname{Re}\big(i\Psi_L^T\sigma_2\Psi_L - i\Psi_R^T\sigma_2\Psi_R\big).$$

Here, $\tilde{\Psi} = \Psi^T C$ is the Majorana conjugate from Definition 1.6.4. It follows that the Dirac mass term vanishes for spinors that only have one non-zero Weyl component Ψ_L or Ψ_R, contrary to the Majorana mass term.

In the general case there is another substantial difference in the behavior of the Dirac and the Majorana mass term when they are to be extended to twisted spinor bundles $S \otimes E$. As we saw in the previous section, for the Dirac mass term such an extension is always possible if only we twist both the left and right handed spinor bundles with the same associated vector bundle E. The Hermitian scalar product on $S \otimes E$ then is simply the tensor product of the Dirac form on S and a Hermitian scalar product on E.

The same cannot be done automatically for the Majorana mass term. Indeed, the complex bilinear Majorana form on S usually does not combine satisfactorily with a Hermitian scalar product on E. For this to work for $E = P \times_\rho V$ we would need a G-invariant complex bilinear form on the vector space V. But even for one dimensional Lie groups such a form need not exist:

3.6.5. Lemma. *Let*

$$\rho_k : U(1) \to U(1)$$
$$\alpha \mapsto \alpha^k$$

be the complex representation of $U(1)$ on $\mathbb{C}$ with winding number k. Let B be a ρ_k-invariant complex bilinear form on $\mathbb{C}$. Then $B = 0$.

proof. By assumption, $B(z, z) = B(\alpha^k z, \alpha^k z) = \alpha^{2k}B(z, z)$ for any $\alpha \in U(1)$ and any $z \in \mathbb{C}$. But then $B(z, z) = 0$ for each z, so $B = 0$ by polarization. □

The takeaway from this is that in general there is no obvious way of extending the Majorana mass term to charged fermions.

CHAPTER 4

The Higgs mechanism

As was pointed out repeatedly in the previous chapter (cf. Remarks 3.2.9, 3.5.8), assigning a gauge invariant mass term to gauge bosons or fermions is a non-trivial task that goes beyond the methods we have employed so far. In this chapter we will look at the Higgs mechanism, which rectifies this situation.

4.1. Symmetry breaking and mass generation

Let (M, g) be a pseudo-Riemannian manifold and $P \to M$ a principal bundle with compact structure group G, the gauge group, of dimension $r \in \mathbb{N}$. Suppose $\rho\colon G \to \mathrm{GL}(W)$ is a complex (or real) representation and let $E := P \times_\rho W$ be the associated complex (or real) vector bundle. We further assume that W possesses a G-invariant Hermitian (or Euclidean) scalar product $\langle .\,, . \rangle_W$ and denote by $\langle .\,, . \rangle_E$ the associated bundle metric on E (via [**25**, Theorem 2.4.10]). If W is complex, then we use the notation $\langle\langle .\,, . \rangle\rangle_W := \mathrm{Re}\langle .\,, . \rangle_W$ for the corresponding Euclidean scalar product on the underlying real vector space structure in W.

In the context of the notation and objects specified above, we introduce the following notions.

4.1.1. Definition. A section Φ of E is called *Higgs field*, E is the *Higgs bundle*, and W is the *Higgs vector space*.

Recall from Definition 3.4.7 that the Higgs field is said to be a charged scalar if the induced Lie algebra representation $\rho_*\colon \mathfrak{g} \to \mathrm{L}(W)$ is non-trivial.

Upon a choice of an orthonormal basis we may assume that $W = \mathbb{C}^n$ with standard scalar product $\langle v, w \rangle_W = v^\dagger \cdot w$.

Let $V\colon \mathbb{R} \to \mathbb{R}$ define a *Higgs potential* (cf. Definition 3.4.3), so that the Higgs Lagrangian for any Higgs field Φ and connection 1-form (or gauge field) $A \in \Omega^1(P, \mathfrak{g})$ is given by

$$\mathscr{L}_H[\Phi, A] = \langle d_A\Phi, d_A\Phi \rangle_E - V(\langle \Phi, \Phi \rangle_E)$$

and in combination with the Yang-Mills Lagrangian for the gauge field (cf. Section 3.2) it gives the Yang-Mills-Higgs Lagrangian

$$\mathscr{L}_H[\Phi, A] + \mathscr{L}_{YM}[A] = \langle d_A\Phi, d_A\Phi \rangle_E - V(\langle \Phi, \Phi \rangle_E) - \frac{1}{2}\langle F^A_M, F^A_M \rangle_{\mathrm{Ad(P)}}.$$

Note that gauge invariance of the Higgs term $V(\langle \Phi, \Phi \rangle_E)$ follows from the G-invariance of the scalar product on W (cf. the proof of Theorem 3.4.9).

4.1.2. Definition. A *vacuum configuration* (or *vacuum*) for the Yang-Mills-Higgs Lagrangian is a pair (Φ_0, A_0), where Φ_0 is a Higgs field and A_0 is a connection, such that the following hold:

1) A_0 is a flat connection, i.e., $F^{A_0} = 0$.
2) Φ_0 is covariantly constant, i.e., $d_{A_0}\Phi_0 = \nabla^{A_0}\Phi_0 = 0$.
3) At every point $x \in M$, $V(\langle \Phi_0(x), \Phi_0(x) \rangle_E)$ is a minimum value of V.

To simplify the discussion in the sequel, we now suppose in addition that

M is connected and simply connected.

Since on a simply connected base manifold, only a principial bundle isomorphic to a trivial bundle can possess a flat connection (cf. [**25**, Theorem 3.5.11]), we further assume that

the principal G-bundle P is trivial (implying that also E is trivial).

4.1.3. Definition. An element $w_0 \in W$ is called a *vacuum vector*, if $V(\langle w_0, w_0\rangle_W)$ is a minimum value of V. The subset $W_0 \subseteq W$ of all vacuum vectors will be called the *vacuum manifold* for the Higgs potential.

4.1.4. Proposition. *Let (Φ_0, A_0) be a vacuum configuration, then there exists a global gauge $s_0\colon M \to P$, called a* vacuum gauge*, and a vacuum vector $w_0 \in W$ such that*

$$A_0^{s_0} := A_0 \circ Ts_0 = 0 \quad \textit{and} \quad \Phi_0 = [s_0, w_0]. \tag{4.1}$$

Conversely, given a global gauge s_0 of the principal bundle P and a vacuum vector $w_0 \in W$, there exists a unique vacuum configuration (Φ_0, A_0) satisfying (4.1)*.*

proof. Let (Φ_0, A_0) be a vacuum configuration. Since $F^{A_0} = 0$, [**25**, Theorem 3.5.11] tells that P is isomorphic to the trivial bundle with the canonical flat connection and then point 1.) in the proof of [**25**, Theorem 3.3.9] shows that there exists a global A_0-horizontal section s_0 of P. As for any connection 1-form the horizontal subspaces are precisely the kernels, we have $A_0^{s_0} = 0$. With respect to the global gauge s_0, the section Φ_0 of E corresponds to a unique smooth map $\phi_0\colon M \to W$ and the covariant derivative translates into the standard derivative of ϕ_0 (insert $A_0^{s_0} = 0$ into [**25**, (3.4.15)]). Thus, property 2) in Definition 4.1.2 implies that $\phi_0(x) = w_0$ for some fixed vector $w_0 \in W$. By property 3) in the same definition, w_0 must be a vacuum vector.

For the proof of the converse, suppose that s_0 is a global section of P and $w_0 \in W$ is a vacuum vector. By [**25**, (2.2.1)], $Th : s_0(x) \cdot g \mapsto TR_g(Ts_0(T_xM))$ defines a smooth distribution on P with $TR_{\tilde g}(Th_pP) = Th_{p\cdot\tilde g}P$ for each $\tilde g \in G$, i.e., a connection. Let A_0 be the connection one-form corresponding to Th according to [**25**, Theorem 3.1.4]. Then the kernel of A_0 is equal to $\mathrm{im}(T_xs_0)$ at any $s_0(x)$, so the first equation in (4.1) is satisfied. Moreover, by [**25**, Definition 3.1.3 (i)] this requirement also completely determines $\ker A_0$ at *any* point $s_0(x) \cdot g \in P$. This means that the first equation in (4.1) uniquely determines a connection form A_0 on P. In addition, the second equation in (4.1) uniquely determines a section Φ_0 of E, and as in the first part of the proof it follows that Φ_0 is covariantly constant. Thus points 2) and 3) of Definition 4.1.2 are satisfied. Finally, appealing again to point 1.) in the proof of [**25**, Theorem 3.3.9], we obtain that A_0 is flat, thus also property 1) in Definition 4.1.2 holds. □

Based on this proposition, we assume that, in addition to all specifications so far, we are given

a global vacuum gauge $s_0\colon M \to P$, a vacuum $w_0 \in W$,
and the corresponding unique vacuum configuration (Φ_0, A_0).

We have the representation ρ of G on W, which is unitary by the assumptions stated at the beginning of the current section. Recall from [**25**, Remark 1.1.6] that for any $w \in W$, the isotropy group (or stabilizer) of w, defined by $G_w := \{g \in G \mid \rho(g)w = w\}$ is closed and (either a discrete or) a Lie subgroup of G.

4.1.5. Definition. The isotropy group $H := G_{w_0} \subseteq G$ of the vacuum vector w_0 is called the *unbroken subgroup of the vacuum configuration*. The gauge theory, i.e., the whole set-up of mathematical objects specified in the current section, is said to be *spontaneously broken*, if $H \neq G$. In any case, H is compact since G is compact.

We will henceforth impose two further conditions in the context of gauge theory as discussed in the current section:

$$\text{The gauge theory is spontaneously broken, i.e., } H \neq G \text{ for } H := G_{w_0} \subseteq G,$$

and

$$V\colon \mathbb{R} \to \mathbb{R} \text{ possesses a global minimum, but none is located at } 0.$$

4.1.6. Remark. In physics, a process like the one causing the Higgs field to change from value 0 to a non-zero vacuum vector value is called *(spontaneous) symmetry breaking* (cf., e.g. [**10**, Sections 8.4 and 15.6]). Since $G_0 = G$ and $H = G_{w_0}$ is, by assumption, a proper subgroup of G, this means that some of the "original" full symmetry has "disappeared" in the state corresponding to $w_0 \neq 0$. The Higgs field Φ_0 defined by s_0 and w_0 is called the *Higgs condensate* and serves as a background field, with non-zero vacuum (expectation) value, for the dynamics of all other fields or particles.

4.1.7. Example. The group and representation used in the standard model of *electroweak interaction* are $G = \mathrm{SU}(2) \times \mathrm{U}(1)$ and $W = \mathbb{C}^2$ with the action of an element $(A, e^{i\alpha}) \in \mathrm{SU}(2) \times U(1)$ given in terms of a parameter $n_Y \in \mathbb{N}$ by

$$\forall w \in \mathbb{C}^2\colon \quad \rho(A, e^{i\alpha})w := e^{i\alpha n_Y} Aw.$$

The Higgs potential is (associated with) the map $\tilde{V}\colon W \to \mathbb{R}$, $w \mapsto V(\|w\|^2)$, where $V\colon \mathbb{R} \to \mathbb{R}$ is the polynomial function

$$V(t) := -\mu t + \lambda t^2 \quad (t \in \mathbb{R})$$

with constants $\lambda, \mu > 0$. It is an elementary exercise to show that this form of the Higgs potential as a map $\tilde{V}\colon W \to \mathbb{R}$ is determined by the following requirements: It should be a polynomial of order at most 4 in the components of w, satisfy $\tilde{V}(0) = 0$, have a minimum value away from 0, and be invariant under the action of G by the representation ρ. The model parameters λ and μ for the Higgs potential can be determined from experiments (see [**19**, page 451 and Section 8.3.4]).

Writing $V(t) = (\sqrt{\lambda}t - \frac{\mu}{2\sqrt{\lambda}})^2 - \frac{\mu^2}{4\lambda}$ we see that V reaches its (global) minimum at $t = \frac{\mu}{2\lambda}$ and thus we have for $w_0 \in W$ that

$$w_0 \text{ is a vacuum vector} \quad \Leftrightarrow \quad \|w_0\| = \sqrt{\frac{\mu}{2\lambda}}.$$

We obtain that the vacuum manifold W_0 is the sphere around 0 in $\mathbb{C}^2$ with radius $\sqrt{\mu/(2\lambda)}$. The restriction of ρ to $\mathrm{SU}(2) \times \{1\}$ gives just the standard action of $\mathrm{SU}(2)$ on $\mathbb{C}^2$, which is transitive on the unit sphere with $\mathrm{SU}(2)$-isotropy group of e_1 being $\mathrm{SU}(1) = \{1\}$ ([**19**, Theorem 3.3.2,4.]), hence e_1 has isotropy group isomorphic to $\mathrm{U}(1)$ for the full G-action. Therefore G acts transitively on W_0 and the isotropy group G_{w_0} for any $w_0 \in W_0$ is isomorphic to $\mathrm{U}(1)$ (since $G_{Aw_0} = AG_{w_0}A^{-1}$ for any $A \in \mathrm{SU}(2)$ and $\mathrm{SU}(2) \cdot \{w_0\}$ is the entire sphere of radius $\|w_0\|$). Let us choose the specific vacuum vector

$$w_0 = \begin{pmatrix} 0 \\ \sqrt{\frac{\mu}{2\lambda}} \end{pmatrix},$$

then $(A, e^{i\alpha}) \in H := G_{w_0}$ is determined by the condition $e^{i\alpha n_Y} A \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Recall that the matrices in $\mathrm{SU}(2)$ can be parameterized in the form $\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$ with $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$. Putting $A = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$, the stabilizer condition reads $e^{i\alpha n_Y} \begin{pmatrix} -\bar{b} \\ \bar{a} \end{pmatrix} = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ and thus implies $b = 0$ and $a = e^{i\alpha n_Y}$, leaving precisely the free real parameter α. We may rescale this by $n_Y\alpha = \delta/2$ and describe the unbroken subgroup of the vacuum configuration in the form

$$H = \{(\begin{pmatrix} e^{i\frac{\delta}{2}} & 0 \\ 0 & e^{-i\frac{\delta}{2}} \end{pmatrix}, e^{i\frac{\delta}{2n_Y}}) \mid \delta \in \mathbb{R}\} \subseteq G = \mathrm{SU}(2) \times \mathrm{U}(1). \tag{4.2}$$

With respect to the global section s_0 of P, we may write any Higgs field Φ in the form

$$\Phi = [s_0, \phi] \quad \text{with } \phi\colon M \to W$$

and we will also call the map ϕ a Higgs field. The Higgs condensate is the special case with constant value w_0 for ϕ. In physics, one considers perturbations of the Higgs condensate in the form

$$\phi = w_0 + \Delta\phi, \tag{4.3}$$

where $\Delta\phi$ is called the *shifted Higgs field*, and is then interested in a kind of Taylor expansion of the Higgs potential up to second order in $\Delta\phi$. In case of a polynomial function V defining the Higgs potential, as e.g. in the electroweak theory, certainly the full expansion is used and gives also an explicit form of the so-called self-interaction terms for the Higgs boson in the Lagrangian (cf. [**19**, Equation (8.2)]). In the general case, since w_0 is related to a minimum of the function

$$\tilde{V}\colon W \to \mathbb{R},\ w \mapsto V(\|w\|^2),$$

the second-order approximation is determined by the *Hessian* of $\tilde{V}$. It is defined in abstract terms (cf. [**29**, Chapter 3, Definitiona 48 and Lemma 49]) with respect to the underlying Euclidean structure according to $\langle\langle .\,,.\rangle\rangle_W = \mathrm{Re}\langle .\,,.\rangle_W$ (with flat Levi-Civita connection) as the symmetric $(0,2)$-tensor field mapping a pair (X,Y) of vector fields on W to the function $\langle\langle \nabla_X \operatorname{grad}\tilde{V}, Y\rangle\rangle_W$. On each tangent space $T_wW \cong W$, we therefore have the self-adjoint linear map $v \mapsto \nabla_v \operatorname{grad}\tilde{V}(w)$, which we denote by

$$\mathrm{Hess}(\tilde{V})_w\colon T_wW \to T_wW.$$

The matrix of $\mathrm{Hess}(\tilde{V})_w$ with respect to an orthonormal basis consists of the second order partial derivatives of $\tilde{V}$ at w in the corresponding coordinates (recall that we are in the flat case here). Our goal is to describe the diagonalization of $\mathrm{Hess}(\tilde{V})_{w_0}$ at the vacuum.

Consider the orbit $O_{w_0} := \{\rho(g)w_0 \in W \mid g \in G\}$ of the vacuum under the gauge group. The unbroken subgroup $H = G_{w_0}$ is a compact proper subgroup of G, thus not open in G, and it follows ([**25**, Theorem 1.1.7]) that the map $f\colon G/H \to W$, $gH \mapsto \rho(g)w_0$ is an injective immersion of the quotient manifold G/H into W with image $f(G/H) = O_{w_0}$. Since G is compact, the quotient G/H is compact as well (as continuous image of a compact topological space) and hence the continuous bijective map $\tilde{f}\colon G/H \to O_{w_0}$ induced by f is a homeomorphism. As argued in [**25**, Corollary 1.1.8], $\tilde{f}$ gives O_{w_0} the structure of an immersed submanifold of W diffeomorphic to G/H and we conclude from the homeomorphism property that O_{w_0} is even an embedded (i.e., regular) submanifold of W.

We may now consider the tangent space $T_{w_0}O_{w_0} \subseteq T_{w_0}W$ and have the orthogonal direct sum decomposition

$$W \cong T_{w_0}W = T_{w_0}O_{w_0} \oplus (T_{w_0}O_{w_0})^\perp.$$

Note that $\tilde{V}$ is invariant under the unitary representation ρ of G in W, since $\tilde{V}(w) = V(\|w\|^2)$. In particular, $\tilde{V}$ is constant with minimum value on the orbit O_{w_0} and has therefore vanishing gradient along the orbit, which implies that

$$\forall v \in T_{w_0}O_{w_0}\colon \quad \mathrm{Hess}(\tilde{V})_{w_0}v = 0.$$

This means that $T_{w_0}O_{w_0} \subseteq \ker \mathrm{Hess}(\tilde{V})_{w_0}$ and trivially implies that $T_{w_0}O_{w_0}$ is an invariant subspace for the self-adjoint operator $\mathrm{Hess}(\tilde{V})_{w_0}$. Thus it leaves also $(T_{w_0}O_{w_0})^\perp$ invariant and we may note that

$$\mathrm{Hess}(\tilde{V})_{w_0} \text{ respects the orthogonal splitting } T_{w_0}O_{w_0} \oplus (T_{w_0}O_{w_0})^\perp.$$

Since $\tilde{V}$ has a minimum at w_0, all eigenvalues of $\mathrm{Hess}(\tilde{V})_{w_0}$ have to be non-negative (e.g., because we have a minimum along any curve through w_0 tangent to an eigenvector). We know from the above that any orthonormal basis $e_1, \ldots, e_d$ of $T_{w_0}O_{w_0}$ consists of eigenvectors of $\mathrm{Hess}(\tilde{V})_{w_0}$ for the eigenvalue 0. By the invariance of the above orthogonal splitting, any extension of this basis to an orthonormal basis of $W \cong T_{w_0}W$ consisting of eigenvectors of $\mathrm{Hess}(\tilde{V})_{w_0}$ can add only vectors

$f_1, f_2, \ldots$ from $(T_{w_0}O_{w_0})^\perp$ which are eigenvectors of $\operatorname{Hess}(\tilde{V})_{w_0}$ for non-negative eigenvalues. We summarize this in the following statement.

4.1.8. Proposition. *Let $\dim_{\mathbb{C}} W = n$, so that the real dimension is $2n$. There are orthonormal bases $e_1, \ldots, e_d$ of $T_{w_0}O_{w_0}$ and $f_1, \ldots, f_{2n-d}$ of $(T_{w_0}O_{w_0})^\perp$ consisting of eigenvectors of Hess$(\tilde{V})_{w_0}$. Every e_j is an eigenvector for the eigenvalue 0 ($j = 1, \ldots, d$). For every $l = 1, \ldots, 2n-d$, we can choose $m_l \geq 0$ such that f_l is an eigenvector for the eigenvalue $2m_l^2$.*

Note that $d = \dim O_{w_0} = \dim G/H = \dim G - \dim H = r - \dim H$.

It seems natural from (4.3) to interpret the shifted Higgs field $\Delta\phi$ to have its values in $T_{w_0}W \cong W$. We may then expand $\Delta\phi(x)$ for every $x \in M$ in terms of an orthonormal basis of eigenvectors of $\operatorname{Hess}(\tilde{V})_{w_0}$ according to Proposition 4.1.8 with coefficients depending on x. Thus, there are real-valued functions $\xi_1, \ldots, \xi_d$ and $\eta_1, \ldots, \eta_{2n-d}$ on M such that

$$\Delta\phi = \frac{1}{\sqrt{2}} \sum_{j=1}^{d} \xi_j e_j + \frac{1}{\sqrt{2}} \sum_{l=1}^{2n-d} \eta_l f_l. \tag{4.4}$$

4.1.9. Definition. Consider the real scalar fields on M defined as coefficient functions in the expansion (4.4) for a given Higgs field $\phi = w_0 + \Delta\phi$. Then $\xi_1, \ldots, \xi_d$ are called *Nambu-Goldstone bosons* and $\eta_1, \ldots, \eta_{2n-d}$ are called *Higgs bosons*. As noted above, $d = \dim G/H$, where H is the unbroken subgroup, and n is the complex dimension of the Higgs vector space W.

We thus see that the Nambu-Goldstone bosons describe the perturbation of the Higgs condensate along the orbit of the vacuum vector under the action of G, while the Higgs bosons represent the perturbation perpendicular to the orbit submanifold.

For arbitrary $x \in M$, we may take $\phi(x) = w_0 + \Delta\phi(x)$ as the argument in a Taylor expansion of $\tilde{V}$ at w_0. Recalling that $\operatorname{grad}\tilde{V}(w_0) = 0$, the second-order Taylor polynomial then reads

$$\tilde{V}(w_0) + \frac{1}{2}\langle\langle \operatorname{Hess}(\tilde{V})_{w_0}\Delta\phi(x), \Delta\phi(x)\rangle\rangle_W. \tag{4.5}$$

Inserting the eigenvector expansion (4.4) and using the notation of Proposition 4.1.8 for the eigenvalues of $\operatorname{Hess}(\tilde{V})_{w_0}$, we obtain

$$\tilde{V}(w_0) + \frac{1}{2} \sum_{l=1}^{2n-d} m_l^2 \eta_l(x)^2. \tag{4.6}$$

We learn that, disregarding the constant $\tilde{V}(w_0)$ and up to second-order, the Higgs potential looks like a sum of standard Klein-Gordon Lagrangian terms for the scalar fields η_l, the Higgs bosons, with mass m_l. The Nambu-Goldstone bosons are interpreted to have zero mass.

4.1.10. Example. We take up Example 4.1.7 about the electroweak theory with the 4-dimensional gauge group $G = \mathrm{SU}(2) \times \mathrm{U}(1)$ with Higgs vector space $W = \mathbb{C}^2$ (of 4 real dimensions), the Higgs potential $\tilde{V}(w) = -\mu \|w\|^2 + \lambda \|w\|^4$, vacuum vector $w_0 = \begin{pmatrix} 0 \\ \sqrt{\frac{\mu}{2\lambda}} \end{pmatrix}$, and the 1-dimensional unbroken subgroup H given in (4.2). In the notation of Definition 4.1.9 we have here $n = 2$ and $d = \dim G/H = 3$ and expect to obtain 3 Nambu-Goldstone bosons and one Higgs boson.

For a start we note that

$$b_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, b_2 := \begin{pmatrix} i \\ 0 \end{pmatrix}, b_3 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, b_4 := \begin{pmatrix} 0 \\ i \end{pmatrix}$$

is an orthonormal basis with respect to $\langle\langle . , . \rangle\rangle_W$, because a quick calculation shows

$$\begin{aligned}\langle\langle \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right) \rangle\rangle_W &= \operatorname{Re}\left(\overline{v_1} w_1 + \overline{v_2} w_2\right) = \sum_{k=1,2} \operatorname{Re}\left((\operatorname{Re} v_k - i \operatorname{Im} v_k)(\operatorname{Re} w_k + i \operatorname{Im} w_k)\right) \\ &= \sum_{k=1,2} \operatorname{Re}\left(\operatorname{Re} v_k \operatorname{Re} w_k + \operatorname{Im} v_k \operatorname{Im} w_k + i(\ldots)\right) = \sum_{k=1,2} (\operatorname{Re} v_k \operatorname{Re} w_k + \operatorname{Im} v_k \operatorname{Im} w_k) \\ &= \operatorname{Re} v_1 \operatorname{Re} w_1 + \operatorname{Im} v_1 \operatorname{Im} w_1 + \operatorname{Re} v_2 \operatorname{Re} w_2 + \operatorname{Im} v_2 \operatorname{Im} w_2\end{aligned}$$

and thus $\langle\langle b_j, b_k \rangle\rangle_W = \delta_{jk}$.

We already observed that the action of the subgroup $\mathrm{SU}(2) \times \{1\}$ on w_0 generates the whole sphere of radius $\sqrt{\mu/(2\lambda)}$ around 0, which gave the vacuum manifold W_0 in this case. Since $\mathrm{SU}(2) \times \mathrm{U}(1)$ is generated from the subgroups $\mathrm{SU}(2) \times \{1\}$ and $\{I_2\} \times \mathrm{U}(1)$ and the action of $\{I_2\} \times \mathrm{U}(1)$ is only by phase factor multiplication, we obtain for the orbit[1]

$$O_{w_0} = W_0 = \left\{ w \in \mathbb{C}^2 \mid \|w\| = \sqrt{\frac{\mu}{2\lambda}} \right\}$$

and therefore,

$$T_{w_0} O_{w_0} = \{w_0\}^\perp = \operatorname{span}\{b_1, b_2, b_4\} \quad \text{and} \quad (T_{w_0} O_{w_0})^\perp = \operatorname{span}\{w_0\} = \operatorname{span}\{b_3\}.$$

Let us calculate the Hessian of $\tilde{V}(w) = -\mu \langle\langle w, w \rangle\rangle_W + \lambda \langle\langle w, w \rangle\rangle_W^2$ at w_0: The bilinearity and symmetry of $\langle\langle . , . \rangle\rangle_W$ immediately give

$$\operatorname{grad} \tilde{V}(w) = -2\mu w + 4\lambda \langle\langle w, w \rangle\rangle_W w$$

and then (recalling that $\nabla_v w$ means the differential of the map $w \mapsto w$ at w applied to v)

$$\begin{aligned}\operatorname{Hess}(\tilde{V})_w v &= \nabla_v \operatorname{grad} \tilde{V}(w) = -2\mu \nabla_v w + 4\lambda \langle\langle w, w \rangle\rangle_W \nabla_v w + 4\lambda \left(\nabla_v \langle\langle w, w \rangle\rangle_W\right) w \\ &= (4\lambda \langle\langle w, w \rangle\rangle_W - 2\mu)\, v + 8\lambda \langle\langle w, v \rangle\rangle_W w = \left(4\lambda \|w\|^2 - 2\mu\right) v + 8\lambda \|w\|^2 \langle\langle \frac{w}{\|w\|}, v \rangle\rangle_W \frac{w}{\|w\|} \\ &= \left(4\lambda \|w\|^2 - 2\mu\right) \operatorname{id}(v) + 8\lambda \|w\|^2 P_w(v),\end{aligned}$$

where P_w denotes the orthogonal projection onto $\operatorname{span}\{w\}$. At $w = w_0$ we thus obtain

$$\operatorname{Hess}(\tilde{V})_{w_0} = \left(4\lambda \frac{\mu}{2\lambda} - 2\mu\right) \operatorname{id} + 8\lambda \frac{\mu}{2\lambda} P_{w_0} = 4\mu P_{w_0} = 4\mu P_{b_3}$$

and immediately read off an eigenvector basis in the sense of Proposition 4.1.8 to be given by $e_1 := b_1$, $e_2 := b_2$, $e_3 := b_4$, and $f_1 := b_3$. The matrix of $\operatorname{Hess}(\tilde{V})_{w_0}$ with respect to this basis is $\operatorname{diag}(0, 0, 0, 4\mu)$, so that in the notation of Proposition 4.1.8 we have $2m_1^2 = 4\mu$ for the mass m_1 of the Higgs boson, i.e., $m_1 = \sqrt{2\mu}$.

Finally in this example, let us determine the Higgs field $\phi = w_0 + \Delta\phi$ in terms of the expansion for the shifted Higgs field $\Delta\phi$ according to (4.4) with the three Nambu-Goldstone bosons ξ_1, ξ_2, ξ_3 and the Higgs boson η_1. By definition,

$$\Delta\phi = \frac{1}{\sqrt{2}} \sum_{j=1}^{3} \xi_j e_j + \frac{1}{\sqrt{2}} \eta_1 f_1 = \frac{1}{\sqrt{2}} \left(\xi_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \xi_2 \begin{pmatrix} i \\ 0 \end{pmatrix} + \xi_3 \begin{pmatrix} 0 \\ i \end{pmatrix} + \eta_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_1 + i\xi_2 \\ i\xi_3 + \eta_1 \end{pmatrix}$$

and therefore,

$$\phi = \begin{pmatrix} 0 \\ \sqrt{\frac{\mu}{2\lambda}} \end{pmatrix} + \Delta\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_1 + i\xi_2 \\ i\xi_3 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \sqrt{\frac{\mu}{\lambda}} + \eta_1 \end{pmatrix}, \tag{4.7}$$

where we have collected the Higgs condensate and Higgs boson in the last vector and separated the Nambu-Goldstone bosons in the first vector on the right-hand side.

[1]We knew already from the beginning that $\dim O_{w_0} = \dim G/H = 3$.

Due to the triviality of the principal bundle P and the choice of the global gauge (section) s_0 of P, any gauge transformation on P is simply given by a smooth map $\tau\colon M \to G$, a so-called *physical gauge transformation*. Indeed, recall from the discussion preceding Lemma 3.4.8 that any $f \in \mathcal{G}(P)$ is of the form $f(p) = p\sigma_f(p)$ for some $\sigma_f \in C^\infty(P,G)^G$. Then setting $\tau := \sigma_f \circ s_0$, by Lemma 3.4.8 we obtain that the action of f on a section $\Phi = [s_0, \phi]$ of E with $\phi\colon M \to W$ is given by $f \cdot [s_0(x), \phi(x)] = [s_0(x), \rho(\sigma_f(s_0(x)))\phi(x)]$, i.e., is of the form $\Phi' = [s_0, \phi']$, where

$$\phi'(x) := \rho(\tau(x))\phi(x) \quad (x \in M). \tag{4.8}$$

One easily checks that, conversely, to any $\tau \in C^\infty(M,G)$ there corresponds a unique $\sigma_\tau \in C^\infty(P,G)^G$, namely $\sigma_\tau(s_0(x)\cdot g) = g^{-1}\tau(x)g$.

4.1.11. Definition. A physical gauge transformation $\tau\colon M \to G$ is called a *unitary gauge* for a Higgs field ϕ with respect to the vacuum vector w_0, if all Nambu-Goldstone bosons for the transformed field ϕ' according to (4.8) vanish on all of M. The transformed field ϕ' is then said to be *in unitary gauge*.

Reviewing the constructions leading to the expansion (4.4), we see that a Higgs field $\phi\colon M \to W$ is in unitary gauge, if and only if the shifted Higgs field $\Delta\phi := \phi - w_0$ satisfies

$$\forall x \in M\colon \quad \Delta\phi(x) \in (T_{w_0}O_{w_0})^\perp. \tag{4.9}$$

We see from Equation (4.7) obtained in Example 4.1.10 for the theory of the electroweak interaction that the Higgs field given there will in general not be in unitary gauge. The following is a result about existence of a unitary gauge for this theory.

4.1.12. Theorem. *Let $\phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ be the Higgs field given by (4.7) in the theory of electroweak interaction as described in Examples 4.1.7 and 4.1.10. If ϕ_2 is nowhere vanishing on M, then there exists a unitary gauge $\tau\colon M \to \mathrm{SU}(2)\times\mathrm{U}(1)$ for ϕ with respect to the vacuum vector $w_0 = \begin{pmatrix}0\\ \sqrt{\mu/(2\lambda)}\end{pmatrix}$ such that $\phi' = \begin{pmatrix}0\\ \psi\end{pmatrix}$ with a real-valued function ψ on M.*

proof. *Step 1:* We make the ansatz

$$\tau(x) = (\begin{pmatrix} a(x) & -\overline{b(x)} \\ b(x) & \overline{a(x)}\end{pmatrix}, e^{i\alpha(x)}) \quad (x \in M)$$

with functions $\alpha\colon M \to \mathbb{R}$ and $a, b\colon M \to \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$ and note that then we have

$$\phi'(x) = \rho(\tau(x))\phi(x) = \frac{e^{i\alpha(x)}}{\sqrt{2}}\begin{pmatrix} a(x)\phi_1(x) - \overline{b(x)}\phi_2(x) \\ b(x)\phi_1(x) + \overline{a(x)}\phi_2(x)\end{pmatrix}, \tag{$*$}$$

where $\phi_1(x) = \xi_1(x) + i\xi_2(x)$ and $\phi_2(x) = i\xi_3(x) + \sqrt{\frac{\mu}{\lambda}} + \eta_1(x)$. Since $\phi_2(x) \neq 0$ for all $x \in M$ by assumption, we may put $b := \overline{a}(\xi_1 - i\xi_2)/\overline{\phi_2}$ and thus achieve that the first row of $\rho(\tau(x))\phi(x)$ vanishes. An alternative formulation of this intermediate result would be that without loss of generality, we may already suppose that $\phi = \begin{pmatrix}0\\ \phi_2\end{pmatrix}$.

Step 2 - elementary variant: With the choice of b made in step 1, the second row of the right-hand side in $(*)$ reads

$$\frac{e^{i\alpha}}{\sqrt{2}}(\overline{a}(\xi_1 - i\xi_2)\frac{\phi_1}{\overline{\phi_2}} + \overline{a}\phi_2) = \frac{e^{i\alpha}}{\sqrt{2}}(\xi_1^2 + \xi_2^2 + |\phi_2|^2)\frac{\overline{a}}{\overline{\phi_2}},$$

which we want to become a real-valued function upon choosing α and a, in addition to taking care of the condition $|a|^2+|b|^2 = 1$. Let us put $\alpha := 0$ and $\zeta := \sqrt{\xi_1^2 + \xi_2^2 + |\phi_2|^2}$. By assumption on ϕ_2, we have $\zeta(x) > 0$ for all $x \in M$ and we may thus put $a := \phi_2/\zeta$, which obviously makes the above expression a real-valued function ψ and also gives $|a|^2 + |b|^2 = |a|^2(1 + \frac{|\xi_1 - i\xi_2|^2}{|\phi_2|^2}) = \frac{|a|^2}{|\phi_2|^2}\zeta^2 = 1$.

Step 2 - advanced variant: As mentioned at the end of step 1, we may suppose that $\phi = \begin{pmatrix}0\\ \phi_2\end{pmatrix}$. Since $\phi_2(x) \neq 0$ for all $x \in M$, $\psi(x) := |\phi_2(x)|$ defines a smooth map $M \to]0,\infty[$ and also

$h\colon M \to S^1$, $x \mapsto \phi_2(x)/|\phi_2(x)|$ is smooth. Since M is connected and simply connected (due to one of the many assumptions for the current section), the (smooth) monodromy principle ([**8**, 16.28.8]) guarantees that there is a smooth lift $\tilde{h}\colon M \to \mathbb{R}$ such that $e^{i\tilde{h}(x)} = h(x)$ for all $x \in M$. Therefore, we have $\phi_2 = \psi e^{i\tilde{h}}$ and we may choose the physical gauge $\tau_0\colon M \to \mathrm{SU}(2) \times \mathrm{U}(1)$ in the form $\tau_0(x) := (I_2, e^{-i\tilde{h}(x)})$ to achieve

$$\rho(\tau_0(x))\phi = \begin{pmatrix} 0 \\ \psi(x) \end{pmatrix}.$$

The composition of the gauge transformations in steps 1 and 2 shows that we obtain $\phi' = \left(\begin{smallmatrix} 0 \\ \psi \end{smallmatrix}\right)$ with ψ real-valued.

Step 3: We recall that $(T_{w_0}O_{w_0})^\perp = \mathrm{span}\{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\}$ has been noted in Example 4.1.10 and we have achieved $\Delta\phi'(x) = \left(\begin{smallmatrix} 0 \\ \psi(x) - \sqrt{\mu/(2\lambda)} \end{smallmatrix}\right) \in \mathrm{span}\{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\}$ for all $x \in M$. Therefore, condition (4.9) is satisfied and confirms that ϕ' is in unitary gauge. □

4.1.13. Remark. We note that in the proof of the above theorem, the assumption that ϕ_2 vanishes at no point was used in step 1 as well as in both variants of step 2. Clearly, the explicit formulae using ϕ_2 in the denominator would otherwise not be defined. But also the monodromy principle could not be applied in the advanced variant, because smoothness of ψ and h will fail. (Indeed, consider the example with $M = \mathbb{R}^2$, $\phi_2(x_1, x_2) = x_1 + ix_2$, and suppose we had continuous functions $r, \theta\colon \mathbb{R}^2 \to \mathbb{R}$ such that $\phi_2(x) = r(x)e^{i\theta(x)}$ holds for all $x \in \mathbb{R}^2$. Certainly, $|r(x)| = |\phi_2(x)| \neq 0$ for $x \neq 0$. Let $t > 0$, then $t = \phi_2(t, 0) = r(t, 0)e^{i\theta(t,0)}$ implies $\theta(t, 0) \in \pi\mathbb{Z}$ and $it = \phi_2(0, t) = r(0, t)e^{i\theta(0,t)}$ implies $\theta(0, t) \in \frac{\pi}{2}\mathbb{Z}_{\mathrm{odd}}$. Sending $t \to 0$, continuity of θ at $(0, 0)$ would give $\theta(0) \in (\pi\mathbb{Z}) \cap (\frac{\pi}{2}\mathbb{Z}_{\mathrm{odd}}) = \emptyset$.)

In the remainder of this section, we will describe some of the mathematical ingredients leading in physics to the so-called *mass generation for gauge bosons*. Suppose, in addition to the overall assumptions stated earlier for the current section, that

$$\langle .\,, . \rangle_{\mathfrak{g}} \text{ is an Ad-invariant positive definite scalar product on the Lie algebra } \mathfrak{g}$$

and let $\mathfrak{h} \subseteq \mathfrak{g}$ denote the Lie subalgebra corresponding to the unbroken subgroup $H \subseteq G$ with orthogonal complement $\mathfrak{h}^\perp$ with respect to $\langle .\,, . \rangle_{\mathfrak{g}}$. We have $\dim \mathfrak{g} = r$ and, by $\mathfrak{h}^\perp \cong \mathfrak{g}/\mathfrak{h}$ (as vector spaces), $\dim \mathfrak{h}^\perp = d$.

Recall that the representation $\rho\colon G \to \mathrm{GL}(W)$ induces (by derivation) the Lie algebra representation $\rho_*\colon \mathfrak{g} \to \mathrm{L}(W)$. We combine this with the real scalar product $\langle\langle .\,, . \rangle\rangle_W$ on W to define the following bilinear form on $\mathfrak{g}$.

4.1.14. Definition. The *mass form* (with respect to the vacuum vector $w_0 \in W$) is the positive semi-definite symmetric bilinear form $m\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, defined by

$$m(A, B) := \langle\langle \rho_*(A)w_0, \rho_*(B)w_0 \rangle\rangle_W \quad (A, B \in \mathfrak{g}).$$

Note that the map $A \mapsto \rho_*(A)w_0$, $\mathfrak{g} \to W$ is just the derivative $T_e f_0$ of the orbit map $f_0\colon G \to W$, $g \mapsto \rho(g)w_0$ and recall that H was defined as the isotropy group G_{w_0}. We obviously have $\mathfrak{h} \subseteq \ker T_e f_0$, since $f_0(\exp(tX)) = \rho(\exp(tX))w_0 = w_0$ for all $t \in \mathbb{R}$ and $X \in \mathfrak{h}$. Conversely, if $X \in \ker T_e f_0$ then

$$\frac{d}{dt}\Big|_0 (\rho(\exp(tX))w_0) = 0.$$

Consequently,

$$\begin{aligned} \frac{d}{dt}\Big|_{t=s} (\rho(\exp(tX))w_0) &= \frac{d}{dt}\Big|_0 (\rho(\exp(sX))\rho(\exp(tX))w_0) \\ &= TL_{\rho(\exp(sX))}\Big(\frac{d}{dt}\Big|_0 (\rho(\exp(tX))w_0)\Big) = 0 \end{aligned}$$

for all $s \in \mathbb{R}$. It follows that $\rho(\exp(tX))w_0 = w_0$ for all $t \in \mathbb{R}$, i.e., $\exp(tX) \in H$ for each $t \in \mathbb{R}$. Therefore, $X \in \mathfrak{h}$ (see [**24**, Section 21]).

Altogether, we have $\ker T_e f_0 = \mathfrak{h}$, which implies

$$\forall A \in \mathfrak{h}\colon \rho_*(A)w_0 = 0 \quad \text{and} \quad \mathfrak{h}^\perp \to W,\, A \mapsto \rho_*(A)w_0 \text{ is injective.}$$

The mass form m is thus positive definite on $\mathfrak{h}^\perp \subseteq \mathfrak{g}$, while $m(A,B) = 0$, if $A \in \mathfrak{h}$ or $B \in \mathfrak{h}$. We may therefore directly deduce the following diagonalization result.

4.1.15. Proposition. *There is a basis $\alpha_1, \dots \alpha_r$ of $\mathfrak{g}$, orthonormal with respect to $\langle .\,,.\rangle_{\mathfrak{g}}$ in $\mathfrak{g}$, and a sequence of real numbers $M_1, \dots, M_r$, where $M_a > 0$ for $1 \le a \le d$ and $M_a = 0$ for $d+1 \le a \le r$, such that the following hold:*
(i) $\alpha_1, \dots, \alpha_d$ is a basis of $\mathfrak{h}^\perp$, these are the so-called broken generators*,*
(ii) $\alpha_{d+1}, \dots, \alpha_r$ is a basis of $\mathfrak{h}$, called the unbroken generators*,*
(iii) $m(\alpha_a, \alpha_b) = 0$, if $a \neq b$, and $m(\alpha_a, \alpha_a) = M_a^2/2$ $(a = 1, \dots, r)$.
The numbers M_a $(a \in \{1, \dots, r\})$ are called the masses of the gauge bosons*.*

Thanks to the global section s_0 of P, we may for any gauge field (connection 1-form) $A \in \Omega^1(P, \mathfrak{g})$ consider

$$A^{s_0} := A \circ Ts_0 \in \Omega^1(M, \mathfrak{g})$$

(as in (4.1) for the vacuum configuration). In a local chart on M with tangent basis vectors ∂_μ $(\mu = 1, \dots, \dim M)$ let us define

$$A_\mu := A^{s_0}(\partial_\mu),$$

which is a locally defined smooth map with values in $\mathfrak{g}$, hence can be expanded in terms of the basis $\alpha_1, \dots, \alpha_r$ of $\mathfrak{g}$ with unique coefficient functions A^a_μ in the form

$$A_\mu = \sum_{a=1}^{r} A^a_\mu \alpha_a.$$

(In case of an expansion for A^μ we will write the coefficients as A^μ_a.) Recall from (3.15), (3.11) that in these local coordinates, we can write the covariant derivative and the curvature as

$$\nabla^A_\mu = \partial_\mu + A_\mu \quad \text{and} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],$$

where in ∇^A_μ we have now simplified the notation replacing $\rho_*(A_\mu)$ by A_μ. In physics, the curvature is called *field strength* and we may expand its local expression also according to the Lie algebra basis $\alpha_1, \dots, \alpha_r$ with unique coefficient functions $F^a_{\mu\nu}$ in the form

$$F_{\mu\nu} = \sum_{a=1}^{r} F^a_{\mu\nu} \alpha_a.$$

Now we have everything in place to write out the Yang–Mills–Higgs Lagrangian from (3.22) in local coordinates for the Higgs field and the gauge field A_μ $(\mu = 1, \dots, \dim M)$. We have (cf. (3.10) and the discussion following Definition 3.4.5)

$$\mathscr{L}_{YMH,\mathrm{loc}} := \sum_\mu (\nabla^{A\mu}\phi)^\dagger (\nabla^A_\mu \phi) - \tilde{V} \circ \phi - \frac{1}{4} \sum_{\mu,\nu} \sum_{a=1}^{r} F^{\mu\nu}_a F^a_{\mu\nu}.$$

Introducing again the shifted Higgs field by $\phi = w_0 + \Delta\phi$ and observing

$$\nabla^A_\mu \phi = \partial_\mu \Delta\phi + A_\mu \Delta\phi + A_\mu w_0 \tag{4.10}$$

(and similarly for $\nabla^{A\mu}$), one then tries to expand $\mathscr{L}_{YMH,\mathrm{loc}}$ in terms of powers of the combined field vector $(\Delta\phi, A_1, \dots, A_{\dim M})$. Everything up to second order corresponds to "free" fields, because it contributes only linear terms to the Euler-Lagrange field equations, while the higher-order terms are then considered to represent the interactions between the fields. We will sketch here some basic results for the free field approximation, i.e., up to second order.

Recall that we have already determined the second-order Taylor polynomials for the Higgs potential in (4.5) and (4.6) in terms of the Hessian of the map $w \mapsto \tilde{V}(w) := V(\|w\|^2)$. We will therefore focus now on the other terms in $\mathscr{L}_{YMH,\mathrm{loc}}$ and determine their contributions up to second order.

We see that the middle term on the right-hand side of (4.10) is already of order 2 and gets multiplied in $\mathscr{L}_{YMH,\text{loc}}$ by the terms coming from $(\nabla^{A_\mu}\phi)^\dagger$, which itself is quite similar to (4.10). Thus, only the first-order terms of each factor in the first sum in $\mathscr{L}_{YMH,\text{loc}}$ have to be taken into account and the remaining terms give (introducing the ad-hoc notation $\approx_2$)

$$(\nabla^{A\mu}\phi)^\dagger(\nabla^A_\mu\phi) \approx_2 (\partial^\mu\Delta\phi)^\dagger(\partial_\mu\Delta\phi) + 2\operatorname{Re}(\partial^\mu\Delta\phi)^\dagger(A_\mu w_0) + (A^\mu w_0)^\dagger(A_\mu w_0). \tag{4.11}$$

Reasoning in a similar way for the Lagrangian field strength terms, every product involving contributions from the Lie bracket term in $F_{\mu\nu}$ or $F^{\mu\nu}$ produces order 3 or 4 and may be neglected, therefore

$$F_a^{\mu\nu}F^a_{\mu\nu} \approx_2 (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a). \tag{4.12}$$

Collecting all the contributions up to second order as described so far, we would obtain an explicit expression which we lazily call here $\mathscr{L}_{YMH,\text{loc},2}$ and note that $\mathscr{L}_{YMH,\text{loc}} \approx_2 \mathscr{L}_{YMH,\text{loc},2}$.

In the next step, we additionally suppose that

the Higgs field ϕ is in unitary gauge with respect to the vacuum vector w_0.

Note that according to Theorem 4.1.12, at least in the electroweak theory we know about the existence of a unitary gauge. Then all Nambu-Goldstone bosons disappear and the expansion (4.4) collapses to

$$\Delta\phi = \frac{1}{\sqrt{2}}\sum_{j=1}^{2n-d}\eta_j f_j,$$

which gives the following concrete form for the first term on the right-hand side of (4.11) (recall that the η_j are real):

$$(\partial^\mu\Delta\phi)^\dagger(\partial_\mu\Delta\phi) = \frac{1}{2}\sum_{j=1}^{2n-d}(\partial^\mu\eta_j)(\partial_\mu\eta_j).$$

As for the second term on the right-hand side of (4.11), the unitary gauge implies that $\Delta\phi$ has values in $(T_{w_0}O_{w_0})^\perp$, as noted in (4.9), and this remains true for $\partial^\mu\Delta\phi$. Since $A_\mu w_0 = \rho_*(A_\mu)w_0$ is essentially the derivative of the orbit map (called f_0 above), it has values in $T_{w_0}O_{w_0}$ and (recalling $\langle\langle .,.\rangle\rangle_W = \operatorname{Re}\langle .,.\rangle_W$) we thus have

$$\operatorname{Re}(\partial^\mu\Delta\phi)^\dagger(A_\mu w_0) = 0.$$

The third term on the right-hand side of (4.11) is (real and thus) equal to $\langle\langle\rho_*(A^\mu)w_0, \rho_*(A_\mu)w_0\rangle\rangle_W = m(A^\mu, A_\mu)$. Expanding A^μ and A_μ in the basis $\alpha_1,\dots,\alpha_r$ then yields

$$(A^\mu w_0)^\dagger(A_\mu w_0) = \frac{1}{2}\sum_{a=1}^{d}M_a^2 A^\mu_a A^a_\mu.$$

The second-order Higgs potential approximation is unaffected by the unitary gauge and was already only dependend on the Higgs bosons in formula (4.6). We collect all the simplified terms for $\mathscr{L}_{YMH,\text{loc},2}$ obtained from (4.11), the Higgs potential, and (4.12) and split the final sum over a in the original $\mathscr{L}_{YMH,\text{loc}}$ according to broken and unbroken generators into two sums, one with $1 \le a \le d$ and the other with $d+1 \le a \le r$. The result is

$$\begin{aligned}\mathscr{L}_{YMH,\text{loc}} \approx_2 \; & \frac{1}{2}\sum_\mu\sum_{j=1}^{2n-d}(\partial^\mu\eta_j)(\partial_\mu\eta_j) - V(\|w_0\|^2) - \frac{1}{2}\sum_{l=1}^{2n-d}m_j^2\eta_j^2 \\ & -\frac{1}{4}\sum_{\mu,\nu}\sum_{a=1}^{d}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a) + \frac{1}{2}\sum_\mu\sum_{a=1}^{d}M_a^2 A^\mu_a A^a_\mu \\ & -\frac{1}{4}\sum_{\mu,\nu}\sum_{a=d+1}^{r}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a).\end{aligned}$$

We remark that the constant $V(\|w_0\|^2)$ is irrelevant for the Lagrangian, when (mostly formally) searching for minimizing configurations of the associated action functional. The other terms in the first line have the form of a Klein-Gordon Lagrangian for the $2n-d$ real Higgs bosons, where η_j has mass m_j. The second line is interpreted as the Lagrangian for the d broken gauge bosons $(A^1_\mu)_{\mu=1\ldots,\dim M}, \ldots, (A^d_\mu)_{\mu=1\ldots,\dim M}$, where $(A^a_\mu)_\mu$ has mass $M_a > 0$. The third line is the Lagrangian for $r-d$ unbroken massless gauge bosons $(A^{d+1}_\mu)_{\mu=1\ldots,\dim M}, \ldots, (A^r_\mu)_{\mu=1\ldots,\dim M}$.

To summarize, we have r gauge bosons according to the dimension of the gauge group G and they are encoded geometrically in the $\mathfrak{g}$-valued one-form $A^{s_0} = A \circ Ts_0$ on M, which in turn is obtained from a connection 1-form A on the principal G-bunde P. The d gauge bosons with values in the linear span $\mathfrak{h}^\perp \cong \mathfrak{g}/\mathfrak{h}$ of the broken generators "acquired masses" $M_a > 0$ $(a = 1, \ldots, d)$ in the Lagrangian, because the mass form $m(B,C) = \langle\langle \rho_*(B)w_0, \rho_*(C)w_0 \rangle\rangle_W$ is positive definite there. Observe that it was crucial for this to have a non-zero vacuum vector w_0 and that the mass form term $(A^\mu w_0)^\dagger (A_\mu w_0)$ emerged from the multiplication of covariant derivaties $\nabla^A_\mu \phi$ according to (4.10), which has an explicit coupling between the Higgs field ϕ and the gauge field A. Finally, the $2n-d$ Higgs bosons η_j correspond to the shifted Higgs field directions perpendicular to the orbit $O_{w_0} \cong G/H$ (diffeomorphic) in the representation space W with $n = \dim_{\mathbb{C}} W$. The directions of $\Delta\phi$ along the orbit have been "gauged away" in a unitary gauge.

4.2. Application to the gauge bosons of electroweak interaction theory

We take up Examples 4.1.7 and 4.1.10, and will illustrate here a few main aspects of the Higgs mechanism applied to the gauge fields in the theory of electroweak interaction—recall Theorem 4.1.12 about the existence of a unitary gauge. The gauge group is $\mathrm{SU}(2) \times \mathrm{U}(1)$ and its action on $W = \mathbb{C}^2$ is given by

$$\rho(A, e^{i\alpha})w = e^{i\alpha n_Y} Aw, \quad \text{where } (A, e^{i\alpha}) \in \mathrm{SU}(2) \times U(1), w \in \mathbb{C}^2.$$

The Lie algebra is $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ and recall that we have $\mathfrak{u}(1) = \{z \in \mathbb{C} \mid \overline{z} = -z\} = i\mathbb{R}$ and $\mathfrak{su}(2) = \{C \in \mathrm{L}(\mathbb{C}^2) \mid C^\dagger := \overline{C}^T = -C,\ \mathrm{trace}(C) = 0\}$, where a basis for the latter is given in terms of the Pauli matrices by $i\sigma_1 = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$, $i\sigma_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $i\sigma_3 = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$. We may thus specify a basis of $\mathfrak{g}$ in the form

$$\beta_l := g_W \frac{i}{2}\sigma_l \in \mathfrak{su}(2) \quad (l = 1,2,3), \quad \beta_4 := g' \frac{i}{2n_Y} \in \mathfrak{u}(1) \tag{4.13}$$

with the *coupling constants* $g_W, g' > 0$.

Recall that for any matrix Lie group G we have the following explicit formulae for the actions in the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ of the Lie group and for the Lie algebra representation $\mathrm{ad} = \mathrm{Ad}_*\colon \mathfrak{g} \to \mathrm{L}(\mathfrak{g})$:

$$\forall X, Y \in \mathfrak{g}\, \forall Q \in G\colon \quad \mathrm{Ad}_Q(Y) = QYQ^{-1}, \quad \mathrm{ad}_X(Y) = [X,Y] = XY - YX.$$

In constructing an Ad-invariant scalar product on the Lie algebra of the gauge group for the electroweak theory, we will make use of the *Killing form*, which is defined for any Lie algebra $\mathfrak{g}$ as the bilinear map $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by

$$B(X,Y) := \mathrm{trace}(\mathrm{ad}_X \circ \mathrm{ad}_Y) \quad (X, Y \in \mathfrak{g})$$

Although the following statement is contained in more general results (cf. [**19**, Theorems 2.3.1, 2.3.3, and Corollary 2.4.20]), we do give here an elementary proof as well.

4.2.1. Lemma. *The Killing form B on $\mathfrak{su}(2)$ is symmetric, negative-definite,* Ad*-invariant, and is given explicitly by*

$$B(X,Y) = 4\,\mathrm{trace}(XY) \quad (X, Y \in \mathfrak{su}(2)).$$

proof. Let us first establish the explicit formula for B. We recall from (1.15) the commutation relations $[\sigma_1, \sigma_2] = 2i\sigma_3$, $[\sigma_2, \sigma_3] = 2i\sigma_1$, and $[\sigma_3, \sigma_1] = 2i\sigma_2$. We use the basis $i\sigma_l$ $(l = 1,2,3)$ of

$\mathfrak{su}(2)$ in expanding $X, Y \in \mathfrak{su}(2)$ as $X = \sum_{l=1}^{3} x_l i\sigma_l$ and $Y = \sum_{l=1}^{3} y_l i\sigma_l$ and then also to obtain matrix representations for the operators $\mathrm{ad}_X, \mathrm{ad}_Y \in \mathrm{L}(\mathfrak{su}(2))$. We calculate

$$\begin{aligned}
\mathrm{ad}_Y(i\sigma_1) &= [Y, i\sigma_1] = -y_2[\sigma_2,\sigma_1] - y_3[\sigma_3,\sigma_1] = 2y_2 i\sigma_3 - 2y_3 i\sigma_2,\\
\mathrm{ad}_Y(i\sigma_2) &= [Y, i\sigma_2] = -y_1[\sigma_1,\sigma_2] - y_3[\sigma_3,\sigma_2] = -2y_1 i\sigma_3 + 2y_3 i\sigma_1,\\
\mathrm{ad}_Y(i\sigma_3) &= [Y, i\sigma_2] = -y_1[\sigma_1,\sigma_3] - y_2[\sigma_2,\sigma_3] = 2y_1 i\sigma_2 - 2y_2 i\sigma_1,
\end{aligned}$$

which implies that the (3×3)-matrix $[\mathrm{ad}_Y]$ of ad_Y with respect to the chosen basis in $\mathfrak{su}(2)$ is

$$[\mathrm{ad}_Y] = 2\begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix}$$

and an analogous expression for the matrix $[\mathrm{ad}_X]$. Therefore, the matrix of $\mathrm{ad}_X \circ \mathrm{ad}_Y$ is given by the matrix product

$$[\mathrm{ad}_X]\cdot[\mathrm{ad}_Y] = 4\begin{pmatrix} -x_3y_3 - x_2y_2 & * & * \\ * & -x_3y_3 - x_1y_1 & * \\ * & * & -x_2y_2 - x_1y_1 \end{pmatrix},$$

where we do not care here for the off-diagonal entries. Taking the trace now gives

$$B(X,Y) = -8(x_1y_1 + x_2y_2 + x_3y_3).$$

On the other hand, we have from the basis expansion as a (2×2)-matrix

$$X = \sum_{l=1}^{3} x_l i\sigma_l = \begin{pmatrix} ix_3 & x_2 + ix_1 \\ -x_2 + ix_1 & -ix_3 \end{pmatrix},$$

similarly for Y, and thus obtain

$$X\cdot Y = \begin{pmatrix} -x_3y_3 - x_2y_2 - x_1y_1 + i(y_1x_2 - x_1y_2) & * \\ * & -x_3y_3 - x_2y_2 - x_1y_1 - i(y_1x_2 - x_1y_2) \end{pmatrix},$$

which yields

$$\mathrm{trace}(X\cdot Y) = -2(x_1y_1 + x_2y_2 + x_3y_3) = \frac{1}{4}B(X,Y).$$

Symmetry of B follows directly from the general property $\mathrm{trace}(XY) = \mathrm{trace}(YX)$ of the trace. Recalling that $X^\dagger = -X$ for $X \in \mathfrak{su}(2)$ and with components $X = \left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{smallmatrix}\right)$, we have

$$\begin{aligned}
B(X,X) = 4\,\mathrm{trace}(XX) &= -4\,\mathrm{trace}(X^\dagger X) = -4\,\mathrm{trace}(\begin{pmatrix} \overline{x_{11}} & \overline{x_{21}} \\ \overline{x_{12}} & \overline{x_{22}} \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix})\\
&= -4\,\mathrm{trace}\begin{pmatrix} |x_{11}|^2 + |x_{21}|^2 & * \\ * & |x_{12}|^2 + |x_{22}|^2 \end{pmatrix} = -4\sum_{j,k=1}^{2} |x_{jk}|^2,
\end{aligned}$$

which shows that B is negative-definite.

Finally, Ad-invariance is particularly easy due to the explicit action $\mathrm{Ad}_Q X = QXQ^{-1}$ for every $Q \in \mathrm{SU}(2)$, since

$$B(\mathrm{Ad}_Q X, \mathrm{Ad}_Q Y) = 4\,\mathrm{trace}((QXQ^{-1})(QYQ^{-1})) = 4\,\mathrm{trace}(XY) = B(X,Y).$$

□

4.2.2. Remark. By the above lemma, $-B$ provides us with a Euclidean scalar product on $\mathfrak{su}(2)$. It can be shown that this is, in fact, the unique Ad-invariant Euclidean scalar product on $\mathfrak{su}(2)$ apart from an additional positive factor ([**19**, Theorem 2.5.3]). The latter corresponds to the choice of the coupling constant g_W appearing in the basis β_l $(l = 1,2,3)$ introduced above.

Since the gauge group $\mathrm{SU}(2) \times \mathrm{U}(1)$ is a direct product and $\mathrm{U}(1)$ is commutative, it is now easy to define an Ad-invariant scalar product on the Lie algebra $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ in such a way that this becomes also an orthogonal direct sum decomposition. For any constants $c_1, c_2 > 0$ we may put $\langle (X, ix), (Y, iy) \rangle_{\mathfrak{g}} := -c_1 B(X,Y) + c_2 xy$ for all $X, Y \in \mathfrak{su}(2)$ and $x, y \in \mathbb{R}$ (for convenience, we switch freely between interpretions of X as $(X, 0)$ etc). We make the choice $c_1 = 1/(2g_W^2)$ and $c_2 = 4n_Y^2/g'^2$ and have the following property.

4.2.3. LEMMA. *The basis $\beta_1, \beta_2, \beta_3, \beta_4$ of $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ specified in* (4.13) *is orthonormal with respect to the scalar product*

$$\langle (X, ix), (Y, iy) \rangle_{\mathfrak{g}} := -\frac{2}{g_w^2} \operatorname{trace}(XY) + \frac{4n_Y^2}{g'^2} xy.$$

PROOF. Obviously, $\langle \beta_4, \beta_4 \rangle_{\mathfrak{g}} = 1$ and $\langle \beta_4, \beta_j \rangle_{\mathfrak{g}} = 0$ $(j = 1, 2, 3)$. Recalling that the Pauli matrices are traceless and $\sigma_1\sigma_2 = i\sigma_3$, $\sigma_1\sigma_3 = -i\sigma_2$, $\sigma_2\sigma_3 = i\sigma_1$, we deduce $\langle \beta_j, \beta_k \rangle_{\mathfrak{g}} = 0$, if $j \neq k$. Finally, remembering $\sigma_j^2 = I_2$ $(j = 1, 2, 3)$ we obtain

$$\langle \beta_j, \beta_j \rangle_{\mathfrak{g}} = -\frac{2}{g_w^2} \operatorname{trace}(-\frac{g_w^2}{4}\sigma_j^2) = \frac{1}{2}\operatorname{trace}(I_2) = 1.$$

□

The action of the basis elements on the Higgs vector space $W = \mathbb{C}^2$ according to the Lie algebra representation $\rho_* \colon \mathfrak{g} \to \mathrm{L}(\mathbb{C}^2)$ is easily determined by evaluating for arbitrary $w \in \mathbb{C}^2$ the derivatives of $t \mapsto \rho(\exp(t\beta_l), 1)w$ $(l = 1, 2, 3)$ and of $t \mapsto \rho(I_2, \exp(t\beta_4))w$ at $t = 0$. They give

$$\rho_*(\beta_l)w = g_W \frac{i}{2}\sigma_l w (l = 1, 2, 3), \quad \rho_*(\beta_4)w = g' \frac{i}{2} w.$$

Recall that we had the Higgs vacuum w_0 and vacuum manifold W_0 given by

$$w_0 = \begin{pmatrix} 0 \\ \sqrt{\frac{\mu}{2\lambda}} \end{pmatrix} \quad \text{and} \quad W_0 = \{ w \in \mathbb{C}^2 \mid \|w\| = \sqrt{\frac{\mu}{2\lambda}} \}$$

with the one-dimensional unbroken subgroup H, the isotropy group of w_0 with respect to the ρ-action according to (4.2) in the form

$$H = \{ (\begin{pmatrix} e^{i\frac{\delta}{2}} & 0 \\ 0 & e^{-i\frac{\delta}{2}} \end{pmatrix}, e^{i\frac{\delta}{2n_Y}}) \mid \delta \in \mathbb{R} \} \subseteq \mathrm{SU}(2) \times \mathrm{U}(1).$$

We thus obtain the Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ in the form

$$\text{(4.14)} \qquad \mathfrak{h} = \{ (\begin{pmatrix} i\frac{t}{2} & 0 \\ 0 & -i\frac{t}{2} \end{pmatrix}, i\frac{t}{2n_Y}) \mid t \in \mathbb{R} \} = \{ (t\frac{i}{2}\sigma_3, t\frac{i}{2n_Y}) \mid t \in \mathbb{R} \} = \operatorname{span}\{ \frac{1}{g_W}\beta_3 + \frac{1}{g'}\beta_4 \}.$$

The mass form $m \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is defined by $m(X,Y) = \langle\langle \rho_*(X)w_0, \rho_*(Y)w_0 \rangle\rangle_W = \operatorname{Re}\langle \rho_*(X)w_0, \rho_*(Y)w_0 \rangle_W$ for all $X, Y \in \mathfrak{g}$, so we calculate $\rho_*(\beta_j)w_0$ to find the matrix elements $m(\beta_j, \beta_k)$: Writing $\|w_0\|$ in place of $\sqrt{\mu/(2\lambda)}$ we have

$$\rho_*(\beta_1)w_0 = g_W \frac{i}{2}\sigma_1 w_0 = \frac{g_W \|w_0\| i}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{g_W \|w_0\|}{2} \begin{pmatrix} i \\ 0 \end{pmatrix},$$

$$\rho_*(\beta_2)w_0 = g_W \frac{i}{2}\sigma_2 w_0 = \frac{g_W \|w_0\| i}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{g_W \|w_0\|}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\rho_*(\beta_3)w_0 = g_W \frac{i}{2}\sigma_3 w_0 = \frac{g_W \|w_0\| i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{g_W \|w_0\|}{2} \begin{pmatrix} 0 \\ i \end{pmatrix},$$

which gives

$$m(\beta_j, \beta_k) = g_W^2 \frac{\|w_0\|^2}{4} \delta_{jk} \quad (1 \leq j, k \leq 3);$$

and together with

$$\rho_*(\beta_4)w_0 = g'\frac{i}{2}w_0 = \frac{g'\,\|w_0\|}{2}\begin{pmatrix}0\\ i\end{pmatrix}$$

we find

$$m(\beta_1,\beta_4) = m(\beta_2,\beta_4) = 0, m(\beta_3,\beta_4) = -g_W g'\frac{\|w_0\|^2}{4}, m(\beta_4,\beta_4) = g'^2\frac{\|w_0\|^2}{4}.$$

Therefore, we obtain the matrix

$$(m(\beta_a,\beta_b))_{1\le a,b\le 4} = \frac{\|w_0\|^2}{4}\begin{pmatrix} g_W^2 & 0 & 0 & 0\\ 0 & g_W^2 & 0 & 0\\ 0 & 0 & g_W^2 & -g_W g'\\ 0 & 0 & -g_W g' & g'^2\end{pmatrix}.$$

We show that the matrix of m becomes diagonal with respect to the new basis consisting of

$$\alpha_1 := \beta_1, \alpha_2 := \beta_2, \alpha_3 := \frac{1}{\sqrt{g_W^2+g'^2}}(g_W\beta_3 - g'\beta_4), \alpha_4 := \frac{1}{\sqrt{g_W^2+g'^2}}(g'\beta_3 + g_W\beta_4). \tag{4.15}$$

Clearly, $m(\alpha_j,\alpha_k) = 0$, if $j \le 2$ and $k \ge 3$; calculating $(g_W^2+g'^2)m(\alpha_3,\alpha_4)$ gives

$$\begin{aligned} m(g_W\beta_3 - g'\beta_4, g'\beta_3 + g_W\beta_4) &= g_w g'(m(\beta_3,\beta_3) - m(\beta_4,\beta_4)) + m(\beta_3,\beta_4)(g_W^2 - g'^2)\\ &= \frac{\|w_0\|^2}{4}\left(g_w g'(g_W^2 - g'^2) - g_W g'(g_W^2 - g'^2)\right) = 0; \end{aligned}$$

for $(g_W^2+g'^2)m(\alpha_3,\alpha_3)$ we obtain the expression

$$\begin{aligned} m(g_W\beta_3 - g'\beta_4, g_W\beta_3 - g'\beta_4) &= g_W^2 m(\beta_3,\beta_3) - 2g_W g' m(\beta_3,\beta_4) + g'^2 m(\beta_4,\beta_4)\\ &= \frac{\|w_0\|^2}{4}\left(g_W^4 + 2g_W^2 g'^2 + g'^4\right) = \frac{\|w_0\|^2}{4}(g_w^2+g'^2)^2; \end{aligned}$$

and $(g_W^2+g'^2)m(\alpha_4,\alpha_4)$ is equal to

$$\begin{aligned} m(g'\beta_3 + g_W\beta_4, g'\beta_3 + g_W\beta_4) &= g'^2 m(\beta_3,\beta_3) + 2g_W g' m(\beta_3,\beta_4) + g_W^2 m(\beta_4,\beta_4)\\ &= \frac{\|w_0\|^2}{4}\left(g'^2 g_W^2 - 2g_W^2 g'^2 + g_W^2 g'^2\right) = 0; \end{aligned}$$

thus, we indeed obtain a diagonal form

$$(m(\alpha_a,\alpha_b))_{1\le a,b\le 4} = \frac{\|w_0\|^2}{4}\begin{pmatrix} g_W^2 & 0 & 0 & 0\\ 0 & g_W^2 & 0 & 0\\ 0 & 0 & g_W^2+g'^2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}. \tag{4.16}$$

We claim that (4.16) is a diagonalization of m acording to Proposition 4.1.15.

It remains to show that the basis $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ is orthonormal with respect to $\langle .\,,.\rangle_{\mathfrak{g}}$, $\alpha_1,\alpha_2,\alpha_3$ is a basis of $\mathfrak{h}^\perp$, and α_4 is a basis of $\mathfrak{h}$: Since the basis β_a $(a = 1,2,3,4)$ is orthonormal and α_a $(a = 1,2,3,4)$ is given by (4.15), it is obvious that $\alpha_j \perp \alpha_k$, if $j \le 2$ and $k \ge 3$, and $\langle\alpha_j,\alpha_l\rangle_W = \delta_{jl}$, if $j,l \le 2$; furthermore, $(g_W^2+g'^2)\langle\alpha_3,\alpha_4\rangle_{\mathfrak{g}} = g' g_W(\langle\beta_3,\beta_3\rangle_{\mathfrak{g}} - \langle\beta_4,\beta_4\rangle_{\mathfrak{g}}) = 0$ and clearly $\langle\alpha_3,\alpha_3\rangle_{\mathfrak{g}} = 1 = \langle\alpha_4,\alpha_4\rangle_{\mathfrak{g}}$; thus, the basis α_a $(a = 1,2,3,4)$ is orthonormal.
We had remarked that $\mathfrak{h}$ is the kernel of the map $X \mapsto \rho_*(X)w_0$ just prior to stating Proposition 4.1.15 and the above diagonal form of m shows that this must be the one-dimensional subspace spanned by α_4. But we also directly obtained in (4.14) that $\mathfrak{h}$ is spanned by $\frac{1}{g_W}\beta_3 + \frac{1}{g'}\beta_4 = (g'\beta_3 + g_W\beta_4)/(g' g_W)$, which is proportional to α_4. In any case, $\mathfrak{h} = \operatorname{span}\{\alpha_4\}$ and then clearly $\mathfrak{h}^\perp = \operatorname{span}\{\alpha_1,\alpha_2,\alpha_3\}$.

To summarize, we have the broken generators $\alpha_1,\alpha_2,\alpha_3$, i.e., three massive gauge bosons, with masses

$$M_1 = M_2 = \frac{1}{\sqrt{2}}\|w_0\|\,g_W \quad\text{and}\quad M_3 = \frac{1}{\sqrt{2}}\|w_0\|\sqrt{g_W^2+g'^2}, \tag{4.17}$$

and one massless gauge boson corresponding to the unbroken generator α_4.

4.2.4. Remark. (i) For the physics of the electroweak interaction theory, the gauge bosons, considered as $(\mathfrak{g} \otimes \mathbb{C})$-valued 1-forms on Minkowski space, are distinguished in the complexification $\mathfrak{g} \otimes \mathbb{C}$ as follows (cf. [**19**, Subsections 8.3.3 and 8.5.5]): The two *W-bosons* with their coefficient functions $W_\mu^\pm$ $(\mu = 1, 2, 3, 4)$ according to those of $(\alpha_1 \mp i\alpha_2)/\sqrt{2}$ in the basis expansion and both with mass M_1, the *Z-boson* defined by the direction of α_3 and with mass M_3, and the massless *photon* corresponding to the α_4-part of the $\mathfrak{g} \otimes \mathbb{C}$-basis expansion.

(ii) In the Standard Model of Particle Physics, the Higgs field interacts with all the fermionic matter particles, except for the neutrinos, by Yukawa coupling terms in the Lagrangian. It turns out that after symmetry breaking, these coupling terms produce mass terms for these fermions in the resulting Lagrangian, where the mass factors are, similarly to (4.17), proportional to corresponding Yukawa coupling constants and to the norm of the Higgs condensate (cf. [**19**, Section 8.8]).

APPENDIX A

A geometric scheme for the Standard Model

We summarize here in a very schematic way the basic mathematical ingredients for the geometric aspects of the Standard Model of Particle Physics:

- The base manifold M is Minkowski space, which we denoted by $\mathbb{R}^{1,3}$ in the main text. As spacetime symmetries we consider the proper orthochronous Lorentz group $\mathrm{SO}^+(1,3)$ with double covering $\mathrm{Spin}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C})$.
- There is a unique spin structure $\mathrm{Spin}^+(M)$ and we construct the spinor bundle $S = \mathrm{Spin}^+(M) \times_\alpha \mathbb{C}^4$ associated via the spinor representation $\alpha\colon \mathrm{Spin}^+(1,3) \to \mathrm{GL}(\mathbb{C}^4)$.
- The gauge group and bundle are given by $G = \underbrace{\mathrm{SU}(3)}_{\text{strong}} \times \underbrace{\mathrm{SU}(2) \times \mathrm{U}(1)}_{\text{electroweak}}$ and $P = G \times M$.
- Matter fields are sections (cf. §3.3) of a complex vector bundle E over M (with an action of $\mathrm{Spin}^+(1,3) \times G$) that is given as a direct sum of vector bundles:
$$E \quad = \quad \overbrace{E_1 \oplus E_2 \oplus E_3}^{\text{three fermionic families}} \quad \oplus \quad \overbrace{E_H}^{\text{Higgs boson}} \quad .$$
- For each fermionic family ($l = 1,2,3$), the subbundle E_l is given as a vector bundle associated to the fiber product of principal bundles $(S \times_M P)$ via a unitary representation $\widetilde{\pi}_l$ of G on $\mathbb{C}^{30}$, i.e.,
$$E_l = (S \times_M P) \times_{\alpha\otimes\widetilde{\pi}_l} (\mathbb{C}^4 \otimes \mathbb{C}^{30}).$$
In total, the representations corresponding to all *leptons* and *quarks* adds up to
$$\underbrace{\widetilde{\pi_1} \oplus \widetilde{\pi_2} \oplus \widetilde{\pi_3}}_{\text{dimension 90}} = \underbrace{\pi}_{\text{particles}} \oplus \underbrace{\pi^C}_{\text{antiparticles}}$$
(more details about π and π^C in §B.2).
Remark: The three lepton families e, ν_e; μ, ν_μ; τ, ν_τ do not participate in the strong interaction. The three quark families u, d; c, s; t, b are the basic building blocks for all *hadrons* (which are the particles participating in the strong [and electroweak] interaction and consist of two classes: *Baryons*, like the proton or neutron, are fermionic and consist of three quarks, while the bosonic *mesons* are quark-antiquark pairs.).
- Denoting by $\mathrm{O}(M)$ the orthonormal frame bundle over M and given a unitary representation π_H of G on the Higgs vector space $\mathbb{C}^2$ (cf. §B.2) we have
$$E_H = (\mathrm{O}(M) \times_M P) \times_{\mathrm{triv}\otimes\pi_H} (\mathbb{C} \otimes \mathbb{C}^2),$$
where triv denotes the trivial representation of $\mathrm{O}(1,3)$ (to be considered also as trivial representation of $\mathrm{Spin}^+(1,3)$ to obtain an action of $\mathrm{Spin}^+(1,3) \times G$ on every summand of E).
- The gauge bosons are encoded in a connection one-form with values in the Lie algebra
$$\mathfrak{g} = \underbrace{\mathfrak{su}(3)}_{\text{8 gluons}} \oplus \underbrace{\mathfrak{su}(2) \oplus \mathfrak{u}(1)}_{W^+,\ W^-,\ Z,\ \text{photon}} \quad .$$

APPENDIX B

Representations of the Standard Model gauge group(s)

The gauge group

$$G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$$

of the Standard Model is a product of the matrix Lie groups SU(3), SU(2), and U(1). It is therefore also a matrix Lie group with Lie algebra

$$\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$$

(see the brief discussion on page 88 in [**18**]) and we have $\dim G = \dim \mathfrak{g} = 8 + 3 + 1 = 12$.

All representations of G in the gauge theoretic aspect of the Standard Model are *complex finite-dimensional unitary* representations. They are constructed (cf. [**19**, Section 8.5]) from a direct sum of (outer) tensor product representations $\rho \otimes \sigma \otimes \tau$ on a (Hilbert) tensor product $\mathbb{C}^k \otimes \mathbb{C}^l \otimes \mathbb{C}^m \cong \mathbb{C}^n$, where $\rho\colon \mathrm{SU}(3) \to \mathrm{U}(k)$, $\sigma\colon \mathrm{SU}(2) \to \mathrm{U}(l)$, and $\tau\colon \mathrm{U}(1) \to \mathrm{U}(m)$ are (continuous) *irreducible* unitary representations[1]. For any $A \in \mathrm{SU}(3), B \in \mathrm{SU}(2), c \in \mathrm{U}(1)$, the action of $(A, B, c) \in G$ is given on splitting tensors by

$$((\rho \otimes \sigma \otimes \tau)(A, B, c))u \otimes v \otimes w = \rho(A)v \otimes \sigma(B)w \otimes \tau(c)w \qquad (u \in \mathbb{C}^k, v \in \mathbb{C}^l, w \in \mathbb{C}^m).$$

B.1. The irreducible unitary representations of U(1), SU(2), and SU(3)

B.1.1. The representation classes τ_k ($k \in \mathbb{Z}$) of U(1). Since U(1) is commutative, the irreducible unitary representations are all one-dimensional and hence their equivalence classes are given by the sequence of characters $\tau_k\colon \mathrm{U}(1) \to \mathrm{U}(1)$, where $k \in \mathbb{Z}$ and $\tau_k(c) := c^k$ for all $c \in \mathrm{U}(1)$ (cf. [**12**, Corollary 3.6 and Theorem 4.6]). We note that τ_0 is the trivial representation, $\tau_0(c) = 1$ for all $c \in \mathrm{U}(1)$, and τ_1 is the standard representation, $\tau_1(c) = c$ for all $c \in \mathrm{U}(1)$.

Note that in [**19**, Section 8.5] the irreducible representation classes of U(1) are parameterized by the weak hypercharge y such that $3y \in \mathbb{Z}$ ([**19**, Lemma 8.5.1] and with notation $\mathbb{C}_y$ replacing τ_{3y}.

B.1.2. The representation classes σ_m ($m \in \mathbb{N}_0$) of SU(2). A classification of the irreducible unitary representations of SU(2) can be found in [**12**, Section 5.4] or in [**18**, Sections 4.2 and 4.6]. However, in the Standard Model of Particle Physics only the trivial one-dimensional representation σ_0, with $\sigma_0(B) = 1$ for all $B \in \mathrm{SU}(2)$, and the 2-dimensional standard representation σ_1 with $\sigma_1(B)v = Bv$ for all $B \in SU(2)$ and $v \in \mathbb{C}^2$, are employed. Nevertheless we mention that the whole set of equivalence classes is given by a sequence of unitary representations σ_m ($m \in \mathbb{N}_0$), where the representation space of σ_m has dimension $m + 1$.

B.1.3. The representation classes $\rho_{(m_1,m_2)}$ ($m_1, m_2 \in \mathbb{N}_0$) of SU(3). The equivalence classes of irreducible unitary representations of SU(3) are described in [**18**, Chapter 6] (via a Lie algebra isomorphism between $\mathfrak{sl}(3, \mathbb{C})$ and the complexification of $\mathfrak{su}(3)$). The equivalence classes of unitary representations can be listed in terms of a two-parameter family $\rho_{(m_1,m_2)}$ with $m_1, m_2 \in \mathbb{N}_0$, where the representation space of $\rho_{(m_1,m_2)}$ has dimension $(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2)/2$ ([**18**, Theorem 6.27]). The Standard Model makes use only of the trivial one-dimensional representation $\rho_0 := \rho_{(0,0)}$, with $\rho_0(A) = 1$ for all $A \in \mathrm{SU}(3)$, and of the 3-dimensional standard representation $\rho_1 := \rho_{(1,0)}$, where $\rho_1(A)v = Av$ for all $A \in SU(3)$ and $v \in \mathbb{C}^3$.

[1]It follows then that $\rho \otimes \sigma \otimes \tau$ is an irreducible unitary representation of G (cf. [**12**, Theorem 7.12]).

B.2. The gauge group representation for all matter particles and the Higgs field

On a very abstract level, the representation of $G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ for the matter particles splits into a direct sum

$$\pi \oplus \pi^C,$$

where π incorporates all Fermions and π^C their antiparticles. The representations π and π^C both split into left-handed and right-handed parts,

$$\pi = \pi_L \oplus \pi_R, \quad \pi^C = \pi_L^C \oplus \pi_R^C,$$

where $\pi_L^C = \overline{\pi_R}$, the representation complex conjugate to π_R, and $\pi_R^C = \overline{\pi_L}$. In particular[2],

$$\dim \pi_L^C = \dim \pi_R \quad \text{and} \quad \dim \pi_R^C = \dim \pi_L$$

and hence it suffices to describe π_L and π_R in more detail.

According to three *generations of Fermions* we have

$$\pi_L = \pi_L^1 \oplus \pi_L^2 \oplus \pi_L^3 \quad \text{and} \quad \pi_R = \pi_R^1 \oplus \pi_R^2 \oplus \pi_R^3,$$

where $\pi_L^1 \cong \pi_L^2 \cong \pi_L^3$ and $\pi_R^1 \cong \pi_R^2 \cong \pi_R^3$.

In each generation $(i = 1, 2, 3)$ we have a splitting into a *quark sector* and a *lepton sector*:

$$\pi_L^i = \kappa_L^i \oplus \lambda_L^i \quad \text{and} \quad \pi_R^i = \kappa_R^i \oplus \lambda_R^i.$$

The quark sector representations are given by

$$\kappa_L^i = \rho_1 \otimes \sigma_1 \otimes \tau_1, \qquad \dim \kappa_L^i = 3 \cdot 2 \cdot 1 = 6,$$
$$\kappa_R^i = \big(\rho_1 \otimes \sigma_0 \otimes \tau_4\big) \oplus \big(\rho_1 \otimes \sigma_0 \otimes \tau_{-2}\big), \qquad \dim \kappa_R^i = (3 \cdot 1 \cdot 1) + (3 \cdot 1 \cdot 1) = 6.$$

The lepton sector representations are given by

$$\lambda_L^i = \rho_0 \otimes \sigma_1 \otimes \tau_{-3}, \qquad \dim \lambda_L^i = 1 \cdot 2 \cdot 1 = 2,$$
$$\lambda_R^i = \rho_0 \otimes \sigma_0 \otimes \tau_{-6}, \qquad \dim \lambda_R^i = 1 \cdot 1 \cdot 1 = 1.$$

Counting the dimensions we obtain

$$\dim \pi_L^i = 6 + 2 = 8, \quad \dim \pi_R^i = 6 + 1 = 7,$$

hence

$$\dim \pi_L = 8 + 8 + 8 = 24, \quad \dim \pi_R = 7 + 7 + 7 = 21,$$

which implies

$$\dim \pi^C = \dim \pi = 24 + 21 = 45,$$

and sums up for all matter particles to

$$\dim(\pi \oplus \pi^C) = 45 + 45 = 90,$$

with a contribution[3] of $2(\dim \pi_L^i + \dim \pi_R^i) = 2(8 + 7) = 30$ dimensions from each of the three fermionic generations.

In addition to the Fermions constituting matter, we have a representation π_H for the Higgs field as a further summand, i.e.,

$$\pi \oplus \pi^C \oplus \pi_H,$$

where π_H is two-dimensional and given on $\mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}$ by

$$\pi_H = \rho_0 \otimes \sigma_1 \otimes \tau_3.$$

[2] referring to the dimension of a representation while meaning the dimension of its underlying vector space

[3] Note that $\pi_L^C = \overline{\pi_R} \cong \overline{\pi_R^1} \oplus \overline{\pi_R^2} \oplus \overline{\pi_R^3}$, $\pi_R^C = \overline{\pi_L} = \overline{\pi_L^1} \oplus \overline{\pi_L^2} \oplus \overline{\pi_L^3}$, and $\dim(\overline{\pi_R^i} \oplus \overline{\pi_L^i}) = \dim \pi_R^i + \dim \pi_L^i$.

B.3. The adjoint Lie algebra representation for the gauge fields

The gauge theoretic aspect of the Standard Model is based on the trivial G-bundle over Minkowski space with a connection one-form representing the gauge bosons (or gauge fields), influencing the theory via curvature and covariant derivatives. Upon a choice of a global section (gauge), the connection is given by a one-form on Minkowski space with values in the Lie algebra $\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$, whose adjoint action $\mathrm{ad}_{\mathfrak{g}} \colon \mathfrak{g} \to L(\mathfrak{g})$ determines the zero-order terms in the covariant derivative. Recall that the Lie algebra adjoint representation is given by $\mathrm{ad}_{\mathfrak{g}} := T_e \mathrm{Ad}_G$, where $\mathrm{Ad}_G \colon G \to \mathrm{GL}(\mathfrak{g})$ is the (Lie group) adjoint action on $\mathfrak{g}$. Due to the product structure of $G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ the adjoint action splits in the form

$$\mathrm{Ad}_G(A, B, c) \cdot (X, Y, Z) = (\mathrm{Ad}_{\mathrm{SU}(3)}(A) \cdot X, \mathrm{Ad}_{\mathrm{SU}(2)}(B) \cdot Y, \mathrm{Ad}_{\mathrm{U}(1)}(c) \cdot Z)$$

for all $A \in \mathrm{SU}(3), B \in \mathrm{SU}(2), c \in \mathrm{U}(1)$ and $X \in \mathfrak{su}(3), Y \in \mathfrak{su}(2), Z \in \mathfrak{u}(1)$, which we write by some abuse of notation in short-hand as $\mathrm{Ad}_G = \mathrm{Ad}_{\mathrm{SU}(3)} \times \mathrm{Ad}_{\mathrm{SU}(2)} \times \mathrm{Ad}_{\mathrm{U}(1)}$. Differentiating this map at the group identity $(I_3, I_2, 1) \in \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ we obtain (again by abuse of notation)

$$\mathrm{ad}_{\mathfrak{g}} = \mathrm{ad}_{\mathfrak{su}(3)} \oplus \mathrm{ad}_{\mathfrak{su}(2)} \oplus \mathrm{ad}_{\mathfrak{u}(1)}.$$

In a corresponding decomposition of the gauge field values, the 8-dimensional part from $\mathrm{ad}_{\mathfrak{su}(3)}$ gives the *gluon* gauge field, while the (3+1)-dimensional according to $\mathrm{ad}_{\mathfrak{su}(2)} \oplus \mathrm{ad}_{\mathfrak{u}(1)}$ gives the *weak* and *hypercharge* gauge fields and suitable linear combinations thereof define the two W-bosons, the Z-boson, and the photon.

APPENDIX C

A glimpse at Quantum Field Theory (QFT)

The lecture course focusses on the gauge theoretic aspects and discusses the notion of *field* either in the sense of sections of complex vector bundles (associated to principal bundles via group representations) or of connection one-forms (with values in a Lie algebra). In a local representation, a field is then given in terms of a map from (some open subset of) Minkowski space into a (finite-dimensional) complex vector space or into a (finite-dimensional) Lie algebra. Recall that the groups (and representations) involved concern on the one hand the Lorentz or Spin group implementing space-time symmetries, and on the other hand the compact gauge groups which model the so-called inner symmetries of the fields.

As soon as we pass to *quantum fields*, the local representation of fields ceases to be valid for two important reasons:
1. The fields no longer have values in finite-dimensional vector spaces. The latter have to be replaced by sets of unbounded densely defined operators on an infinite-dimensional Hilbert space.
2. The fields are no longer maps defined on (open subsets of) Minkowski space, but instead have to be considered as distributions (generalized functions) on Minkowski space.
There is also no neat global geometric description of these objects[1] anymore. Note that a nuisance caused by 1. is that the set of all unbounded densely defined operators on an infinite-dimensional Hilbert space is not even a vector space, since addition of operators can only be defined on a common domain. And domain matters are only worse with composition of those linear operators. But such issues are even more delicate because of 2., since distribution theory is purely linear (defined in terms of dual spaces) and does not provide mechanisms for nonlinear combinations of field operators. These indications may give some idea why certain requirements or conditions occur in the set-up for the Gårding-Wightman axioms discussed below. However, these still cannot ensure a complete mathematical treatment and it turns out that as soon as interacting fields are the subject a lot of "compromises with rigor" have to be accepted in approaching results, which are so astonishingly coherent with experimental facts.

In preparing this appendix, we have used or may recommend for further reading the following literature for various aspects of QFT: As sources in mathematical physics [**2**, **5**, **13**, **15**, **17**, **37**] and for theoretical physics including perturbation theoretic results [**6**, **10**, **11**, **30**, **35**, **38**]; for the presentation here, we mostly followed Gerald Folland's book [**13**] and tried to transfer some of its spirit and clarity in providing an admiring mathematical view on QFT as it is applied in theoretical physics.

C.1. Axioms for QFT

C.1.1. Basic notation for the Poincaré group. The Poincaré group $\mathcal{P}$ is the semidirect product $\mathbb{R}^4 \ltimes \mathrm{O}(1,3)$, i.e., $(a, A) \cdot (b, B) = (a + Ab, AB)$ for $a, b \in \mathbb{R}^4$ and $A, B \in \mathrm{O}(1,3)$. By $\mathcal{P}_0$ we denote the connected component of the identity $(0, I)$ in $\mathcal{P}$, thus $\mathcal{P}_0 = \mathbb{R}^4 \ltimes \mathrm{SO}^+(1,3)$ with the proper orthochronous Lorentz group $\mathrm{SO}^+(1,3)$. Recall that the universal covering group of $\mathrm{SO}^+(1,3)$ is $\mathrm{Spin}^+(1,3) = \mathrm{SL}(2,\mathbb{C})$ with the 2-fold covering map $\kappa\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(1,3)$, which is constructed as follows: If $H(2,\mathbb{C})$ denotes the set of all Hermitian complex 2×2-matrices, then

[1]At least not known to the authors of these lecture notes.

we have the isomorphism of real vector spaces $M\colon \mathbb{R}^4 \to H(2,\mathbb{C})$, given by

$$M(x) := \sum_{\nu=0}^{3} x_\nu \sigma_\nu = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \quad \forall x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4,$$

where $\sigma_0 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right), \sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ are the Pauli matrices. A simple calculation shows that $\det M(x) = \eta(x,x)$, where η is the Minkowski metric (with so-called West Coast signature according to $(+,-,-,-)$). For any $A \in \mathrm{SL}(2,\mathbb{C})$ we have

$$\forall x \in \mathbb{R}^4\colon \quad AM(x)A^\dagger \in H(2) \quad \text{and} \quad \det(AM(x)A^\dagger) = \det M(x) = \eta(x,x),$$

hence we can find a unique $\kappa(A) \in \mathrm{SO}^+(1,3)$ such that

$$\forall x \in \mathbb{R}^4\colon \quad AM(x)A^\dagger = M(\kappa(A)x).$$

Thanks to κ we obtain the 2-fold covering map $\mathbb{R}^4 \ltimes \mathrm{SL}(2,\mathbb{C}) \to \mathcal{P}_0$, $(a,A) \mapsto (a,\kappa(A))$.

C.1.2. Remark (Euclidean QFT and C^*-algebraic approaches). (i) The system of so-called Euclidean axioms or Osterwalder-Schrader theory is based on a probability (Borel) measure μ defined on the space of distributions $\mathcal{D}'(\mathbb{R}^d)$ on $\mathbb{R}^d$ ($d \in \mathbb{N}$). In fact, the basic object is its *generating functional* $S\colon \mathcal{D}(\mathbb{R}^d) \to \mathbb{C}$, given by

$$S(f) := \int\limits_{\mathcal{D}'(\mathbb{R}^d)} e^{i\langle u,f\rangle}\, d\mu(u) \quad (f \in \mathcal{D}(\mathbb{R}^d)),$$

which can be considered as a kind of inverse Fourier transform of μ. The set-up of this theory is motivated from a formal "analytic continuation of Minkowski space quantum fields to imaginary time" and was successful in constructing rigorous low-dimensional (i.e., $d = 2$ or 3) models with interactions (cf. [**15**]).

(ii) The axiomatic system of Haag-Kastler(-Araki) starts instead with *local* C^*-algebras (or rather von Neumann algebras) $\mathcal{A}(O)$ assigned to each open subset $O \subseteq \mathbb{R}^4$, such that $\mathcal{A}(O_1) \subseteq \mathcal{A}(O_2)$ if $O_1 \subseteq O_2$, and consider $\mathcal{A} := \bigcup \mathcal{A}(O)$ (with the union over all open subsets O of Minkowski space) as the *algebra of observables*. The axioms include conditions like the following (cf. [**2**, **17**]):
(a) There exists a representation of the Poincaré group as subgroup of all *-automorphisms on $\mathcal{A}$ such that $(a,R)\cdot\mathcal{A}(O) \subseteq \mathcal{A}(RO + a)$ holds for arbitary $(a,R) \in \mathcal{P}$ and every open subset $O \subseteq \mathbb{R}^4$.
(b) *Causality:* The subalgebras $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ commute, if O_1 and O_2 are *space-like separated*, i.e., $\eta(x-y, x-y) < 0$ for any $x \in O_1$ and $y \in O_2$.

C.1.3. Fields as operator-valued distributions on Minkowski space. The following list introduces all basic elements and properties used below for the formulation of the Gårding-Wightman axioms.

(i) Let $(\mathcal{F}, \langle.|.\rangle)$ be a complex Hilbert space with a distinguished dense subspace $\mathcal{D}$ of $\mathcal{F}$.

(ii) Let $\tilde{U}\colon \mathbb{R}^4 \ltimes \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(\mathcal{F})$ be a unitary strongly continuous[2] representation such that $\mathcal{D}$ is invariant under $\tilde{U}$, i.e. $\tilde{U}(a,A)\mathcal{D} \subseteq \mathcal{D}$ for all $(a,A) \in \mathbb{R}^4 \ltimes \mathrm{SL}(2,\mathbb{C})$.

(iii) We suppose to have $K \in \mathbb{N}$ different *types of particles* and for each $j = 1,\ldots,K$ to have $n_j \in \mathbb{N}$ *field components*. If $n_j = 1$, then the field corresponding to particle type j is scalar; with $n_j > 1$ we have vector, tensor, or spinor fields—the number n_j corresponds to the fiber dimension in the geometric formulation with sections of vector bundles associated with certain principal fiber bundles over Minkowski space and representations of their structure group. The number of *inner degrees of freedom* is the total number of field components $N := n_1 + \ldots + n_K$. For every particle type $l = 1,\ldots,K$ we are given an irreducible representation $S_l\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(\mathbb{C}^{n_l})$ (specifying also the so-called particle statistics in the sense of Fermionic type [half-integer spin] or

[2]That is, $\tilde{U}(a,A)$ is a unitary operator for every $(a,A) \in \mathbb{R}^4 \ltimes \mathrm{SL}(2,\mathbb{C})$ and, for every $\xi \in \mathcal{F}$, the map $(a,A) \mapsto \tilde{U}(a,A)\xi$ is continuous $\mathbb{R}^4 \ltimes \mathrm{SL}(2,\mathbb{C}) \to \mathcal{F}$.

Bosonic type [integer spin]) and we denote by $S := S_1 \oplus \cdots \oplus S_K$ the direct sum representation $S\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(\mathbb{C}^N)$.

(iv) The set of bounded operators on $\mathcal{F}$ given by $\{\tilde{U}(a,I) \mid a \in \mathbb{R}^4\}$ is a commutative subgroup of the group of all unitary operators on $\mathcal{F}$ and is generated by the following four one-parameter unitary groups: For each of the standard basis vectors e_0, e_1, e_2, e_3 in $\mathbb{R}^4$ we have the strongly continuous unitary group $t \mapsto \tilde{U}(te_\nu, I)$ and thus, by Stone's theorem, also a unique self-ajoint operator P_ν as generator such that $\tilde{U}(te_\nu, I) = \exp(itP_\nu)$. Spectral theory ([**32**, Theorems VIII.12 and VIII.13]) provides us with a joint spectral measure E for the (commuting) *momentum operators* P_0, P_1, P_2, P_3, which is a σ-additive map from the Borel σ-algebra of $\mathbb{R}^4$ to the set of orthogonal projections in $\mathcal{F}$, normalized by $E(\emptyset) = 0$ and $E(\mathbb{R}^4) = 1$, such that

$$\tilde{U}(a,I) = \int\limits_{\mathbb{R}^4} e^{-i\eta(p,a)}\, dE(p).$$

(v) *Fields* (in fact, field components) $\phi_1, \ldots, \phi_N$ are linear maps from the test function space $\mathscr{S}(\mathbb{R}^4)$ to the set of unbounded operators on $\mathcal{F}$ having $\mathcal{D}$ contained in their domain as well as in the domain of their adjoint operators with the following two additonal properties:
(a) Invariance of the common domain, i.e., for all $f \in \mathscr{S}(\mathbb{R}^4)$ and $n \in \{1, \ldots, N\}$ we have $\phi_n(f)\mathcal{D} \subseteq \mathcal{D}$ and $\phi_n(f)^\dagger \mathcal{D} \subseteq \mathcal{D}$.
(b) Distributional continuity in the sense that for every $\xi, \eta \in \mathcal{D}$, the map $\mathscr{S}(\mathbb{R}^4) \to \mathbb{C}$, $f \mapsto \langle \xi | \phi_n(f) \eta \rangle$ defines an element in $\mathscr{S}'(\mathbb{R}^4)$, i.e., a temperate distribution on $\mathbb{R}^4$.

C.1.4. The Gårding-Wightman axioms. Suppose we have all the objects with properties as specified in C.1.3, then the further axiomatic requirements are (cf. [**2**, **13**, **17**, **31**, **37**]):

(i) There exists a vector $\Omega \in \mathcal{D}$, $\|\Omega\| = 1$, unique up to a phase factor, such that

$$\forall (a,A) \in \mathbb{R}^4 \ltimes \mathrm{SL}(2,\mathbb{C})\colon \quad \tilde{U}(a,A)\Omega = \Omega.$$

We call Ω the *vacuum state* of the theory.

(ii) *Completeness:* The vacuum vector is a cyclic vector for the field algebra, i.e., the linear hull of $\{\phi_{l_1}(f_1) \cdots \phi_{l_m}(f_m)\Omega \mid m \in \mathbb{N}_0, 1 \leq l_1, \ldots, l_m \leq N, f_1, \ldots, f_m \in \mathscr{S}(\mathbb{R}^4)\}$ is dense in $\mathcal{F}$. (In case $m = 0$ we define the empty product of field operators as the identity on $\mathcal{F}$.)

(iii) For arbitary $f \in \mathscr{S}(\mathbb{R}^4)$, $(a,A) \in \mathbb{R}^4 \ltimes \mathrm{SL}(2,\mathbb{C})$, $1 \leq n \leq N$, $\xi \in \mathcal{D}$, and upon defining $((a,A)f)(x) := f(\kappa(A)^{-1}(x-a))$ for every $x \in \mathbb{R}^4$, we have

$$\tilde{U}(a,A)\phi_n(f)\tilde{U}(a,A)^{-1}\xi = \sum_{m=1}^{N} S(A^{-1})_{nm}\, \phi_m\big((a,A)f\big)\xi.$$

Pretending that distributions were functions, the latter could be rewritten in the form

$$\tilde{U}(a,A)\phi_n(x)\tilde{U}(a,A)^{-1} = \sum_{m=1}^{N} S(A^{-1})_{nm}\, \phi_m\big(\kappa(A)x + a\big).$$

(iv) *Spectral condition:* The support of the spectral measure E is contained in the forward light cone $\{(p_0, \vec{p}) \in \mathbb{R}^4 \mid p_0 \geq |\vec{p}|\}$. This is equivalent to saying that with the self-adjoint momentum operators P_ν ($\nu = 0,1,2,3$), both P_0 and $P_0^2 - P_1^2 - P_2^2 - P_3^2$ are positive operators.

(v) *Causality:* If $f, g \in \mathscr{S}(\mathbb{R}^4)$ have their supports space-like separated[3] and $m, n \in \{1, \ldots, N\}$, then as operators on the domain $\mathcal{D}$ the fields $\phi_m(f)$ and $\phi_n(g)$ or $\phi_m(g)^\dagger$, either *commute*

$$[\phi_m(f), \phi_n(g)] = 0 = [\phi_m(f), \phi_n(g)^\dagger]$$

[3]Recall that this means $\eta(x-y, x-y) < 0$, if $f(x)g(y) \neq 0$.

or *anticommute* (notation $\{A, B\} := AB + BA$)

$$\{\phi_m(f), \phi_n(g)\} = 0 = \{\phi_m(f), \phi_n(g)^\dagger\}$$

(depending on m, n, and on the particle species).

Remark: We want to add here a direct quote by Rudolf Haag how he phrases the causality condition in a discussion preparing for the translation of the above axioms into a structure with local C^*-algebras as in C.1.2 ([**17**, page 107]): "Two observables associated with space-like separated regions are compatible. The measurement of one does not disturb the measurement of the other. The operators representing these observables must commute." And then he immediately adds a long comment starting with the following sentence: "To avoid possible confusion it must be stressed that this has nothing to do with the discussion around the Einstein-Podolsky-Rosen paradox and Bell's inequality."

C.1.5. A few vague remarks about rigorous results within axiomatic theories. The spin-statistics theorem (cf. [**37**, Section 4.4] or [**17**, Section II.5]) implies the following: Let the Schwartz functions f and g have space-like separated supports as in the causality axiom. Particles of integer spin must be *Bosons*, that is, each of their field components ϕ_n satisfies the commutation relation $[\phi_n(f), \phi_n(g)^\dagger] = 0$, while particles of half-integer spin are *Fermions*, which means that each of their field components ϕ_n satisfies the anticommutator relation $\{\phi_n(f), \phi_n(g)^\dagger\} = 0$.

The theorem on invariance under PCT ([**37**, Section 4.3]) or CPT ([**17**, Section II.5]) or TCP ([**2**, Section 5.6]) states that a QFT is not influenced by the combined operation of space inversion or parity P, time inversion T, and charge conjugation C. (The invariance is in general not true for a single operation or an arbitrary pairwise combination, as is illsutrated by the non-axiomatic theory of weak interaction, which is known to be not P-invariant, and evidence from meson decays about violation of CP-invariance.)

The reconstruction theorem ([**37**, Section 3.4] or [**5**, Section 8.3]) shows that a theory can be "recovered from knowing all its vacuum expectation values". This means that the so-called *hierarchy of Wightman distributions* $w^r_{l_1,\ldots,l_r} \colon f_1 \otimes \cdots \otimes f_r \mapsto \langle \Omega | \phi_{l_1}(f_1) \cdots \phi_{l_r}(f_r)\Omega\rangle$ on $\mathbb{R}^{4r}$ $(r \in \mathbb{N})$ determines $\mathcal{F}$ and the action of all field operators $\phi_n(f)$ uniquely (up to a unitary isomorphism).

The rigorous construction of some theories with nontrivial interactions (cf. [**37**, Appendix] or [**15**]) has been successful in space-time dimensions 2 and 3, although none so far in dimension 4, where at least many *rigorous free quantum field theories* do exist and thus show that the axioms are consistent (see, e.g., [**13**, Chapter 5]).

C.2. Perturbation theory for QFT with interactions

C.2.1. Dynamics (time evolution). As always in any quantum theory, the dynamics of time evolution is implemented in the form of a strongly continuous one-parameter unitary group $(U(t))_{t\in\mathbb{R}}$ on the Hilbert space $\mathcal{F}$. By Stone's theorem there is a unique self-adjoint (densely defined) generator H, the *Hamilton operator*, such that

$$\forall t \in \mathbb{R}\colon \quad U(t) = e^{-itH}.$$

The basic additional requirements about the dynamics in terms of H are the following:

(i) $\mathcal{D}$ is contained in the domain of H and $H\mathcal{D} \subseteq \mathcal{D}$,

(ii) the vacuum vector Ω shall be a basis of the eigenspace of H for the smallest eigenvalue of H.

In attempts to construct a QFT with interaction, the typical formal structure of H is

$$H = H_0 + H_I,$$

where H_0 is the self-adjoint Hamiltonian of a free QFT (i.e., without interactions, or considerably well understood) and H_I is the so-called *interaction Hamiltonian*, considered a perturbation of H_0, and it is hoped that (or ignored whether) $H_0 + H_I$ is a densely defined self-adjoint operator on $\mathcal{F}$ as well.

The interaction Hamiltonian H_I is often rather formally constructed by integration over *a Lagrangian density (operator)* $\mathcal{L}_I$ or a *Hamilton density (operator)* $\mathcal{H}_I := -\mathcal{L}_I$, symbolically, $H_I = -\int_{\mathbb{R}^3} \mathcal{L}_I \, dx = \int_{\mathbb{R}^3} \mathcal{H}_I \, dx$. The total *Lagrangian density*

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

is usually written down in terms of functions (or local representations of sections of appropriate vector bundles) instead of actually constructing operator-valued distributions (which in most cases would not even exist anyways), where typically,

$\mathcal{L}_0$ is bilinear in the fields and represents the *kinetic terms*

$\mathcal{L}_I$ is a polynomial of degree 3 or higher in the fields (or of some other form, maybe a potential, that does not fit into $\mathcal{L}_0$) and represents the *interaction terms*.

Recall that any "Hilbert space based quantum theory" usually distinguishes two ways of implementing the dynamics in terms of the unitary group $(U(t))_{t\in\mathbb{R}}$, namely the *Schrödinger picture*, where vector states $\xi \in \mathcal{F}$ evolve according to

$$\xi(t) := U(t)\xi \quad (t \in \mathbb{R}), \qquad \text{or, if } \xi \text{ belongs to the domain of } H\text{:} \quad \dot{\xi}(t) = -iH\xi(t),\ \xi(0) = \xi,$$

and the *Heisenberg picture*, where an *observable* R, i.e., R is self-adjoint with domain containing $\mathcal{D}$, evolves in time as given by

$$R(t) := U(-t)RU(t) \quad (t \in \mathbb{R}), \qquad \text{or, if } R\mathcal{D} \subseteq \mathcal{D}\text{:} \quad \dot{R}(t) = i[H, R(t)].$$

C.2.2. Interaction (or Dirac) picture. We consider $H = H_0 + H_I$ and, for the pure illustrative purpose of this appendix, will from now on assume the naive viewpoint described in the following quote from Gerald Folland [**13**, page 124]: "For psychological comfort, the reader may wish to think of the case where H_0 is a self-adjoint operator and H_I is a *bounded* self-adjoint operator, in which case H is a self-adjoint operator with the same domain as H_0. However, this is far from the case we shall need in the sequel."

We distinguish the two time evolutions given by the free Hamiltonian H_0 and by the total Hamiltonian H in the following notation for the respective unitary groups defined on $\mathcal{F}$:

$$U_0(t) := e^{-itH_0}, \quad U(t) := e^{-itH} = e^{-it(H_0+H_I)} \quad (t \in \mathbb{R}).$$

In the sequel, the free time evolution of an observable R will be denoted by

$$R(t) := U_0(-t)RU_0(t) \quad (t \in \mathbb{R}),$$

while in the time dependence of a state ξ we will focus on a comparison of the evolution with respect to H with the free evolution and define

$$\xi(t) := U_0(-t)U(t)\xi = V(t)\xi \quad (t \in \mathbb{R}),$$

where we have also used the following family[4] of unitary operators

$$V(t) := U_0(-t)U(t) \quad (t \in \mathbb{R}).$$

We show that with the above notation for the time dependence of observables and states, the so-called *transition probabilities* or *matrix elements* are correctly conserved: If $\xi_1, \xi_2 \in \mathcal{D}$, then

$$\begin{aligned}\langle \xi_1(t) | R(t)\xi_2(t)\rangle &= \langle V(t)\xi_1 | U_0(-t)RU_0(t)V(t)\xi_2\rangle \\ &= \langle \xi_1 | \big(U_0(-t)U(t)\big)^\dagger U_0(-t)RU_0(t)U_0(-t)U(t)\xi_2\rangle = \langle \xi_1 | U(-t)U_0(t)U_0(-t)RU(t)\xi_2\rangle \\ &= \langle \xi_1 | U(-t)RU(t)\xi_2\rangle = \langle U(t)\xi_1 | RU(t)\xi_2\rangle,\end{aligned}$$

where the two expressions on the last line are the transition amplitudes in the Heisenberg and Schrödinger pictures for the full theory with time evolution according to the total Hamiltonian H.

[4]which is in general not a one-paramter group

The basic hope of perturbative quantum theories is to get somehow useful approximations for $V(t)$, since $U_0(t)$ is usually well-known and one could then still get good results for $U(t) = U_0(t)V(t)$, at least in terms of matrix elements (transition amplitudes).

C.2.3. Remark: Non-existence of the interaction picture. According to a theorem by Rudolf Haag ([**5**, Section 9.4] or [**37**, Section 4.5]), from the unitarity of $V(t)$ and $V(0) = I$, one could construct a complete isomorphism between the interaction theory (with H) and the free theory (with H_0) for all times, which means in particular, that all predictions about transition amplitudes would agree. The latter clearly is absurd, since true interactions change the measurements as compared to the free theory. Thus, it cannot be mathematically strictly true that $V(t)$ exists. On the other hand, already over decades now the results of perturbative QFT are often in amazingly good agreement with experiments. Apart from this pragmatic argument there are even conceptual reasons, considering various asymptotic cut-offs and renormalization techniques, for not rejecting the perturbative methods (cf. [**10**, Chapter 10], which even contains a subsection title "How to stop worrying about Haag's theorem").

C.2.4. The Dyson series. By definition, we have $V(t) = U_0(-t)U(t)$, in particular, $V(0) = I$, and on the domain $\mathcal{D}$ we may differentiate to get (recalling that H_0 and $U_0(s) = e^{-isH_0}$ commute for every $s \in \mathbb{R}$)

$$\frac{d}{dt}V(t) = iH_0U_0(-t)U(t) - iU_0(-t)HU(t) = \frac{1}{i}U_0(-t)(-H_0 + H)U(t)$$
$$= \frac{1}{i}U_0(-t)H_IU_0(t)U_0(-t)U(t) = \frac{1}{i}H_I(t)V(t).$$

Therefore, $V(t)$ is the solution of the following initial value problem (on the domain $\mathcal{D}$)

$$\dot{V}(t) = \frac{1}{i}H_I(t)V(t), \quad V(0) = I.$$

The basic idea now is to make succsessive approximations in the form

$$V(t) = V(0) + \int_0^t \dot{V}(\tau)\,d\tau = I + \frac{1}{i}\int_0^t H_I(\tau)V(\tau)\,d\tau$$
$$= I + \frac{1}{i}\int_0^t \Big(H_I(\tau) + \frac{1}{i}\int_0^\tau H_I(\tau)H_I(\tau')V(\tau')\,d\tau'\Big)\,d\tau$$
$$= I + \frac{1}{i}\int_0^t H_I(\tau_1)\,d\tau_1 + \frac{1}{i^2}\int_0^t\int_0^{\tau_2} H_I(\tau_2)H_I(\tau_1)V(\tau_1)\,d\tau_1\,d\tau_2$$
$$= I + \frac{1}{i}\int_0^t H_I(\tau_1)\,d\tau_1 + \frac{1}{i^2}\int_0^t\int_0^{\tau_2} H_I(\tau_2)H_I(\tau_1)\,d\tau_1\,d\tau_2 + \ldots,$$

which suggests to consider the *Dyson series* associated with V and given by

$$V(t) \sim I + \sum_{n=1}^\infty \frac{1}{i^n}\int_0^t\int_0^{\tau_n}\cdots\int_0^{\tau_2} H_I(\tau_n)H_I(\tau_{n-1})\cdots H_I(\tau_1)\,d\tau_1\cdots d\tau_n =: I + \sum_{n=1}^\infty V_n(t).$$

(In the [unrealistic] case that H_I is a bounded operator, the series converges in operator norm.)

A simple observation about multiple integrals of matrix-valued functions: Let $f\colon [0,t] \to M_d(\mathbb{C})$ be continuous and define $T_2\colon [0,t]^2 \to M_d(\mathbb{C})$ by $T_2(\tau_1, \tau_2) := f(\tau_2)f(\tau_1)$, if $\tau_1 \leq \tau_2$, and $T_2(\tau_1, \tau_2) := f(\tau_1)f(\tau_2)$, if $\tau_1 > \tau_2$. By symmetry of T_2 with respect to the diagonal $\tau_1 = \tau_2$ in $[0,t] \times [0,t]$, we obtain

$$\int_0^t \int_0^{\tau_2} f(\tau_2) f(\tau_1)\, d\tau_1\, d\tau_2 = \frac{1}{2} \int_0^t \int_0^t T_2(\tau_1, \tau_2)\, d\tau_1\, d\tau_2.$$

Similarly, for any dimension $n \geq 2$, writing $T_n(\tau_1, \ldots, \tau_n) = f(\tau_{l_n}) \cdots f(\tau_{l_1})$, if $\tau_{l_1} \leq \cdots \leq \tau_{l_n}$, we obtain

$$\int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_2} f(\tau_n) f(\tau_{n-1}) \cdots f(\tau_1)\, d\tau_1 \cdots d\tau_n = \frac{1}{n!} \int_{[0,t]^n} T_n(\tau_1, \tau_2, \ldots, \tau_n)\, d(\tau_1, \ldots, \tau_n).$$

Note that the functions T_n arranged the ordering of the time-dependent matrix factors in such a way, that the earlier times always got to the right (and thus are applied first when acting on a vector). This is just a special case of the general scheme of a so-called *time odered product* in QFT, where for any finite family of operators $R_1(t)$, …, $R_n(t)$ with time-dependence and distinct times $t_1, \ldots, t_n$, one writes[5]

$$\mathcal{T}(R_1(t_1) \cdots R_n(t_n)) = R_{l_n}(t_{l_n}) \cdots R_{l_1}(t_{l_1}), \text{ if } \{l_1, \ldots, l_n\} = \{1, \ldots, n\} \text{ such that } t_{l_1} < \cdots < t_{l_n}.$$

Employing this notation, we have by extrapolation of the above observations

$$V_2(t) = \frac{1}{i^2} \int_0^t \int_0^{\tau_2} H_I(\tau_2) H_I(\tau_1)\, d\tau_1\, d\tau_2 = \frac{1}{2i^2} \int_0^t \int_0^t \mathcal{T}(H_I(\tau_1) H_I(\tau_2))\, d\tau_1\, d\tau_2$$

and more generally,

$$\begin{aligned} V_n(t) = \frac{1}{i^n} \int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_2} H_I(\tau_n) H_I(\tau_{n-1}) \cdots H_I(\tau_1)\, d\tau_1 \cdots d\tau_n \\ = \frac{1}{i^n n!} \int_0^t \cdots \int_0^t \mathcal{T}(H_I(\tau_1) \cdots H_I(\tau_n))\, d\tau_1 \cdots d\tau_n. \end{aligned}$$

With some further boldness, we may consider $\mathcal{T}$ to be extended linearly, therefore, being applied in the usual way to sums of products of (unbounded) operators (hopefully with a common and invariant domain?) or equally well to integrals … and go on to elegantly write

$$\begin{aligned} V(t) \sim I + \sum_{n=1}^{\infty} V_n(t) = I + \mathcal{T}\Big(\sum_{n=1}^{\infty} \frac{1}{i^n n!} \int_0^t \cdots \int_0^t H_I(\tau_1) \cdots H_I(\tau_n)\, d\tau_1 \cdots d\tau_n \Big) \\ =: \mathcal{T}\left(\exp\left(\frac{1}{i} \int_0^t H_I(\tau)\, d\tau \right) \right), \end{aligned}$$

which gives the most compact form of the Dyson series.

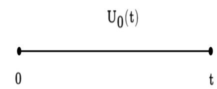
C.2.5. The lower order terms in the approximation of $U(t)$. Recall that $U(t) = U_0(t)V(t)$ and therefore

$$U(t) \sim U_0(t) + \sum_{n=1}^{\infty} U_0(t) V_n(t)$$

represents the series associated via the Dyson series to the time evolution with interaction.

[5]Beware of the fact that the notation $\mathcal{T}(R_1(t_1) \cdots R_n(t_n))$ is misleading, because this is not the value of a map $\mathcal{T}$ applied to the operator $R_1(t_1) \cdots R_n(t_n)$, but rather assigning an operator to certain n-tuples $(R_1(t_1), \ldots, R_n(t_n))$ (cf. [**6**, page 136]). Moreover, we silently ignore the issue of defining the product in case $t_l = t_k$ for some $l \neq k$, because this corresponds to a subset of Lebesgue measure 0 in the integration (and do not engage in arguing about measurability with respect to the time parameters).

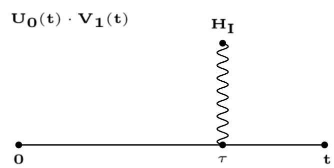
The lowest order term is just the free time evolution $U_0(t)$ between the initial time 0 and the "current" time t, symbolically this is expressed in a very simple drawing.



The first-order term is

$$U_0(t)V_1(t) = \frac{1}{i}\int_0^t U_0(t)H_I(\tau)\,d\tau = \frac{1}{i}\int_0^t U_0(t)U_0(-\tau)H_I U_0(\tau)\,d\tau = \frac{1}{i}\int_0^t U_0(t-\tau)H_I U_0(\tau)\,d\tau,$$

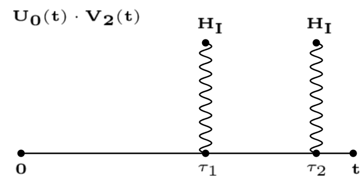
in which the integrand, read from right to left, represents first free evolution from time 0 up to time τ, then interaction according to H_I, then again free time evolution for the duration $t-\tau$, pictographically this is shown here to the right.



The second-order term is

$$U_0(t)V_2(t) = \frac{1}{i^2}\int_0^t\int_0^{\tau_2} U_0(t)H_I(\tau_2)H_I(\tau_1)\,d\tau_1\,d\tau_2$$

$$= \frac{1}{i^2}\int_{\{0<\tau_1<\tau_2<t\}} U_0(t-\tau_2)H_I U_0(\tau_2-\tau_1)H_I U_0(\tau_1)\,d(\tau_1,\tau_2)$$



and its integrand is represented in the picture.

C.2.6. The scattering matrix. A basic intuition of any scattering experiment is that the spatial region of interaction is bounded and moreover, for $t \to \mp\infty$ no particle has yet participated or can still participate in the interaction process. Heuristically, these asymptotically *incoming* or *outgoing states* may be considered as states of free particles, and one is interested in the *scattering transformation* between the incoming and the outgoing states.

Suppose that $\xi_{\text{in}} \in \mathcal{F}$ and $\eta_{\text{out}} \in \mathcal{F}$ represent incoming and outgoing asymptotically free states in a scattering experiment according to the interacting Hamiltonian $H = H_0 + H_I$. Recall that we denoted the unitary group for the free time evolution by $U_0(t) = e^{-itH_0}$, the dynamics including the interaction by $U(t) = e^{-itH}$, and that the time dependence of a state $\zeta \in \mathcal{F}$ in the interaction picture was given by $\zeta(t) = V(t)\zeta = U_0(-t)U(t)\zeta$. If asymptotically with respect to time, η_{out} emerges from ξ_{in} in the scattering process, it seems reasonable that over long times in the whole dynamics both η_{out} and ξ_{in} are connected via some common intermediate state $\zeta \in \mathcal{F}$, i.e., $\eta_{\text{out}} = V(t_2)\zeta$ and $\xi_{\text{in}} = V(-t_1)\zeta$ for some $t_1, t_2 \gg 0$. We therefore expect to find

$$\eta_{\text{out}} = V(t_2)\zeta = V(t_2)V(-t_1)^{-1}\xi_{\text{in}}$$

for very large $t_1, t_2 > 0$. We may get rid of the dependence on $t_1, t_2 \gg 0$ by taking limits and thus arrive at the "definition" for the *scattering transformation* (or S-transform)

$$S := \lim_{t_1,t_2\to\infty} V(t_2)V(-t_1)^{-1},$$

which for practical purposes is often considered in the form of its so-called *scattering matrix* (or S-matrix), i.e., the number scheme $\langle \eta | S\xi \rangle$ for all $\xi, \eta \in \mathcal{F}$ (or being members of some complete orthonormal system in $\mathcal{F}$).

Remarks: (i) Scattering theory has been studied intensively and rigorously also in axiomatic settings of QFT (cf. [**2**, **5**]).

(ii) It is customary to write $S = I + iT$, where T is called the *transfer matrix* and contains all the nontrivial parts of the scattering, because the identity I means "incoming state = outgoing state" and thus corresponds to the free theory without interactions. In a so-called irreducible scattering experiment one considers only incoming states ξ_{in} that are perpendicular[6] to the outgoing states η_{out}. In this situation we always have

$$\langle \eta_{\text{out}} | S \xi_{\text{in}} \rangle = \langle \eta_{\text{out}} | (S - I) \xi_{\text{in}} \rangle = i \langle \eta_{\text{out}} | T \xi_{\text{in}} \rangle.$$

In a realistic szenario, only the momenta of the states ξ_{in} or η_{out} are known or determined with certain accuracy and they often correspond in an L^2-picture to wave packets with highly concentrated momentum distributions[7]. Schematically and in an idealized situation with sharp momenta, the transfer matrix turns out to be of the form

$$\langle \eta_{\text{out}} | T \xi_{\text{in}} \rangle = (2\pi)^4 \, \delta(\sum_{\text{in}} p - \sum_{\text{out}} p) \, M(\xi_{\text{in}} \to \eta_{\text{out}}),$$

where the δ-factor reflects conservation of momenta and $M(\xi_{\text{in}} \to \eta_{\text{out}})$ is called the *Feynman amplitude*. It turns out that $|M(\xi_{\text{in}} \to \eta_{\text{out}})|^2$ can be related directly either to differential cross sections or also to decay rates (inverse life time) of particles in scattering experiments.

C.2.7. Perturbation series for the scattering operator. We may slightly rewrite the limit expression defining the scattering operator by

$$S = \lim_{t_1, t_2 \to \infty} V(t_2) V(-t_1)^{-1} = \lim_{\substack{t \to \infty, \\ t_0 \to -\infty}} V(t) V(t_0)^{-1} = \lim_{\substack{t \to \infty, \\ t_0 \to -\infty}} V(t, t_0),$$

where we put $V(t, t_0) := V(t) V(t_0)^{-1}$. Note that $V(t_0, t_0) = I$ and differentiating $V(t, t_0)$ with respect to t gives $\frac{d}{dt} V(t, t_0) = -i H_I(t) V(t) V(t_0)^{-1} = -i H_I(t) V(t, t_0)$, which "asks again for" a Dyson series as formal solution

$$V(t, t_0) \sim \mathcal{T}\left(\exp\left(\frac{1}{i} \int_{t_0}^{t} H_I(\tau) \, d\tau \right) \right) = I + \sum_{n=1}^{\infty} \frac{1}{i^n n!} \int_{[t_0, t]^n} \mathcal{T}\Big(H_I(\tau_1) \cdots H_I(\tau_n) \Big) \, d(\tau_1, \ldots, \tau_n).$$

Sending now $t_0 \to -\infty$ and $t \to \infty$ gives

$$S \sim I + \sum_{n=1}^{\infty} \frac{1}{i^n n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathcal{T}\Big(H_I(\tau_1) \cdots H_I(\tau_n) \Big) \, d\tau_1 \cdots d\tau_n.$$

Suppose H_I was given in terms of a Hamiltonian or Lagrangian density by $H_I = \int_{\mathbb{R}^3} \mathcal{H}_I(x) \, dx = -\int_{\mathbb{R}^3} \mathcal{L}_I(x) \, dx$. Therefore, in this case we have at least formally $H_I(t) = U_0(-t) H_I U_0(t) = \int_{\mathbb{R}^3} U_0(-t) \mathcal{H}_I(x) U_0(t) \, dx = \int_{\mathbb{R}^3} \mathcal{H}_I(t, x) \, dx = -\int_{\mathbb{R}^3} \mathcal{L}_I(t, x) \, dx$ with the obvious "definition" of $\mathcal{H}_I(t, x)$ and $\mathcal{L}_I(t, x)$ as the free time evolution of the operator-valued "function values" $\mathcal{H}_I(x)$ and $\mathcal{L}_I(x)$ at $x \in \mathbb{R}^3$ (of course, these are at best operator-valued distributions, not functions). Combining then the time integrations in the Dyson series with the spatial integration of the Hamiltonian density, we obtain in summary a Dyson series involving integrals over Minkowski space (using here now $x_1, \ldots, x_n$ to denote points in $\mathbb{R}^4$)

$$S \sim I + \sum_{n=1}^{\infty} \frac{1}{i^n n!} \int_{\mathbb{R}^4} \cdots \int_{\mathbb{R}^4} \mathcal{T}\Big(\mathcal{H}_I(x_1) \cdots \mathcal{H}_I(x_n) \Big) \, dx_1 \cdots dx_n,$$

where the time ordering refers to the time components in $x_1, \ldots, x_n \in \mathbb{R}^4$. An important aspect for QFT of this last representation of S is its indication of Lorentz invariance.

[6]In the sense of the Hilbert space $\mathcal{F}$ of all quantum states, this has nothing to do with the spatial geometry!

[7]Here, the word distribution is used (mainly) in the sense of probability/statistics.

C.2.8. Creation and annihilation operators. In the sequel we will employ notation as in the physics literature with annihilation and creation operators treated like operator-valued functions on momentum space (similarly as with the fields ϕ from the Wightman axioms that are mostly written like $\phi(x)$ etc). However, we emphasize that here we are on a sound mathematical basis with explicit constructions of these operators (see, e.g. [**13**, Section 4.5]) on the so-called *Fock space* $\mathcal{F}$ generated from a given *one-particle Hilbert space* $\mathcal{H}$, where often we simply have $\mathcal{H} = L^2(\mathbb{R}^3)$ or $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^m$, if we have to incorporate m additional field components relating to internal symmetries.

We indicate at least in a rough sketch how to rigorously construct annihilation and creation operators. First, recall that a Hilbert space tensor product $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ is defined as the completion of the algebraic tensor product with respect to the norm stemming from the unique scalar product defined by sesquilinear extension from $\langle u_1 \otimes \cdots \otimes u_k | v_1 \otimes \cdots \otimes v_k \rangle_k := \langle u_1 | v_1 \rangle \cdots \langle u_k | v_k \rangle$. In case $\mathcal{H}_1 = \ldots = \mathcal{H}_k = \mathcal{H}$ we write $\bigotimes^k \mathcal{H}$ for the k-fold Hilbert space tensor product and use the convention $\bigotimes^0 \mathcal{H} := \mathbb{C}$. In the second step, the Fock space is given by the Hilbert space orthogonal direct sum $\mathcal{F} := \sum_{k=0}^{\infty} \bigotimes^k \mathcal{H}$, which again includes completion of the algebraic direct sum in the obvious norm stemming from the obvious scalar product, in particular, $\bigotimes^k \mathcal{H} \perp \bigotimes^l \mathcal{H}$ $(k \neq l)$. To be more precise, the above construction has to be carried out with the symmetric tensor product in case of Bosons and with the antisymmetric tensor product in case of Fermions, but we ignore this aspect in favor of brevity.

Let $f \in \mathcal{H}$, then we may define two operators $B(f)^\dagger$ and $B(f)$ on the dense subspace of finite linear combinations of splitting tensors of all orders in $\mathcal{F}$ by the assigments

$$B(f)^\dagger (u_1 \otimes \cdots \otimes u_k) := f \otimes u_1 \otimes \cdots \otimes u_k, \quad B(f)(u_1 \otimes \cdots \otimes u_k) := \langle f | u_1 \rangle \, u_2 \otimes \cdots \otimes u_k.$$

We see that $B(f)^\dagger$ maps k-particle states into states with $k+1$ particles while $B(f)$ decreases the number of particles when acting on product states. These are the prototypes of the so-called *creation* and *annihilation operators* (and it turns out that $B(f)^\dagger$ indeed is the adjoint of $B(f)$). Furthermore, we see that $f \mapsto B(f)^\dagger$ is linear while $f \mapsto B(f)$ is conjugate linear and showing continuity of the matrix elements with respect to f is also easy.

We have already mentioned that, even with the same one-particle Hilbert space, we have to distinguish between the two types of Fock spaces for Fermions and Bosons. Each of these can accommodate an arbitrary number of particles of the same species—in the Fermionic case non of these can be in the same state as is guaranteed by the antisymmetry of the tensor product. If we have a theory with $K \in \mathbb{N}$ different particle species, then one has for each $j = 1, \ldots, K$ a single-particle state space $\mathcal{H}_j$, which also has to take care of the n_j field components (scalar, vector, tensor, spinor type). If $\mathcal{F}_j$ denotes the Fock space for species number j, then the Hilbert state space for the whole system (at least for a non-interacting theory) is $\mathcal{F} := \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_K$ and the creation and annihilation operators corresponding to a certain species j act nontrivially only on "their" factor $\mathcal{F}_j$ and act as the identity on all other factors in the tensor product.

The prototypes of creation and annihilation operators have been defined above for any $f \in L^2(\mathbb{R}^3)$ and one can now easily imagine to take f as a function with support of its Fourier transform $\widehat{f}$ concentrated very close to $p \in \mathbb{R}^3$, which shall give some intuition for the notation being introduced now. Let $p \in \mathbb{R}^3$ represent momentum and σ and π denote some discrete variables, where σ labels spin states and π is a label for the particle species (in a broad sense, so that any discrete parameters are incoropated into π; distinct antiparticles also have a different π-label). We write $a(p, \sigma, \pi)$ or $a^\dagger(p, \sigma, \pi)$ for the operator of annihilation or creation of a particle of species π with spin state σ and momentum p. The *canonical commutation relations* (CCR) and the *canonical anticommutation relations* (CAR) that hold for the corresponding operator-valued distributions ([**13**, Equations (4.48) and (4.56)]) are translated with the current notation into

$$a^\dagger(p,\sigma,\pi) a^\dagger(p',\sigma',\pi') = \pm a^\dagger(p',\sigma',\pi') a^\dagger(p,\sigma,\pi), \quad a(p,\sigma,\pi) a(p',\sigma',\pi') = \pm a(p',\sigma',\pi') a(p,\sigma,\pi),$$

$$\text{(CCR/CAR)} \qquad a(p,\sigma,\pi) a^\dagger(p',\sigma',\pi') = \pm a^\dagger(p',\sigma',\pi') a(p,\sigma,\pi) + (2\pi)^3 \delta(p-p') \delta_{\sigma\sigma'} \delta_{\pi\pi'},$$

where $\pm$ becomes $+$, if at least one of π and π' is Bosonic, and it becomes $-$, if both π and π' are Fermions (cf. [**13**, Equation (6.33)]).

C.2.9. The S-matrix in terms of vacuum expectation values. In the Fock space, we have the special vector $|0\rangle := (1, 0, 0, \dots) \in \mathbb{C} \oplus \sum_{k=1}^{\infty} \bigotimes^k \mathcal{H} = \mathcal{F}$, which is the *no-particle state* and can serve as the *vacuum vector for a free QFT*. It turns out that a dense subspace of $\mathcal{F}$ is generated as the linear hull of all vectors being polynomials of creation operators applied to $|0\rangle$. This suggests for scattering experiments to specify the incoming and outgoing states in the form

$$|\text{in}\rangle := \xi_{\text{in}} = \prod_{l=1}^{k_{\text{in}}} a^\dagger(p_l^{\text{in}}, \sigma_l^{\text{in}}, \pi_l^{\text{in}})|0\rangle \quad \text{and} \quad |\text{out}\rangle := \eta_{\text{out}} = \prod_{j=1}^{k_{\text{out}}} a^\dagger(p_j^{\text{out}}, \sigma_j^{\text{out}}, \pi_j^{\text{out}})|0\rangle.$$

Using the fact that creation and annihilation operators are adjoints of one another, we may thus rewrite the S-matrix in the following form

$$\langle \text{out}|S|\text{in}\rangle := \langle \eta_{\text{out}}|S\xi_{\text{in}}\rangle = \langle 0| \prod_{j=k_{\text{out}}}^{1} a(p_j^{\text{out}}, \sigma_j^{\text{out}}, \pi_j^{\text{out}}) \cdot S \cdot \prod_{l=1}^{k_{\text{in}}} a^\dagger(p_l^{\text{in}}, \sigma_l^{\text{in}}, \pi_l^{\text{in}})|0\rangle.$$

Now we recall that S "is approximated" by a sum of terms $\int_{\mathbb{R}^{4n}} \mathcal{T}(\mathcal{H}_I(x_1) \cdots \mathcal{H}_I(x_n))\, d(x_1, \dots, x_n)$ and assume that $\mathcal{H}_I(x)$ is a *polynomial in the field operators* $\phi_1(x), \dots, \phi_N(x)$. Then we obtain

a series representation associated with $\langle \text{out}|S|\text{in}\rangle = \langle \eta_{\text{out}}|S\xi_{\text{in}}\rangle$,

where the term of order n is $\dfrac{1}{i^n n!}$ times an expression of the form

$(\star)$

$$\int_{\mathbb{R}^{4n}} \langle 0| \prod_{j=k_{\text{out}}}^{1} a(p_j^{\text{out}}, \sigma_j^{\text{out}}, \pi_j^{\text{out}}) \cdot \mathcal{T}\big(\phi_{i_1}(x_1)^{r_1} \cdots \phi_{i_n}(x_n)^{r_n}\big) \cdot \prod_{l=1}^{k_{\text{in}}} a^\dagger(p_l^{\text{in}}, \sigma_l^{\text{in}}, \pi_l^{\text{in}})|0\rangle\, d(x_1, \dots, x_n).$$

The field operators are usually constructed as Fourier integral representations with annihilation and creation operators and are, in principle, of the schematic form (the symbol π appears here in a double role, all but one occurences refer to a particle species)

$$\phi_\pi(t, x) = \sum_\sigma \int_{\mathbb{R}^3} f(q) \Big(u(q, \sigma, \pi) a(q, \sigma, \pi) e^{-it\sqrt{|q|^2 + m^2}} e^{i\langle x, q\rangle} + v(q, \sigma, \pi) a^\dagger(q, \sigma, \bar{\pi}) e^{it\sqrt{|q|^2 + m^2}} e^{-i\langle x, q\rangle} \Big) \frac{dq}{(2\pi)^3},$$

where $\bar{\pi}$ denotes the antiparticle for the species of type π, f is a scalar function, m is the mass of the particle species π and $\bar{\pi}$, and the functions u, v have values in some $\mathbb{C}^d$, for example to represent spinors or polarization directions. Note that taking the adjoint turns the above representation into one for the antiparticle $\bar{\pi}$.

Upon "plugging in" the Fourier integral representations of the field operators, the fundamental terms $(\star)$ for the calculation of the S-matrix are basically multiple integrals of vacuum expectations

$(\star\star)$

$$\langle 0|\mathcal{T}(A_1 \cdots A_d)|0\rangle$$

with $d \in \mathbb{N}$ and

$$A_j = A_j^a + A_j^c \quad (j = 1, \dots d),$$

where A_j^a, A_j^c is a sum or integral of annihilation operators, creation operators, respectively. Here, we have suppressed the dependences on any variables (and indices) referring to points in $\mathbb{R}^4$ or $\mathbb{R}^3$. Moreover, the convention about the time ordering $\mathcal{T}$ has to be adapted: First, to the Fermionic case, with additional minus signs appearing in interchanges of any two Fermionic operators; however, this is without effect for interaction Hamiltonians $\mathcal{H}_I$ containing Fermions only in pairs, which happens to be the case in all relevant models. Second, to the case of operators without time dependence (e.g., the annihilation and creation operators originating from ξ_{in} and η_{out} remain in their respective order).

Any annhiliation operator acting on the vacuum $|0\rangle$ gives the zero vector, i.e., $a(\ldots)|0\rangle = 0$, which leads to some simplifications in the vacuum expectation series thanks to the consequences

$$\langle 0|A_j^c \cdots |0\rangle = 0 \quad \text{and} \quad \langle 0| \cdots A_l^a|0\rangle = 0.$$

C.2.10. Wick or normally ordered products and contractions. We learn from the observation at the end of the previous subsection that, in evaluating vaccum expectations of products $A_1 \cdots A_n$, it would be desirable to have all creation operators to the left of all annihilation operators. The corresponding rearrangement (including the expansion into a sum of products) of the product will be denoted by $:A_1 \cdots A_n:$ and is called *Wick (or normally) ordered product*, with an additional factor of -1 for every term that had two Fermionic operators interchanged. The simplest case is $n = 2$ with at least one Bosonic factor, where we have

$$:A_1A_2: = :(A_1^a + A_1^c)(A_2^a + A_2^c): = A_1^aA_2^a + A_1^cA_2^a + A_2^cA_1^a + A_1^cA_2^c$$

and comparing this with $A_1A_2 = A_1^aA_2^a + A_1^cA_2^a + A_1^aA_2^c + A_1^cA_1^c$ we see that

$$:A_1A_2: = A_1A_2 - [A_1^a, A_2^c].$$

If both A_1 and A_2 are Fermions, then the Wick ordered product is instead

$$:A_1A_2: = :(A_1^a + A_1^c)(A_2^a + A_2^c): = A_1^aA_2^a + A_1^cA_2^a - A_2^cA_1^a + A_1^cA_2^c = A_1A_2 - \{A_1^a, A_2^c\},$$

where $\{B, C\}$ denotes the *anticommutator* $BC + CB$.

In any case, the whole point of the Wick ordering is that we have, by construction,

$$\langle 0|\, :A_1 \cdots A_n:\, |0\rangle = 0.$$

In the example with $n = 2$ we thus obtain

$$\langle 0|A_1A_2|0\rangle = \langle 0|\, :A_1A_2:\, |0\rangle + \langle 0|\, [A_1^a, A_2^c]\, |0\rangle = \langle 0|\, [A_1^a, A_2^c]\, |0\rangle$$

in case with at least one Boson, while for two Fermionic factors,

$$\langle 0|A_1A_2|0\rangle = \langle 0|\, :A_1A_2:\, |0\rangle + \langle 0|\, \{A_1^a, A_2^c\}\, |0\rangle = \langle 0|\, \{A_1^a, A_2^c\}\, |0\rangle.$$

Recall that the CCR or CAR for annihilation and creation operators always give a (distributional) scalar multiple of I for the commutator or anticommutator appearing in the last term of each of the two equations above, hence the expectation values evaluate essentially to that (distributional) scalar factor (which can then be integrated over).

A reasoning very similar as in the above example shows that for two factors A_1 and A_2, the difference between the time ordered product $\mathcal{T}(A_1A_2)$ and the Wick ordered product $:A_1A_2:$ always gives a (distributional) scalar multiple of the identity I. Therefore, one can define the *contraction* of A_1 and A_2 to be the unique (distributional) scalar $\overbrace{A_1A_2}$ such that

$$\mathcal{T}(A_1A_2) -\, :A_1A_2: = \overbrace{A_1A_2} \cdot I.$$

It turns out that in evaluating the vacuum expectation values in $(\star\star)$, which were derived as the main ingredients in the series for the S-matrix in $(\star)$, contractions "is all we need" thanks to

Wick's theorem: $\mathcal{T}(A_1 \cdots A_n)$ is equal to a sum of terms with k contraction pairs, where $0 \leq k \leq \lfloor n/2 \rfloor$, and all other factors in Wick ordering.

The basic case $n = 2$ is just the definition of the contraction, i.e., $\mathcal{T}(A_1A_2) = \overbrace{A_1A_2} +\, :A_1A_2:$. With $n = 3$, one can figure out that Wick's theorem gives four terms

$$\mathcal{T}(A_1A_2A_3) = :A_1A_2A_3: + \overbrace{A_1A_2}\, A_3 + \overbrace{A_1A_3}\, A_2 + \overbrace{A_2A_3}\, A_1,$$

and for $n = 4$ we show only three of the ten terms in

$$\mathcal{T}(A_1A_2A_3A_4) = :A_1A_2A_3A_4: + \overbrace{A_1A_2}\, :A_3A_4: + \ldots + \overbrace{A_1A_4}\,\overbrace{A_2A_3}\,.$$

The important consequence for vacuum expectation values is that

$$\langle 0|\mathcal{T}(A_1 \cdots A_n)|0\rangle = 0, \quad \text{if } n \text{ is odd,}$$

and, with appropriately counting and summing over all possible contractions of pairs,

$$(\star\star\star) \qquad \langle 0|\mathcal{T}(A_1 \cdots A_n)|0\rangle = \sum \overbrace{A_{j_1} A_{j_2}} \cdots \overbrace{A_{j_{2l-1}} A_{j_{2l}}}, \quad \text{if } n = 2l.$$

To summarize, we "only" need to calculate all possible contractions $\overbrace{A_j A_k}$, where A_j and A_k are creation operators, annihilation operators, or field operators. We have the following three different cases to consider:

(i) Both A_j and A_k are creation or annihilation operators: As can be easily deduced from the above examples for expectation values in case of two factors, this boils down simply to the CCR or CAR and produces essentially only 0 or δ distributions of differences of momentum variables.

(ii) One of A_j and A_k is a field and the other is an operator of creation or annihilation: With a Fourier integral representation of the field ϕ_π as in Subsection C.2.9 and applying the CCR/CAR in the integrand one can deduce that

$$\overbrace{a(p,\sigma,\pi')\phi_\pi(t,x)} = f(p)v(p,\sigma,\pi)e^{i(t\sqrt{|p|^2+m^2}-\langle x,p\rangle)}\delta_{\pi\pi'}$$

and

$$\overbrace{\phi_\pi(t,x)a^\dagger(p,\sigma,\pi')} = f(p)u(p,\sigma,\pi)e^{-i(t\sqrt{|p|^2+m^2}-\langle x,p\rangle)}\delta_{\pi\pi'},$$

while the other two contractions $\overbrace{a^\dagger(p,\sigma,\pi')\phi_\pi(t,x)}$ and $\overbrace{\phi_\pi(t,x)a(p,\sigma,\pi')}$ vanish (because $\langle 0|a^\dagger \cdots |0\rangle = 0 = \langle 0| \cdots a|0\rangle$).

(iii) Both factors A_j and A_k are field operators: The Fourier integral representation describes a field ϕ_π in terms of annihilation operators for the species π and creation operators for the antiparticle $\bar{\pi}$. If A_j does not correspond to the antiparticle of A_k, the CCR or CAR always give 0. Therefore, the only nontrivial combination is with $A_j = \phi_\pi(t,x)$ and $A_k = \phi_\pi^\dagger(s,y)$ (with $t,s \in \mathbb{R},\ x,y \in \mathbb{R}^3$) and we will drop now the reference to the particle species π. Recall that, by definition, $\overbrace{A_j A_k} = \mathcal{T}(A_j A_k) - :A_j A_k:$ and $\langle 0| :\phi\phi^\dagger: |0\rangle = 0$, thus we are left with

$$\overbrace{\phi(t,x)\phi^\dagger(s,y)} = \langle 0|\mathcal{T}(\phi(t,x)\phi^\dagger(s,y))|0\rangle.$$

The right-hand side expresses an amplitude, whose modulus squared[8] is to represent a probability density, namely, in case $t > s$, for the creation out of the vacuum of a particle at location y and time s and annihilation of the same at location x and time t; in case $t < s$ it will be creation of an antiparticle out of the vacuum at location x and time t and annihilation at location y and time s. In this sense, the above describes either a particle propagating from y to x or an antiparticle propagating from x to y. The latter justifies the notion of *propagator* for any expression of the type $\langle 0|\mathcal{T}(\phi(t,x)\phi^\dagger(s,y))|0\rangle$.

C.2.11. Propagators are fundamental solutions (or Green functions). Recall that in the interaction picture and for the Dyson series, the time dependence of the observables and fields is understood according to the free time evolution. The basic strategy in the construction of the free fields is to consider the operator-valued analogue of the Fourier integral representations of solutions to a guiding (Lorentz invariant) partial differential equation, e.g., the Klein-Gordon equation or the Dirac equation. Therefore, the coefficients in the Fourier integral representations of the fields still have this information about the solution of the field equation encoded and taking expectation values brings this back from the operator level to functions or distributions. This is

[8] Although these objects are not functions, but distributions (in the sense of generalized functions; but also with a probabilistic interpretation).

the reason why the propagators that emerged above from contractions of field operators turn out to be, in fact, nothing but Green functions or fundamental solutions for the corresponding field equation underlying the construction of the free field operators.

C.2.12. Term by term evaluation of the series for the S-matrix. According to ($\star$), ($\star\star$), and ($\star\star\star$) the S-matrix element $\langle \text{out}|S|\text{in}\rangle = \langle \eta_{\text{out}}|S\xi_{\text{in}}\rangle$ is associated with a series of multiple integrals with integrands that are, apart from constants, products of factors of one of the following types (with ad-hoc notational shortcuts, e.g., suppress the variables σ and π in the creation or annihilation operators and use only simple indices instead; in the notation for the fields, we drop parameters referring to particle species and indices for spinor components etc):

(a) $\overbrace{a_j(p_j^{\text{out}})a_k^\dagger(p_k^{\text{in}})} = (2\pi)^3\delta_{jk}\,\delta(p_j^{\text{out}} - p_k^{\text{in}})$;

(b) $\overbrace{a_j(p_j^{\text{out}})\phi(t_l, x_l)} = 0$, if ϕ does not create particles of type j, and otherwise we have
$$\overbrace{a_j(p_j^{\text{out}})\phi(t_l, x_l)} = e^{-i\left(t_l\sqrt{|p_j^{\text{out}}|^2+m_j^2} - \langle x_l, p_j^{\text{out}}\rangle\right)}u(p_j^{\text{out}}),$$
where m_j now denotes the mass of particle j;

(c) $\overbrace{\phi(t_l, x_l)a_k^\dagger(p_k^{\text{in}})} = 0$, if ϕ does not annihilate particles of type k, and otherwise we have
$$\overbrace{\phi(t_l, x_l)a_k^\dagger(p_k^{\text{in}})} = e^{i\left(t_l\sqrt{|p_k^{\text{in}}|^2+m_k^2} - \langle x_l, p_k^{\text{in}}\rangle\right)}u(p_k^{\text{in}});$$

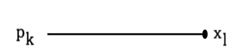
(d) $\overbrace{\phi(t_r, x_r)\phi^\dagger(t_l, x_l)} =: -i\Delta(t_r - t_l, x_r - x_l)$ is just the propagator for this field, i.e., some fundamental solution (Green function).

Note that (b) and (c) yield Fourier transforms upon integration with respect to (t_l, x_l), while (a) represents a subprocess without interaction, since only $p_j^{\text{out}} = p_k^{\text{in}}$ contributes here.

Towards Feynman rules and Feynman diagrams: The only variables in the above evaluation process that are not integrated over are the 3-dimensional momenta p_j^{out} and p_k^{in}. Recall that for any particle of 3-momentum $\vec{p}$ and mass m, the corresponding 4-momentum is $(\sqrt{|\vec{p}|^2 + m^2}, \vec{p})$. To simplify notation, we will henceforth write simply p_j for a 4-momentum corresponding to p_j^{out} etc. These 4-momenta $p_1, p_2, \dots$ are used as labels for the so-called *external vertices*. (The distinction whether some p_j belongs to an incoming or outgoing particle can be indicated in the Feynman diagrams by other means.) We will use labels $x_1, x_2, \dots \in \mathbb{R}^4$ for the so-called *internal vertices* and each of these correspond to one integral over Minkowski space.

For every contraction there will be an *edge* in the graph connecting the appropriate vertices. In the following simple example, the formula on the left corresponds to the picture on the right:

$$\overbrace{\phi(x_l)a_k^\dagger(p_k^{\text{in}})}$$

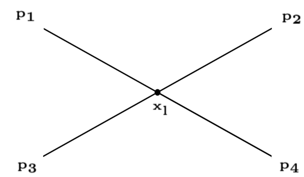


There are conventions about the style of line drawing for different species of particles, e.g., in Quantum Electrodynamics (QED) electrons have solid lines while photon edges are wavy. In addition, there are conventions for adding momentum flow directions and orientation of arrows (particle vs. antiparticle) and rules for assigning diagrams or parts of diagrams (and vice versa) to the factors in the integrands described in (a), (b), (c), (d) above and how to combine and evaluate these. (In particular, (a) produces disconnected diagrams, because the edge between the vertices p_j and p_k will not be attached to anything else.) Proceeding in this way, one obtains a kind of isomorphism between the Dyson series of integrals for the S-matrix and a cascade of ever more complex Feynman diagrams.

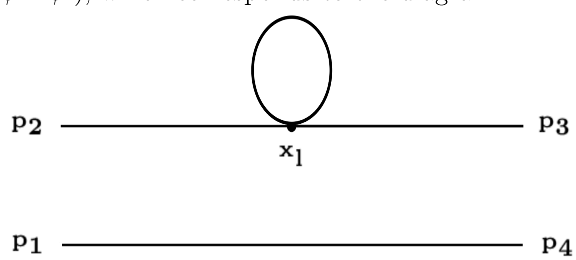
Now in retrospect, since all the ϕ-occurrences in the middle of formula ($\star$) derive immediately from the Lagrange or Hamilton density $\mathcal{L}_I(x) = -\mathcal{H}_I(x)$, one may more or less jump directly from the specification of the latter to calculations with Feynman diagrams. In a very nice schematic figure of QFT in [**11**, Abbildung 4.1, Seite 106], this short-cut is emphasized by bold double arrows labeled "Einfache Rezepte" (simple recipies).

Example with ϕ^4-interaction: If ϕ is a real scalar field and $\mathcal{H}_I(x) = \phi(x)^4 = -\mathcal{L}_I(x)$, then already at order $n = 1$ in the Dyson series the factor $\phi(x_l)^4$ in the integrand will, according to Wick's theorem produce, a sum of terms with contractions, one of these will be

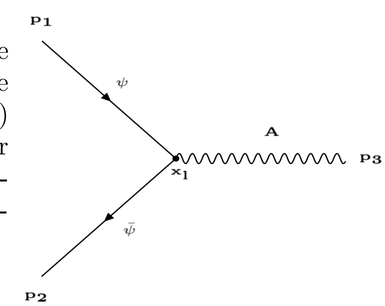
$\overbrace{a(p_4)\phi(x_l)}\,\overbrace{a(p_2)\phi(x_l)}\,\overbrace{\phi(x_l)a^\dagger(p_3)}\,\overbrace{\phi(x_l)a^\dagger(p_1)}$
and define the following Feynman diagram



while one of the other terms will be of the form $\overbrace{a(p_4)a^\dagger(p_1)}\,\overbrace{\phi(x_l)a^\dagger(p_2)}\,\overbrace{\phi(x_l)\phi(x_l)}\,\overbrace{a(p_3)\phi(x_l)}$ (since ϕ is real, we have $\phi = \phi^\dagger$), which corresponds to the diagram



A simplified QED-like interaction: Let ψ be complex and A be real, so that $\overline{\psi}$ represents the antiparticle of ψ. Consider the interaction Lagrangrian $\mathcal{L}_I(x) = -e\overline{\psi}(x)A(x)\psi(x) = -\mathcal{H}_I(x)$ with a coupling constant $e \in \mathbb{R}$. In this case, the factor $A(x_l)\overline{\psi}(x_l)\psi(x_l)$ in the integrand at order $n = 1$, upon contractions, will contribute, among others, the following diagram corresponding to the product $\overbrace{a(p_3)A(x_l)}\,\overbrace{\overline{\psi}(x_l)a^\dagger(p_2)}\,\overbrace{\psi(x_l)a^\dagger(p_1)}$.



C.2.13. A few further remarks: (i) The typical interaction Lagrangians in physics involve some explicit *coupling constant(s)* and the constributions of the terms in the Dyson series are usually sorted by powers of it. One is then often interested in perturbation calculation up to a certain order in the coupling constant. However, as is well-known and notorious, convergence of the Dyson series cannot be expected and even several of the terms of relatively low order do not give finite results (e.g., loops in Feynman diagrams often produce divergent integrals). To cope with these, various ingenious techniques have been developed in the art of *renormalization* with impressive successes in producing highly accurate predictions of measurements.

(ii) As some readers might have experienced themselves, one often has a much more explicit expression for fundamental solutions of the Klein-Gordon or the Dirac equation upon applying Fourier transforms. Since these provide the crucial propagators for QFT, there are much more elegant *momentum space Feynman rules* and these are, in fact, more important in real applications.

(iii) We recall from C.2.2 that the time dependence of operators in the interaction picture is given according to the free time evolution $U_0(t)$. In particular, throughout the perturbation theoretic considerations starting with Subsection C.2.2 and up to now, a field ϕ did not represent the

Heisenberg picture according to time evolution $U(t)$. For this reason, one often finds a notation like ϕ_{0j} for the interaction picture time evolution to distinguish it from the field ϕ_j in the sense of the Gårding-Wightman axioms. We now adopt this notation here.

Let Ω denote the vacuum vector for the interacting theory and $|0\rangle$ be the vacuum vector of the free theory (see C.2.9). We briefly mention two results, accessible even from axiomatic theories, that relate (a) the S-matrix with expectation values of the interacting fields ϕ_j with respect to the interaction vacuum Ω and (b) the latter with free field vacuum expectations:

(a) The *Lehmann-Symanzik-Zimmerman (LSZ) reduction:* For $n \in \mathbb{N}$, denote by $W_n(p_1, \ldots, p_n)$ the Fourier transform of $\tilde{w}_n$, a "time-ordered variant" of the Wightman distribution $w^n_{1,\ldots,n}$ (compare with the discussion of the reconstruction theorem in C.1.5), i.e., $\tilde{w}_n$ is the temperate distribution on $\mathbb{R}^{4n}$ that maps splitting tensors $f_1 \otimes \cdots \otimes f_n$ with $f_l \in \mathscr{S}(\mathbb{R}^4)$ $(l = 1, \ldots, n)$ to $\langle \Omega | \phi_1(f_1) \cdots \phi_n(f_n) \Omega \rangle$. Then we have

$$\langle \mathrm{out} | S | \mathrm{in} \rangle = i^{k_{\mathrm{out}} + k_{\mathrm{in}}} \prod_{j=1}^{k_{\mathrm{out}}} \frac{\overline{u_j}(p_j^{\mathrm{out}})}{\widehat{\Delta_j}(p_j^{\mathrm{out}})} \prod_{l=1}^{k_{\mathrm{in}}} \frac{u_l(p_l^{\mathrm{in}})}{\widehat{\Delta_l}(-p_l^{\mathrm{in}})} W_{k_{\mathrm{out}} + k_{\mathrm{in}}}(p_1^{\mathrm{out}}, \ldots, p_{k_{\mathrm{out}}}^{\mathrm{out}}, -p_1^{\mathrm{in}}, \ldots, -p_{k_{\mathrm{in}}}^{\mathrm{in}}),$$

where $\widehat{\Delta_r}$ denotes the Fourier transform of the propagator for the field ϕ_r and $\overline{u_j}$, u_l stem from the coefficients in the Fourier representation of the fields.

(b) The *Formula of Gell-Mann and Low:* In the sense of time-ordered exponentials similar to the Dyson series, or rather via the notion of generating functionals, or alternatively via boundary values of analytic distributions, the vacuum expectations of the interacting theory (left-hand side) can be expressed solely in terms of the free fields, the interaction Lagrangian of the free fields, and the vacuum of the free theory by

$$\langle \Omega | \phi_1(x_1) \cdots \phi_n(x_n) \Omega \rangle = \frac{\langle 0 | \mathcal{T}(\phi_{01}(x_1) \cdots \phi_{0n}(x_n) e^{i \int_{\mathbb{R}^4} \mathcal{L}_I[\phi_0]}) | 0 \rangle}{\langle 0 | \mathcal{T}(\exp(i \int_{\mathbb{R}^4} \mathcal{L}_I[\phi_0])) | 0 \rangle}.$$

Bibliography

[1] R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, tensor analysis, and applications*. Applied Mathematical Sciences, vol. 75. Springer-Verlag, New York, Second, 1988.

[2] H. Araki. *Mathematical theory of quantum fields*. International Series of Monographs on Physics, vol. 101. Oxford University Press, New York, 1999. Translated from the 1993 Japanese original by Ursula Carow-Watamura.

[3] H. Baum. *Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltigkeiten*. Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], vol. 41. BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1981. With English, French and Russian summaries.

[4] D. Bleecker. *Gauge theory and variational principles*. Global Analysis Pure and Applied Series A, vol. 1. Addison-Wesley Publishing Co., Reading, Mass., 1981.

[5] N. N. Bogolubov, A. A. Logunov, A. I. Oksak, and I. T. Todorov. *General principles of quantum field theory*. Mathematical Physics and Applied Mathematics, vol. 10. Kluwer Academic Publishers Group, Dordrecht, 1990.

[6] B. G. Chen, D. Derbes, D. Griffiths, B. Hill, R. Sohn, and Y.-S. Ting, editors. *Quantum Field Theory Lectures of Sidney Coleman*. World Scientific Publishing, 2019.

[7] A. Deitmar and S. Echterhoff. *Principles of harmonic analysis*. Universitext. Springer, Cham, Second, 2014.

[8] J. Dieudonné. *Treatise on analysis. Vol. III*. Academic Press, New York-London, 1972. Translated from the French by I. G. MacDonald, Pure and Applied Mathematics, Vol. 10-III.

[9] J. Dieudonné. *Treatise on analysis. Vol. II*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1976. Enlarged and corrected printing, Pure and Applied Mathematics, 10-II.

[10] A. Duncan. *The conceptual framework of quantum field theory*. Oxford University Press, Oxford, 2012.

[11] L. Edelhäuser and A. Knochel. *Tutorium Quantenfeldtheorie*. Springer-Verlag, Berlin Heidelberg, 2016.

[12] G. B. Folland. *A course in abstract harmonic analysis*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, Second, 2016.

[13] G. B. Folland. *Quantum field theory*. Mathematical Surveys and Monographs, vol. 149. American Mathematical Society, Providence, RI, 2008. A tourist guide for mathematicians.

[14] Th. Friedrich. *Dirac operators in Riemannian geometry*. Graduate Studies in Mathematics, vol. 25. American Mathematical Society, Providence, RI, 2000. Translated from the 1997 German original by Andreas Nestke.

[15] J. Glimm and A. Jaffe. *Quantum physics*. Springer-Verlag, New York, Second edition, 1987. A functional integral point of view.

[16] W. Greub. *Multilinear algebra*. Springer-Verlag, New York-Heidelberg, Second, 1978. Universitext.

[17] R. Haag. *Local quantum physics*. Texts and Monographs in Physics. Springer-Verlag, Berlin, Second edition, 1996. Fields, particles, algebras.

[18] B. C. Hall. *Lie groups, Lie algebras, and Representations*. Graduate Texts in Mathematics. Springer International Publishing Switzerland, Second, 2015.

[19] M. J. D. Hamilton. *Mathematical gauge theory*. Universitext. Springer, Cham, 2017. With applications to the standard model of particle physics.

[20] F. R. Harvey. *Spinors and calibrations*. Perspectives in Mathematics, vol. 9. Academic Press, Inc., Boston, MA, 1990.

[21] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2001.

[22] H. Kragh. *Dirac*. Cambridge University Press, Cambridge, 1990. A scientific biography.

[23] M Kunzinger. *Differential Geometry i*, 2008. Lecture notes, University of Vienna, Summer Term 2008.

[24] M Kunzinger. *Lie Groups*, 2019. Lecture notes, University of Vienna, Winter Term 2019.

[25] M Kunzinger. *Principal fiber bundles*, 2020. Lecture notes, University of Vienna, Summer Term 2020.

[26] H. B. Lawson Jr. and M.-L. Michelsohn. *Spin geometry*. Princeton Mathematical Series, vol. 38. Princeton University Press, Princeton, NJ, 1989.

[27] J. M. Lee. *Introduction to topological manifolds*. Graduate Texts in Mathematics, vol. 202. Springer, New York, Second, 2011.

[28] J. M. Lee. *Introduction to smooth manifolds*. Graduate Texts in Mathematics, vol. 218. Springer, New York, Second, 2013.

[29] B. O'Neill. *Semi-Riemannian geometry*. Pure and Applied Mathematics, vol. 103. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983. With applications to relativity.

[30] M. E. Peskin and D. V. Schroeder. *An introduction to quantum field theory*. Addison-Wesley Publishing Company, Advanced Book Program, Reading, MA, 1995.

[31] M. Reed and B. Simon. *Methods of modern mathematical physics II: Fourier analysis, self-adjointness*. Academic Press, New York, 1975.

[32] M. Reed and B. Simon. *Methods of modern mathematical physics I: Functional analysis*. Academic Press, New York, second, 1980.

[33] S. Roman. *Advanced linear algebra*. Graduate Texts in Mathematics, vol. 135. Springer, New York, Third, 2008.

[34] G. Rudolph and M. Schmidt. *Differential geometry and mathematical physics. Part II*. Theoretical and Mathematical Physics. Springer, Dordrecht, 2017. Fibre bundles, topology and gauge fields.

[35] M. D. Schwartz. *Quantum field theory and the standard model*. Cambridge University Press, Cambridge, 2014.

[36] S. S. Schweber. *QED and the men who made it: Dyson, Feynman, Schwinger, and Tomonaga*. Princeton Series in Physics. Princeton University Press, Princeton, NJ, 1994.

[37] R. F. Streater and A. S. Wightman. *PCT, spin and statistics, and all that*. Princeton Landmarks in Physics. Princeton University Press, Princeton, NJ, 2000. Corrected third printing of the 1978 edition.

[38] S. Weinberg. *The quantum theory of fields. Vol. I*. Cambridge University Press, Cambridge, 2005. Foundations.

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