THE PROBABILITY THAT A CHARACTER VALUE IS ZERO FOR THE SYMMETRIC GROUP

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INTRODUCTION

Let χ be chosen at random from the irreducible characters of the symmetric group S_n and let g be chosen at random from the group itself. What is the probability that $\chi(g) = 0$? In this short note we give a remarkable asymptotic answer of one. Throughout the paper "at random" means uniformly at random.

Theorem 1. *If χ is chosen at random from the irreducible characters of S_n and g is chosen at random from S_n, then $\chi(g) = 0$ with probability $P(S_n) \to 1$ as $n \to \infty$.*

It will follow that the same must be true for the alternating group A_n.

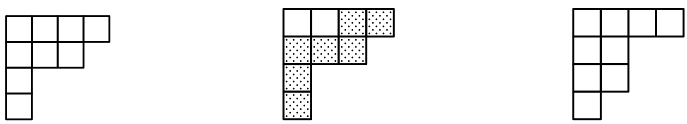
Theorem 2. *If χ is chosen at random from the irreducible characters of A_n and g is chosen at random from A_n, then $\chi(g) = 0$ with probability $P(A_n) \to 1$ as $n \to \infty$.*

We prove these results in Section 1 and make some remarks in Section 2.

1. PROOFS

Theorem 1 is a direct consequence of the Murnaghan–Nakayama rule and two classical results about random partitions and random permutations. We give a second proof without the Murnaghan–Nakayama rule in Section 2.

Recall that a partition of n is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_\ell = n$. The Young diagram of λ is the left-justified array with λ_1 boxes in first row, λ_2 boxes in the second row, and so on; see Figure 1(a). The total number of partitions of n is denoted by p_n.



(a) (b) (c)

FIGURE 1. The (a) diagram, (b) border, and (c) conjugate of $(4, 3, 1, 1)$.

A permutation $g \in S_n$ factors into disjoint cycles, and the cycle lengths determine g up to conjugation. Write K_λ for the conjugacy class of g where λ is the

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partition of n whose parts are the cycle lengths for g. In particular, the number of conjugacy classes (resp. irreducible characters) of S_n is equal to p_n. Write χ^λ for the irreducible S_n-character associated to the partition λ of n in the usual way [5].

The character values $\chi^\lambda(g)$ can be computed using border strips. The border of a partition λ is the set of boxes in the Young diagram that have no southeast neighbor, as shown in Figure 1(b), and a border strip of λ is a connected subset of border boxes whose complementary set of boxes $\lambda \setminus \beta$ is a valid Young diagram. The height $\mathrm{ht}(\beta)$ of a border strip is one less than the number of rows that it occupies. If $g \in S_n$ has a k-cycle x then $g = xy$ for some disjoint $y \in S_{n-k}$ and the Murnaghan–Nakayama rule [5, Thm. 2.4.7] says that

$$\chi^\lambda(g) = \sum\nolimits_\beta (-1)^{\mathrm{ht}(\beta)} \chi^{\lambda \setminus \beta}(y)$$

where β runs over all border strips of λ with exactly k boxes. If λ has no border strip of size k then $\chi^\lambda(g) = 0$. In particular $\chi^\lambda(g) = 0$ if $k \geq \ell(\lambda) + \lambda_1$.

We use the Murnaghan–Nakayama rule in tandem with two other old results to show that $P(S_n)$ tends to one. We use the classical result of Erdős and Lehner [1] which tells us that, if $f(n)$ is any function which tends to infinity with n, then for all but at most $o(p_n)$ (as $n \to \infty$) partitions λ of n the number of parts $\ell(\lambda)$ and the largest part λ_1 satisfy

$$c\sqrt{n}(\log n - f(n)) \;\leq\; \lambda_1, \ell(\lambda) \;\leq\; c\sqrt{n}(\log n + f(n)) \tag{1}$$

where c is some explicit positive constant. We also use the following result of Goncharov [3] about the number of cycles m of an element in S_n:

$$\text{Prob.}\left\{\alpha < \frac{m - \log n}{\sqrt{2\log n}} < \beta\right\} \to \pi^{-\frac{1}{2}} \int_\alpha^\beta e^{-t^2}\,dt, \qquad n \to \infty.$$

First proof of Theorem 1. Let $B(n)$ be the set of partitions λ of n that satisfy (1) when $f(n) = \log n$, so that $|B(n)|/p_n$ tends to one as n tends to infinity.

Goncharov's result tells us that all but at most $o(n!)$ permutations in S_n have $\log n + o(\log n)$ cycles, so all but at most $o(n!)$ have a cycle of size at least $n/(2\log n)$.

Let $C(n)$ be the set of elements in S_n that have a cycle of size at least $n/(2\log n)$. Partitions in $B(n)$ have border strips of size at most $4c\sqrt{n}\log n$, which is smaller than $n/(2\log n)$ for n sufficiently large, so by the Murnaghan–Nakayama rule

$$P(S_n) \geq \frac{|B(n)||C(n)|}{p_n n!}$$

for n sufficiently large, and the right side tends to 1 by the previous paragraphs. □

Recall the usual construction of the irreducible characters of A_n by restricting down from S_n. Let λ be a partition of n and let λ' be the conjugate partition, so that the Young diagram for λ' is the transpose of the diagram for λ; see Figure 1(c). We say that λ is self-conjugate if $\lambda = \lambda'$. Then the following hold [5, Thm. 2.5.7]:

(i) If $\lambda \neq \lambda'$ then the restrictions of χ^λ and $\chi^{\lambda'}$ to A_n are equal and irreducible.
(ii) If $\lambda = \lambda'$ then the restriction of χ^λ to A_n is a sum of two distinct irreducible characters.
(iii) Each irreducible character of A_n arises in this way from a unique pair λ, λ'.

Proof of Theorem 2. First note that at most $o(p_n)$ partitions of n are self-conjugate; a well-known result [5, p. 67] says that the number of self-conjugate partitions of n equals the number of partitions of n into distinct odd parts, and there are at most $o(p_n)$ of the latter because there are at most $o(p_n)$ partitions of n in total that have fewer than $\sqrt{n}$ parts by Erdős–Lehner with $f(n) = \log\log n$ in (1) for example.

Write $\mathrm{Irr}(S_n)$ as the disjoint union $X_1 \cup X_2$ where X_1 is the set irreducible characters associated to self-conjugate partitions of n and let $\mathrm{Irr}(A_n) = Y_1 \cup Y_2$ be the corresponding partition of $\mathrm{Irr}(A_n)$ according to (i)–(iii) above, so that the maps $Y_1 \to X_1$ and $X_2 \to Y_2$ given by induction and restriction are double covers. Then $|X_1|/|\mathrm{Irr}(S_n)|$ and $|Y_1|/|\mathrm{Irr}(A_n)| \to 0$ as $n \to \infty$.

For $X \subseteq \mathrm{Irr}(G)$ and $S \subseteq G$ write $P(X, S)$ for the proportion of pairs (χ, g) in $X \times S$ that satisfy $\chi(g) = 0$. Theorem 1 says $P(\mathrm{Irr}(S_n), S_n) \to 1$ (as $n \to \infty$), so by the previous paragraph $P(X_2, S_n) \to 1$, and since A_n covers half of S_n it follows that $P(X_2, A_n) \to 1$. Since $P(X_2, A_n) = P(Y_2, A_n)$ by (i) and (iii) we thus have that $P(Y_2, A_n) \to 1$. Hence $P(\mathrm{Irr}(A_n), A_n) \to 1$ by the previous paragraph. □

2. Remarks

2.1. Empirical evidence suggests that many other groups have a high proportion of character values equal to zero as well, and one might conjecture that the following question has a positive answer, perhaps even for a wider class of groups. For a finite group G write $P(G)$ for the probability that $\chi(g) = 0$ when χ is chosen at random from the irreducible characters and g is chosen at random from the group.

Question 1. *Let P_ϵ be the proportion of finite simple groups G of size less than n which satisfy $P(G) > 1 - \epsilon$. Then is it true that for every $\epsilon > 0$ one has that $P_\epsilon \to 1$ as $n \to \infty$?*

It would be interesting to show that $P(G) > \epsilon$ with probability $\to 1$ as $n \to \infty$ even for small ϵ. The following estimate for $P(G)$ is a direct consequence of Gallagher's estimate [2, p. 127] for the number of zeros in a given column of a character table. We give a proof of Proposition 3 for the reader's convenience, then we use the proposition to prove Theorem 1 without appealing to Murnaghan–Nakayama.

Proposition 3. *Let Ω be a set of classes of a finite group G. Then*

$$P(G) \geq Q(G, \Omega) - R(G, \Omega), \tag{2}$$

where $Q(G, \Omega)$ is the proportion of G covered by Ω, and $R(G, \Omega)$ is the proportion of classes which belong to Ω. Moreover, the right-hand side of (2) is largest when Ω is the set of larger than average classes.

Proof. The character values $\chi(g)$ for G are sums of roots of unity lying in a cyclotomic extension $E/\mathbb{Q}$ whose Galois group $\mathcal{G}$ is abelian and commutes with complex conjugation, so if the algebraic integer $|\chi(g)|^2$ is positive then it is *totally positive* in the sense that $\sigma(|\chi(g)|^2)$ is positive for every embedding $\sigma : E \hookrightarrow \mathbb{C}$. Let $\mathrm{Av} : E \to \mathbb{C}$ denote the average of the embeddings $\sigma \in \mathcal{G}$. If $\chi(g)$ is not zero then the product $\prod \sigma(|\chi(g)|^2)$ over all $\sigma \in \mathcal{G}$ is at least one because it is a nonzero rational algebraic integer. Hence by the theorem of arithmetic and geometric means $\mathrm{Av}(|\chi(g)|^2) \geq 1$ for $\chi(g) \neq 0$. (See for example [2, p. 127], [4, p. 40], [6, p. 37].)

For $g \in G$, the usual column orthogonality relation [4, p. 21] tells us that $\sum |\chi(g)|^2 = |C_G(g)|$ where the sum is over all $\chi \in \mathrm{Irr}(G)$ and $C_G(g)$ is the centralizer of g. Hence

$$\sum_{\chi} \mathrm{Av}\left(|\chi(g)|^2\right) = |C_G(g)|.$$

The number of terms on the left side is the total number of conjugacy classes $|\mathrm{Cl}(G)|$, and $\mathrm{Av}(|\chi(g)|^2)$ is at least one if $\chi(g)$ is not zero, so at least $|\mathrm{Cl}(G)| - |C_G(g)|$ irreducible characters vanish at g, and thus at every conjugate of g. This is Gallagher's result [2, p. 127], and it implies that

$$P(G) \geq \frac{1}{|\mathrm{Cl}(G)|\,|G|} \sum_{K \in \Omega} \left(|\mathrm{Cl}(G)| - |C_G(g)|\right) |K|$$

where $g \in K$. Rewriting $|C_G(g)|$ as $|G|/|K|$ gives

$$P(G) \geq \sum_{K \in \Omega} |K|/|G| - |\Omega|/|\mathrm{Cl}(G)|. \qquad \square$$

Remark 1. Averaging in the proof of Proposition 3 is superfluous when the character values for G are rational integers, which happens if and only if each $g \in G$ is conjugate to g^m for all m relatively prime to $|G|$ (see [4, p. 31]), as in the case when G is S_n. We now use Proposition 3 with $G = S_n$ to prove Theorem 1 directly from the above results of Erdős–Lehner and Goncharov:

Second proof of Theorem 1. Let Ω_n be the set of S_n-classes K_λ such that the largest part of λ is greater than $2c\sqrt{n}\log n$, so that Erdős–Lehner with $f(n) = \log n$ in (1) tells us that $R(S_n, \Omega_n) \to 0$ as $n \to \infty$.

To see that $Q(S_n, \Omega_n) \to 1$ as $n \to \infty$ recall from the first proof of Theorem 1 that Goncharov's result implies that all but at most $o(n!)$ elements of S_n have a cycle of size at least $n/(2\log n)$. Hence for n sufficiently large, all but at most $o(n!)$ elements of S_n have a cycle greater than $2c\sqrt{n}\log n$ as n tends to infinity. $\square$

Remark 2. Proposition 3 used $\mathrm{Av}(|\chi(g)|^2) \geq 1$ for nonzero $\chi(g)$. A result of [7] Siegel tells us that in fact $\mathrm{Av}(|\chi(g)|^2) \geq 3/2$ if $|\chi(g)| \neq 0, 1$; see [6, p. 37] and cf. [4, p. 46]. The stronger inequality gives a slightly better estimate for $P(G)$.

2.2. We also ask about choosing $\chi(g)$ at random from the character table of S_n.

Question 2. *Let χ be chosen at random from the irreducible characters of S_n and let K be chosen at random from the conjugacy classes of S_n. What can be said about the probability that $\chi(g_K) = 0$ as $n \to \infty$? (Here $g_K \in K$ is arbitrary.)*

One might conjecture that the probability converges to $1/e$, or perhaps even $1/3$. It would also be interesting then to investigate similar asymptotic questions about the nonzero entries. For example, we ask the following.

Question 3. *Does the ratio of positive to negative entries of the character table of S_n tend to one as n tends to infinity?*

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