CONGRUENCES IN CHARACTER TABLES OF SYMMETRIC GROUPS

ALEXANDER R. MILLER

ABSTRACT. If λ and μ are two non-empty Young diagrams with the same number of squares, and $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are obtained by dividing each square into d^2 congruent squares, then the corresponding character value $\chi_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ is divisible by $d!$.

1. Introduction

For any partition $\lambda = 1^{m_1}2^{m_2}\dots n^{m_n}$ of an integer n, let χ_λ be the corresponding irreducible character of the symmetric group S_n, let $\chi_\lambda(\mu)$ be the value at any $\sigma \in S_n$ of cycle type μ, and, fixing once and for all a positive integer d, define partitions

$$d.\lambda = d^{m_1}(2d)^{m_2}\dots(nd)^{m_n}, \qquad \boldsymbol{\lambda} = d^{dm_1}(2d)^{dm_2}\dots(nd)^{dm_n},$$

so $d.\lambda$ is obtained by scaling the parts of λ, and $\boldsymbol{\lambda}$ is obtained by subdividing the squares of the Young diagram of λ. The purpose of this paper is to prove:

Theorem 1. *For any two partitions λ and μ of a positive integer,*

$$\chi_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) \equiv 0 \pmod{d!}. \tag{1.1}$$

More generally, for any partition λ of a positive integer n, and any partition μ of dn,

$$\chi_{\boldsymbol{\lambda}}(d.\mu) \equiv 0 \pmod{d!}. \tag{1.2}$$

For any two partitions λ and μ of a positive integer not divisible by d,

$$\chi_{\boldsymbol{\lambda}}(d^2.\mu) = 0. \tag{1.3}$$

Explicit results like these are rare. Previous results include J. McKay's characterization of partitions λ of n satisfying $\chi_\lambda(1^n) \equiv 0 \pmod 2$ [10], I. G. Macdonald's generalization for $\chi_\lambda(1^n) \equiv 0 \pmod p$ [8], the corollary of Murnaghan–Nakayama that $\chi_\lambda(\mu) = 0$ under certain conditions involving hook lengths [9], and the relation between ordinary and modular vanishing given by the fact that Frobenius' formula for $\chi_\lambda(\mu)$ [4] implies, for any prime p, that $\chi_\lambda(\mu) \equiv \chi_\lambda(\nu) \pmod p$ whenever ν can be obtained from μ by breaking some part into p equal parts.

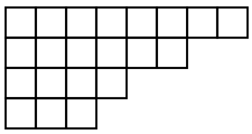
There are also general results of Burnside, J. G. Thompson, and P. X. Gallagher, with Burnside proving that zeros exist for nonlinear irreducible characters of a finite group [1], J. G. Thompson modifying Burnside's argument with a result of C. L. Siegel [15] to show that each irreducible character is zero or root of unity on more than a third of the group [7], and P. X. Gallagher proving similarly that more than a third of the irreducible characters are zero or root of unity on a larger than average class [5]. For large symmetric groups S_n it was shown a few years ago [11], using estimates of Erdős–Lehner [3] and Goncharoff [6], that $\chi(\sigma) = 0$ for all but an $o(1)$ proportion of pairs $\chi, \sigma \in \mathrm{Irr}(S_n) \times S_n$, and conjectured [12] that, for any prime p, $\chi_\lambda(\mu) \equiv 0 \pmod p$ for all but an $o(1)$ proportion of pairs of partitions λ, μ of n. Theorem 1 with $d \geq p$ implies $\chi_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) \equiv 0 \pmod p$ for *all* pairs of partitions λ, μ of n.

We prove (1.1) and (1.2) by showing that in the Murnaghan–Nakayama formula for computing $\chi_{\boldsymbol{\lambda}}(d.\mu)$ as a weighted sum over certain rim hook tableaux, the relevant rim hook tableaux admit an action of S_d that is both free and weight-preserving. This is done by first translating from rim hook tableaux to some new objects we call *cascades*, which are a matrix analogue of Comét's classical one-line binary notation for partitions, and which can be viewed as collections of lattice paths with weight defined in terms of crossings. As a benefit of independent interest, we obtain a lattice-path version of Murnaghan–Nakayama in Proposition 1. Then in Theorem 2 we establish an explicit weight-preserving free action of S_d on cascades. As a corollary we obtain (1.1) and (1.2), while (1.3) will come from Proposition 1.

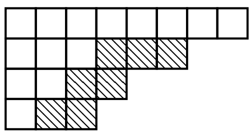
2. Preliminaries

By *partition* of an integer $n \geq 0$ we mean an integer sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ satisfying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l \geq 1$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_l = n$. We say λ has *size* n with l *parts*, writing $|\lambda| = n$ and $\ell(\lambda) = l$. The alternative shorthand $\lambda = 1^{m_1} 2^{m_2} \ldots n^{m_n}$ means λ is the partition with m_1 1's, m_2 2's, and so on, e.g. $(4, 2, 1, 1) = 1^2 2^1 4^1$.

We identify λ with its *shape* or *Young diagram*, i.e. the left-justified array with λ_1 squares in the first row, λ_2 squares in the second row, and so on, e.g. the partition $(8, 6, 4, 3)$ is identified with the following shape:



By *rim hook* ρ of λ we mean the union of a non-empty sequence of squares in λ such that each square is directly to the left or directly below the previous square and $\lambda \setminus \rho$ is a Young diagram, e.g. the following is a rim hook of size 7 in $(8, 6, 4, 3)$:



By *rim hook tableau* T we mean a labeling of the squares of a non-empty Young diagram λ with integers $1, 2, \ldots, m$ such that the squares with label $\geq i$ form a Young diagram T_i and, for $1 \leq i \leq m$, the squares labeled i form a (non-empty) rim hook of size α_i in T_i. We say T has *shape* λ and *content* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, we write

$$T = \mathrm{Tab}(T_1, T_2, \ldots, T_{m+1}),$$

so $T_1 = \lambda$ and $T_{m+1} = \emptyset$, and we define the *weight* of T by

$$\mathrm{wt}(T) = \prod_{i=1}^{m} (-1)^{\#\{\text{rows of } T \text{ occupied by } i\} - 1}. \tag{2.1}$$

An example rim hook tableau of shape $(8,6,4,3)$ and content $(4,4,6,3,2,2)$ is

6	5	3	3	2	1	1	1
6	5	3	2	2	1		
4	4	3	2				
4	3	3					

which has weight $(-1)^{2-1+3-1+4-1+2-1+2-1+2-1}$.

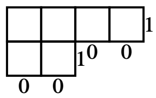
Denoting by $\mathcal{T}(\lambda,\alpha)$ the set of all rim hook tableaux of shape λ and content $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_m)$, the mapping $T \mapsto (T_1,T_2,\ldots,T_{m+1})$ takes $\mathcal{T}(\lambda,\alpha)$ bijectively onto the set of all partition sequences $\lambda = \lambda^1,\lambda^2,\ldots,\lambda^{m+1} = \emptyset$ in which each succeeding λ^i is obtained from the previous partition λ^{i-1} by removing a rim hook of size α_{i-1}, so in this way rim hook tableaux serve as shorthand for the various ways of going from λ to $\emptyset$ by successively removing rim hooks of prescribed size.

The *Murnaghan–Nakayama formula* [13, 14] gives, for any two partitions λ and μ of a positive integer, and any sequence α that can be rearranged to μ,

$$\chi_\lambda(\mu) = \sum_{T\in\mathcal{T}(\lambda,\alpha)} \mathrm{wt}(T). \tag{2.2}$$

3. Cascades

By the *word* of a partition λ we mean the binary sequence $w(\lambda)$ obtained from λ by writing 0 under each column, 1 alongside each row, and reading clockwise, e.g. the word of $(4,2)$ is 001001:



By the *shape* of a binary sequence $\beta = (\beta_1,\beta_2,\ldots,\beta_k)$ we mean the partition

$$\mathrm{sh}(\beta) = 1^{m_1}2^{m_2}\ldots$$

where m_i is the number of 1's in β with exactly i 0's to the left, e.g. both 001001 and 10010010 have shape $(4,2)$. The word of a non-empty partition λ is the unique binary sequence of shape λ that starts with 0 and ends with 1; the word of the empty partition is the empty sequence.

The standard fact that we require goes back to Comét in the 1950's (cf. [2]) and can be stated as follows:

Lemma 1. *For any finite binary sequence β and integer k, the mapping $\beta' \mapsto \mathrm{sh}(\beta')$ takes $\mathcal{B}$, the set of β' obtainable from β by swapping a* 0 *with a right-lying* 1 *exactly k positions away, bijectively onto the set of shapes obtainable from* $\mathrm{sh}(\beta)$ *by removing a rim hook of size k, and moreover, the number of rows occupied by the rim hook* $\mathrm{sh}(\beta)\setminus\mathrm{sh}(\beta')$ *equals the number of* 1*'s lying weakly between the swapped* 0-1 *pair.* $\square$

For example, if λ is the partition $(8,6,4,3)$ and ρ is the rim hook of λ shown in §2, and if $\beta = 11000101001001$, so that $\mathrm{sh}(\beta) = \lambda$, then the shape $\lambda\setminus\rho$ corresponds to $\beta' = 11010101000001$.

3.1. Our main tool is the following:

Definition 1. A *cascade* is a binary matrix C with rows $C_i = (C_{i1}, C_{i2}, \dots, C_{il})$, $1 \le i \le m$, such that

1) $C_{11} = 0$ and $C_{1l} = 1$,
2) for each row C_i with $1 \le i \le m-1$, there is a unique pair $a_i < b_i$ such that

$$C_{ia_i} = 0, \quad C_{ib_i} = 1, \quad C_{i+1} = (C_{i\tau(1)}, C_{i\tau(2)}, \dots, C_{i\tau(l)}) \quad \text{for} \quad \tau = \tau_{C,i} = (a_i\ b_i),$$

3) $C_m = (1, 1, \dots, 1, 0, 0, \dots, 0)$.

The *shape* of C is the shape of C_1.
The *content* of C is the sequence

$$(b_1 - a_1, b_2 - a_2, \dots, b_{m-1} - a_{m-1}).$$

A *crossing* in C is a pair (i, j) such that

$$1 \le i \le m-1, \quad C_{ij} = 1, \quad \text{and} \quad a_i < j < b_i.$$

The *weight* of C is defined by

$$\mathrm{wt}(C) = (-1)^{\mathrm{cr}(C)}, \quad \text{where} \quad \mathrm{cr}(C) = \#\{\text{crossings in } C\}.$$

The *permutation* associated to C is

$$\pi_C = \begin{pmatrix} 1 & 2 & \dots & k \\ \sigma_C(i_1) & \sigma_C(i_2) & \dots & \sigma_C(i_k) \end{pmatrix},$$

where $i_1 < i_2 < \dots < i_k$ are the positions of the 1's in the first row of C, and

$$\sigma_C = \tau_{C,m-1}\tau_{C,m-2}\dots\tau_{C,1}.$$

We denote by $\mathcal{C}(\lambda, \alpha)$ the set of cascades of shape λ and content α.

Lemma 2. *The mapping*

$$\Theta : C \mapsto \mathrm{Tab}(\mathrm{sh}(C_1), \mathrm{sh}(C_2), \dots, \mathrm{sh}(C_{\#\mathrm{rows}(C)})) \tag{3.1}$$

takes the set of cascades bijectively onto the set of rim hook tableaux, and it preserves shape, content, and weight.

Proof. This follows from Comét's observation in Lemma 1, the standard facts in §2 about rim hook tableaux, and the fact that there is a unique binary sequence β of a given non-empty shape such that β starts with 0 and ends with 1. In particular,

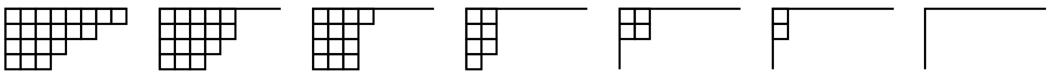
$$\Theta^{-1} : T \mapsto \mathrm{Mat}(w_\lambda(T_1), w_\lambda(T_2), w_\lambda(T_3), \dots, w_\lambda(T_{m+1})), \tag{3.2}$$

where $\lambda = \mathrm{sh}(T_1)$, m is the largest label in T, $w_\lambda(T_i)$ is the sequence obtained from $w(T_i)$ by appending to the start $\ell(\lambda) - \ell(T_i)$ many 1's and to the end $\lambda_1 - T_{i1}$ many 0's, and where $\mathrm{Mat}(r_1, r_2, \dots, r_k)$ with $r_i = (r_{i1}, r_{i2}, \dots)$ means the matrix (r_{ij}). □

Example. Consider the following cascade C:

$$\begin{pmatrix} 0&0&0&1&0&1&0&0&1&0&0&1\\ 0&0&0&1&0&1&0&1&1&0&0&0\\ 0&0&0&1&1&1&0&1&0&0&0&0\\ 0&1&0&1&1&1&0&0&0&0&0&0\\ 1&1&0&0&1&1&0&0&0&0&0&0\\ 1&1&0&1&1&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0 \end{pmatrix} \tag{3.3}$$

The shape is $(8, 6, 4, 3)$, the content is $(4, 4, 6, 3, 2, 2)$, the weight is $(-1)^{1+2+3+1+1+1}$. The row shapes $\mathrm{sh}(C_k)$ are:



The corresponding rim hook tableau $\mathrm{Tab}(\mathrm{sh}(C_1), \mathrm{sh}(C_2), \ldots, \mathrm{sh}(C_7))$ is:

6	5	3	3	2	1	1	1
6	5	3	2	2	1		
4	4	3	2				
4	3	3					

The associated permutation π_C is the transposition $(2\ 4)$ in S_4.

3.2. We define a *path* in a cascade $C = (C_1, C_2, \ldots, C_m)$ to be a sequence of column positions $p = (p_1, p_2, \ldots, p_m)$, one position p_i for each row C_i, such that

$$C_{1p_1} = 1 \quad \text{and} \quad p_{i+1} = \tau_{C,i}(p_i) \quad \text{for} \quad 1 \le i \le m-1.$$

We say p *starts* at p_1 and *ends* at p_m. There is exactly one path for each 1 in the first row of C, and we agree to always number the paths $p^1, p^2, p^3, \ldots$ according to relative start position, so that $p_1^1 < p_1^2 < p_1^3 < \ldots$. With this convention,

$$\pi_C(i) = p_m^i, \quad i = 1, 2, \ldots. \tag{3.4}$$

By a *crossing* of paths p, p' in C we mean a pair (i, j) with $1 \le i \le m-1$ such that

$$p_i = j, \quad p_i < p_i', \quad \text{and} \quad p_{i+1}' < p_{i+1}.$$

Lemma 3. *For a cascade C with paths $p^1, p^2, \ldots, p^k$,*

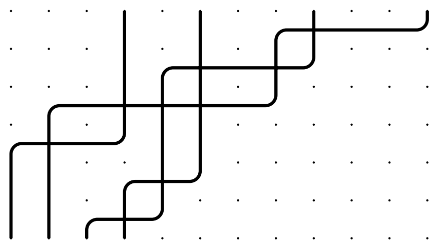
$$\{\text{crossings in } C\} = \dot{\bigcup}_{1 \le i < j \le k} \{\text{crossings of } p^i \text{ and } p^j\}. \tag{3.5}$$

Proof. By comparing definitions. $\square$

3.3. It is often convenient to visualize a cascade by constructing an associated graph.

Definition 2. The *diagram* or *graph* of a cascade is obtained by replacing each 1 by a node, each 0 by an empty space "$\cdot$", and then connecting any two nodes x, y that occupy adjacent rows and either share a single column or occupy the two columns where the two rows differ.

Example. The diagram of the cascade in (3.3) is:



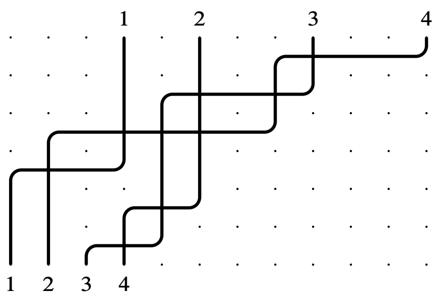
The paths of the cascade are

$$p^1 = (4, 4, 4, 4, 1, 1, 1), \quad p^2 = (6, 6, 6, 6, 6, 4, 4),$$
$$p^3 = (9, 9, 5, 5, 5, 5, 3), \quad p^4 = (12, 8, 8, 2, 2, 2, 2).$$

There are 9 crossings in total, e.g. p^3 and p^4 cross 3 times. And the permutation

$$\pi_C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

can be read off from the diagram by numbering the nodes in the top row, from left to right, $1, 2, \dots$, doing the same in the bottom row, and then chasing through the diagram from top to bottom:



3.4. Denote by $\operatorname{sgn}(\sigma)$ the *sign* of a permutation σ, so that

$$\operatorname{sgn}(\sigma) = (-1)^{\iota(\sigma)}, \quad \iota(\sigma) = \#\{\text{pairs } i < j \text{ with } \sigma(j) < \sigma(i)\}.$$

Lemma 4. *For any cascade C, we have*

$$\operatorname{wt}(C) = \operatorname{sgn}(\pi_C). \tag{3.6}$$

Proof. Consider the paths $p^1, p^2, \ldots, p^k$ in $C = (C_1, C_2, \ldots, C_m)$, numbered so $\pi_C(i) = p^i_m$, and let $\mathrm{cr}(p^i, p^j)$ be the number of crossings of p^i and p^j, so by (3.5),

$$\mathrm{cr}(C) = \sum_{1 \le i < j \le k} \mathrm{cr}(p^i, p^j). \tag{3.7}$$

Fix a pair $i < j$, so p^i starts left of p^j. If $\pi_C(j) < \pi_C(i)$, then p^i ends to the right of p^j, so p^i and p^j must have an odd number of crossings; if $\pi_C(i) < \pi_C(j)$, then p^i ends to the left of p^j, so p^i and p^j must have an even number of crossings. Hence

$$\iota(\pi_C) \equiv \sum_{1 \le i < j \le k} \mathrm{cr}(p^i, p^j) \pmod 2. \tag{3.8}$$

By (3.7) and (3.8), we have $\mathrm{cr}(C) \equiv \iota(\pi_C) \pmod 2$, so $\mathrm{wt}(C) = \mathrm{sgn}(\pi_C)$. □

As a corollary, we have the following useful reformulation of Murnaghan–Nakayama:

Proposition 1. *For any two partitions λ and μ of a positive integer, and any sequence α that can be rearranged to μ, we have*

$$\chi_\lambda(\mu) = \sum_{C \in \mathcal{C}(\lambda, \alpha)} \mathrm{wt}(C), \quad \mathrm{wt}(C) = (-1)^{\mathrm{cr}(C)} = \mathrm{sgn}(\pi_C), \tag{3.9}$$

where $\mathcal{C}(\lambda, \alpha)$ is the set of cascades of shape λ and content α.

Proof. By Lemmas 2 and 4. □

4. Proof of Theorem 1

An action on cascades. The main object of this section is to prove the following:

Theorem 2. *Let λ be a partition of a positive integer n, so $\boldsymbol{\lambda}$ is a partition of $d^2 n$, and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a sequence of positive d-divisible integers summing to $d^2 n$.*

Define a pairing $\sigma.C$ on $S_d \times \mathcal{C}(\boldsymbol{\lambda}, \alpha)$ by

$$(\sigma, C) \mapsto C\Phi(\sigma)^{-1}, \tag{4.1}$$

where $\Phi(\sigma)$ is the block-diagonal matrix

$$\Phi(\sigma) = \begin{pmatrix} \phi(\sigma) & & & \\ & \phi(\sigma) & & \\ & & \ddots & \\ & & & \phi(\sigma) \end{pmatrix}$$

with $\lambda_1 + \ell(\lambda)$ copies of the d-by-d permutation matrix $\phi(\sigma) = (\delta_{i\sigma(j)})$ on the diagonal.

(i) *The pairing $\sigma.C$ is an action of S_d on $\mathcal{C}(\boldsymbol{\lambda}, \alpha)$,*
(ii) *the action is free,*
(iii) *the action is weight-preserving, i.e.* $\mathrm{wt}(\sigma.C) = \mathrm{wt}(C)$ *for all σ and C.*

Proof. Assume $\mathcal{C}(\boldsymbol{\lambda}, \alpha) \neq \emptyset$. Let $l = \lambda_1 + \ell(\lambda)$ and $L = dn + d\ell(\lambda)$.

The word of λ starts with 0, ends with 1, and consists of λ_1 0's and $\ell(\lambda)$ 1's, so the sequence $w(\lambda)$ has length l. The word of $\boldsymbol{\lambda}$ is obtained by replacing in $w(\lambda)$ each 0 by d consecutive 0's and each 1 by d consecutive 1's, so $w(\boldsymbol{\lambda})$ starts with d 0's, ends with d 1's, has length L, and writing $w(\boldsymbol{\lambda}) = (w_1, w_2, \ldots, w_L)$,

$$w_{1+dk} = w_{2+dk} = \ldots = w_{d+dk}, \quad 0 \le k \le L/d - 1. \tag{4.2}$$

In particular, each $C \in \mathcal{C}(\boldsymbol{\lambda}, \alpha)$ has L columns, so the matrix multiplication on the right-hand side of (4.1) makes sense, and multiplying C on the right by $\Phi(\sigma)^{-1}$ permutes the first d columns of C, the next d columns of C, and so on: denoting by $\mathrm{Col}_i(C)$ the i-th column of C, we have

$$\mathrm{Col}_{i+dk}(C) = \mathrm{Col}_{\sigma(i)+dk}(\sigma.C) \tag{4.3}$$

for $1 \le i \le d$ and $0 \le k \le L/d - 1$.

(i). Fix $C \in \mathcal{C}(\boldsymbol{\lambda}, \alpha)$ and $\sigma \in S_d$. Let $C' = \sigma.C$. By (4.2) and (4.3),

$$C'_1 = C_1. \tag{4.4}$$

The last row of C is $C_m = (1, \ldots, 1, 0, \ldots, 0)$, with $d\ell(\lambda)$ 1's, so by (4.3),

$$C'_m = C_m. \tag{4.5}$$

By (4.4), (4.5), and C being a cascade, C' satisfies the first and third cascade conditions.

Let C'_i and C'_{i+1} be two consecutive rows in C'. Since C is a cascade, the rows C_i and C_{i+1} differ in exactly two positions, a_i and b_i with $a_i < b_i$, and

$$C_{i,a_i} = 0, \quad C_{i,b_i} = 1, \quad C_{i+1,a_i} = 1, \quad C_{i+1,b_i} = 0.$$

Since the difference $\alpha_i = b_i - a_i$ is positive and divisible by d,

$$a_i = r_i + ds_i \quad \text{and} \quad b_i = r_i + dt_i \tag{4.6}$$

for some non-negative integers r_i, s_i, t_i with $1 \le r_i \le d$ and $s_i < t_i$. Setting

$$a'_i = \sigma(r_i) + ds_i \quad \text{and} \quad b'_i = \sigma(r_i) + dt_i, \tag{4.7}$$

and using (4.3), we have that C'_i and C'_{i+1} differ in exactly positions a'_i and b'_i, and

$$C'_{i,a'_i} = 0, \quad C'_{i,b'_i} = 1, \quad C'_{i+1,a'_i} = 1, \quad C'_{i+1,b'_i} = 0.$$

Since $s_i < t_i$, we also have that $a'_i < b'_i$. So C' satisfies the second condition of a cascade. Hence C' is a cascade.

By (4.4), the shape of the cascade C' is $\boldsymbol{\lambda}$. The content of C' is $(b'_1 - a'_1, b'_2 - a'_2, \ldots)$, which by (4.6) and (4.7) equals α. So $C' \in \mathcal{C}(\boldsymbol{\lambda}, \alpha)$. This concludes the proof of (i).

(ii). Let $z_i(C)$ be the number of 0's in the i-th column of a cascade $C \in \mathcal{C}(\boldsymbol{\lambda}, \alpha)$. Let

$$z(C) = (z_1(C), z_2(C), \ldots, z_d(C)). \tag{4.8}$$

By the cascade conditions, and the positivity and d-divisibility of the α_i's, we have

$$z_i(C) \neq z_j(C) \quad \text{for} \quad 1 \leq i < j \leq d. \tag{4.9}$$

By (4.3),

$$z(\sigma.C) = (z_{\sigma^{-1}(1)}(C), z_{\sigma^{-1}(2)}(C), \ldots, z_{\sigma^{-1}(d)}(C)). \tag{4.10}$$

From (4.9) and (4.10), for each $C \in \mathcal{C}(\boldsymbol{\lambda}, \alpha)$, we have

$$\sigma.C = C \text{ if and only if } \sigma = 1. \tag{4.11}$$

This concludes the proof of (ii).

(iii). Fix a cascade $C \in \mathcal{C}(\boldsymbol{\lambda}, \alpha)$ and a permutation $\sigma \in S_d$, so $\sigma.C \in \mathcal{C}(\boldsymbol{\lambda}, \alpha)$ by (i). Let $p^1, p^2, \ldots, p^{d\ell(\lambda)}$ be the paths in C, so $p_1^1 < p_1^2 < \ldots$ and

$$\pi_C(i) = p_m^i, \tag{4.12}$$

and let $q^1, q^2, \ldots, q^{d\ell(\lambda)}$ be the paths in $\sigma.C$, so $q_1^1 < q_1^2 < \ldots$ and

$$\pi_{\sigma.C}(i) = q_m^i. \tag{4.13}$$

Let γ be the permutation in S_L given by

$$\gamma(i + dk) = \sigma(i) + dk, \quad 1 \leq i \leq d, \quad 0 \leq k \leq L/d - 1. \tag{4.14}$$

By (4.3), the sequences

$$\sigma.p^i = (\gamma(p_1^i), \gamma(p_2^i), \ldots, \gamma(p_m^i)), \quad 1 \leq i \leq d\ell(\lambda), \tag{4.15}$$

are the paths of $\sigma.C$, in some order. Let ω be the permutation in $S_{d\ell(\lambda)}$ given by

$$\omega(i + dk) = \sigma(i) + dk, \quad 1 \leq i \leq d, \quad 0 \leq k \leq \ell(\lambda) - 1. \tag{4.16}$$

Then by (4.2), for each i,

$$q^{\omega(i)} = \sigma.p^i. \tag{4.17}$$

Since $C_m = (1, \ldots, 1, 0, \ldots, 0)$ with $d\ell(\lambda)$ 1's, we also have $\gamma(p_m^i) = \omega(p_m^i)$, so

$$q_m^{\omega(i)} = \omega(p_m^i). \tag{4.18}$$

By (4.12), (4.13), and (4.18), the permutation $\pi_{\sigma.C}$ takes $\omega(i)$ to $\omega(\pi_C(i))$ for each i, i.e.

$$\pi_{\sigma.C} = \omega \pi_C \omega^{-1}. \tag{4.19}$$

So $\pi_{\sigma.C}$ and π_C have the same sign. By Lemma 4, we conclude that

$$\mathrm{wt}(\sigma.C) = \mathrm{wt}(C) \tag{4.20}$$

for all $\sigma \in S_d$ and $C \in \mathcal{C}(\boldsymbol{\lambda}, \alpha)$. This concludes the proof of (iii) and Theorem 2. □

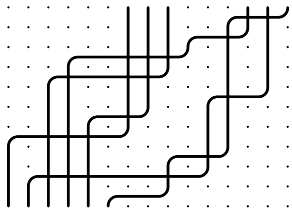
It is worth remarking that Theorem 2 and Lemma 2 together give a weight-preserving free action on rim hook tableaux:

Corollary 1. *For any partition λ of a positive integer n, and any sequence α of positive d-divisible integers summing to d^2n, there is a well-defined action of S_d on $\mathcal{T}(\boldsymbol{\lambda}, \alpha)$ given by $\sigma.T = \Theta(\sigma.\Theta^{-1}(T))$, and this action is both free and weight-preserving.* $\square$

Example. With $d = 3$ and $\lambda = (3, 2)$, the following shows an S_d-orbit of a cascade C and corresponding rim hook tableau T of shape $\boldsymbol{\lambda}$ and content $(3, 3, 6, 6, 3, 3, 6, 9, 3)$.

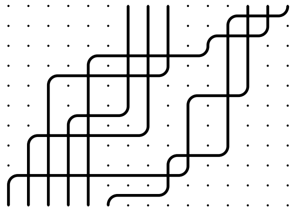
σ | diagram of $\sigma.C$ | $\sigma.T$

1



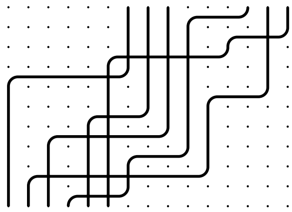
10	10	10	9	9	8	5	5	1
9	9	9	9	8	8	5	2	1
9	7	7	6	4	3	2	2	1
9	7	6	6	4	3			
9	7	4	4	4	3			
7	7	4	3	3	3			

$(1\,2)$



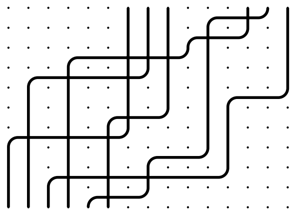
10	10	10	9	8	8	5	2	1
9	9	9	9	8	5	5	2	1
9	7	7	7	4	3	3	2	1
9	7	6	6	4	3			
9	7	6	4	4	3			
9	7	4	4	3	3			

$(1\,3)$



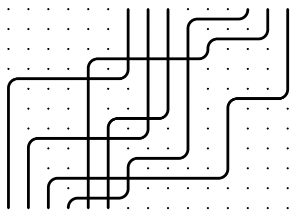
10	9	9	9	9	5	5	5	2
10	9	8	8	8	3	3	2	2
10	9	7	7	7	3	1	1	1
9	9	7	6	6	3			
9	7	7	6	4	3			
4	4	4	4	4	3			

$(2\,3)$



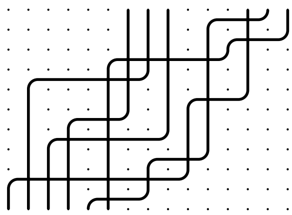
10	10	9	9	9	9	5	5	5
10	9	9	8	8	8	2	2	1
9	9	7	6	6	3	2	1	1
9	7	7	6	4	3			
7	7	4	4	4	3			
7	4	4	3	3	3			

$(1\,2\,3)$



10	9	9	9	9	9	5	5	5
10	9	8	8	8	3	2	2	2
10	9	7	7	6	3	1	1	1
9	9	7	6	6	3			
7	7	7	4	4	3			
4	4	4	4	3	3			

$(1\,3\,2)$



10	10	9	9	8	5	5	2	2
10	9	9	8	8	5	3	2	1
9	9	7	7	7	3	3	1	1
9	7	7	6	4	3			
9	7	6	6	4	3			
9	4	4	4	4	3			

Proof of Theorem 1. For (1.2), let λ be a partition of a positive integer n, and let μ be a partition of dn. By Proposition 1, we have

$$\chi_{\boldsymbol{\lambda}}(d.\mu) = \sum_{C \in \mathcal{C}(\boldsymbol{\lambda}, d.\mu)} \mathrm{wt}(C),$$

and by Theorem 2 there exists a weight-preserving free action of S_d on $\mathcal{C}(\boldsymbol{\lambda}, d.\mu)$. So $\chi_{\boldsymbol{\lambda}}(d.\mu)$ is divisible by $d!$. This completes the proof of (1.2).

(1.1) is a special case of (1.2): let λ and μ be partitions of a positive integer n, write $\mu = 1^{m_1}2^{m_2}\ldots n^{m_n}$, and define $\nu = 1^{dm_1}2^{dm_2}\ldots n^{dm_n}$, so that ν is a partition of dn with $d.\nu = \boldsymbol{\mu}$, and hence by (1.2), $\chi_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ is divisible by $d!$.

For (1.3), let λ and μ be partitions of an integer n not divisible by d. Suppose that there exists a cascade $C \in \mathcal{C}(\boldsymbol{\lambda}, d^2.\mu)$, let D be the matrix with columns

$$\mathrm{Col}_1(C), \mathrm{Col}_{d+1}(C), \mathrm{Col}_{2d+1}(C), \ldots,$$

occurring in that order, and let C' be the matrix obtained from D by deleting redundant rows. Then $C' \in \mathcal{C}(\lambda, d.\mu')$ for some partition μ', hence $n = d|\mu'|$. So $\mathcal{C}(\boldsymbol{\lambda}, d^2.\mu) = \emptyset$, hence by Proposition 1, $\chi_{\boldsymbol{\lambda}}(d^2.\mu)$ equals 0. $\square$

References

[1] W. Burnside, On an arithmetical theorem connected with roots of unity, and its application to group-characteristics. Proc. Lond. Math. Soc. **1** (1904) 112–116.

[2] S. Comét, Notations for partitions. Math. Tables Aids Comput. **9** (1955) 143–146.

[3] P. Erdős and J. Lehner, The distribution of the number of summands in the partitions of a positive integer. Duke Math J. **8** (1941) 335–345.

[4] G. Frobenius, Über die Charaktere der symmetrischen Gruppe. Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, 1900, 516–534.

[5] P. X. Gallagher, Degrees, class sizes and divisors of character values. J. Group Theory **15** (2012) 455–467.

[6] V. L. Goncharoff, Sur la distribution des cycles dans les permutations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **35** (1942) 267–269.

[7] I. M. Isaacs, *Character theory of finite groups*. Dover, New York, 1994.

[8] I. G. Macdonald, On the degrees of the irreducible representations of symmetric groups, Bull. London Math. Soc. **3** (1971) 189–192.

[9] I. G. Macdonald, *Symmetric Functions and Hall polynomials*, 2nd ed. Oxford University Press, 1995.

[10] J. McKay, Irreducible representations of odd degree. J. Algebra **20** (1972) 416–418.

[11] A. R. Miller, The probability that a character value is zero for the symmetric group. Math. Z. **277** (2014) 1011–1015.

[12] A. R. Miller, On parity and characters of symmetric groups. J. Combin. Theory Ser. A **162** (2019) 231–240.

[13] F. D. Murnaghan, The characters of the symmetric group. Amer. J. Math. **59** (1937) 739–753.

[14] T. Nakayama, On some modular properties of the irreducible representations of a symmetric group I, II. Jap. J. Math. **17** (1941) 165–184 and 411–423.

[15] C. L. Siegel, The trace of totally positive and real algebraic integers. Ann. of Math. **46** (1945) 302–312.

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT WIEN, VIENNA, AUSTRIA
E-mail address: alexander.r.miller@univie.ac.at