Reduction of the Kolmogorov n-width for transport dominated fluid-structure interaction problem

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Introduction and motivation

Given a problem of interest, let $\mathcal{M}$ be the solution manifold. Assumption of the reduced basis method: $\mathcal{M}$ can be well approximated by a sequence of finite dimensional spaces, meaning the Kolmogorov n-width D_n of $\mathcal{M}$ decays fast as a function of n. For convection dominated problems D_n can decay quite slowly, and this translates in a poor efficiency of the reduced basis method. Here we show how we have tried to overcome this situation, explaining the idea presented in [3], whose main feature is the presence of a preprocessing step, which is carried out right after the offline step.

FSI problem

Problem: simulate the displacement in the time interval $[0, T]$ of a thin structure Σ_t at the top of a 2D rectangle filled with a fluid Ω_t^f We adopt an Arbitrary Lagrangian Eulerian (**ALE**) formulation.

$$\begin{cases} J\rho_f(\partial_t \mathbf{u}_f + F^{-1}(\mathbf{u}_f - \partial_t d_f \mathbf{e}_y)\cdot\nabla)\mathbf{u}_f) - \text{div}(J\sigma^f F^{-T}) = \mathbf{b}_f & \text{in } \Omega^f \times [0,T], \\ \text{div}(JF^{-1}\mathbf{u}_f) = 0 & \text{in } \Omega^f \times [0,T], \\ \rho_s h_s \partial_{tt} d_s - c_0 \partial_{xx} d_s + c_1 d_s = -\sigma^f \mathbf{n}\cdot\mathbf{n} & \text{in } \Sigma, \end{cases}$$

$$\begin{cases} d_f = d_s & \text{in } \Sigma_t, & \text{continuity of the displacement} \\ \mathbf{u}_f = \partial_t d_s \mathbf{e}_y & \text{in } \Sigma_t, & \text{continuity of the velocity.} \end{cases}$$

To impose the continuity conditions we use two Lagrange multipliers λ_u and λ_d.

Transport phenomenon

γ_n

Figure: snapshots for p_f at different timesteps. Let γ_n be the abscissa of the peak of the pressure wave at timestep t_n: γ_n changes in time, since the peak of the wave travels along the domain. We have a transport phenomenon.

Kolmogorov n-width

Kolmogorov n-width (of $\mathcal{M}_{p_f}$)

$$D_n(\mathcal{M}_{p_f}, ||\cdot||) = \inf_{E_n} \sup_{p\in\mathcal{M}_{p_f}} \inf_{q\in E_n} ||p-q||_X.$$

E_n is any linear subspace of dimension n embedded in X.

Link between D_n and POD

$$\int_0^T ||p_f(\cdot;t) - \Pi_{POD}p_f(\cdot;t)||_X\, dt = \sum_{i>N_{POD}} \lambda_i,$$

Preprocessing step (for $\mathcal{M}_{p_f}$)[3]

Family of **smooth** and **invertible** mappings $\mathcal{F} = \{F : \Omega^f \to \Omega^f\}$ s.t. $\forall t \in [0;T]$, $\exists F_t \in \mathcal{F}$ s. t.

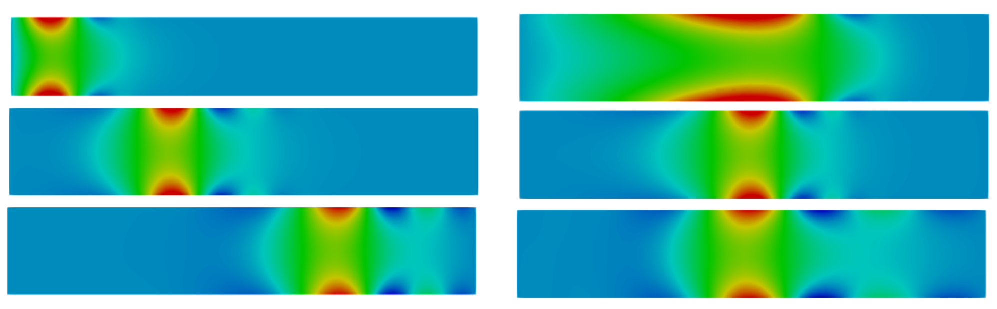
$$\mathcal{M}_{p_f,\mathcal{F}} = \{p_f(F_t^{-1}(\cdot), t); t\in[0;T]\}$$

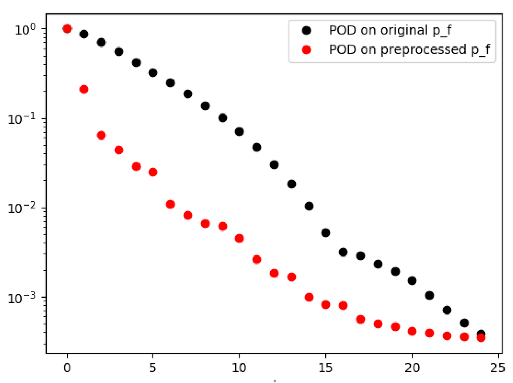
has a smaller Kolmogorov n-width.

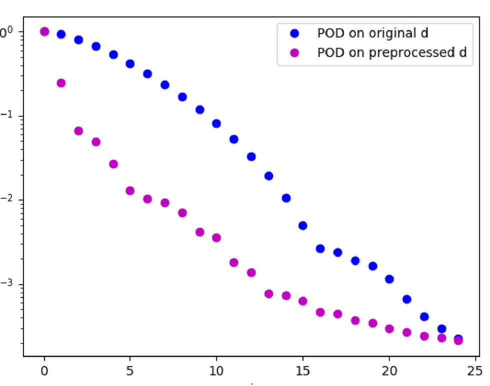
After preprocessing: $\{\Phi_k^{p_f}\}_{k=1}^N$ s. t. span$\{\Phi_k^{p_f}\}_k$ approaches to a given accuracy $\mathcal{M}_{p_f,\mathcal{F}}$, N small. At each time step, look for coordinates $\{\alpha_k^{n+1}\}$ and a suitable map $F_{n+1} \in \mathcal{F}$ such that $p_f(\cdot, t^{n+1})$ is well approximated by:

$$p_N^{n+1} = \sum_{k=1}^N \alpha_k^{n+1}\Phi_k^{p_f} \circ F_{n+1}.$$

Results







Left column: p_f before and after the preprocessing. The peaks of the waves are all aligned at the same point.
Right column: POD on $\mathcal{M}_{p_f}$, $\mathcal{M}_d$ and on $\mathcal{M}_{p_f,\mathcal{F}}$ and $\mathcal{M}_{d,\mathcal{F}}$.
After the preprocessing, we reach a magnitude of 10^{-3} with less than 15 modes.

References

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[4] Ballarin F., Maday Y., Nonino M., Rozza G., article in preparation.

https://github.com/mathLab/RBniCS
https://github.com/mathLab/multiphenics

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