POD–Galerkin reduced order methods for inverse and multi-physics problems in fluid dynamics

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Introduction and Motivation

Parametrized inverse problems, such as **optimal flow control** problems (OFCP($\boldsymbol{\mu}$)), **data assimilation**, and **multi-physics** applications, play an ubiquitous role in several fields of application, yet are usually very **demanding** from a computational standpoint. POD–Galerkin reduction allow us to solve them in a low-dimensional framework and in a **fast** and **reliable** way. Following [1, 2], we present some fluid-structure interaction problems in view further applications in multi-physics in cardiovascular modeling [4], employing a novel preprocessing proposed in [3]. Following a similar methodology, we also propose two applications to optimal flow control problems, for cardiovascular modeling and environmental marine applications [5], respectively.

Fluid-structure interaction problems

Problem: simulate the displacement in the time interval $[0, T]$ of a thin structure Ω_t^s at the top of a 2D rectangle filled with a fluid Ω_t^f.

Model:

$$\begin{cases} \rho_f(\partial_t \mathbf{u} + (\mathbf{u} - \partial_t \mathbf{d}) \cdot \nabla \mathbf{u}) - \mathrm{div}\sigma^f(\mathbf{u}, p) = \mathbf{b}_f, & \text{in } \Omega_t^f \times [0,T] \\ \mathrm{div}\mathbf{u} = 0, & \text{in } \Omega_t^f \times [0,T] \\ \rho_s \partial_t \mathbf{u} - \mathrm{div}\mathbf{P}(\mathbf{d}, p) = \mathbf{b}_s, & \text{in } \Omega^s \\ \partial_t \mathbf{d} - \mathbf{u} = \mathbf{0}, & \text{in } \Omega^s, \\ \partial_t \mathbf{d} - \mathrm{div}\sigma^e(\mathbf{d}) = 0, & \text{in } \Omega_t^f \times [0,T]. \end{cases}$$

Discretization: we adopt an ALE formulation, which results in a non-linear system of equations to be solved with monolithic approach.

Solution manifold preprocessing [3]: once we have truth solutions $(\mathbf{u}, p, \mathbf{d})$ we define a map $F\colon \Omega \to \Omega$, smooth and invertible, so that the manifold of the preprocessed snapshots, obtained composing the original snapshots with the map F, features a lower Kolmogorov n-width.

Preliminary results for FSI (with Y. Maday[2])

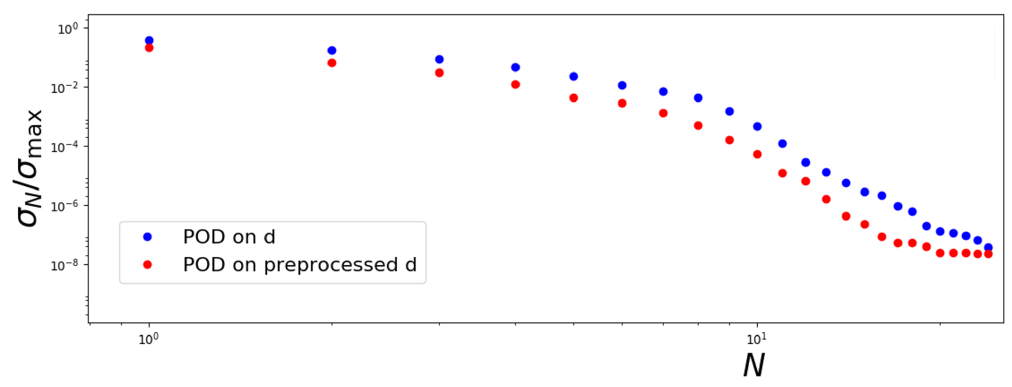


Figure: decay of the first singular values for the original and for the preprocessed displacements.

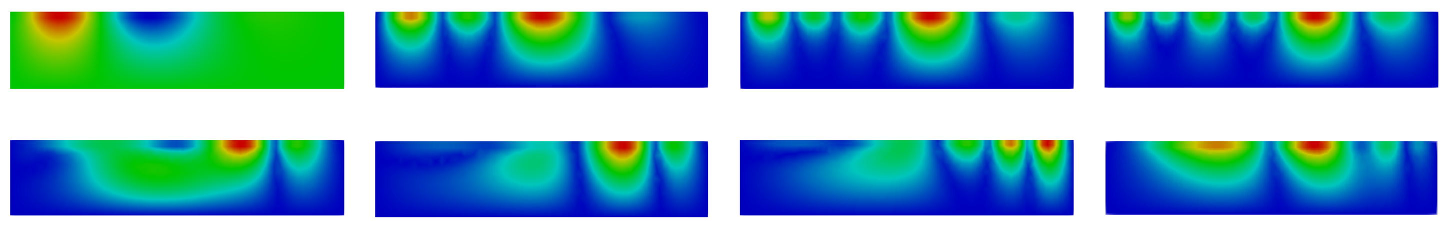


Figure: bases functions from 1 to 4 for original (first row) and for preprocessed displacements (second row): bases after preprocessing are more suitable to capture the transport effect by a reduction of the "frequency" of oscillations.

Optimal flow control for cardiovascular haemodynamics (with P. Triverio, L. Jimenez-Juan[3])

Problem: Find optimal pair $(y(\boldsymbol{\mu}), u(\boldsymbol{\mu}))$ of state and control such that $\min_{(y,u)} \mathcal{J}(y(\boldsymbol{\mu}), u(\boldsymbol{\mu}))$ is satisfied subject to $\mathcal{F}(y(\boldsymbol{\mu}), u(\boldsymbol{\mu}); \boldsymbol{\mu}) = 0$.

Solution: Numerical approximation of solution to coupled optimality system via one-shot approach:

$$\nabla \mathcal{J}(y(\boldsymbol{\mu}), u(\boldsymbol{\mu})) + \nabla \mathcal{F}(y(\boldsymbol{\mu}), u(\boldsymbol{\mu}))\lambda = 0, \quad \mathcal{F}(y(\boldsymbol{\mu}), u(\boldsymbol{\mu})) = 0$$

In cardiovascular haemodynamics: State-constraints $\mathcal{F}(y, u; \boldsymbol{\mu})$ are Navier-Stokes equations. Cost-functional $\mathcal{J}(y, u; \boldsymbol{\mu})$ represents cardiovascular quantities of interest e.g. blood flow velocity, pressure drop, wall shear stress or viscous energy dissipation.

Test case: *Viscous Energy Dissipation and Pressure-Tracking with Distributed Control*

$$\mathcal{J}(\boldsymbol{v}, p, \boldsymbol{u}) = \frac{\nu}{2}\int_\Omega |\nabla \boldsymbol{v}|^2 + \frac{1}{2\nu}\int_\Omega (p - p_d)^2 + \frac{\alpha}{2}\int_\Omega |\boldsymbol{u}|^2$$

$$\begin{cases} -\nu\Delta \boldsymbol{v} + \boldsymbol{v}(\nabla \cdot \boldsymbol{v}) + \nabla p = \boldsymbol{u} & \text{in } \Omega \\ \nabla \cdot \boldsymbol{v} = 0 & \text{in } \Omega \\ \boldsymbol{v} = \boldsymbol{g}(\mu_{in}) & \text{on } \Gamma_{in} \\ \boldsymbol{v} = \boldsymbol{0} & \text{on } \Gamma_D \\ -p\boldsymbol{n} + \nu \nabla \boldsymbol{v} \cdot \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma_N \end{cases}$$

here, $\boldsymbol{v}, p$ and $\boldsymbol{u}$ denote velocity, pressure and control respectively and ν is the viscosity. Moreover, Ω is simplified domain for arterial bifurcation.

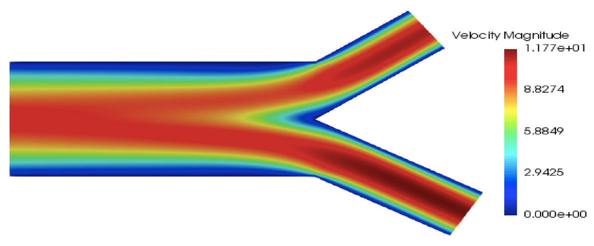


Figure: Velocity

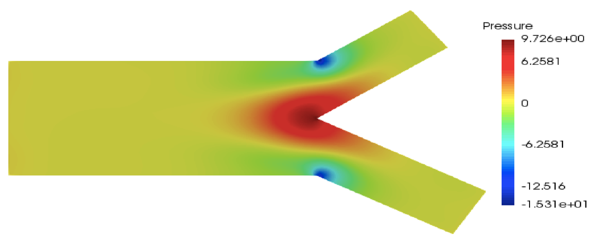
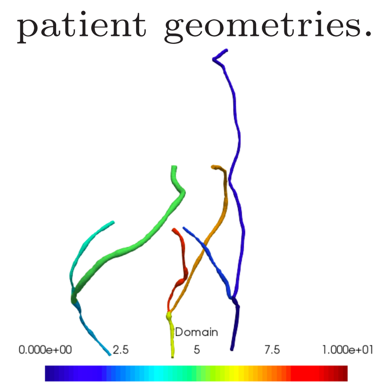


Figure: Pressure

$\mathcal{J}$ **reduction:** $\sim O(10^3)$

Work in progress: Reduced order optimal flow control on real-patient geometries.



Geometry for triple coronary artery bypass grafts

Reduced OFCP($\boldsymbol{\mu}$) in environmental sciences

1) Loss of pollutant in the Gulf of Trieste, Italy: concentration of the pollutant y under a safeguard y_d. Parameter $\boldsymbol{\mu} \in [0.5, 1] \times [-1, 1] \times [-1, 1]$ describes regional winds action.

$\to$ **Model**:

$$\min_{(y,u)\in Y\times U} \frac{1}{2}\int_{\Omega_{OBS}} (y - y_d)^2 + \frac{\alpha}{2}\int_{\Omega_u} u^2$$

$$\text{s.t.} \int_\Omega (\mu_1 \nabla y \cdot \nabla q + [\mu_2, \mu_3] \cdot \nabla y q)) = L\, u \int_{\Omega_u} q.$$

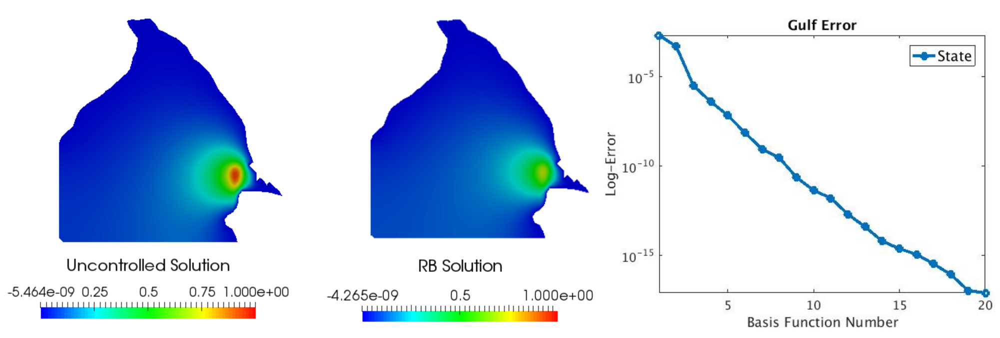
2) Nonlinear solution tracking North Atlantic Ocean: make the solution (ψ) similar to a current profile based on experimental data (Gulf Stream dynamics). Parameter $\boldsymbol{\mu} \in [0.07^3, 1] \times [10^{-4}, 1] \times [10^{-4}, 0.045^2]$ describing the Ocean dynamic.

$\to$ **Model:**

$$\min_{(\psi,u)\in Y\times U} \frac{1}{2}\int_{\Omega_{OBS}} (\psi - \psi_d)^2 + \frac{\alpha}{2}\int_{\Omega_u} u^2$$

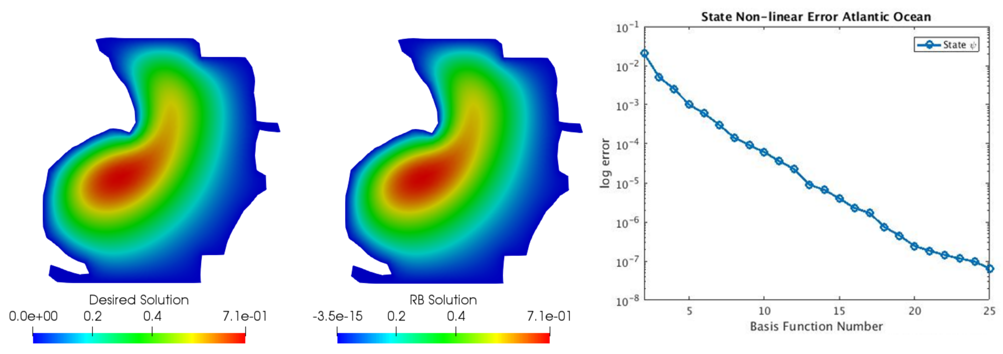
$$\text{s.t.}\, \mu_3 \frac{\partial \psi}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial q}{\partial x} = f - \mu_1 \Delta \psi + \mu_2 \Delta^2 \psi.$$

Results environmental app.(with R. Mosetti[4])



1) Gulf pollutant control

Left plot: Finite Element uncontrolled concentration.
Center plot: Reduced Order controlled concentration.
Right plot: Convergence error vs N ($\sim 10^{-8}$).
Dimension Comparison FE vs RB: 5939 vs 201.



2) Nonlinear Ocean dynamic

Left plot: Finite Element stream-function profile.
Center plot: Reduced Order stream-function profile.
Right plot: Convergence error vs N ($\sim 10^{-7}$).
Dimension Comparison FE vs RB: 6490 vs 225.

References

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https://gitlab.com/RBniCS/RBniCS.git
https://gitlab.com/multiphenics/multiphenics.git

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