Ambient and Conformal Holonomy

Matthias Hammerl

University of Vienna

Workshop on Conformal and CR Geometry
Banff International Research Station
August 1 2012

Joint work with A. Čap (Univ. of Vienna), A.R. Gover (Auckland Univ.) and R. Graham (Univ. of Washington)

Plan

Conformal structures

Let $(M, \mathcal{C})$ be a conformal structure of dimension $n \geq 3$. Here $\mathcal{C} = [g]$ denotes a conformal class of metrics, with g some representative.

Given that a transformation $g \to \hat{g} = e^{2f} g$ modifies the corresponding Levi-Civita covariant derivatives, there exists no torsion-free covariant derivative on the tangent bundle TM that preserves all metrics in $\mathcal{C}$.

To associate a natural covariant derivative to $\mathcal{C}$ one employs one of the following two techniques:

1. Cartan resp. tractor approach (Élie Cartan, Tracy Thomas)
2. Ambient metric approach (Fefferman-Graham)

Conformal holonomy

Each approach delivers a covariant derivative (on some extended bundle or space) for $\mathcal{C}$, and in particular yields a notion of conformal holonomy.

In this talk I am going to review both constructions and discuss a recently developed method to compare the resulting holonomy groups.

The conformal standard tractor bundle

The tractor approach associates a natural vector bundle $\mathcal{T}$ to $\mathcal{C}$ that is of dimension $n+2$: it carries a metric h of signature $(p+1, q+1)$ together with a compatible covariant derivative, the tractor covariant derivative $\nabla^{\mathcal{T}}$.

To construct this tractor bundle one has 3 options:

1. Describe it as the associated bundle to the Cartan geometry that equivalently encodes $(M, \mathcal{C})$.
2. Give its defining relations directly in terms of a $g \in [g] = \mathcal{C}$ and provide transformation formulas for $g \rightsquigarrow \hat{g} = e^{2f} g$.
3. Realize it as a suitable space of tensors on the Fefferman-Graham ambient space.

Fefferman-Graham ambient metrics

The conformal structure $\mathcal{C} = [g]$ can be understood as the ray-subbundle $\mathcal{C} \subset S^2 T^* M$ that consists of all metrics in the given conformal class. $\mathcal{C}$ carries a tautological symmetric form $g_0 \in S^2 T^* \mathcal{C}$, and with respect to dilation $\delta_s^*(g) = s^2 g$ for $g \in \mathcal{C}$ this form is homogeneous of degree 2.

The Fefferman-Graham ambient metric $\tilde{g}$ lives on the $n+2$ dimensional ambient space $\tilde{M} = \mathcal{C} \times (-1, 1)$ and extends the degenerate form g_0 on $\mathcal{C}$ to a $(p+1, q+1)$ metric on $\tilde{M}$ that is homogeneous of degree 2.

Fefferman-Graham ambient metrics

To obtain a unique $\tilde{g}$ it is necessary to employ a normalization condition:

- For $n = p + q$ odd $\tilde{g}$ is uniquely determined as an infinite order jet along $\mathcal{C}$ by the normalization condition that $\mathrm{Ric}(\tilde{g})$ vanishes to infinite order along $\mathcal{C}$.
- For $n = p + q$ even, $\tilde{g}$ is unique up addition of terms of order higher than $\frac{n}{2}$ under the normalization condition that $\mathrm{Ric}(\tilde{g})$ vanishes to order $\frac{n}{2} - 2$ along $\mathcal{C}$ and to order $\frac{n}{2} - 1$ in tangential directions along $\mathcal{C}$.

Tractor holonomy vs. ambient holonomy

Tractor holonomy: The conformal holonomy of $(M, \mathcal{C})$ is defined as the holonomy of the associated tractor covariant derivative: $\mathrm{Hol}(\mathcal{C}) := \mathrm{Hol}(\nabla^{\mathcal{T}}) \subset \mathrm{SO}(p+1, q+1)$.

In general it is not possible to form a conformally invariant holonomy group on the ambient level: Since the ambient covariant derivative $\tilde{\nabla}$ on $T\tilde{M}$ is only well-defined formally along the cone $\mathcal{C} \subset \tilde{M}$, forming the corresponding holonomy $\mathrm{Hol}(\tilde{\nabla})$ (even at a point in the cone) won't yield a conformally invariant object.

For a real-analytic conformal structure $(M, \mathcal{C})$ **in odd dimension** $n = p + q$ the ambient space $(\tilde{M}, \tilde{g})$ is a well-defined pseudo-Riemannian structure of signature $(p+1, q+1)$, and in particular one can form $\mathrm{Hol}(\tilde{\nabla}) \subset \mathrm{SO}(p+1, q+1)$.

Realization of tractors as ambient vector fields

It is possible to recover the tractor bundle $\mathcal{T}$, the tractor metric h and the tractor covariant derivative ∇ in the ambient picture [Čap-Gover, '03]:

- A section $s \in \Gamma(\mathcal{T})$ corresponds to an ambient vector field $\tilde{s} \in \mathfrak{X}_{\mathcal{C}}(\tilde{M})$ along $\mathcal{C}$ which is homogeneous of degree -1.
- The restriction of the ambient metric $\tilde{g}$ to $\mathfrak{X}_{\mathcal{C}}$ corresponds to the tractor metric h.
- The tractor covariant derivative $\nabla_\xi s$ of a tractor $s \in \Gamma(\mathcal{T})$ in direction of a vector field $\xi \in \mathfrak{X}(M)$ corresponds to the ambient covariant derivative of the ambient vector field $\tilde{s} \in \mathfrak{X}_{\mathcal{C}}(\tilde{M})$ in direction of a homogeneous lift $\tilde{\xi} \in \mathfrak{X}(\mathcal{C})$.

Graham-Willse: Lifting of parallel tractors to parallel ambient tensors

Theorem (Graham-Willse, 2011)

Let $(M, \mathcal{C})$ *be a real-analytic conformal structure of signature* (p, q)*,* $n = p + q$ *odd. Let* $\mathcal{V}$ *be some tensor power of the conformal tractor bundle and* $s \in \Gamma(\mathcal{V})$ *a section of that bundle which is parallel with respect to the tractor connection. Then there exists a canonical lift of* s *to an ambient tensor field* $\tilde{s}$ *that is well-defined in a neighborhood of the cone* $\mathcal{C}$ *and parallel with respect to the ambient Levi-Civita covariant derivative.*

A truncated version of this theorem exists for the even-dimensional case and the result also holds formally (on the jet-level) at the cone $\mathcal{C} \subset \tilde{M}$ if $\mathcal{C}$ is not necessarily real-analytic.

Example: Conformal structures induced by generic rank 2 distributions on 5-manifolds

Let M be a manifold of dimension 5 endowed with a generic rank 2 distribution $\mathcal{D} \subset TM$. Then, according to [Nurowski, '05], there exists a canonical conformal structure $\mathcal{C}$ of signature $(2,3)$ on M that is induced from $\mathcal{D}$.

In the framework of parabolic geometries this can be understood as an extension of structure group of the G_2-structure $\mathcal{D}$ to an $\mathrm{SO}(3,4)$-structure on the same manifold via a generalized Fefferman-Type construction:

Conformal G_2-structures

It was shown in joint work [H.-Sagerschnig, '11] that one has:

Theorem

Let $(M, \mathcal{C})$ be a conformal structure of signature $(2,3)$. Then the following are equivalent:

- *$(M, \mathcal{C})$ is induced from a generic rank 2 distribution $\mathcal{D} \subset TM$*
- *The conformal holonomy $\mathrm{Hol}(\mathcal{C}) \subset \mathrm{SO}(3,4)$ is contained in $G_2 \subset \mathrm{SO}(3,4)$.*

Application of Graham-Willse's-result for conformal G_2-structures

- Since $G_2 \subset \mathrm{SO}(3,4)$ can be realized as the stabilizer of a suitable generic 3-vector in $\Lambda^3\mathbb{R}^7$, the holonomy reduction $\mathrm{Hol}(\mathcal{C}) \subset G_2$ can be equivalently characterized by the existence of a (suitably generic) parallel tractor 3-form $\Phi \in \Lambda^3\mathcal{T}$.
- Employing Graham-Willse's result, this yields a canonical ambient 3-form on $\tilde{M}$ that is parallel with respect to the ambient Levi-Civita derivative.
- In particular, $\mathrm{Hol}(\tilde{\nabla}) \subset G_2$.

We note however that it does not follow from this chain of reasoning that $\mathrm{Hol}(\mathcal{C}) = G_2 \Rightarrow \mathrm{Hol}(\tilde{\nabla}) = G_2$.

Infinitesimal holonomy

We have already discussed that the holonomy $\mathrm{Hol}(\tilde{g})$ of the ambient metric is in general not a conformally well-defined object since $\tilde{g}$ is only conformally invariant as an infinite-order (or truncated) jet along the cone $\mathcal{C} \subset \tilde{M}$.

The natural type of holonomy one should therefore employ is **infinitesimal ambient holonomy along the cone $\mathcal{C}$:**

$$\tilde{\mathfrak{hol}} = \mathrm{span}\big(\{\big(\tilde{\nabla}_{\xi_1} \cdots \tilde{\nabla}_{\xi_{l-2}} \tilde{R}(\xi_{l-1}, \xi_l)\big)_{|\mathcal{C}}, \xi_1, \ldots, \xi_l \in \mathfrak{X}(\tilde{M})\}\big) \subset \Gamma_{\mathcal{C}}(\mathrm{End}(T\tilde{M})).$$

The corresponding invariant of the tractor bundle $(\mathcal{T}, \nabla^{\mathcal{T}})$ is **infinitesimal tractor holonomy:**

$$\mathfrak{hol} = \mathrm{span}\big(\{\big(\nabla_{\xi_1} \cdots \nabla_{\xi_{l-2}} R(\xi_{l-1}, \xi_l)\big), \xi_1, \ldots, \xi_l \in \mathfrak{X}(M)\}\big) \subset \Gamma(\mathrm{End}(\mathcal{T})).$$

Equality of infinitesimal ambient holonomy with infinitesimal tractor holonomy:

Theorem

Let $(M, \mathcal{C})$ be a conformal structure of signature (p, q) with $n = p + q$ odd. Then infinitesimal tractor holonomy coincides with infinitesimal ambient holonomy: $\mathfrak{hol} = \tilde{\mathfrak{hol}}$.

- It follows in particular for $\mathcal{C}$, and then also $\tilde{g}$, real-analytic, that then $\mathrm{Hol}(\mathcal{C}) = \mathrm{Hol}(\nabla^{\mathcal{T}}) = \mathrm{Hol}(\tilde{\nabla})$.
- For $n = p + q$ even one has to employ a truncated version of infinitesimal ambient holonomy that only involves derivatives up to order $\frac{n}{2}$ and can then show analogously that this space is contained in infinitesimal tractor holonomy $\mathfrak{hol}$.

Equality of ambient and tractor holonomy

That $\mathfrak{hol} \subset \tilde{\mathfrak{hol}}$. can be shown directly by invoking the realization of the tractor covariant derivative in terms of the ambient covariant derivative.

Outline of the inclusion $\tilde{\mathfrak{hol}} \subset \mathfrak{hol}$:
We first introduce the truncated infinitesimal holonomy spaces

$$\tilde{\mathfrak{hol}}_k = \mathrm{span}\Big(\{\big(\tilde{\nabla}_{\xi_1} \cdots \tilde{\nabla}_{\xi_{l-2}} \tilde{R}(\xi_{l-1}, \xi_l)\big)_{|\mathcal{C}} | \xi_1, \ldots, \xi_l \in \mathfrak{X}(\tilde{M}), 2 \leq l \leq k\}\Big).$$

and then proceed inductively:

Outline of the inclusion $\tilde{\mathfrak{hol}}_{\mathfrak{k}} \subset \mathfrak{hol}$:

1. For $k = 2$ there are no derivatives of the ambient curvature tensor $\tilde{R}$ involved, and it is possible to express the ambient curvature tensor $\tilde{R}$ in terms of the tractor curvature tensor R and its covariant derivative ∇R. It is then seen directly that the span of the image of $\tilde{R}$ lies in the image of $R, \nabla R$. 'pause
2. The inductive step

$$\tilde{\mathfrak{hol}}_k \subset \mathfrak{hol} \Rightarrow \tilde{\mathfrak{hol}}_{k+1} \subset \mathfrak{hol} \tag{1}$$

is more involved:

Some ingredients for the proof of the inclusion $\tilde{\mathfrak{hol}}_{\mathfrak{k}} \subset \mathfrak{hol}$:

One needs to express the ambient covariant derivative $\tilde{\nabla}\tilde{R}$ of ambient curvature in terms of tractor data:

- This invokes a formula by [Gover-Peterson, '03] which expresses the ambient covariant derivative $\tilde{\nabla}$ of an arbitrary homogeneous ambient tensor $\tilde{V}$ along the cone $\mathcal{C}$ in terms of the tractor covariant derivative ∇ and the fundamental D-operator.
- A detailed inspection of the resulting terms and invocation of the inductive hypothesis shows that they can either be written as tractor covariant derivatives $\nabla_{\xi_1} \cdots \nabla_{\xi_{l-2}} R(\xi_{l-1}, \xi_l) \in \mathfrak{hol}$ or as commutators (in the endomorphism-slot) of such terms. In the latter case, one employs that $\mathfrak{hol}$ is a filtered Lie algebra: $[\mathfrak{hol}_i, \mathfrak{hol}_j] \subset \mathfrak{hol}_{i+j}$.

References:

- C. Fefferman and C.R. Graham ('11)
  The Ambient Metric.
  American Mathematical Society.
- A.R. Gover and L.J. Peterson ('03)
  Conformally invariant powers of the Laplacian, Q-curvature, and tractor calculus.
  Comm. Math. Phys., 235(2):339–378.
- C.R. Graham and T. Willse ('11)
  Parallel tractor extension and ambient metrics of holonomy split G_2.
  arXiv:1109.3504