Fefferman-type constructions for parabolic geometries

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The original Fefferman construction [Fefferman,'76] describes a natural conformal structure on a circle bundle over a CR-structure. It was shown by Sparling and [Graham, '87] that a conformal structure is the Fefferman-space of some CR-structure if and only if it admits a light-like conformal Killing field which also satisfies additional (conformally invariant) properties.

The characterizing property can alternatively be understood as a *holonomy reduction* of the conformal structure: It was shown in [Čap-Gover, '10] that a conformal structure $(M, \mathcal{C})$ is locally the Fefferman-space of a CR-structure if and only if its conformal holonomy satisfies
$\mathrm{Hol}(\mathcal{C}) \subset \mathrm{SU}(p+1, q+1) \subset \mathrm{SO}(2p+2, 2q+2)$.

A generalization of the original Fefferman-construction was described in [Čap, '05], and in recent years a number of constructions have been discussed in that framework:

- The original construction was treated by [Čap-Gover, '10]
- A construction of [Biquard, '00] of conformal structures from quaternionic contact structures was treated by [Alt, '10]
- Nurowski's conformal structures that are associated to generic rank 2 distributions on 5-manifolds and Bryant's [Bryant, '06] conformal structures associated to generic rank 3 distributions on 6-manifolds were discussed in joint work [H.-Sagerschnig, '10, '11]

In all cited cases the Fefferman-type construction is *normal*: a usually non-trivial computation shows that the construction is compatible with the canonical normalization condition one has for a parabolic geometry.

This has immediate strong geometric consequences, since this implies that the holonomy of the conformal structure reduces. It also allows one to derive a holonomy-based characterization of the induced Fefferman-spaces.

Via the BGG-machinery for parabolic geometries this characterization can also be understood in terms of solutions of natural overdetermined equations on the manifold.

Basic facts about parabolic geometries

Parabolic geometries are Cartan geometries of type (G, P), with P a parabolic subgroup of a Lie group G: A parabolic geometry of the given type on a manifold M is described by a principal P-bundle $\mathcal{G} \to M$ that is endowed with a Cartan connection form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$.

The Cartan connection form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ has to satisfy the usual properties: it has to be P-equivariant, reproduce fundamental vector fields of that P-action and provide an absolute parallelism $\omega : T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$.

The curvature of ω can be regarded as a function $\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, and the geometry is flat if and only if it is locally isomorphic to the homogeneous model G/P.

Example: Conformal (spin) structures as parabolic geometries

When $G = \mathrm{SO}(p+1, q+1)$ and P the stabilizer of a null-ray $R \subset \mathbb{R}^{p+1,q+1}$, $G/P \cong S_p \times S_q$. With g_p, g_q the standard metrics on $S_p \subset \mathbb{R}^p$ and $g_q \subset \mathbb{R}^q$, G/P is endowed with the conformal class $\mathcal{C}$ of signature (p, q)-metrics which contains $(g_p, -g_q)$. This is the homogeneous model of conformal structures in that signature.

If in addition one has a spin structure on the manifold, the geometry is modeled on $G = \mathrm{Spin}(p+1, q+1)$ and $P \subset G$ is again the stabilizer of a null-ray.

If $(M, \mathcal{C})$ is the conformal structure that is described by the Cartan geometry $(\mathcal{G}, \omega)$ of type (G, P) on M, then flatness of ω implies that $(M, \mathcal{C})$ is locally isomorphic to $(S^p \times S^q, [g_p, -g_q])$, which says that $\mathcal{C}$ is locally conformally flat.

Basic facts about parabolic geometries

The main feature of parabolic geometries is that they allow uniform regularity and normality conditions: if these conditions are satisfied, the parabolic structure is an equivalently description of an underlying geometric structure, like a projective, conformal or CR-structure.

This normalization condition employs the Kostant co-differential $\partial^* : \Lambda^{k+1}(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \to \Lambda^k(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$: It is algebraic and defines a complex, $\partial^* \circ \partial^* = 0$. The Cartan connection ω is normal if and only if $\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ has values in the kernel of ∂^*.

Fefferman-type constructions

A Fefferman-type construction [Čap, '05] is a natural procedure that starts with a parabolic geometry of a type (G, P) on a manifold M and associates a parabolic geometry of another type $(\tilde{G}, \tilde{P})$ on a (possibly larger) manifold $\tilde{M}$.

The algebraic input for this is an inclusion of Lie groups $i : G \hookrightarrow \tilde{G}$ with the property that $Q := G \cap \tilde{P} \subset P$. The Fefferman-type construction $(G, P) \rightsquigarrow (\tilde{G}, \tilde{P})$ is always possible if G acts (locally) transitive on the homogeneous space $\tilde{G}/\tilde{P}$.

Example: The Fefferman-construction for CR-structures

To describe the original Fefferman construction we use that CR structures (of hypersurface type) can be equivalently encoded as parabolic geometries of type $(G, P) = (\mathrm{SU}(p+1, q+1), P)$, where P is the stabilizer of a complex null-line in $\mathbb{C}^{p+1,q+1}$: for some element $r \in \mathbb{C}^{p+1,q+1}$ with $\langle r, r \rangle = 0$, P is defined as the stabilizer of $\mathbb{C}r \subset \mathbb{C}^{p+1,q+1}$.

The Fefferman construction employs the standard embedding $G = \mathrm{SU}(p+1, q+1) \hookrightarrow \mathrm{SO}(2p+2, 2q+2) = \tilde{G}$: recall that the parabolic subgroup $\tilde{P} \subset \tilde{G}$ can be described as the stabilizer of the null-ray $\mathbb{R}_+ r \subset \mathbb{R}^{2p+2,2q+2}$.

By linear algebra, $G = \mathrm{SU}(p+1, q+1)$ acts transitively on the space $\tilde{G}/\tilde{P}$ of null-rays in $\mathbb{R}^{2p+2,2q+2}$: The isotropy subgroup in $\mathrm{SU}(p+1, q+1)$ of this action is $Q = G \cap \tilde{P}$, which is the stabilizer in $\mathrm{SU}(p+1, q+1)$ of the null-ray $\mathbb{R}_+ r$: thus $P/Q = S^1$.

Example: The Fefferman-construction for CR-structures

Now if $(\mathcal{G}, \omega)$ is a parabolic geometry of type $(\mathrm{SU}(p+1, q+1), P)$ that describes a CR-structure on M, then the Fefferman-space is obtained as $\tilde{M} = \mathcal{G}/Q = \mathcal{G} \times_P P/Q$, which is an S^1-bundle over M.

$\mathcal{G} \to \tilde{M}$ is a Q-principal bundle, which can be extended to the $\tilde{P}$-principal bundle $\tilde{\mathcal{G}} = \mathcal{G} \times_Q \tilde{P}$. It is easy to see that ω defines a canonical Cartan connection form $\tilde{\omega}$ on $\mathcal{G}$ by extension, and therefore $\tilde{M}$ naturally carries a conformal structure $\mathcal{C}$ of signature $(2p+1, 2q+1)$.

General procedure for a Fefferman-type construction

The first step is to form the correspondence space $\tilde{M} := \mathcal{G}/Q = \mathcal{G} \times_P P/Q$. $\mathcal{G} \to \tilde{M}$ is then a Q-principal bundle endowed with the Cartan connection form ω of type (G, Q).

The second step is to form the extended Cartan bundle $\tilde{\mathcal{G}} = \mathcal{G} \times_Q \tilde{P}$ and canonically extend ω to a Cartan connection form $\tilde{\omega}$ on $\tilde{\mathcal{G}}$. Then $(\tilde{\mathcal{G}}, \tilde{\omega})$ is a Cartan geometry of type $(\tilde{G}, \tilde{P})$ on $\tilde{M}$.

The Fefferman-type construction $(G, P) \rightsquigarrow (\tilde{G}, \tilde{P})$ is called normal if normality of ω automatically implies normality of $\tilde{\omega}$. Normality is a very strong condition, and if satisfied, the special (or characterizing) properties of the structures induced by the Fefferman-type construction can often be easily identified by holonomy techniques:

Background: the holonomy of a parabolic geometry

To form the holonomy $\mathrm{Hol}(\omega)$ of a parabolic geometry $(\mathcal{G}, \omega)$, one extends $\mathcal{G}$ to a principal G-bundle $\hat{G} = \mathcal{G} \times_P G$ and canonically extends ω to the a principal connection form $\hat{\omega}$ on $\hat{G}$. Then $\mathrm{Hol}(\omega) := \mathrm{Hol}(\hat{\omega})$.

The geometric meaning of a reduced holonomy $\mathrm{Hol}(\omega) \subset H \subset G$ for some subgroup $H \subset G$ is already visible on the homogeneous (flat) model G/P: The holonomy-reduction to $H \subset G$ decomposes G/P into the disjoint union of its H-orbits,

$$G/P = \bigcup_{HgP \in H \backslash G/P} H \cdot gP/P.$$

Each orbit $H \cdot gP/P \cong H/(H \cap gPg^{-1})$ is again the homogeneous model of a Cartan geometry of type $(H, H \cap gPg^{-1})$.

Theorem (Čap-Gover-H., '11)

Let $(\mathcal{G}, \omega)$ *be a parabolic geometry of type* (G, P) *with* $\mathrm{Hol}(\omega) \subset H \subset G$. *Then there exists a natural decomposition*

$$M = \bigcup_{HgP \in H \backslash G / P} M_{HgP}$$

of M *that is parametrized by the set of* H*-orbits on* G/P*, which is the double co-set space* $H \backslash G / P$.
Each $M_{HgP} \subset M$ *is an initial submanifold that carries a canonical Cartan geometry of type* $(H, H \cap gPg^{-1})$.

This decomposition is called the curved orbit decomposition of M with respect to the given holonomy reduction. When H acts transitively on G/P this shows that M carries a global reduced geometry of type $(H, H \cap P)$.

Holonomies of Fefferman spaces

We consider now a normal Fefferman-type construction $(G, P) \rightsquigarrow (\tilde{G}, \tilde{P})$: This starts with a geometry $(\mathcal{G}, \omega)$ of type (G, P) and associates a geometry $(\tilde{\mathcal{G}}, \tilde{\omega})$ of type $(\tilde{G}, \tilde{P})$.

It follows immediately from the construction that with $\hat{i} : \hat{\mathcal{G}} \hookrightarrow \hat{\tilde{\mathcal{G}}}$ the canonical embedding of the extended G-principal bundle into the extended $\hat{G}$-principal bundle, $\hat{i}^*(\hat{\tilde{\omega}}) = \hat{\omega}$, and therefore $\mathrm{Hol}(\tilde{\omega}) = \mathrm{Hol}(\omega)$.

Of course this has geometric meaning only in the case where the construction is normal: In this case, $\mathrm{Hol}(\tilde{\omega})$ is the well-defined holonomy of the parabolic geometry on $\tilde{M}$. In particular, this implies that if ω is non-flat, $(\tilde{\mathcal{G}}, \tilde{\omega})$ is a non-flat parabolic geometry on $\tilde{M}$ with holonomy contained in $G \subset \tilde{G}$.

Induced solutions of BGG-equations

In many cases the inclusion $G \hookrightarrow \tilde{G}$ is realized as the stabilizer of an element in a $\tilde{G}$-representation V. It is well known that the *tractor bundle* $\mathcal{V} = \tilde{\mathcal{G}} \times_{\tilde{P}} V$ carries the *tractor connection* ∇ that is naturally induced from the Cartan connection form $\tilde{\omega}$. Then $\mathrm{Hol}(\tilde{\omega}) \subset G$ is equivalent to the existence of a parallel section $s \in \Gamma(\mathcal{V})$ of a suitable type.

By the general theory of BGG-operators on parabolic geometries as developed by [Čap-Slovak-Souček, '01], such a parallel section s is equivalent to a normal solution of the first BGG-operator $\Theta_0 : \mathcal{H}_0 \to \mathcal{H}_1$ associated to $\mathcal{V}$.

Example: Characterization of the Fefferman-space of a CR-structure

Let $\mathbb{J} \in \mathfrak{so}(2p+2, 2q+2)$ be the orthogonal complex structure on $\mathbb{R}^{2p+2,2q+2}$ corresponding to multiplication with i on $\mathbb{C}^{p+1,q+1}$. The isotropy group of $\mathbb{J}$ in $\mathrm{SO}(2p+2, 2q+2)$ is $\mathrm{SO}(2p+2, 2q+2)_{\mathbb{J}} = \mathrm{U}(p+1, q+1)$.

It was shown by [Čap-Gover, Leitner], that conformal holonomy $\mathrm{Hol}(\omega) \subset \mathrm{U}(p+1, q+1)$ already implies locally that $\mathrm{Hol}(\omega) \subset \mathrm{SU}(p+1, q+1)$. Therefore a parallel orthogonal complex structure, which is a section of $\tilde{\mathcal{G}} \times_{\tilde{P}} \mathfrak{so}(2p+2, 2q+2)$ can be used to (locally) characterize $\mathrm{SU}(p+1, q+1)$-holonomy: the corresponding normal BGG-solution is a light-like conformal Killing field on $(M, \mathcal{C})$ that inserts trivially into Weyl curvature and Cotton tensor.

Summary of some constructions

- $\mathrm{SU}(p+1, q+1) \hookrightarrow \mathrm{SO}(2p+2, 2q+2)$: [normal]
  CR-structure $\rightsquigarrow$
  signature $(2p+1, 2q+1)$-conformal structure on S^1-bundle
  $+$ lightlike conformal Killing field (with additional properties)
- $G_2 \hookrightarrow \mathrm{Spin}(3,4)$: [normal]
  generic rank 2-distribution on 5-manifold $\rightsquigarrow$
  signature $(2,3)$-conformal spin structure $+$ generic twistor spinor
- $\mathrm{SL}(n+1) \hookrightarrow \mathrm{Spin}(n+1, n+1)$: [non-normal for $n \geq 3$]
  projective structure on n-manifold $\rightsquigarrow$
  signature (n, n)-conformal spin structure $+$ twistor spinor

Projective $\rightsquigarrow$ split signature conformal

This is recent joint work with K. Sagerschnig. The original motivation for this Fefferman-type construction was work by [Dunajski-Tod, '10]:

Extending a construction due to [Walker, '54], which associates a pseudo-Riemannian split signature (n, n)-metric to an affine torsion-free connection on an n-manifold, they associate a conformal split signature (n, n)-metric to a projective class of torsion-free affine connections on an n-manifold. Using a normal form for the induced metrics it is also shown that they admit a twistor spinor. For $n = 2$ this construction was also observed in work by [Nurowski-Sparling, '03]. The precise relation between the cited works and the construction here has been shown recently by [Šilhan-Žádník, '11].

Projective $\rightsquigarrow$ split signature conformal

This Fefferman-type construction is based on an inclusion $\mathrm{SL}(n+1) \hookrightarrow \mathrm{Spin}(n+1, n+1)$:

Denote by $\Delta = \Delta_+^{n+1,n+1} \oplus \Delta_-^{n+1,n+1}$ the real 2^{n+1}-dimensional spin representation of $\tilde{G} = \mathrm{Spin}(n+1, n+1)$. Then we fix two pure spinors $s_F \in \Delta_-^{n+1,n+1}, s_E \in \Delta_\pm^{n+1,n+1}$ with non-trivial pairing - here s_E lies in $\Delta_+^{n+1,n+1}$ if n is even or $\Delta_-^{n+1,n+1}$ if n is odd.

These assumptions guarantee that the kernels $E, F \subset \mathbb{R}^{n+1,n+1}$ of s_E, s_F with respect to Clifford multiplication are complementary maximally isotropic subspaces.

Then
$G := \{g \in \mathrm{Spin}(n+1, n+1) : g \cdot s_E = s_E, g \cdot s_F = s_F\} \cong \mathrm{SL}(n+1)$, defines an embedding $G = \mathrm{SL}(n+1) \overset{i}{\hookrightarrow} \mathrm{Spin}(n+1, n+1)$.

Induced structure

One computes $\tilde{M} = \mathcal{G} \times_Q P/Q \cong (T^*M \otimes \mathcal{E}[2])/\{0\}$. Here we use the notation $\mathcal{E}[w]$ for suitably weighted (projective) version of the density bundle.

The invariant spinors s_E and s_F give rise to pure spin tractors: The spin tractor bundle of $(M, \mathcal{C})$ is $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$, where $\mathcal{S}_\pm = \tilde{\mathcal{G}} \times_{\tilde{P}} \Delta_\pm^{n+1,n+1} = \mathcal{G} \times_Q \Delta_\pm^{n+1,n+1}$. Since $s_E \in \Delta_\pm^{n+1,n+1}$ and $s_F \in \Delta_-^{n+1,n+1}$ are Q-invariant, they induce canonical sections $\mathbf{s}_E \in \Gamma(\mathcal{S}_\pm)$ and $\mathbf{s}_F \in \Gamma(\mathcal{S}_-)$.

Twistor spinors

With respect to a choice of metric $g \in \mathcal{C}$ the spin tractor bundle can be written as the sum of weighted spin bundles: $[\mathcal{S}]_g = S[-\frac{1}{2}] \oplus S[\frac{1}{2}]$. Parallel sections of $\mathcal{S}$ with respect to the normal conformal tractor connection are in 1 : 1-correspondence with solutions of of the twistor spinor equation $\chi \mapsto D\chi + \frac{1}{2n}\gamma \not{D}\chi$: If $\tau \oplus \chi$ is parallel, then χ is a twistor spinor, and converesely, if χ satisfies the twistor spinor equation, then $\frac{1}{\sqrt{2n}}\not{D}\chi \oplus \chi$ is parallel.

The conformal Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ obtained via the Fefferman construction induces a tractor connection ∇ on each conformal tractor bundle; by construction the spin tractors $\mathbf{s}_E, \mathbf{s}_F$ are parallel with respect to the induced tractor connections on the respective spin tractor bundles. But these are not necessarily the normal conformal tractor connections!

The normal case $n = 2$:

Proposition

The Fefferman-type construction $\mathrm{SL}(3) \hookrightarrow \mathrm{Spin}(3,3)$ *is normal.*

Corollary

The split-signature conformal structures obtained from two-dimensional projective structures have the following properties:

1. *The conformal holonomy* $\mathrm{Hol}(\tilde{\omega})$ *is contained in* $\mathrm{SL}(3)$.
2. *The spin tractor bundle has two sections* $\mathbf{s}_E$ *and* $\mathbf{s}_F$ *with non-trivial pairing that are parallel with respect to the normal tractor connection, i.e.* $\nabla^{\mathcal{S}_+,nor}\mathbf{s}_E = 0$ *and* $\nabla^{\mathcal{S}_-,nor}\mathbf{s}_F = 0$. *Thus they correspond to two pure twistor spinors* $\chi_E \in \Gamma(\mathbf{S}_+[\frac{1}{2}])$ *and* $\chi_F \in \Gamma(\mathbf{S}_-[\frac{1}{2}])$.

The non-normal case: $n \geq 3$

Proposition

For $n \geq 3$ The conformal Cartan connection form $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ induced by the normal projective Cartan connection form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is normal if and only ω is flat, in which case also $\tilde{\omega}$ is flat.

Since for $n \geq 3$ the induced Cartan connection form $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ is not already the normal conformal connection form, one needs a modification $\Psi \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{p}})$ with the property that $\tilde{\omega}^{nor} = \tilde{\omega} + \Psi$ is normal. While it is difficult to obtain an explicit formula for this modification Ψ, it is possible to show specific properties; in particular, the normalized connection still preserves one of the pure tractor spinors:

The non-normal case: $n \geq 3$

Theorem

$\mathbf{s}_F \in \Gamma(\mathcal{S}_-)$ is parallel with respect to the normal conformal spin tractor connection $\nabla^{\mathcal{S}_-,nor}\mathbf{s}_F = 0$. In particular, the conformal spin structure $(M, \mathcal{C})$ carries a canonical (pure) twistor spinor $\chi_F \in \Gamma(\mathbf{S}_-[\frac{1}{2}])$.

Corollary

The conformal holonomy $\mathrm{Hol}(\mathcal{C})$ is contained in the isotropy subgroup of $s_F \in \Delta_-^{n+1,n+1}$ in $\mathrm{Spin}(n+1, n+1)$; this is $SL(n+1) \ltimes \Lambda^2(\mathbb{R}^{n+1})^* \subset \mathrm{Spin}(n+1, n+1)$.

Outlook

The following questions are currently treated in joint work with K. Sagerschnig, J. Šilhan and V. Žádník:

- In the normal case $n = 2$: [Dunajski-Tod, '10] showed that the original projective structure on M is metrizable if and only if the induced signature $(2, 2)$ conformal structure includes a Kähler or para-Kähler metric. Using the description of these geometric solutions as parallel sections of suitably modified tractor connections we show this correspondence as a direct consequence of normality.
- For the non-normal case $n \geq 3$, it is an open problem to characterize the induced conformal structures. The twistor spinor described above will play a role in this, but additional characterizing data is necessary.