A non-normal Fefferman-type construction of split-signature conformal structures

Matthias Hammerl

University of Vienna, Faculty of Mathematics

32^{nd} Winter School geometry and Physics. Srní, Jan 19 2012

Joint work with K. Sagerschnig (ANU Canberra),
J. Šilhan and V. Žádník (Masaryk University).

The Fefferman-construction

The original Fefferman construction [Fefferman,'76] canonically associated a conformal structure on a circle bundle over a CR-structure. It was shown by Sparling and discussed by [Graham, '87] that a conformal structure is the Fefferman-space of some CR-structure if and only if it admits a light-like conformal Killing field which also satisfies additional (conformally invariant) properties.

The characterizing property can alternatively be understood as a *holonomy reduction* of the conformal structure: It was shown in [Čap-Gover, '10] that a conformal structure $(M, \mathcal{C})$ is locally the Fefferman-space of a CR-structure if and only if its conformal holonomy satisfies
$\mathrm{Hol}(\mathcal{C}) \subset \mathrm{SU}(p+1, q+1) \subset \mathrm{SO}(2p+2, 2q+2)$.

Generalization: Fefferman-type constructions

A generalization of the original Fefferman-construction was described in [Čap, '05], and in recent years a number of constructions have been discussed in that framework:

- The original construction was treated by [Čap-Gover, '10]
- A construction of [Biquard, '00] of conformal structures from quaternionic contact structures was treated by [Alt, '10]
- Nurowski's conformal structures that are associated to generic rank 2 distributions on 5-manifolds and Bryant's [Bryant, '06] conformal structures associated to generic rank 3 distributions on 6-manifolds were discussed in [H.-Sagerschnig, '10, '11]

In all cited cases the Fefferman-type construction is *normal*, which allows one to derive a holonomy-based characterization of the induced structures.

A non-normal construction

Here we discuss a (generically) *non-normal Fefferman-type construction*. We associate a split signature (n, n) conformal spin structure to a projective structure of dimension n.

The original motivation for this Fefferman-type construction was work by [Dunajski-Tod, '10]:

Extending a construction due to [Walker, '54], which associates a pseudo-Riemannian split signature (n, n)-metric to an affine torsion-free connection on an n-manifold, they associate a conformal split signature (n, n)-metric to a projective class of torsion-free affine connections on an n-manifold. Using a normal form for the induced metrics it is also shown that they admit a twistor spinor. For $n = 2$ this construction was also observed in work by [Nurowski-Sparling, '03].

Parabolic geometries and Fefferman-type constructions

Parabolic geometries are Cartan geometries of type (G, P), with P a parabolic subgroup of a Lie group G: A parabolic geometry of the given type an a manifold M is described by a principal P-bundle $\mathcal{G} \to M$ that is endowed with a Cartan connection form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{p})$.

Parabolic geometries allow uniform regularity and normality conditions, and if these conditions are satisfied, the parabolic structure is an equivalently description of an underlying geometric structure, like projective, conformal or CR-structures.

A Fefferman-type construction [Čap, '05] is a natural procedure that starts with a parabolic geometry of a type (G, P) on a manifold M and associates a parabolic geometry of another type $(\tilde{G}, \tilde{P})$ on a (possibly larger) manifold $\tilde{M}$.

Parabolic geometries and Fefferman-type constructions

The algebraic premise for this construction is an inclusion of Lie groups $i : G \hookrightarrow \tilde{G}$ with the property that $Q := G \cap \tilde{P} \subset P$. The Fefferman-type construction $(G, P) \rightsquigarrow (\tilde{G}, \tilde{P})$ is then possible if G acts locally transitive on the homogeneous space $\tilde{G}/\tilde{P}$.

The first step is to form the correspondence space $\tilde{M} := \mathcal{G}/Q = \mathcal{G} \times_P P/Q$. $\mathcal{G} \to \tilde{M}$ is then a Q-principal bundle endowed with the Cartan connection form ω of type (G, Q).

The second step is to form the extended Cartan bundle $\tilde{\mathcal{G}} = \mathcal{G} \times_Q \tilde{P}$ and canonically extend ω to a Cartan connection form $\tilde{\omega}$ on $\tilde{\mathcal{G}}$. Then $(\tilde{\mathcal{G}}, \tilde{\omega})$ is a Cartan geometry of type $(\tilde{G}, \tilde{P})$ on $\tilde{M}$.

Normality and holonomy

The Fefferman-type construction $(G, P) \rightsquigarrow (\tilde{G}, \tilde{P})$ is called *normal* if normality of ω automatically implies normality of $\tilde{\omega}$. This has immediate strong consequences, which are best visible by holonomy methods:

To form the *holonomy* $\mathrm{Hol}(\omega)$ of a parabolic geometry $(\mathcal{G}, \omega)$, one extends $\mathcal{G}$ to a principal G-bundle $\hat{G} = \mathcal{G} \times_P G$ and canonically extends ω to the a principal connection form $\hat{\omega}$ on $\hat{G}$. Then $\mathrm{Hol}(\omega) := \mathrm{Hol}(\hat{\omega})$.

It immediately follows from the Fefferman-type construction that $\mathrm{Hol}(\tilde{\omega}) = \mathrm{Hol}(\omega)$, and if the construction is normal, $\mathrm{Hol}(\tilde{\omega})$ is the well-defined holonomy of the parabolic geometry on $\tilde{M}$.

In particular, this implies that if ω is non-flat, $(\tilde{\mathcal{G}}, \tilde{\omega})$ is a non-flat parabolic geometry on $\tilde{M}$ with holonomy contained in $G \subset \tilde{G}$.

Induced solutions of BGG-equations

In many cases the inclusion $G \hookrightarrow \tilde{G}$ is realized as the stabilizer of an element in a $\tilde{G}$-representation V. It is well known that the *tractor bundle* $\mathcal{V} = \tilde{\mathcal{G}} \times_{\tilde{P}} V$ carries the *tractor connection* ∇ that is naturally induced from the Cartan connection form $\tilde{\omega}$. Then $\mathrm{Hol}(\tilde{\omega}) \subset G$ is equivalent to the existence of a parallel section $s \in \Gamma(\mathcal{V})$ of a suitable type.

By the general theory of BGG-operators on parabolic geometries as developed by [Čap-Slovak-Souček, '01], such a parallel section s is equivalent to a normal solution of the first BGG-operator $\Theta_0 : \mathcal{H}_0 \to \mathcal{H}_1$ associated to $\mathcal{V}$.

This $1 : 1$-correspondence is realized by a natural (tensorial) projection $\Pi_0 : \mathcal{V} \to \mathcal{H}_0$ and the *first BGG-splitting operator* $L_0 : \Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{V})$ of $\mathcal{V}$.

Examples of normal Fefferman-type constructions of conformal structures

- $\mathrm{SU}(p+1, q+1) \hookrightarrow \mathrm{SO}(2p+2, 2q+2)$:
  CR-structure $\rightsquigarrow$
  signature $(2p+1, 2q+1)$-conformal structure on S^1-bundle
  $+$ lightlike conformal Killing field (with additional properties)
- $\mathrm{Sp}(n+1, 1) \hookrightarrow \mathrm{SO}(4n+4, 4)$:
  quaternionic contact structure $\rightsquigarrow$
  signature $(4n+3, 3)$ conformal structure
  $+$ 2 orthogonal lightlike conformal Killing fields
- $G_2 \hookrightarrow \mathrm{Spin}(3, 4)$:
  generic rank 2-distribution on 5-manifold $\rightsquigarrow$
  signature $(2, 3)$-conformal spin structure $+$ generic twistor spinor

$\mathrm{SL}(n+1) \hookrightarrow \mathrm{Spin}(n+1, n+1)$

This Fefferman-type construction is based on an inclusion $\mathrm{SL}(n+1) \hookrightarrow \mathrm{Spin}(n+1, n+1)$:

Denote by $\Delta = \Delta_+^{n+1,n+1} \oplus \Delta_-^{n+1,n+1}$ the real 2^{n+1}-dimensional spin representation of $\tilde{G} = \mathrm{Spin}(n+1, n+1)$. Then we fix two pure spinors $s_F \in \Delta_-^{n+1,n+1}, s_E \in \Delta_\pm^{n+1,n+1}$ with non-trivial pairing - here s_E lies in $\Delta_+^{n+1,n+1}$ if n is even or $\Delta_-^{n+1,n+1}$ if n is odd.

These assumptions guarantee that the kernels $E, F \subset \mathbb{R}^{n+1,n+1}$ of s_E, s_F with respect to Clifford multiplication are complementary maximally isotropic subspaces.

Then
$G := \{g \in \mathrm{Spin}(n+1, n+1) : g \cdot s_E = s_E, g \cdot s_F = s_F\} \cong \mathrm{SL}(n+1)$,
defines an embedding $G = \mathrm{SL}(n+1) \stackrel{i}{\hookrightarrow} \mathrm{Spin}(n+1, n+1)$.

Fefferman-space $\tilde{M}$ and induced structure

One computes $\tilde{M} = \mathcal{G} \times_Q P/Q \cong (T^*M \otimes \mathcal{E}[2])/\{0\}$. Here we use the notation $\mathcal{E}[w]$ for suitably weighted (projective) version of the density bundle.

The invariant spinors s_E and s_F give rise to pure spin tractors:

The spin tractor bundle of $(M, \mathcal{C})$ is $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$, where $\mathcal{S}_\pm = \tilde{\mathcal{G}} \times_{\tilde{P}} \Delta_\pm^{n+1,n+1} = \mathcal{G} \times_Q \Delta_\pm^{n+1,n+1}$. Since $s_E \in \Delta_\pm^{n+1,n+1}$ and $s_F \in \Delta_-^{n+1,n+1}$ are Q-invariant, they induce canonical sections $\mathbf{s}_E \in \Gamma(\mathcal{S}_\pm)$ and $\mathbf{s}_F \in \Gamma(\mathcal{S}_-)$.

The conformal Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \mathfrak{g})$ induces a tractor connection ∇ on each conformal tractor bundle; the spin tractors $\mathbf{s}_E, \mathbf{s}_F$ are parallel with respect to the induced tractor connections on the respective spin tractor bundles. But these are not necessarily the normal conformal tractor connection!

Normality of the induced conformal Cartan connection

Proposition

For $n = 2$ the Fefferman-type construction $\mathrm{SL}(3) \hookrightarrow \mathrm{Spin}(3,3)$ is normal. For $n \geq 3$ The conformal Cartan connection form $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ induced by the normal projective Cartan connection form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is normal if and only ω is flat, in which case also $\tilde{\omega}$ is flat.

Outline of the argument:

The normalization condition on a conformal structure automatically implies that it is also torsion-free, i.e., that $\tilde{\kappa} : \tilde{\mathcal{G}} \to \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \mathfrak{p}$ has values in $\tilde{p} \subset \tilde{\mathfrak{g}}$. If the Fefferman-type construction $\mathrm{SL}(n+1) \rightsquigarrow \mathrm{Spin}(n+1, n+1)$ is normal, this forces the curvature of the projective structure κ to have values in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{q}$, since $\mathfrak{q} = \mathfrak{g} \cap \tilde{\mathfrak{p}}$. Now one has to treat two seperate cases:

Normality

In the case where $n = 2$ the projective curvature consists only of the projective Cotton tensor, and has values in $\mathfrak{p}_+ \subset \mathfrak{q}$. Then a detailed discussion of the normalization condition indeed implies normality of the Fefferman-type construction.

However when $n \geq 3$, the projective curvature is uniquely determined by the projective Weyl tensor, and this has values in a P-module larger than $\mathfrak{q}$. But then, if the curvature κ doesn't vanish, it immediately follows from equivariancy-properties that κ has values outside of $\mathfrak{q} \subset \mathfrak{p}$.

The normal case $n = 2$:

Proposition

1. *The conformal holonomy* $\mathrm{Hol}(\tilde{\omega})$ *is contained in* $\mathrm{SL}(3)$.
2. *The spin tractor bundle has two sections* $\mathbf{s}_E$ *and* $\mathbf{s}_F$ *with non-trivial pairing that are parallel with respect to the normal tractor connection, i.e.* $\nabla^{\mathcal{S}_+,nor}\mathbf{s}_E = 0$ *and* $\nabla^{\mathcal{S}_-,nor}\mathbf{s}_F = 0$. *Thus they correspond to two pure twistor spinors* $\chi_E \in \Gamma(\mathbf{S}_+[\frac{1}{2}])$ *and* $\chi_F \in \Gamma(\mathbf{S}_-[\frac{1}{2}])$.

Here $\mathbf{S}[\frac{1}{2}] = \mathbf{S}_+[\frac{1}{2}] \oplus \mathbf{S}_-[\frac{1}{2}]$ is the weighted conformal spin bundle on $\tilde{M}$ which is associated to the real $4 = 2^2$-dimensional spin representation of $\mathrm{CSpin}(2,2) = \mathbb{R}_+ \times \mathrm{Spin}(2,2)$. The second part of the proposition is then a consequence of the
$1:1$-correspondence between parallel spin tractors in $\mathcal{S}_\pm$ and twistor spinors $\chi \in \mathbf{S}_\pm[\frac{1}{2}]$, satisfying $D\chi + \frac{1}{4}\gamma \not{D}\chi = 0$.

The non-normal case: Normalization and preserved tractor spinor

Since for $n \geq 3$ the induced Cartan connection form $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ is not already the normal conformal connection form, one needs a modification $\Psi \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{p}})$ with the property that $\tilde{\omega}^{nor} = \tilde{\omega} + \Psi$ is normal, i.e., the curvature function $\tilde{\kappa}^{nor}$ of the modified Cartan connection form has to lie in the kernel of the Kostant co-differential $\tilde{\partial}^* : \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$.

The inductive normalization procedure that's necessary for a full computation of the modification Ψ makes it difficult to obtain an explicit formula for this map. It turns out however that certain properties of Ψ can be obtained without an explicit form. In particular, the normalized connection can be shown to still preserve one of the pure tractor spinors:

The non-normal case: Normalization and preserved tractor spinor

Theorem

$\mathbf{s}_F \in \Gamma(\mathcal{S}_-)$ is parallel with respect to the normal conformal spin tractor connection $\nabla^{\mathcal{S}_-,nor}\mathbf{s}_F = 0$. In particular, the conformal spin structure $(M, \mathcal{C})$ carries a canonical (pure) twistor spinor $\chi_F \in \Gamma(\mathbf{S}_-[\frac{1}{2}])$.

Corollary

The conformal holonomy $\mathrm{Hol}(\mathcal{C})$ is contained in the isotropy subgroup of $s_F \in \Delta_-^{n+1,n+1}$ in $\mathrm{Spin}(n+1, n+1)$; this is $SL(n+1) \ltimes \Lambda^2(\mathbb{R}^{n+1})^* \subset \mathrm{Spin}(n+1, n+1)$.

Metrizability and induced (para)-Kähler metrics

- A projective structure $[D]$ on M is called *metrizable* if there is an affine connection $\overline{D} \in [D]$ which is is the Levi-Civita-connection of some (pseudo)-Riemannian metric.
- It was observed by [Sinjukov, '79] and [Mikeš, '96] that the metrizability of a given projective class of affine connections $[D]$ is governed by the projectively invariant, overdetermined equation

$$D_c\sigma^{ab} - \frac{1}{n+1}(D_c\sigma^{p(a})\delta_c^{b)} = 0.$$

Metrizability and induced (para)-Kähler metrics

- This is the first BGG-equation associated to the projective tractor bundle $S^2\mathcal{T}$. In our specific situation $n = 2$, we write the corresponding first BGG-operator as

$$\Theta_0^{S^2\mathcal{T}} : \mathbf{E}^{(ab)}[-2] \to \mathbf{E}_0{}_c^{ab}[-2],$$
$$\sigma^{ab} \mapsto D_c\sigma^{ab} - \frac{2}{3}(D_p\sigma^{p(a})\delta_c^{b)}$$

- By general methods for prolongations of first BGG-equations [H.-Somberg-Souček-Šilhan,'11] the equation is of finite type and solutions are in 1 : 1-correspondence with parallel tractor sections with respect to a *prolongation connection* $\overline{\nabla}$:

$$\{\sigma^{ab} \in \Gamma(\mathbf{E}^{(ab)}[-2]) \,|\, \Theta_0^{S^2\mathcal{T}}(\sigma) = 0\} \leftrightarrow \{\Phi \in \Gamma(S^2\mathcal{T}) \,|\, \overline{\nabla}^{S^2\mathcal{T}}\Phi = 0\}$$

Metrizability and induced (para)-Kähler metrics

- It was observed by [Dunajski-Tod, '10] that the projective class of affine connections $[D]$ is metrizable if and only if the induced conformal structure $\mathcal{C}$ contains a metric $\bar{g} \in \mathcal{C}$ which is (para)-Kähler.
- Using methods from the BGG-machinery, we can show that this result is based on a natural inclusion

$$\iota : \Gamma(S^2\mathcal{T}) \hookrightarrow \Gamma(\Lambda^3_+\widetilde{\mathcal{T}}) \cong \Gamma(S_2\mathcal{S}_+)$$

of the projective tractor bundle that governs metrizability into the conformal tractor bundle governing the existence of a suitable (para)-Kähler form.

- The first BGG-operator controlling the existence of (para)-Kähler metrics in the conformal class $(\widetilde{M}, \mathcal{C})$ is

$$\Theta_0^{\Lambda^3_+\widetilde{\mathcal{T}}} : \widetilde{\mathbf{E}}_{[ab]+}[3] \to \widetilde{\mathbf{E}}_c \boxtimes \widetilde{\mathbf{E}}_{[ab]+},$$
$$\tilde{\sigma} \mapsto D_c\tilde{\sigma}_{ab} - D_{[c}\tilde{\sigma}_{ab]} + \frac{2}{3}\mathbf{g}_{c[a}D^p\tilde{\sigma}_{b]p}$$

where $\boxtimes$ denotes the Cartan product and $\widetilde{\mathbf{E}}_{[ab]+} \cong S^2S_+[1]$ is the bundle of self-dual two forms and $\Lambda^3_+\widetilde{\mathcal{T}} \cong S^2\widetilde{\mathcal{S}}_+$ the corresponding tractor bundle.

- Again, we have a prolongation of the equation $\Theta_0^{\Lambda^3_+\widetilde{\mathcal{T}}}(\tilde{\sigma}) = 0$:

$$\left\{\widetilde{\sigma}_{ab} \in \Gamma(\widetilde{\mathbf{E}}_{[ab]}[3]) \,|\, \Theta_0^{\Lambda^3_+\widetilde{\mathcal{T}}}(\widetilde{\sigma}) = 0\right\} \leftrightarrow \left\{\widetilde{\Phi} \in \Gamma(\Lambda^3_+\widetilde{\mathcal{T}}) \,|\, \overline{\nabla}^{\Lambda^3_+\widetilde{\mathcal{T}}}\widetilde{\Phi} = 0\right\}.$$

Our goal is to show

Theorem

The projective structure $[D]$ on M is metrizable with a Riemannian (or Lorentzian) metric if and only if there exists a Kähler (or para-Kähler) metric in the conformal class $\mathcal{C}$ on $\widetilde{M}$.

Here we will sketch one direction of this result, going from metrizability to the existence of a (para)-Kähler metric in the conformal class. In terms of the prolonged systems, employing the respective prolongation connections on the tractor bundles, this can be phrased as:

Proposition

Let $\Phi \in \Gamma(S^2\mathcal{T})$ satisfy $\overline{\nabla}^{S^2\mathcal{T}}\Phi = 0$ and put $\widetilde{\Phi} := \iota(\Phi) \in \Gamma(\Lambda^3_+\widetilde{\mathcal{T}})$. Then $\overline{\nabla}^{\Lambda^3_+\widetilde{\mathcal{T}}}\widetilde{\Phi} = 0$.

Explicit form of the prolongation connections:

This result depends only minimally on the explicit formula of the prolongation connections, which are

$$\overline{\nabla}_c^{S^2\mathcal{T}} \begin{pmatrix} \nu \\ \mu^a \\ \sigma^{ab} \end{pmatrix} = \nabla_c^{S^2\mathcal{T}} \begin{pmatrix} \nu \\ \mu^a \\ \sigma^{ab} \end{pmatrix} + \begin{pmatrix} A_{rcp}\, \sigma^{pr} \\ 0 \\ 0 \end{pmatrix} \quad \text{and}$$

$$\overline{\nabla}_c^{\Lambda^3_+\widetilde{\mathcal{T}}} \begin{pmatrix} \widetilde{\nu}_{ab} \\ \widetilde{\mu}_a \\ \widetilde{\sigma}_{ab} \end{pmatrix} = \nabla_c^{\Lambda^3\widetilde{\mathcal{T}}_+} \begin{pmatrix} \widetilde{\nu}_{ab} \\ \widetilde{\mu}_a \\ \widetilde{\sigma}_{ab} \end{pmatrix} + \begin{pmatrix} \widetilde{A}_{c(pa)} \widetilde{\sigma}^p{}_b + \frac{1}{2} \widetilde{C}_c{}^p{}_{ab}\, \widetilde{\mu}_p \\ 0 \\ 0 \end{pmatrix}.$$

The only point that matters is that the modifications from the natural tractor connections only occur in the top slots.

Sketch of proof:

- Only the top slot of $\nabla^{S^2\mathcal{T}}\Phi$ is nonzero, and since ι is compatible with the filtration, we obtain that also only the top slot of $\nabla^{\Lambda^3_+\widetilde{\mathcal{T}}}\widetilde{\Phi}$ can be nonzero.
- It follows that $\tilde{\partial}^* \nabla^{\Lambda^3_+\widetilde{\mathcal{T}}}\widetilde{\Phi} = 0$. Denote $\widetilde{\sigma} \in \Gamma(\Lambda^2_+ T^*\widetilde{M}[3])$ the projecting slot of $\widetilde{\Phi}$. Then by the general principles for the BGG-machinery, this implies $\widetilde{\Phi} = L_0^{\Lambda^3_+\widetilde{\mathcal{T}}}(\widetilde{\sigma})$.
- From the observation that the projecting slot of $\nabla^{\Lambda^3_+\widetilde{\mathcal{T}}}\widetilde{\Phi} = \nabla^{\Lambda^3_+\widetilde{\mathcal{T}}} L_0^{\Lambda^3_+\widetilde{\mathcal{T}}}(\widetilde{\sigma})$ vanishes, we obtain that $\Theta_0^{\Lambda^3_+\widetilde{\mathcal{T}}}(\widetilde{\sigma}) = 0$, i.e. $\widetilde{\sigma}$ is a conformal Killing 2-form.
- Since $\overline{\nabla}^{\Lambda^3_+\widetilde{\mathcal{T}}}$ is the prolongation connection, we conclude that $\overline{\nabla}^{\Lambda^3_+\widetilde{\mathcal{T}}}\widetilde{\Phi} = 0$.