Holography of BGG-Solutions

Matthias Hammerl

University of Greifswald

January 2016 - Srní Winter School Geometry and Physics

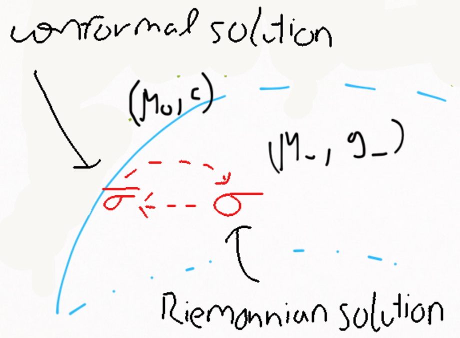
Joint work with Travis Willse (University of Vienna)

Introductory picture: Holography of solutions

Let $M = M_0 \cup M_-$ be a (real-analytic) *Poincaré-Einstein* manifold:

- $(M_0, \mathbf{c})$ is a conformal n-dimensional space.
- (M_-, g_-) an Einstein metric with $\mathrm{Ric}(g_-) = -ng_-$.
- $(M_0, \mathbf{c})$ is the *conformal infinity* of (M_-, g_-).

Our goal is to relate overdetermined differential equations on M_- with corresponding equations on the conformal infinity M_0:



Conformally invariant overdetermined Equations on $(M_0, \mathbf{c})$

Conformally invariant overdeterimend equations on the boundary $(M_0, \mathbf{c})$ are governed by *first BGG-operators*

$$\overline{\Theta}_0 : \Gamma(\overline{\mathcal{H}}_0) \to \Gamma(\overline{\mathcal{H}}_1).$$

Examples are:

- **Conformally-Einstein equation:** The equation governing the existence of an Einstein-metric in the conformal class:

$$\mathbf{tf}(\overline{D}_a \overline{D}_b \sigma + \overline{\mathrm{P}}_{ab}\sigma) = 0.$$

- **Conformal Killing form equation:**

$$\overline{D}_c \overline{\varphi}_{a_1 \cdots a_k} = \overline{D}_{[a_0} \overline{\varphi}_{a_1 \cdots a_k]} + \frac{k}{n-k+1} \overline{g}_{c[a_1} \overline{g}^{pq} \overline{D}_{|p} \overline{\varphi}_{q| a_2 \cdots a_k]}$$

- **Twistor spinor equation:**

$$\overline{D}_c \overline{\chi} + \frac{1}{n} \overline{\gamma}_c \overline{\not{D}} \overline{\chi} = 0.$$

Projectively invariant overdetermined Equations on (M_-, g_-)

We regard overdetermined equations on the Riemannian structure (M, g_-) which are projectively invariant and are therefore governed by *first BGG-operators* $\Theta_0 : \Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{H}_1)$ of the projective structure **p** spanned by the Levi-Civita covariant derivative D of g_-. Examples are:

- **Projectively-Ricci-flat equation:** The equation governing the existence of a Ricci-flat affine connection in the projective class of the Levi-Civita-connection D:

$$D_\alpha D_\beta \sigma + \sigma \mathrm{P}_{\alpha\beta} = 0$$

- **Killing form equation:**

$$D_\gamma \varphi_{\alpha_1 \cdots \alpha_k} = D_{[\alpha_0} \varphi_{\alpha_1 \cdots \alpha_k]}$$

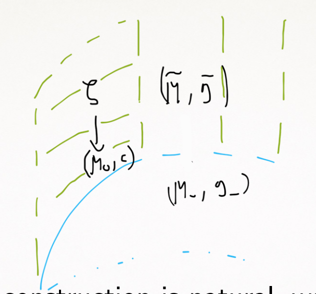
- **Killing spinor equation:**

$$D_\alpha \chi = \lambda \gamma_\alpha \chi \quad \text{for some } \lambda \in \mathbb{R}.$$

Ambient approach to holography

The conformal structure **c** can be understood as the ray-bundle $\mathcal{C}$ that consists of all metrics in the given conformal class.

For n odd and **c** real-analytic, the *Fefferman-Graham ambient metric* $\tilde{g}$ is a Ricci-flat signature $(n+1,1)$ metric on an $n+2$ dimensional *ambient space* $\tilde{M} = \mathcal{C} \times (-1,1)$.



Since the ambient metric construction is natural, we obtain a (first, trivial) categorical holographic correspondence between symmetries:

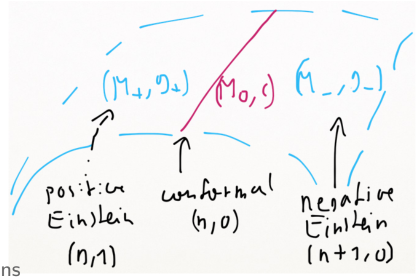
conformal Killing field $\bar{\xi}$ on M_0 $\Leftrightarrow$ Killing field ξ of g_-.

Klein-Einstein

Instead of working with Poincaré-Einstein structures, it will be useful for our purposes to work with the closely related class of *Klein-Einstein* structures, which are more directly related to Fefferman-Graham ambient metrics: (Fefferman-Graham, 2011; Čap-Gover-H., 2012)

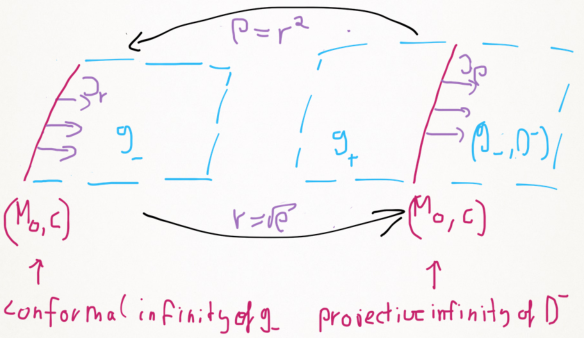
Let $M = M_+ \cup M_0 \cup M_-$ be a (real-analytic) *Klein-Einstein* manifold:

- $(M_0, \mathbf{c})$ is a conformal n-dimensional space.
- (M_+, g_+) an Einstein metric with $\mathrm{Ric}(g_+) = ng_+$.
- (M_-, g_-) an Einstein metric with $\mathrm{Ric}(g_-) = -ng_-$.
- $(M_0, \mathbf{c})$ is the *projective infinity* of (M_-, g_-).



Relationship between Poincaré-Einstein and Klein-Einstein

One has that *even* Poincaré-Einstein structures correspond to Klein-Einstein structures via a "change of variables" (Fefferman-Graham, 2011).



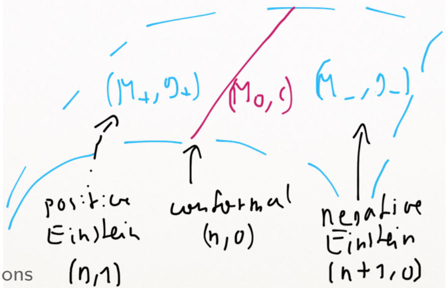
Klein-Einstein manifolds as projective structures

Let M be an $n+1$-dimensional manifold endowed with a projective structure $\mathbf{p}$ whose *normal projective tractor connection* $\nabla^{\mathcal{T}}$ preserves a signature $(n+1, 1)$ *tractor metric* $\mathbf{g}$. Equivalently, the *projective holonomy* is reduced to

$$\mathrm{Hol}(\mathbf{p}) \subseteq \mathrm{SO}(n+1,1) \subseteq \mathrm{SL}(n+2).$$

Via the general *curved orbit decomposition* ([Čap-Gover-H], 2014) for holonomy reduced structures, one obtains (generically) a decomposition of M with properites as above into *curved* $\mathrm{SO}(n+1,1)$-orbits

$$(M_+, g_+) \cup (M_0, \mathbf{c}) \cup (M_-, g_-):$$

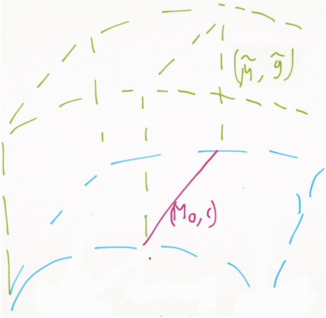


Cone construction over a Klein-Einstein manifold

Thomas cone construction: The projective $n+1$-dimensional structure $(M, \mathbf{p})$ can be equivalently encoded as a canonical Ricci-flat connection $\widetilde{\nabla}$ on an $n+2$-dimensional cone $\widetilde{M}$ over M.

Metric cone: The (sub-)cone $\mathcal{C} \subset \widetilde{M}$ over the conformal piece $M_0 \subset M$ can be identified with the *metric cone* formed by the conformal class of metrics $\mathbf{c}$.

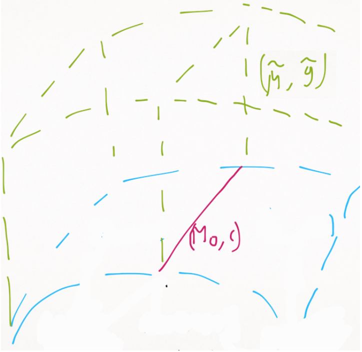
Metric on $\widetilde{M}$: The signature $(n+1, 1)$ parallel tractor metric $\mathbf{g}$ gives rise to a $\widetilde{\nabla}$-parallel metric $\widetilde{g}$ of the same signature on $\widetilde{M}$.



Fefferman-Graham ambient space

The Thomas cone over a Klein-Einstein structure thus provides a concrete realisation of:

- A Fefferman-Graham ambient space $\widetilde{M}$ with
- Ricci-flat, $\widetilde{\nabla}$-parallel Fefferman-Graham ambient metric $\widetilde{g}$.



The Klein-Einstein structure $(M, \mathbf{p})$ is an equivalent description of an (abstract) ambient space $(\widetilde{M}, \widetilde{g})$.

Holonomies

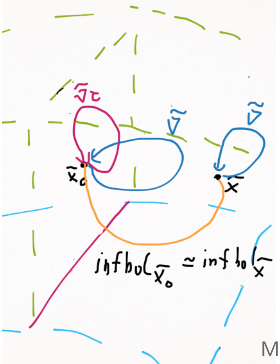
It was shown in (Čap-Gover-Graham-H. 2015) that 'infinitesimal ambient holonomy = infinitesimal conformal holonomy', which can be expressed in our (projective Klein-Einstein, 'abstract ambient' - setting) as:

$$\mathrm{inf.hol.}_{x_0}^{M_0}(\overline{\nabla}^{conf}) = \mathrm{inf.hol.}_{x_0}^{M}(\nabla^{proj}) \subset \mathfrak{so}(n+1,1)$$

at each point $x_0 \in M_0$ This infinitsimal holonomy subalgebra is naturally isomorphic to infinitesimal ambient holonomy $\mathrm{inf.hol.}^{\widetilde{M}}(\widetilde{\nabla})$.

Since our structure is real analytic:

- Infinitesimal holonomy is already the full (Lie algebraic) holonomy, which generates (at least locally at every point) the full holonomy.
- In particular, infinitesimal holonomy is the same (or naturally isomorphic) at every point.



Relationships between BGG-Solutions on M_0, M_-

Let V be an (irreducible) $\mathrm{SL}(n+2)$-representation and $\overline{V} \subset V$ a $\mathrm{SO}(n+1,1)$-submodule. Examples are:

- $\bar{V} = V = \mathbb{R}^{n+2}$
- $\bar{V} = \mathfrak{so}(n+1,1)$, $V = \mathfrak{sl}(n+2)$
- $\bar{V} = V = \Lambda^{k+1}\mathbb{R}^{n+2}$
- $\bar{V} = S^2_0\mathbb{R}^{n+2}$, $V = S^2\mathbb{R}^{n+2}$.

By general tractor machinery and holonomy principles:

- $\bar{V}$ gives rise to a conformal tractor bundle $\overline{\mathcal{V}}$ endowed with normal tractor connection $\overline{\nabla}^{\overline{\mathcal{V}}}$ for $(M_0, \mathbf{c})$, defined along M_0.
- V gives rise to a projective tractor bundle $\mathcal{V}$ endowed with normal tractor connection $\nabla^{\mathcal{V}}$ on $(M, \mathbf{p})$
- Since $(M, \mathbf{p})$ has holonmy reduced to $\mathrm{SO}(n+1,1)$, $\overline{\mathcal{V}}$ is globally defined on M, and then respective projective/conformal tractor connections on this bundle agree along M_0 (Čap-Gover-H., 2012).

BGG-Solutions

Via the general principles of the BGG-machinery (Čap-Slovak-Souček, 2001):

- $(\overline{\mathcal{V}}, \overline{\nabla}^{\overline{\mathcal{V}}})$ gives rise to a conformally invariant (first) BGG-operator $\overline{\Theta}_0$.
- $(\mathcal{V}, \nabla^{\mathcal{V}})$ gives rise to a projectively invariant (first) BGG-operator Θ_0.
- *Normal solutions* of the first BGG-operator $\overline{\Theta_0}$ resp. Θ_0 are defined as parallel sections of $\overline{\mathcal{V}}$ resp. $\mathcal{V}$:

$$\nabla^{\mathcal{V}} - \text{parallel sections} \quad \xrightarrow{\Pi_0} \quad \{\text{normal solutions}\} \subset \ker \Theta_0^{\mathcal{V}} \quad (L_0 \text{ back})$$

- For general solutions, one can employ the HSSŠ-'prolongation connection' $\hat{\nabla}^{\mathcal{V}}$ (H.-Somberg-Souček-Šilhan, 2012):

$$\hat{\nabla}^{\mathcal{V}} - \text{parallel sections} \quad \xrightarrow{\Pi_0} \quad \ker \Theta_0^{\mathcal{V}} \quad (L_0 \text{ back})$$

General Restriction & Extension for Normal BGG-Solutions

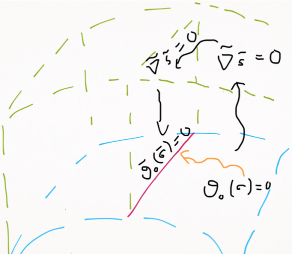
Since the connections $\overline{\nabla}^{\overline{\mathcal{V}}}$ resp. $\nabla^{\mathcal{V}}$ are induced from the normal conformal resp. projective tractor connections, infinitesimal holonomies agree again:

$$\mathrm{inf.hol.}_{x_0}^{M_0}(\overline{\nabla}^{\overline{\mathcal{V}}}) \cong \mathrm{inf.hol.}_{x}^{M_-}(\nabla^{\mathcal{V}}) \quad (x_0 \in M_0, x \in M_-).$$

It follows in particular that

$$\{\overline{\nabla}^{\overline{\mathcal{V}}}-\text{paralllel sections on } M_0\} \cong \{\nabla^{\mathcal{V}}-\text{parallel sections of type } \overline{V} \text{ on } M_-\}$$

and thus $\ker_{nor} \overline{\Theta}_0^{\overline{V}} \cong \ker_{nor,\overline{V}} \Theta_0^{V}$, and we have $1:1$-correspondence between normal solutions of type $\overline{V}$ on the interior and normal solutions on the boundary.



Examples

For some very simple classes of BGG-equations, every solution is normal, and in particular the correspondences between solutions can be considered:

$\bar{V}$	$\widetilde{M}$	M_0	M_-
$\mathbb{R}^{n+2}$	Parallel field	Einstein metric $\overline{g}_E$ in **c**	Ricci-flat affine connection projectively equivalent to D^{g_-}
$\Delta^{n+1,1}$	Parallel spinor	twistor spinor $\overline{\chi}$	Killing spinor χ

We next discuss the case $\bar{V} = V = \Lambda^{k+1}$, which is the first interesting case of a BGG-operator with (generally) non-normal solutions. The corresponding solutions are *conformal Killing forms* on M_0 resp. Killing forms on M_0.

Prolongations of Projective / Conformal Killing Forms

- According to (H.-Somberg-Souček-Šilhan, 2012), one has

$$\{\text{Killing forms}\} \cong \ker \Theta_0 \cong \{\hat{\nabla}-\text{parallel tractors}\}$$

  where $\hat{\nabla} = \nabla + \Psi$ is the *prolongation connection* of this BGG-equation with Ψ the *prolongation modification* of the normal tractor connection.
  $\rightsquigarrow$ *Simple explicit formula available*.

- Similarly, one has

$$\{\textit{Conformal Killing forms}\} \cong \ker \overline{\Theta}_0 \cong \{\hat{\overline{\nabla}}-\text{parallel tractors}\},$$

  where $\hat{\overline{\nabla}} = \overline{\nabla} + \overline{\Psi}$ is the *prolongation connection* of this BGG-equation with $\overline{\Psi}$ the *prolongation modification* of the normal tractor connection.
  $\rightsquigarrow$ *Explicit formula available*. (Gover-Šilhan 2008; HSSŠ, 2012)

Comparison of projective and conformal prolongation connections

Since the projective prolongation connection $\hat{\nabla}$ is defined on all of M, we can in particular restrict it to a linear connection $\hat{\nabla}_{M_0}$ on conformal tractors in $\Lambda^{k+1}\overline{\mathcal{T}}$ along M_0.

A conformal tractor $\bar{s} \in \Lambda^{k+1}\overline{\mathcal{T}}$ is parallel with respect to the restricted, projective prolongaton connection $\hat{\nabla}_{M_0}$ if and only if the following conditions hold:

$$\overline{\Theta}_0(\overline{\varphi}) = 0 \tag{cKf}$$

$$C_{c[a_2}{}^{pq}\overline{\sigma}_{|pq|a_3\cdots a_k]} = 0, \ \ldots . \tag{I}$$

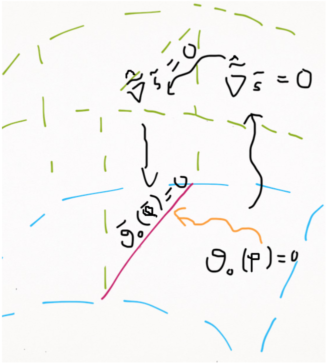
- This is the conformal Killing equation cKF together with an additional integrability condition (I).
- In particular, the restriction of the projective prolongation connection $\hat{\nabla}$ to M_0 does not coincide with the conformal prolongation connection $\hat{\overline{\nabla}}$.

Restriction process: $\varphi \rightsquigarrow \overline{\varphi}$

The projective prolongation connection $\hat{\nabla} = \nabla + \Psi$ of the (projective) Killing form equation on M can be equivalently described by a suitable/corresponding 'ambient prolongation connection' $\hat{\tilde{\nabla}}$.

Let $\varphi_{\alpha_1 \cdots \alpha_k}$ be a Killing form of g_- on M_-.

1. One regards the corresponding tractor as a suitably 'ambient prolongation connection' $\hat{\tilde{\nabla}}$-parallel (multi-vector-)field $\tilde{s}$,
2. restricts this field to $\mathcal{C}$,
3. and obtains (via the obove comparison of prolongation connections) a conformal Killing form $\overline{\varphi}_{a_1 \cdots a_k}$ on M_0 satisfying additional integrability conditions (I).



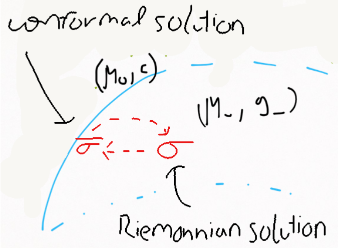
Extension process: $\bar{\phi} \rightsquigarrow \phi$

Conversely, let $\overline{\varphi}_{a_1 \cdots a_k}$ be a conformal Killing form on M_0 satisfying (I).

1. Let $\widetilde{s} \in \Gamma_{\mathcal{C}}(\Lambda^{k+1} T\widetilde{M})$ be the corresponding ambient field defined along $\mathcal{C}$.
2. By the above comparison of prolongation connections, $\widetilde{s}$ is parallel with respect to the 'ambient prolongation connection' $\hat{\hat{\nabla}}$ along $\mathcal{C}$ in $\mathcal{C}$-directions.
3. It is shown via an infinitesimal holonomy-computation that one obtains a global $\hat{\hat{\nabla}}$-parallel ambient field $\widetilde{s}$.
4. In particular, since $\hat{\hat{\nabla}}$ governs the (projective) Killing form equation globally on M, we obtain (via restriction), a Killing form $\varphi_{\alpha_1 \cdots \alpha_k}$ on M_-.

Summary of BGG-Relationships

M_0	M_-
Einstein metric $\bar{g}_E$ in **c**	Ricci-flat affine connection projectively equivalent to D^{g_-}
conformal Killing form $\bar{\varphi}$ satisfying (I)	Killing form φ
twistor spinor $\bar{\chi}$	Killing spinor χ



• For $k = 1$ (I) is always satisfied and we recover the (categorical) relationship between conformal Killing fields $\bar{\xi}$ on M_0 and Killing fields ξ of g_- on M_-.

• All respective objects correspond to parallel fields on ambient space $\widetilde{M}$.