Conformal Patterson–Walker metrics and Fefferman–Graham ambient spaces

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Fefferman-Graham ambient spaces

- Let $(M, [g])$ be a *conformal geometry* of signature (p, q) with $p + q = m$ the dimension of M.
  A Fefferman-Graham ambient space of $(M, [g])$ is a (pseudo-)Riemannian space $(\mathbf{M}, \mathbf{g})$ of signature $(p + 1, q + 1)$ which is Ricci-flat and gives an equivalent encoding of $[g]$.
- This description has been fundamental for constructing and classifying conformal invariants (Fefferman-Graham, 1984) and for constructing and studying conformally invariant differential operators (Graham-Jenne-Mason-Sparling, 1992).

Let $g \in [g]$ be some representative metric in the conformal class. The Fefferman-Graham ambient space can then be written as

$$\mathbf{M} = \underbrace{\mathbb{R}_+ \times M \times \mathbb{R}}_{(t,x,\rho)},$$

- where $\mathbb{R}_+ \times M \subseteq \mathbf{M}$ is regarded as the ray bundle of metrics in the conformal class $[g]$ parametrized by $(t, x) \mapsto t^2 g$ and
- $\rho \in \mathbb{R}$ is a new transversal coordinate.

Let x denote local coordinates on M. Then an *ansatz* for the Fefferman-Graham ambient metric $\mathbf{g}$ is

$$\mathbf{g} = t^2 g_{ij}(x, \rho) dx^i \odot dx^j + 2\rho dt \odot dt + 2t dt \odot d\rho, \qquad \text{(FG)}$$

where

$$g(x, 0) = g_{ij}(x, 0) dx^i dx^j$$

is the representative metric g.

The Fefferman-Graham metric $\mathbf{g}$ is homogeneous of degree 2 with respect to the *Euler field* $t\partial_t$ on $\mathbf{M}$.

To show existence of a Fefferman-Graham ambient metric $\mathbf{g}$ for given g, the *ansatz* (FG) determines an iterative procedure to determine $g_{ij}(x, \rho)$ as a Taylor series in ρ satisfying

$$\mathrm{Ric}(\mathbf{g}) = 0 \text{ to infinite order at } \rho = 0.$$

- For m odd existence (and a natural version of uniqueness) of $\mathbf{g}$ as an infinity-order series expansion in ρ is guaranteed for general $g_{ij}(x)$.
- For $m = 2n$ even, the procedure for determining the expansion in ρ for $g_{ij}(x, \rho)$ such that $\mathrm{Ric}(\mathbf{g}) = 0$ is generically obstructed at order n.

In general, it is not known whether a 'global' ambient space $(\mathbf{M}, \mathbf{g})$ satisfying $\mathrm{Ric}(\mathbf{g}) = 0$ on all on $\mathbf{M}$ and not only asymptotically exists always in the odd-dimensional case or in the even, obstruction-flat situation.

Results which provide global Fefferman–Graham ambient metrics, where $\mathbf{g}$ can be constructed in a natural way from g and satisfies $\mathrm{Ric}(\mathbf{g})$ globally and not just asymptotically at $\rho = 0$ are rare, both in the odd- and even-dimensional situation.

- A special instance where ambient metrics can at least be shown to exist properly occurs for g real–analytic, and m either being odd or m even and with obstruction tensor $\mathcal{O}$ of g vanishing.
- The simplest case of geometric origin for which one has global ambient metrics consists of locally conformally flat structures $(M, [g])$, where $(\mathbf{M}, \mathbf{g})$ exists and is unique up to diffeomorphisms.
- A well known geometric case is formed by conformal structures $(M, [g])$ which contain an Einstein metric g: If $\mathrm{Ric}(g) = 2\lambda(m-1)g$, then $\mathbf{g}$ on $\mathbb{R}_+ \times M \times \mathbb{R}$ can be written directly in terms of g as

$$\mathbf{g} = t^2(1+\lambda\rho)^2 g + 2\rho dt \odot dt + 2t dt \odot d\rho. \qquad \text{(E)}$$

Examples for global ambient metrics

- In work by Thomas Leistner and Pawel Nurowski (2010) it was shown that (odd-dimensional) *pp-waves* admit global and explicit ambient metrics.
- Ambient metrics have also been constructed for particular families of conformal structures which are induced by
  - ... *generic* 2*-distributions on* 5*-manifolds* (Leistner-Nurowski 2012).
  - ... *generic* 3*-distributions on* 6 *manifolds* (Anderson-Leistner-Nurowski 2015).

  and
  - for particular families of of conformal structures for which the equation $\mathrm{Ric}(\mathbf{g}) = 0$ becomes a linear PDE (Anderson-Leistner–Lischewski-Nurowski 2016).
- An explicit ambient metric for an example of an *homogeneous conformal structure* was obtained by (Willse 2014).

We expand the geometric class of metrics for which canonical ambient metrics exist globally and in a canonical realization to *Patterson–Walker metrics*.

Patterson–Walker metrics

- Let N be a smooth manifold and $p : T^*N \to N$ its co-tangent bundle. The vertical subbundle $V \subseteq T(T^*N)$ of this projection is canonically isomorphic to T^*N.
- An affine connection D on N determines a complementary horizontal distribution $H \subseteq T(T^*N)$ that is isomorphic to TN via the tangent map of p.

The Patterson–Walker metric associated to a torsion-free affine connection D on N is the pseudo-Riemannian split-signature (n, n)-metric g on T^*N fully determined by the following conditions:

- both V and H are isotropic with respect to g,
- the value of g with one entry from V and another entry from H is given by the natural pairing between $V \cong T^*N$ and $H \cong TN$.

$\rightsquigarrow$ It follows that V is parallel with respect to the Levi-Civita connection of the just constructed metric. Hence Patterson–Walker metrics are special cases of Walker metrics, which are metrics admitting a parallel isotropic distribution.

Local Formula for Patterson–Walker metrics

- Let D be a torsion-free affine connection on N which preserves a volume form.
- Denote local coordinates on N by x^A and the induced canonical fibre coordinates on T^*N by p_A.
- Let ${\Gamma_A}^C{}_B$ denote the Christoffel symbols of D.

Then

$$g = 2\,\mathrm{d}x^A \odot \mathrm{d}p_A - 2\,{\Gamma_A}^C{}_B\, p_C\, \mathrm{d}x^A \odot \mathrm{d}x^B \qquad \text{(PW)}$$

is the Patterson–Walker metric induced on T^*N by D.

Properties of the induced Patterson–Walker space (M, g):

- (M, g) carries a parallel pure spinor $\chi \in \Gamma(\mathcal{S}_-)$,

$$\widetilde{D}\chi = 0.$$

  $\rightsquigarrow$ equivalent encoding of the parallel maximally isotropic distribution $V \subset TM$.

- (M, g) carries a homothety $k \in \mathfrak{X}(M)$,

$$\mathcal{L}_k g = 2\, g.$$

- Any infinitesimal symmetry v^A of the affine connection D induces a Killing field $\widetilde{v}^a$ of g.

Given an affine connection D on N we may *weaken* it to its projective equivalence class $[D]$ and regard $(N, [D])$ as a projective structure:

We recall that two affine connections D, D' on N are called *projectively related* if there exists a 1-form $\Upsilon \in \Omega^1(N)$ with

$$D'_X Y = D_X Y + \Upsilon(X)Y + \Upsilon(Y)X \qquad \text{(P)}$$

for all $X, Y \in \mathfrak{X}(N)$.

It is an obviously interesting question to ask how the association

$$N \rightsquigarrow T^*N, \ D \rightsquigarrow g$$

behaves with respect to a projective change from D to D'.

- In general, for projectively related metrics D, D', the associated Patterson–Walker metrics on $M = T^*N$ will fail to be conformally equivalent.

- In work by Dunajski-Tod (2010) the Patterson–Walker construction was generalized to a projectively invariant setting in dimension $n = 2$.
- In work by Nurowski-Sparling (2003), a construction of conformal structures of signature $(2, 2)$ using Cartan connections was presented.
- In joint work (HSŠTZ, arXiv:1510.03337) it was shown that the association from a projective structure to a conformal split signature structure can be understood a s a Fefferman-type construction based on a group inclusion $\mathrm{SL}(n+1) \hookrightarrow \mathrm{Spin}(n+1, n+1)$.
- In joint work (HSŠTZ, arXiv:1604.08471) we provided a 'direct' approach, primarily based on suitable 'spin calculus', giving explicit formuli relating geometric properties and objects on N with corresponding objects on the conformal space M.

Preliminaries for the construction: Projective Densities and Scales:
For projective structures on an oriented manifold N it is often useful to employ suitably calibrated *projective density bundles of weight* w,

$$\mathcal{E}(w) := (\wedge^n TN)^{-\frac{w}{n+1}} .$$

For the special case of *weight* $w = 1$ we call the ray bundle $\mathcal{E}_+(1) \subseteq \mathcal{E}(1)$ the bundle of projective scales.

Let $[D]$ be a projective class which contains volume-preserving (also called special) connections. Then projective scales $s \in \mathcal{E}_+(1)$ correspond to a special affine connections $D \in [g]$.

We define

$$M = T^*N(2) = T^*N \otimes \mathcal{E}(2)$$

the (projectively) weighted co-tangent bundle of N.

Conformal Patterson–Walker metrics

Given a projective scale $s \in \mathcal{E}_+(1)$ we obtain a trivialization/identification of $T^*N(2) \cong T^*N$. With D the special affine connection corresponding to the scale s, we have the
induced Patterson–Walker metric g_s on $M = T^*N(2)$.

Proposition

If $s' = e^f s$ is another projective scale, then $g_{s'} = e^{2f} g_s$.
Thus, the projectively related affine connections D, D' on N induce conformally related Patterson–Walker metrics $g_s, g_{s'}$ on $M = T^*N(2)$, and we obtain a natural association

$$(N, [D]) \rightsquigarrow (M, [g]).$$

Properties of conformal Patterson–Walker metrics:

- $(M, [g])$ carries a pure *twistor spinor* χ with (maximally isotropic, n-dimensional) integrable kernel $\ker \chi$.
- $(M, [g])$ carries a nowhere-vanishing *conformal Killing field* $k \in \ker \chi$

In addition, one can show the following:

- The Lie-derivative of χ with respect to the conformal Killing field k is

$$\mathcal{L}_k \chi = -\frac{1}{2}(n+1)\chi \,. \qquad \text{(L)}$$

- The following integrability condition is satisfied for all $v^r, w^s \in \ker \chi$:

$$\widetilde{W}_{abcd} v^a w^d = 0 \,. \qquad \text{(W)}$$

Then:

- These conditions characterize conformal Patterson–Walker metrics.
- Under those conditions there always exist (at least locally) Patterson–Walker metrics $g \in [g]$, which satisfy $\widetilde{D}\chi = 0$.

The Thomas cone connection

A much simpler analog of ambient spaces of conformal structures is available for projective structures due to Tracy Thomas (1934):

- The Thomas cone associated to a projective manifold (N, D) is the natural ray bundle $\mathcal{C} := \mathcal{E}_+(1) = (\Lambda^n TN)^{-\frac{1}{n+1}}$.
- The Thomas cone connection ∇ is a canonical affine, Ricci-flat connection on $\mathcal{C}$.

Let $s : N \to \mathcal{E}_+(1)$ be the scale corresponding to an affine connection $D \in [D]$, providing a trivialization $\mathcal{E}_+(1) \cong \mathbb{R}_+ \times N$ via $(x^0, x) \mapsto s(x)x^0$. In this trivialization the Thomas cone connection is given by

$$\nabla_X Y = D_X Y - \frac{1}{n-1} \mathrm{Ric}(X, Y) Z, \; \nabla Z = \mathrm{id}_{T\mathcal{C}} \qquad \text{(T)}$$

where $X, Y \in \mathfrak{X}(N)$ and $Z = x^0 \partial_{x^0}$ is the Euler field on $\mathcal{C}$.

It is in fact easy to see directly from formula (T) that the thus defined affine connection ∇ on the Thomas cone $\mathcal{C}$ is independent of the choice of scale and Ricci-flat.

Combining the constructions

- Given a projective structure $(N, [D])$ on an n-dimensional manifold N, we can form the Thomas cone $(\mathcal{C}, \nabla)$ and consider the associated Patterson–Walker metric $\mathbf{g}$ on $\mathbf{M} = T^*\mathcal{C} = T^*\mathcal{E}_+(1)$.
- Obviously: $\dim \mathcal{C} = (n+1)$, so $\mathrm{sig}(\mathbf{g}) = (n+1, n+1)$.
- Since ∇ is Ricci-flat, so is its Patterson–Walker metric $\mathbf{g}$.

In particular, we may be tempted to investigate whether $(\mathbf{M}, \mathbf{g})$ is in fact the Fefferman–Graham ambient metric space associated to the conformal class $(M, [g])$:

$$\begin{array}{ccc} (\mathcal{C}, \nabla) & \rightsquigarrow & (\mathbf{M}, \mathbf{g}) \qquad \ldots \mathrm{Ricci-flat, split-signature}\ (n+1, n+1) \\ \uparrow\wr & & \\ (N, [D]) & \rightsquigarrow & (M, [g]) \qquad \ldots \mathrm{conformal,\ split-signature}\ (n, n) \end{array}$$

We also have the induced homothety $\mathbf{k}$ on $\mathbf{M}$, which might be suspected to be a canonical candidate for the Euler-field of the ambient space.

Procedure:

- Compute the Thomas cone connection ∇ on $\mathcal{C}$ for given D.
- Compute the Patterson–Walker metric $\mathbf{g}$ on $T^*\mathcal{C}$ associated to ∇.
- Perform (locally) an appropriate coordinate change which shows that the resulting split-signature $(n+1, n+1)$ pseudo-Riemannian metric $\mathbf{g}$ is a Fefferman–Graham ambient metric.

Concretely:

- We use a local coordinate patch on N which induces coordinates $x^A,\ y_A$ on the co-tangent bundle T^*N and coordinates x^0, x^A, y_A, y_0 on $T^*\mathcal{C} \cong \mathbb{R}_+ \times T^*N \times \mathbb{R}$.
- Then the Patterson–Walker metric $\mathbf{g}$ associated to the Thomas cone connection ∇ is

$$\begin{aligned}\mathbf{g} =& 2dx^A \odot dy_A + 2dx^0 \odot dy_0 - \frac{4}{x^0} y_B dx^0 \odot dx^B \\ & - 2y_C \Gamma_A{}^C{}_B dx^A \odot dx^B + 2\frac{x^0 y_0}{n-1} \operatorname{Ric}_{AB} dx^A \odot dx^B.\end{aligned} \tag{1}$$

- We employ the change of coordinates $t = x^0,\ \rho = \frac{y_0}{x^0},\ p_A = \frac{y_A}{(x^0)^2}$.

Theorem (Local Statement)

For a given torsion–free, volume–preserving affine connection D with Christoffel symbols $\Gamma_A{}^C{}_B$,

$$
\text{`}\mathbf{g} = 2\rho dt \odot dt + 2t dt \odot d\rho, \qquad \text{(PW-A)}
$$

$$
+ t^2\Big(2dx^A \odot dp_A - 2p_C \Gamma_A{}^C{}_B dx^A \odot dx^B + \frac{2\rho}{n-1} \operatorname{Ric}_{AB} dx^A \odot dx^B\Big),
$$

is the Fefferman–Graham ambient metric of the Patterson–Walker metric

$$
g = 2dx^A \odot dp_A - 2p_C \Gamma_A{}^C{}_B dx^A \odot dx^B. \qquad \text{(PW)}
$$

- Once one has the above formula, it can also be proved directly: One checks Ricci-flatness of (PW-A) for any given Christoffel symbols Γ^A_{BC}, satisfying
$$\Gamma^A_{BC} = \Gamma^A_{CB},\ \partial_A \gamma^P_{BP} - \partial_B \Gamma^P_{AP}$$
where the first condition corresponds to torsion–freeness of D and second condition to volume–preservation of D.
- It follows in particular that the Fefferman-Graham obstruction tensor $\mathcal{O}$ vanishes for any Patterson–Walker metric.

Properties of the ambient metric $\mathbf{g}$

- As a Patterson–Walker metric $(\mathbf{M}, \mathbf{g})$ carries a naturally induced homothety
$$\mathbf{k} = 2p_A\partial_{p_A} + 2\rho\partial_\rho$$
of degree 2.
- The infinitesimal affine symmetry Z of the affine connection ∇ lifts to the Killing field
$$\xi = t\partial_t - 2p_A\partial_{p_A} - 2\rho\partial_\rho.$$
- The Euler field of the Fefferman–Graham ambient metric $\mathbf{g}$ can be written as the sum $\xi + \mathbf{k}$ of this Killing field and the homothety $\mathbf{k}$:
$$t\partial_t = \xi + \mathbf{k}.$$
- $T\mathbf{M}$ carries the maximally isotropic $(n+1)$-dimensional subspace spanned by $\{\partial_{p_A}, \partial_\rho\}$ which is preserved by $\boldsymbol{\nabla}$. This subspace can be equivalently described by a $\boldsymbol{\nabla}$-parallel pure spinor $\mathbf{s}$ on $\mathbf{M}$.
- In particular,
$$\mathrm{Hol}(\mathbf{g}) \subseteq \mathrm{SL}(n+1) \ltimes \Lambda^2\mathbb{R}^{n+1,n+1}.$$

Theorem (Global statement)

Given a projective structure $(N, [D])$ on an n-dimensional manifold N, the geometric constructions indicated in the following diagram commute:

Thomas cone **Ambient space**

$$\begin{array}{ccc} (\mathcal{C}, \nabla) & \rightsquigarrow & (\mathbf{M}, \mathbf{g}) \\ \uparrow & & \uparrow \\ (N, [D]) & \rightsquigarrow & (M, [g]) \end{array}$$

In particular, the induced conformal structure $[g]$ admits a globally Ricci-flat Fefferman–Graham ambient metric $\mathbf{g}$ which is itself a Patterson–Walker metric.

Q-Curvature

- Q-curvature Q_g of a given metric g on an even-dimensional manifold is a Riemannian scalar invariant with a particularly simple (linear) transformation law with respect to conformal change of metric (Thomas Branson 1993).
- Computation of Q-curvature is notoriously difficult since it typically requires knowledge of the Fefferman-Graham ambient metric:
  - Formulas in terms of underlying data can in principle be obtained algorithmically for each given dimension, but the resulting formulas are not (at the moment) accessible to human inspection.
  - An explicit form of a Fefferman–Graham ambient metric $\mathbf{g}$ for a given metric g allows a computation of Q_g. Using the fact that $\mathbf{g}$ is actually a Patterson–Walker metric, this computation is particularly simple.

Theorem

The Patterson–Walker metric g associated to a volume–preserving, torsion–free affine connection D has vanishing Q-curvature Q_g.

Computation:

- According to (Fefferman-Hirachi, 2003), we have to compute

$$Q_g = \left(-\mathbf{\Delta}^n \log(t)\right)_{|\{1\}\times T^*N\times\{0\}},$$

  - where $\mathbf{\Delta}$ is the ambient Laplacian on $\mathbf{M} = \mathbb{R}_+ \times T^*N \times \mathbb{R}$,
  - $t : \mathbf{M} \to \mathbb{R}_+$ is the first coordinate projection and
  - the subscript denotes restriction to $T^*N \hookrightarrow \mathbf{M}$.
- To show that Q-curvature vanishes for g, it is in particular sufficient to show that $\mathbf{\Delta}\log(t) = 0$.
- We observe that the function $t : \mathbf{M} \to \mathbb{R}_+$ is horizontal since it is just the pullback of the coordinate function $x^0 : \mathcal{C} \to \mathbb{R}_+$ on the Thomas cone $\mathcal{C} \cong \mathbb{R}_+ \times N$.
- The explicit formula for the Christoffel symbols of a Patterson–Walker metric shows that $\mathbf{\Delta}$ vanishes on any horizontal function. Thus in particular $\mathbf{\Delta}\log(t) = 0$, and then also $Q_g = 0$.

Thank you for your attention!