Holonomy Reductions of Parabolic Geometries and Curved Orbit Decompositions

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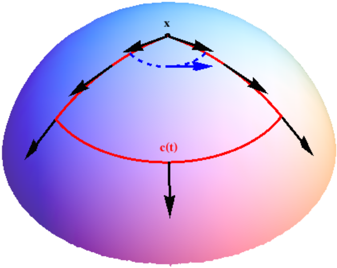
Outline

The holonomy group

Let M be an n-dimensional manifold endowed with an affine connection ∇.
Let $x \in M$ be some (fixed) point and $c : [0,1] \to M$ a closed (piece-wise) C^1-curve with $c(0) = c(1) = x$.



The connection ∇ allows us to define a *parallel transport* along c,

$$\mathrm{Pt}_c^t : T_x M \to T_{c(t)} M,$$

which is a vector field along c for any given $v \in T_x M$: it is the unique solution of the ODE-problem

$$\nabla_{\dot{c}(t)} \mathrm{Pt}_c(v) = 0, \ \mathrm{Pt}_c^0(v) = v$$

Parallel transport along closed curves generates the *holonomy group*:

$$\mathsf{Hol}(\nabla) := \{\mathrm{Pt}_c^1 \mid c \in C^1([0,1], M), \ c(0) = c(1) = x\} \subset \mathrm{GL}(T_x M).$$

Riemannian holonomy

Let g be a Riemannian metric on an (oriented) manifold M and ∇^g its Levi-Civita connection. Then

$$\mathrm{Hol}(g) := \mathrm{Hol}(\nabla^g) \subset \mathrm{SO}(n)$$

Generically one has *full holonomy* $\mathrm{Hol}(g) = \mathrm{SO}(n)$.
Fundamental results in Riemannian holonomy theory:

- When $\mathrm{Hol}(g) = H_1 \times H_2$, then (locally) $(M, g) = (M_1, g_1) \times (M_2, g_2)$.
- When $\mathrm{Hol}(g)$ is irreducible and g is not symmetric, then *Berger's list of Riemannian holonomy groups* [1955] provides a finite list for possible types of holonomy groups.
- All holonomies in Berger's list have been constructed. Specific *exceptional* holonomies like $G_2 \subset \mathrm{SO}(7)$ and $\mathrm{Spin}(7) \subset \mathrm{Spin}(8)$ have proven most difficult:
  - [Bryant-Salomon 1987] constructed *local examples*
  - [Joyce 1994/1996] constructed *compact examples*

Reduced holonomy via principal bundles

The holonomy group of a Riemannian metric g is reduced to

$$H \subset \mathrm{SO}(n)$$

if and only if the *orthogonal frame bundle* $\mathcal{P}$ of g (together with its *principal connection form*) has a reduction

$$\begin{array}{ccc} \mathcal{H} & \hookrightarrow & \mathcal{P} \\ \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle \mathrm{SO}(n)} \\ M & \xrightarrow{id} & M \end{array}$$

$\mathrm{Hol}(g)$ is the *smallest subgroup* of $\mathrm{SO}(n)$ to which $\mathcal{P}$ can be reduced.

Example: For $n = 2m$ even-dimensional, the following are equivalent:

- The Riemannian metric g is *Kähler*, i.e., there exists a compatible complex structure $\mathbb{J}$: $\nabla^g \mathbb{J} = 0,\ g(J\cdot, J\cdot) = g(\cdot, \cdot)$
- $\mathrm{Hol}(g) \subset \mathrm{U}(m)$
- There exists a reduction of the full orthogonal frame bundle (endowed with its connection form) to $\mathrm{U}(m) \subset \mathrm{SO}(2m)$.

Conformal structures

We say that two Riemannian metrics g and $\hat{g}$ on M are *conformally related* if there is a function $f \in \mathrm{C}^\infty(M, \mathbb{R}_{>0})$ such that $\hat{g} = fg$. This defines an equivalence relation for Riemannian metrics, and $\mathcal{C} = [g]$ is called a *conformal structure* on M.

While each metric $g \in [g]$ has its canonical Levi-Civita connection ∇^g, there is no natural affine connection on M for the whole structure $\mathcal{C}$. The following problems thus become more difficult:

- Defining conformal invariants and in particular developing notions of curvature and holonomy
- Defining conformally invariant differential operators

Example:
The n-dimensional Riemannian sphere (S^n, g_{rd}) has full holonomy $\mathrm{SO}(n)$, but the corresponding conformal structure $(S^n, [g_{rd}])$ is *locally conformally flat* via stereographic projection.

Specific examples of overdetermined problems:

Confomally invariant overdetermined problems:

- Let g be a Riemannian metric on a manifold M. Can we rescale g conformally to $\hat{g} = fg$ for some positive function f such that $\hat{g}$ is Einstein,

$$\mathrm{Ric}(\hat{g}) = \lambda \hat{g} \ ?$$

- If M is even-dimensional, can one rescale a Riemannian metric g conformally to a Kähler metric?

Projectively invariant overdetermined problems:

Two torsion-free affine connections ∇ and $\hat{\nabla}$ on TM are *projectively equivalent* if they have the same unparameterized geodesics. For given affine connection ∇ one thus obtains a *projective structure* $[\nabla]$.

- If ∇ is an affine torsion-free connection on M, is it metrizable? I.e, can one describe its geodesics by an Riemannian metric?
- Does the affine connection allow a projectively equivalent Ricci-flat connection?

Prolongation approaches

To associate a natural connection to a conformal structure one employs one of the following two techniques: Each approach delivers, on some extended bundle or space a connection, and in particular yields a notion of conformal holonomy:

- Cartan resp. tractor approach (Élie Cartan, Tracy Thomas)
  $\rightsquigarrow$ *Generalizes to Cartan geometries, and in particular to general parabolic geometries*
- Ambient metric approach (Fefferman-Graham)
  $\rightsquigarrow$ *Specific for conformal structures*

In the present talk we present a general holonomy reduction theory for Cartan geometries developed in joint work with A. Čap (Univ. of Vienna) and A.R. Gover (Auckland Univ.)

For the specific case of conformal structures and the resulting notion of *ambient holonomy* we discuss current joint work with A. Čap, A.R. Gover and R. Graham (Univ. of Washington).

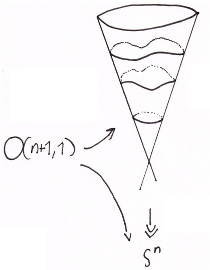
Klein-model-geometries

A *Klein geometry* is a pair (G, P) with G a Lie group and $P \subset G$ a closed subgroup. G is then regarded as the *automorphism group* of the homogeneous space G/P.

• Euclidean geometry can be understood as the study of the invariants under the action of the Euclidean group $\mathrm{Euc}(n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$ on Euclidean space $\mathbb{E}^n \cong \mathbb{R}^n \cong \mathrm{Euc}(n)/\mathrm{O}(n)$.

• Projective space $\mathbb{P}^n$ can be described as $\mathrm{SL}(n+1)/P$, with $G = \mathrm{SL}(n+1)$ acting transitively on the space of lines in $\mathbb{R}^{n+1}$ and $P \subset \mathrm{SL}(n+1)$ the stabilizer of a line.

• The *conformal n-sphere* is realized by regarding the (transitive) action of the Lorentz-group $G = \mathrm{O}(n+1, 1)$ on the light-cone $\mathcal{C} \subset \mathbb{R}^{n+1,1}$ and having P the stabilizer of some $\mathbb{R}_+$-ray in this cone.



Cartan Geometries

Élie Cartan, in his seminal works from ca. 1910, generalized Klein's approach to geometry to allow a description of curved structures which is based on the underlying natural symmetry-group:

Homogeneous Space	**(Curved) Cartan geometry**
$\mathbb{E}^n \cong \mathrm{Euc}(n)/O(n)$	n-dimensional Riemannian structure - Allows measurement of distances on curved space.
$\mathbb{S}^n \cong \mathrm{SO}(n+1)/P$	n-dimensional conformal structure - Allows measurement of angles on curved space.
$\mathbb{P}^n \cong \mathrm{SL}(n+1)/P$	n-dimensional projective structure - Generalizes the notion of "straight lines" to "unparameterized geodesics".

Cartan and parabolic geometries

A Cartan geometry of type (G, P) on a manifold M is a P-principal bundle $\mathcal{G} \to M$ endowed with a Cartan connection form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ which generalizes properties from the homogeneous model G/P:

$$\begin{array}{ccc} G & \dashrightarrow & G \\ & & \downarrow{\scriptstyle P} \\ & & G/P \end{array} \qquad \begin{array}{ccc} \mathrm{Aut} & \dashrightarrow & \mathcal{G} \\ & & \downarrow{\scriptstyle P} \\ & & M \end{array}$$

- For P a parabolic subgroup of a simple Lie group G, $(\mathcal{G}, \omega)$ is a *parabolic geometry*.
- Parabolic geometries allow uniform regularity and normality conditions, and if these conditions are satisfied, the parabolic structure is an equivalent description of an underlying geometric structure.

Holonomy of Cartan connections

The Cartan geometry $(\mathcal{G}, \omega)$ with structure group P naturally extends to a G-principal bundle $(\hat{G}, \hat{\omega})$:

$$\begin{array}{ccc} \mathcal{G} & \hookrightarrow & \hat{\mathcal{G}} \\ \downarrow P & & \downarrow G \\ M & \xrightarrow{id} & M \end{array}$$

This allows us to define the holonomy of the Cartan connection form ω via the extended principal connection form:

$$\mathrm{Hol}(\omega) := \mathrm{Hol}(\hat{\omega}) \subset G.$$

- For any G-representation V we can form the associated *tractor bundle* $\mathcal{V} = \mathcal{G} \times_P V$ endowed with its induced *tractor connection* ∇^V and parallel sections of $\mathcal{V}$ correspond to G-equivariant maps $s : \hat{\mathcal{G}} \to V$ that are constant on horizontal curves.
- It follows in particular that for connected M one has a well defined global G-type $\mathcal{O} := s(\hat{\mathcal{G}})$ which is a G-orbit in V.
- Moreover, one has a point-wise P-type for each $x \in M$: $s(\mathcal{G}_x) \in P \backslash \mathcal{O}$, which is a P-orbit in $\mathcal{O} \subset V$.

Holonomy-reductions of type $\mathcal{O} = G/H$

The minimal way to obtain a reduction to some G-orbit $\mathcal{O} = G/H$ is:

Definition

A *holonomy reduction of* $(\mathcal{G}, \omega)$ *to* $H \hookrightarrow G$ is a parallel section of $\mathcal{G} \times_P \mathcal{O}$.

Given a reduction to $\mathcal{O}$, we define for given $\bar{\alpha} = P \cdot \alpha \in P \backslash \mathcal{O}$

$$M_{\bar{\alpha}} := \{x \in M \mid s(\mathcal{G}_x) = \bar{\alpha}\}.$$

P-type decomposition of the homogeneous model

We regard the homogeneous model G/P and consider a holonomy reduction of type $\mathcal{O} = G/H$, i.e., a reduction $H \hookrightarrow G$.

The map which assocates to each point $gP \in G/P$ its P-type, factorizes to the natural bijection

$$H\backslash G/P \overset{\sim}{\to} P\backslash \mathcal{O} = P\backslash G/H$$

between double co-set spaces defined via inversion.

$$\{H\text{–orbits in } G/P\} \leftrightarrow \{P\text{–types in } \mathcal{O}\}$$

The curved orbit decomposition theorem

Theorem (Čap-Gover-H. 2014)

Let $(\mathcal{G}, \omega)$ be a Cartan geometry of type (G, P) with a given holonomy reduction of type $\mathcal{O} = G/H$. Let $\bar{\alpha} \in P \backslash \mathcal{O}$ be such that $M_{\bar{\alpha}} \neq \emptyset$. Then $M_{\bar{\alpha}} \subset M$ is an initial submanifold of M and carries an induced Cartan geometry of type $(G_\alpha, G_\alpha \cap P)$.

The decomposition

$$M = \bigcup_{\bar{\alpha} \in P \backslash \mathcal{O}} M_{\bar{\alpha}}$$

is the *curved orbit decomposition* of M with respect to the given holonomy reduction:

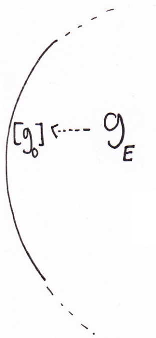
$$\{H\text{−orbits in } G/P\} \leftrightarrow \{\text{curved orbits in } M\}$$

- When H acts transitively on G/P this shows that M carries a global reduced geometry of type $(H, H \cap P)$.
- The remaining work for analyzing reductions in specific cases is to see what the normalization conditions that were employed for the original Cartan connection form imply for the reduced structure.

Einstein-reductions of conformal structures

Let $(M, [g])$ with $[g] = \{fg \mid f \in \mathrm{C}^\infty(M, \mathbb{R}_{>0})\}$ be an n-dimensional Riemannian signature conformal structure.

- For a reduction to $\mathrm{SO}(n+1) \hookrightarrow \mathrm{SO}(n+1,1)$ we obtain a single curved orbit of type $\mathrm{SO}(n+1)/\mathrm{SO}(n)$, which carries an Einstein metric with positive scalar curvature.
- For a reduction to $\mathrm{SO}(n,1) \hookrightarrow \mathrm{SO}(n+1,1)$ we obtain an open curved orbit of type $\mathrm{SO}(n,1)/\mathrm{SO}(n)$, which carries an Einstein metric with negative scalar curvature and a closed curved orbit of type $\mathrm{SO}(n,1)/\bar{P}$ which carries a conformal structure.



We recover a description of **almost Einstein structures** [Gover, 2010]. The reduction $\mathrm{SO}(n,1) \hookrightarrow \mathrm{SO}(n+1,1)$ yields examples of *Poincaré-Einstein* manifolds with an Einstein metric on an interior part of a manifold and a *conformal structure at infinity*.

Einstein-reductions of projective structures

We consider an n-dimensional projective structure (M, p) with p an equivalence class of affine torsion-free connections on M.

We regard a reduction with G-type $\mathcal{O} = \mathrm{SL}(n+1)/SO(p,q)$ with $p + q = n + 1$, which is the space of all signature (p, q) inner products.

As a P-space, $\mathcal{O}$ decomposes into three pieces

$$P\backslash\mathcal{O} = (P\backslash\mathcal{O})_+ \cup (P\backslash\mathcal{O})_0 \cup (P\backslash\mathcal{O})_-$$

Orbit	**Dim.**	**Type**	**Description**
M_+	n	$\mathrm{SO}(p,q)/\mathrm{SO}(p-1,q)$	Einstein metric
M_-	n	$\mathrm{SO}(p,q)/\mathrm{SO}(p,q-1)$	Einstein metric
M_0	$n-1$	$\mathrm{SO}(p,q)/\bar{P}$	conformal structure

This yields *Klein-Einstein*-structures [Čap-Gover-H., 2012], which can be regarded as a *projective compactification* of a conformal structure.

Geometric construction of spaces with reduced holonomy

For group inclusions $G \hookrightarrow \tilde{G}$ with G acting transitively on $\tilde{G}/\tilde{P}$ one can employ a *Fefferman-type construction* [Čap 2005] to construct a geometry of type $(\tilde{G}, \tilde{P})$ from a geometry of type (G, P):

$$\begin{array}{ccccccc}
 & \textit{Correspondence} & & \textit{Extension} & & \textit{Normalization} & \\
(\mathcal{G}, \omega) & \rightsquigarrow & (\mathcal{G}, \omega) & \rightsquigarrow & (\tilde{G}, \tilde{\omega}) & \rightsquigarrow & (\tilde{G}, \tilde{\omega}^{nor}) \\
\downarrow P & & \downarrow Q & & \downarrow \tilde{P} & & \downarrow \tilde{P} \\
M & & \tilde{M} & & \tilde{M} & & \tilde{M}
\end{array}$$

In case the last *normalization step* is trivial, one has

$$\mathrm{Hol}(\tilde{\omega}) = \mathrm{Hol}(\omega) \subset G \tag{H}$$

Condition (H) then characterizes the resulting *Fefferman-type spaces*:

$$\begin{array}{ccc}
 & \textit{Holonomy-reduction} & \\
 & \longleftarrow\text{-}\,\text{-}\,\text{-} & \\
(\mathcal{G}, \omega) & \rightsquigarrow & (\tilde{\mathcal{G}}, \tilde{\omega}) \\
 & \textit{Fefferman-type-construction} &
\end{array}$$

Holonomy reduction $\mathrm{SU}(p+1, q+1) \hookrightarrow \mathrm{SO}(2p+2, 2q+2)$:

Let $\mathcal{O}$ be the set of all orthogonal complex structures on $\mathbb{R}^{2p+2,2q+2}$. With $\mathbb{J} \in \mathcal{O}$ one has $\mathcal{O} = \mathrm{SO}(2p+2, 2q+2)/H$ for $H = G_{\mathbb{J}} = \mathrm{U}(p+1, q+1)$ and $\mathrm{U}(p+1, q+1)$ acts transitively on $\mathrm{SO}(2p+2, 2q+2)/P$. It was shown by [Čap-Gover 2010, Leitner 2008], that conformal holonomy $\mathrm{Hol}(\omega) \subset \mathrm{U}(p+1, q+1)$ already implies locally that $\mathrm{Hol}(\omega) \subset \mathrm{SU}(p+1, q+1)$.

Given an additional integrability condition, the resulting reduced Cartan geometry locally factorizes to a *CR-structure*, and the conformal geometry is completely determined by that CR-structure via the classical Fefferman-construction [Fefferman 1976, Graham 1987, Čap-Gover 2010].

Holonomy reduction $G_2 \hookrightarrow \mathrm{Spin}(3,4)$

Let $\mathcal{O}$ be the set of all non-isotropic spinors in the 8-dimensional real spin representation $\Delta_{\mathbb{R}}^{3,4}$ of $\mathrm{Spin}(3,4)$. The stabilizer of such a spinor provides an embedding of G_2 into $\mathrm{Spin}(3,4)$, and since G_2 is seen to act transitively on $\mathrm{Spin}(3,4)/P$ there is only a single P-type in $P\backslash\mathcal{O}$.

The resulting holonomy reduction of conformal spin structures of signature $(2,3)$ describes the geometry of a *generic rank 2-distribution* and the original conformal structure is completely determined by this rank 2-distribution via the Fefferman-type construction $G_2 \hookrightarrow \mathrm{Spin}(3,4)$ [H.-Sagerschnig, 2011] .

Characterization in terms of BGG-solutions

For every G-representation V one has a naturally associated overdetermined differential operator, the *first BGG-operator* $\Theta_0 : \mathcal{H}_0 \to \mathcal{H}_1$, which defines the *first BGG-equation* $\Theta_0(\sigma) = 0$ [Čap-Slovak-Souček 2001, Calderbank-Diemer 2000].

$$\nabla^{\mathcal{V}} - \text{parallel sections} \subset \Gamma(\mathcal{V}) \qquad \textit{prolonged system}$$

$$\Pi_0 \downarrow \quad \uparrow L_0$$

$$\text{normal solutions} \subset \ker \Theta_0 \qquad \textit{overdetermined system}$$

- General solutions of $\Theta_0(\sigma) = 0$ correspond to sections of $\mathcal{V}$ which are parallel with respect to a (modified) *prolongation connection* [H.-Somberg-Souček-Šilhan 2012], and therefore don't induce holonomy reductions of the parabolic structure.
- *Normal solutions* (following [Leitner, 2005]) of $\Theta_0(\sigma) = 0$ are those which correspond to parallel sections of $\mathcal{V}$. In particular, normal solutions are equivalent to holonomy reductions.

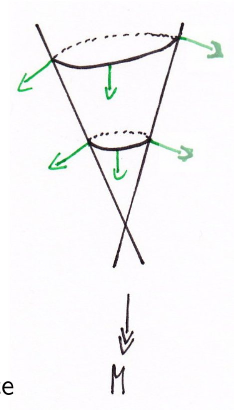
Conformal Holonomy Characterizations

- $\mathrm{SU}(p+1, q+1) \hookrightarrow \mathrm{SO}(2p+2, 2q+2)$:
  CR-structure $\rightsquigarrow$ signature $(2p+1, 2q+1)$-conformal structure on S^1-bundle
  \+ lightlike conformal Killing field
  [Fefferman 1976, Graham 1987, Čap-Gover 2010]
- $G_2 \hookrightarrow \mathrm{Spin}(3, 4)$:
  generic rank 2-distribution on 5-manifold $\rightsquigarrow$
  signature $(2, 3)$-conformal spin structure
  \+ generic twistor spinor
  [H.-Sagerschnig 2011]
- $\mathrm{SL}(3) \hookrightarrow \mathrm{Spin}(3, 3)$:
  projective 2-dimensional structure $\rightsquigarrow$
  split signature $(2, 2)$-conformal spin structure
  \+ 2 compatible twistor spinors
  [Dunajski-Tod 2010, H.-Tagavi–Chabert-Žádník-Šilhan-Sagerschnig]

Fefferman-Graham ambient metrics

A conformal structure $\mathcal{C} = [g]$
can be understood as the ray-subbundle $\mathcal{C} \subset S^2 T^* M$
which consists of all metrics in the given conformal class.

The *Fefferman-Graham ambient metric*
$\tilde{g}$ is a signature $(n+1, 1)$ metric
on $n+2$-dimensional ambient space $\tilde{M} = \mathcal{C} \times (-1, 1)$
and extends a tautological (degenerate) form $\mathbf{g}_0$ on $\mathcal{C}$.



- For $n = p + q$
  odd $\tilde{g}$ is uniquely determined as an infinite
  order jet along $\mathcal{C}$ by the normalization condition
  that $\mathrm{Ric}(\tilde{g})$ vanishes to infinite order along $\mathcal{C}$.
- The holonomy $\mathrm{Hol}(\tilde{g})$ of the ambient metric is
  in general not a conformally well-defined object since
  $\tilde{g}$ is only conformally invariant as an infinite-order
  (or truncated) jet along the cone $\mathcal{C} \subset \tilde{M}$.

Lifting of parallel tractors to parallel ambient tensors

Theorem (Graham-Willse, 2011)

Let $(M, \mathcal{C})$ be a real-analytic conformal structure of signature (p, q), $n = p + q$ odd. Let $\mathcal{V}$ be some tensor power of the conformal tractor bundle and $s \in \Gamma(\mathcal{V})$ a section of that bundle which is parallel with respect to the tractor connection. Then there exists a canonical lift of s to an ambient tensor field $\tilde{s}$ that is well-defined in a neighborhood of the cone $\mathcal{C}$ and parallel with respect to the ambient Levi-Civita covariant derivative.

A truncated version of this theorem exists for the even-dimensional case and the result also holds formally (on the jet-level) at the cone $\mathcal{C} \subset \tilde{M}$ if $\mathcal{C}$ is not necessarily real-analytic.

Application of Graham-Willse's-result for G_2-structures

Theorem (H.-Sagerschnig, 2011)

Let $(M, \mathcal{C})$ be a conformal structure of signature $(2,3)$. Then the following are equivalent:

- *$(M, \mathcal{C})$ is induced from a generic rank 2 distribution $\mathcal{D} \subset TM$*
- *The conformal holonomy $\mathrm{Hol}(\mathcal{C}) \subset \mathrm{SO}(3,4)$ is contained in $G_2 \subset \mathrm{SO}(3,4)$.*

- Since $G_2 \subset \mathrm{SO}(3,4)$ can be realized as the stabilizer of a suitable generic 3-vector in $\Lambda^3 \mathbb{R}^7$, the holonomy reduction $\mathrm{Hol}(\mathcal{C}) \subset G_2$ can be equivalently characterized by the existence of a (suitably generic) parallel tractor 3-form $\Phi \in \Lambda^3 \mathcal{T}$.
- Employing Graham-Willse's result, this yields a canonical ambient 3-form on $\tilde{M}$ that is parallel with respect to the ambient Levi-Civita derivative.
- In particular, $\mathrm{Hol}(\tilde{\nabla}) \subset G_2$.

Ambient holonomy = conformal holonomy

- The natural type of holonomy one should employ is infinitesimal ambient holonomy along the cone $\mathcal{C}$:

$$\tilde{\mathfrak{hol}} = \mathrm{span}\Big(\big\{\big(\tilde{\nabla}_{\xi_1}\cdots\tilde{\nabla}_{\xi_{l-2}}\tilde{R}(\xi_{l-1},\xi_l)\big)_{|\mathcal{C}}, \xi_1,\ldots,\xi_l \in \mathfrak{X}(\tilde{M})\big\}\Big) \subset \Gamma_{\mathcal{C}}(\mathrm{End}(T\tilde{M}))$$

- Since $\mathrm{Hol}(\mathcal{C}) := \mathrm{Hol}(\nabla^{\mathcal{T}}) \subset \mathrm{SO}(p+1,q+1)$, infinitesimal tractor holonomy is given by

$$\mathfrak{hol} = \mathrm{span}\Big(\big\{\big(\nabla_{\xi_1}\cdots\nabla_{\xi_{l-2}}R(\xi_{l-1},\xi_l)\big), \xi_1,\ldots,\xi_l \in \mathfrak{X}(M)\big\}\Big) \subset \Gamma(\mathrm{End}(\mathcal{T}))$$

Theorem (Čap-Gover-Graham-H.)

Let $(M,\mathcal{C})$ be a conformal structure of signature (p,q) with $n=p+q$ odd. Then infinitesimal tractor holonomy coincides with infinitesimal ambient holonomy: $\mathfrak{hol} = \tilde{\mathfrak{hol}}$.

Fefferman-Graham ambient holonomy

- For a real-analytic conformal structure $(M, \mathcal{C})$ in odd dimension $n = p + q$ the ambient space $(\tilde{M}, \tilde{g})$ is a well-defined pseudo-Riemannian structure of signature $(p+1, q+1)$. In particular, in this case the holonomy $\mathrm{Hol}(\tilde{\nabla}) \subset \mathrm{SO}(p+1, q+1)$ of the ambient Levi-Civita connection $\tilde{\nabla}$ is a conformal invariant.
  It follows in particular from the theorem that for $\mathcal{C}$, and then also $\tilde{g}$, real-analytic, that $\mathrm{Hol}(\mathcal{C}) = \mathrm{Hol}(\nabla^{\mathcal{T}}) = \mathrm{Hol}(\tilde{\nabla})$.
- For $n = p + q$ even one has to employ a truncated version of infinitesimal ambient holonomy that only involves derivatives up to order $\frac{n}{2}$ and can then show analogously that this space is contained in infinitesimal tractor holonomy $\mathfrak{hol}$.