# Coupling solutions of BGG-equations in conformal spin geometry

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# Setting

Let $M$ be a smooth manifold and $(\mathcal{G} \to M, \omega)$ a parabolic geometry of type $(G, P)$. Here $G$ a is semi-simple Lie group and $P \subset G$ a parabolic subgroup. $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ the Cartan connection form of the geometry with values in the Lie algebra $\mathfrak{g}$ of $G$. Geometries of interest could for instance be projective structures, conformal structures or CR-structures.

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We are interested in overdetermined operators on such geometries which appear as the first operators in the BGG-sequence

$$\mathcal{H}_0 \overset{\Theta_0}{\to} \mathcal{H}_1 \overset{\Theta_1}{\to} \dots \overset{\Theta_{n-1}}{\to} \mathcal{H}_n$$

of natural differential operators as constructed by [Cap-Slovak-Souček].

# Outline

1. BGG-equations, the prolongation connection and coupling
2. Coupling in conformal spin geometry
3. Generic twistor spinors and cKf decomposition

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# Tractor bundles

For every $G$-representation $V$ one associates the tractor bundle $\mathcal{V} = \mathcal{G} \times_P V$.

$\mathcal{V}$ carries its canonical tractor connection, denoted by $\nabla = \nabla^V$, and this gives rise to a sequence

$$\mathcal{C}_0 \xrightarrow{\nabla} \mathcal{C}_1 \xrightarrow{d^\nabla} \mathcal{C}_2 \xrightarrow{d^\nabla} \cdots$$

on the chain spaces $\mathcal{C}_k = \Omega^k(M, \mathcal{V})$.

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Moreover, one has the (algebraic) Kostant co-differential $\partial^* : \mathcal{C}_{k+1} \to \mathcal{C}_k$, $\partial^* \circ \partial^* = 0$, which yields the complex

$$\mathcal{C}_0 \stackrel{\partial^*}{\leftarrow} \mathcal{C}_1 \stackrel{\partial^*}{\leftarrow} \mathcal{C}_2 \stackrel{\partial^*}{\leftarrow} \cdots .$$

This complex gives rise to spaces $\mathcal{Z}_k = \ker \partial^*$ of cycles, $\mathcal{B}_k = \mathrm{im}\ \partial^*$ of borders and homologies $\mathcal{H}_k = \mathcal{Z}_k / \mathcal{B}_k$. The canonical surjections are denoted $\Pi_k : \mathcal{Z}_k \to \mathcal{H}_k$.

# The BGG-operators and the prolongation connection

The BGG-machinery of [Čap-Slovak-Souček] is based on canonical differential splitting operators $L_k : \Gamma(\mathcal{H}_k) \to \Gamma(\mathcal{Z}_k)$: A section $s \in \Gamma(\mathcal{Z}_k)$ is of the form $L_k\sigma, \sigma \in \Gamma(\mathcal{H}_k)$ if and only if $d^\nabla s \in \ker \partial^*$.

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In particular, one can form the first BGG-operator $\Theta_0 = \Pi_1 \circ \nabla \circ L_0$, $\Theta_0 : \Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{H}_1)$.

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If a section $s \in \Gamma(\mathcal{V})$ is $\nabla$-parallel, we see that $\sigma = \Pi_0(s) \in \Gamma(\mathcal{H}_0)$ lies in the kernel of $\Theta_0$. We say that $\sigma$ is a normal solution of $\Theta_0(\sigma) = 0$.

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In [H.-Somberg-Souček-Šilhan, 2010] a natural modification $\tilde{\nabla} = \nabla + \Psi$ with $\Psi \in \Omega^1(M, \mathrm{End}(\mathcal{V}))$ was constructed which has the property that

**Proposition (HSSS, 2010)**

*The solutions $\sigma \in \mathcal{H}_0$ of $\Theta_0(\sigma) = 0$ are in $1:1$-correspondence with the $\tilde{\nabla}$-parallel sections of $\mathcal{V}$.*

We call $\tilde{\nabla}$ the prolongation connection of $\Theta_0$.

# An efficient condition for checking normality of $\sigma \in \Gamma(\mathcal{H}_0), \Theta_0(\sigma) = 0$

The Cartan curvature form $K \in \Omega^2(M, \mathcal{A})$ of the Cartan connection $\omega$ has values in the adjoint tractor bundle $\mathcal{A} := \mathcal{G} \times_P \mathfrak{g}$, which naturally acts on the tractor bundle $\mathcal{V}$ via $\bullet : \mathcal{A} \to \mathrm{End}(\mathcal{V})$.

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*A solution $\sigma$ of $\Theta_0(\sigma) = 0$ is normal if and only if $\partial^*\big(K \bullet (L_0 \sigma)\big) = 0$.*

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**Proof:**

Since $\nabla$ is the natural connection induced by $\omega$ on $\mathcal{V}$ the curvature of $\nabla$ is $R = K \bullet \in \Omega^2(M, \mathrm{End}(\mathcal{V}))$.

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Denote $s = L_0\sigma$. Now if $\partial^*(K\bullet s) = 0$, then since $R = d^\nabla \circ \nabla$ we have $d^\nabla(\nabla s) \in \ker \partial^*$. Thus

$$\nabla s = L_1(\Pi_1(s)) = L_1(\Theta_0\sigma) = 0.$$

# Coupling maps

Let now $V, V'$ and $W$ be $G$ representations and $C : V \times V' \to W$ be a $G$-equivariant bilinear map. The corresponding tractor map is denoted

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It induces the (differential) coupling map $\mathbf{c} : \Gamma(\mathcal{H}_0^V) \times \Gamma(\mathcal{H}_0^{V'}) \to \Gamma(\mathcal{H}_1^W)$

$$(\sigma, \sigma') \mapsto \Pi_0^W\big(\mathbf{C}(L_0^V(\sigma), L_0^{V'}(\sigma'))\big).$$

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Since $\mathbf{C} : \mathcal{V} \times \mathcal{V}' \to \mathcal{W}$ is algebraic and natural, we have that for all $s \in \Gamma(\mathcal{V})$ ,$s' \in \Gamma(\mathcal{V}')$,

$$\nabla^W \mathbf{C}(s, s') = \mathbf{C}(\nabla^V s, s') + \mathbf{C}(s, \nabla^{V'} s').$$

In particular, if $\sigma \in \mathcal{H}_0^V$ and $\sigma' \in \mathcal{H}_0^{V'}$ are normal solutions of $\Theta_0^V$ resp. $\Theta_0^{V'}$, then $\eta := \mathbf{c}(\sigma, \sigma')$ is a normal solution of $\Theta_0^W$.

# Coupling for $|1|$-graded parabolic geometries with $\Theta_0^W$ of first order

The operators $\Theta_0^V$ and $\Theta_0^{V'}$ have prolongation connections $\tilde{\nabla}^V = \nabla^V + \Psi^V$, $\tilde{\nabla}^{V'} = \nabla^{V'} + \Psi^{V'}$. We write $s = L_0^V \sigma, s' = L_0^{V'} \sigma'$.

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By definition, $t := \mathbf{C}(s, s')$ is a lift of $\eta = \mathbf{c}(\sigma, \sigma') = \Pi_0^W(t)$, but one doesn't necessarily have $t = L_0^W \eta$:
$\nabla^W t \in \mathcal{C}_1^W = \Omega^1(M, \mathcal{W})$ need not lie in $\mathcal{Z}_1^W = \ker \partial^*$.

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That $\Theta_0^W$ is an operator of first order implies that one has a canonical extension of $\Pi_1^W : \mathcal{Z}_1^W \to \mathcal{H}_1^W$ to a map $\Pi_{1,\odot}^W : \mathcal{C}_1^W \to \mathcal{H}_1^W$ and then $\Theta_0^W(\eta) = \Pi_{1,\odot}^W(\nabla^W t)$.

# Coupling for |1|-graded parabolic geometries with $\Theta_0^W$ of first order

Now, for $\sigma \in \ker \Theta_0^V$ we have (equivalently) that $s = L_0^V$ satisfies

$$0 = \tilde{\nabla}^V s = \nabla^V s + \Psi^V s,$$

so $\nabla^V s = -\Psi^V s$, and analogously for $\sigma' \in \ker \Theta_0^{V'}$.

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Therefore $\nabla^W t = \nabla^W \mathbf{C}(s, s') = \mathbf{C}(\nabla^V s, s') + \mathbf{C}(s, \nabla^{V'} s') = -\mathbf{C}(\Psi^V s, s') - \mathbf{C}(s, \Psi^{V'} s')$.

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Thus, using $\Theta_0^W(\eta) = \Pi^W_{1,\odot}(\nabla^W t)$ we have

**Proposition**

*For* $\sigma \in \ker \Theta_0^V, \sigma' \in \ker \Theta_0^{V'}$ *and* $\eta = \mathbf{c}(\sigma, \sigma')$ *one has* $\Theta_0^W(\eta) = -\Pi^W_{1,\odot}\big(\mathbf{C}(\Psi^V s, s') + \mathbf{C}(s, \Psi^{V'} s')\big)$.

In particular, this yields necessary and sufficient coupling conditions.

# Outline

# Conformal spin structures

A conformal spin structure of signature $(p, q)$ on an $n = p + q$-manifold $M$ is a reduction of structure group of $TM$ from $\mathrm{GL}(n)$ to $\mathrm{CSpin}(p, q) = \mathbb{R}_+ \times \mathrm{Spin}(p, q)$.

This induces a conformal class $\mathcal{C}$ of pseudo-Riemannian signature $(p, q)$-metrics on $M$. The associated bundle to the spin representation $\Delta^{p,q}$ of $\mathrm{CSpin}(p, q)$ with $\mathbb{R}_+$ acting trivially is the (unweighted) conformal spin bundle $\mathcal{S}$.

We will often employ the conformal density bundles $\mathcal{E}[w]$, $w \in \mathbb{R}$, which are associated to the 1-dimensional $\mathbb{R}_+$ representations $c \mapsto c^w$. We also employ abstract index notation $\mathcal{E}_a = \Omega^1(M), \mathcal{E}^a = \mathfrak{X}(M)$.

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A conformal spin structure of signature $(p, q)$ is described by a parabolic geometry of type $(\mathrm{Spin}(p + 1, q + 1), P)$, with $P \subset G = \mathrm{Spin}(p + 1, q + 1)$ the stabilizer of an isotropic ray in the standard representation on $\mathbb{R}^{p+1,q+1}$.

# The almost Einstein scale operator $\Theta_0^{\mathbb{R}^{p+1,q+1}}$

With $T = \mathbb{R}^{p+1,q+1}$ the standard representation of $\mathrm{Spin}(p+1, q+1)$, one obtains the standard tractor bundle $\mathcal{T} = \mathcal{G} \times_P \mathbb{R}^{p+1,q+1}$ together with its tractor metric $\mathbf{h}$. It has a semidirect composition series $\mathcal{T} = \mathcal{E}[1] \oplus\!\!\!\!+ \mathcal{E}_a[1] \oplus\!\!\!\!+ \mathcal{E}[-1]$.

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With respect to the Levi-Civita connection $D$ of a metric $g \in \mathcal{C}$ the first BGG-operator of $T$ is

$$\Theta_0^T : \mathcal{E}[1] \to \mathcal{E}_{(ab)}[2],$$
$$\sigma \mapsto \mathbf{tf}(DD\sigma + P\sigma),$$

with $P \in \mathcal{E}_{(ab)}$ the Schouten tensor of $g$.

In this case the standard tractor connection $\nabla^T$ is already the prolongation connection.

# The conformal Killing form operator $\Theta_0^{\Lambda^{k+1}\mathbb{R}^{p+1,q+1}}$

Now let $V = \Lambda^{k+1}\mathbb{R}^{p+1,q+1}$ for $k \geq 1$ be an exterior power of the standard representation and $\mathcal{V} = \mathcal{G} \times_P V$ the associated tractor bundle. $\mathcal{V}$ has a semidirect composition series
$\mathcal{E}_{[a_1\cdots a_k]}[k+1] \oplus\!\!\!\!+ (\mathcal{E}_{[a_1\cdots a_{k+1}]}[k+1] \oplus \mathcal{E}_{[a_1\cdots a_{k-1}]}[k-1]) \oplus\!\!\!\!+ \mathcal{E}_{[a_1\cdots a_k]}[k-1]$.

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The first BGG-operator of $V$ is

$$\Theta_0^V : \mathcal{E}_{[a_1\cdots a_k]}[k+1] \to \mathcal{E}_{c[a_1\ldots a_k]}[k+1],$$

$$\sigma_{a_1\cdots a_k} \mapsto D_c\sigma_{a_1\cdots a_k} - D_{[a_0}\sigma_{a_1\cdots a_k]} - \frac{k}{n-k+1} g_{c[a_1} g^{pq} D_{|p}\sigma_{q|a_2\cdots a_k]}$$

and its solutions are the conformal Killing forms.

# The prolongation connection of $\Theta_0^{\Lambda^{k+1}\mathbb{R}^{p+1,q+1}}$

The prolongation connection is $\tilde{\nabla}^V = \nabla^V + \Psi^V$ for $\Psi^V \in \Omega^1(M, \mathrm{End}(\mathcal{V}))$. Here it is enough to know its part of lowest homogeneity, which is

$$\bar{\Psi}^V \in \mathrm{Hom}(\mathcal{E}_{[a_1\cdots a_k]}[k+1], \mathcal{E}_c \otimes \big(\mathcal{E}_{[a_1\cdots a_{k+1}]}[k+1] \oplus \mathcal{E}_{[a_1\cdots a_{k-1}]}[k-1]\big)),$$
$$\sigma \mapsto L(\sigma) \oplus R(\sigma)$$

with

$$L(\sigma) = \frac{k+1}{2} C_{[a_0 a_1}{}^{p}{}_{|c}\sigma_{p|a_2\cdots a_k]} + \frac{(k-1)(k+1)}{2n} g_{c[a_0} C_{a_1 a_2}{}^{pq}\sigma_{|pq|a_3\cdots a_k]}$$
$$R(\sigma) = \frac{(k-1)(n-2)}{2(n-k)n} C_{c[a_2}{}^{pq}\sigma_{|pq|a_3\cdots a_k]} - \frac{(k-1)(k-2)}{2(n-k)n} C_{[a_2 a_3}{}^{pq}\sigma_{|cpq|a_4\ldots a_k]}.$$

# The twistor spinor operator $\Theta_0^{\Delta^{p+1,q+1}}$

With $\Delta^{p+1,q+1}$ the spin representation of $\mathrm{Spin}(p+1,q+1)$ we form the associated spin tractor bundle $\Sigma := \mathcal{G} \times_P \Delta^{p+1,q+1}$. Recall the the (unweighted) spin bundle $\mathcal{S}$ of the conformal structure. Then $\Sigma$ has a semidirect composition series $\Sigma = \mathcal{S}[\frac{1}{2}] \oplus \mathcal{S}[-\frac{1}{2}]$.

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With respect to the Levi-Civita connection $D$ of a metric $g \in \mathcal{C}$ the first BGG-operator of $\Delta^{p+1,q+1}$ is $\Theta_0^{\Delta^{p+1,q+1}} : \Gamma(\mathcal{S}[\frac{1}{2}]) \to \Gamma(\mathcal{E}_c \otimes \mathcal{S}[\frac{1}{2}])$,

$$\chi \mapsto D_c \chi + \frac{1}{n} \gamma_c \not{D} \chi,$$

where $\gamma \in \mathcal{E}_c \otimes \mathrm{End}(\mathcal{S})$ the Christoffel symbol of $\mathcal{S}$ and $\not{D}\chi = g^{pq} \gamma_p D_q \chi$.

The solutions of $\Theta_0^{\Delta^{p+1,q+1}}(\chi) = 0$ are twistor spinors, and the tractor connection $\nabla$ coincides with the prolongation connection.

# Clifford action and invariant pairing

For every $g \in \mathcal{C}$ we obtain identifications $\mathcal{T} \cong \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$ and $\mathcal{S} \cong \mathcal{S}[\frac{1}{2}] \oplus \mathcal{S}[-\frac{1}{2}]$.

Then the tractor Clifford multiplication is given by

$$\Gamma : \mathcal{T} \otimes \mathcal{S} \to \mathcal{S},$$
$$(\sigma, \varphi_a, \rho) \cdot (\chi, \tau) = (\varphi_a \cdot \chi - \sqrt{2}\sigma\tau, -\varphi_a \cdot \tau + \sqrt{2}\rho\chi).$$

With $\mathbf{b} : \mathcal{S}[\frac{1}{2}] \otimes \mathcal{S}[\frac{1}{2}] \to \mathbb{R}$ the invariant pairing of the (weighted) conformal spin bundle the tractor spinor pairing is

$$\mathbf{B} : \Sigma \otimes \Sigma \to \mathbb{R}, \quad \mathbf{B}\big((\chi, \tau), (\chi', \tau')\big) = \mathbf{b}(\chi, \tau') + (-1)^{p+1}\mathbf{b}(\chi', \tau).$$

# Wedge coupling of conformal Killing forms

Given $s \in \Gamma(\Lambda^{k+1}\mathcal{T})$ and $s' \in \Gamma(\Lambda^{k'+1}\mathcal{T})$ we form $\mathbf{C}^\wedge(s, s') := s \wedge s'$.

We obtain the coupling map
$\mathbf{c}^\wedge : \mathcal{E}_{[a_1 \cdots a_k]}[k+1] \times \mathcal{E}_{[a_1 \cdots a_{k'}}[k'+1] \to \mathcal{E}_{[a_1 \cdots a_{k+k'+1}]}[k+k'+2]$

$$(\sigma_{a_1 \cdots a_k}, \sigma'_{a_1 \cdots a_{k'}}) \mapsto (k+1)\sigma_{[a_1 \cdots a_k} D_{a_{k+1}} \sigma'_{a_{k+2} \cdots a_{k+k+1}]} + (-1)^{(k+1)(k'+1)}(k'+1)\sigma'_{[a_1 \cdots a_{k'}} D_{a_{k'+1}} \sigma_{a_{k'+2} \cdots a_{k+k'+1}]}.$$

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$$(\sigma_{a_1 \cdots a_k}, \sigma'_{a_1 \cdots a_{k'}}) \mapsto (k+1)\sigma_{[a_1 \cdots a_k} D_{a_{k+1}} \sigma'_{a_{k+2} \cdots a_{k+k+1}]} + (-1)^{(k+1)(k'+1)}(k'+1)\sigma'_{[a_1 \cdots a_{k'}} D_{a_{k'+1}} \sigma_{a_{k'+2} \cdots a_{k+k'+1}]}.$$

Assume that $\sigma$ and $\sigma'$ are conformal Killing forms. Then the coupled $k+k'+1$-form $\eta = \mathbf{c}^{\wedge}(\sigma, \sigma')$ is a conformal Killing form if and only if

$$(-1)^{k+1} C_{[a_1 a_2}{}^{p}{}_{|c}\sigma_{p|a_3 \cdots a_{k+1}} \sigma'_{a_{k+3} \cdots a_{k+k'+1}]} + \sigma_{[a_1 \cdots a_k} C_{a_{k+1} a_{k+2}}{}^{p}{}_{|c} \sigma'_{p|a_{k+3} \cdots a_{k+k'+1}]} \overset{\odot}{=} 0.$$

This generalizes formulas of [Gover-Šilhan, 2008].

# Contraction coupling of conformal Killing forms

Let now $k' > k$. We employ the tractor metric $\mathbf{h}$ to form a contraction map $\mathbf{C}^\vee : \Lambda^{k+1}\mathcal{T} \times \Lambda^{k'+1}\mathcal{T} \to \Lambda^{k'-k}\mathcal{T}$.

The coupling map is then
$\mathbf{c}^\vee : \mathcal{E}_{[a_1 \cdots a_k]}[k+1] \times \mathcal{E}_{[a_1 \cdots a_{k'}]}[k'+1] \to \mathcal{E}_{[a_1 \cdots a_{k'-k-1}]}[k'-k]$,

$$\begin{aligned}(\sigma_{a_1 \cdots a_k}, \sigma'_{a_1 \cdots a_{k'}}) \mapsto (k+1)\sigma^{p_1 \cdots p_k} D^q \sigma'_{q p_1 \cdots p_k a_1 \cdots a_{k'-k-1}} \\ + (n-k'+1)\sigma'_{p_0 \cdots p_k a_1 \cdots a_{k'-k-1}} D^{p_0} \sigma^{p_1 \cdots p_k}.\end{aligned}$$

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$$(\sigma_{a_1\cdots a_k}, \sigma'_{a_1\cdots a_{k'}}) \mapsto (k+1)\sigma^{p_1\cdots p_k} D^q \sigma'_{qp_1\cdots p_k a_1\cdots a_{k'-k-1}} \\ + (n-k'+1)\sigma'_{p_0\cdots p_k a_1\cdots a_{k'-k-1}} D^{p_0}\sigma^{p_1\cdots p_k}.$$

If $\sigma$ and $\sigma'$ are conformal Killing forms the coupled $k'-k-1$-form $\eta = \mathbf{c}(\sigma,\sigma')$ is also be a conformal Killing form if and only if

$$(n-k')C^{p_0p_1}{}_{qc}\sigma^{qp_2\cdots p_k}\eta_{p_0\cdots p_k a_1\cdots a_{k'-k-1}} \\ -(k'-1)\sigma^{p_1\cdots p_k} C_{cp_1}{}^{q_1q_2}\eta_{q_1q_2p_2\cdots p_k a_1\cdots a_{k'-k-1}} \overset{\odot}{=} 0.$$

# Twistor spinor coupling

Let $X, X' \in \Gamma(\Sigma)$ and fix a $k \geq 0$. We define an element in $\Lambda^{k+1}\mathcal{T} \cong \Lambda^{k+1}\mathcal{T}^*$ by

$$\mathbf{C}^k(X, X')(\Phi) = \mathbf{B}(\Phi \cdot X, X') \ \forall \ \Phi \in \Lambda^{k+1}\mathcal{T}.$$

This yields the invariant pairing from spinors to forms,

$$\begin{aligned} \mathbf{c}^k : \Gamma(S[\tfrac{1}{2}]) \times \Gamma(S[\tfrac{1}{2}]) &\to \mathcal{E}_{[a_1 \cdots a_k]}[k+1], \\ (\chi, \chi') &\mapsto \mathbf{b}(\chi, \gamma_{[a_1} \cdots \gamma_{a_k]}\chi'). \end{aligned}$$

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Since the prolongation connection of $\Sigma$ coincides with the tractor connection this (well known) map always produces a conformal Killing $k$-form from two given twistor spinors.

# Conformal Killing forms - twistor spinor coupling

Let $k \geq 0$. The tractor Clifford multiplication provides a map

$$\mathbf{C}^{\gamma} : \Lambda^{k+1}\mathcal{T} \otimes \Sigma \to \Sigma$$

and the corresponding coupling map is

$$\mathbf{c}^{\gamma} : \mathcal{E}_{[a_1 \cdots a_k]}[k+1] \times \Gamma(S[\frac{1}{2}]) \to \Gamma(S[\frac{1}{2}]),$$
$$\varphi \times \chi \mapsto (-1)^{k+1}\frac{2(k+1)}{n}\varphi \cdot \not{D}\chi + (d\varphi)\cdot\chi + \frac{k(k+1)}{(n-k+1)}(\delta\varphi)\cdot\chi.$$

Here $d\varphi = D_{[a_0}\varphi_{a_1\cdots a_k]}$ is the exterior derivative of $\varphi$ and $\delta\varphi = -g^{pq}D_p\varphi_{qa_2\cdots a_k}$ is the divergence of $\varphi$.

# Conformal Killing forms - twistor spinor coupling

This will be a twistor spinor iff $k \leq 1$ or $k \geq 2$ and $C_{ca_1}{}^{pq}\sigma_{pqa_2\cdots a_k}\gamma^{a_1\cdots a_k}\chi \overset{\odot}{=} 0$.

In particular, since this corresponds to the cases $k = 0$ or $k = 1$, pairing an almost Einstein scale or a conformal Killing field with a twistor spinor always yields another twistor spinor.

One can show that if the compatibility condition is satisfied, then in fact $L_0^{\Lambda^{k+1}}(\sigma) \cdot L_0^{\Sigma}(\chi)$ is $\nabla^{\Sigma}$-parallel.

# Outline

# Generic twistor spinors

We start with an algebraic observation. Take a $k \geq 0$ and the map

$$C : \Delta^{p+1,q+1} \times \Delta^{p+1,q+1} \to \Lambda^{k+1}\mathbb{R}^{p+1,q+1},$$

realized with respect to the $\mathrm{Spin}(p+1, q+1)$-invariant pairing $B \in \Delta^{p+1,q+1*} \otimes \Delta^{p+1,q+1*}$.

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realized with respect to the $\mathrm{Spin}(p+1, q+1)$-invariant pairing $B \in \Delta^{p+1,q+1*} \otimes \Delta^{p+1,q+1*}$.

For a fixed $X \in \Delta^{p+1,q+1}$ we can form

$$i_X C : \Delta^{p+1,q+1} \to \Lambda^{k+1}\mathbb{R}^{p+1,q+1},$$

which is $G := \mathrm{Spin}(p+1, q+1)_X$-invariant.

# Generic twistor spinors

## Lemma

*Assume that $B(X,X) \neq 0$. Then, after some suitable rescaling, one has that the map*

$$P : \Lambda^{k+1}\mathbb{R}^{p+1,q+1} \to \Lambda^{k+1}\mathbb{R}^{p+1,q+1},$$
$$\Phi \mapsto i_X C(\Phi \cdot X)$$

*satisfies $P \circ P = \pm P$.*

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## Lemma

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*satisfies* $P \circ P = \pm P$*.*
*Then* $\ker P = \ker \Gamma X$ *and we obtain a* $G$*-invariant decomposition*

$$\Lambda^{k+1}\mathbb{R}^{p+1,q+1} = \ker \Gamma X \oplus \mathrm{im}\ P.$$

**Definition**

We say that a twistor spinor $\chi \in \Gamma(\mathcal{S}[\frac{1}{2}])$ is generic if $\mathbf{b}(\chi, \not{D}\chi) \neq 0$. This is the case if and only if the corresponding parallel tractor $X = L_0^{\Delta^{p+1,q+1}}\chi$ satisfies $\mathbf{B}(X, X) \neq 0$.

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Now for a twistor spinor $\chi$ the coupling map $\mathfrak{X}(M) \times \mathcal{S}[\frac{1}{2}] \to \mathcal{S}[\frac{1}{2}]$ can be rewritten into

$$\xi \times \chi \mapsto D_\xi \chi - \frac{1}{4}(D_{[a}\xi_{b]}) \cdot \chi + \frac{1}{2n}(D_p \xi^p)\chi.$$

For a conformal Killing field $\xi \in \mathfrak{X}(M)$ this is just the Lie derivative of the (weighted) spinor $\chi$ with respect to $\xi$. Our algebraic observation from above tells us that every generic twistor spinor $\chi$ provides a decomposition

$$\mathrm{cKf}(M, \mathcal{C}) = \mathrm{cKf}_\chi(M, \mathcal{C}) \oplus \mathrm{cKf}_\chi^\perp(M, \mathcal{C})$$

of conformal Killing fields into a part which also preserves $\chi$ and a complement.

# Generic twistor spinors in low dimension [HS, 2010]

Let $(M, \mathcal{C}, \chi)$ be a conformal spin structure of signature $(2, 3)$ with a generic twistor spinor $\chi$.

Genericity of $\chi$ implies that $\mathcal{D}_\chi = \ker \gamma\chi$ is a generic rank 2 distribution on $M$: One has that the subbundle $[\mathcal{D}_\chi, \mathcal{D}_\chi]$ of $TM$ spanned by Lie brackets of sections of $\mathcal{D}_\chi$ is 3-dimensional and $TM = [\mathcal{D}_\chi, [\mathcal{D}_\chi, \mathcal{D}_\chi]]$.

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**Proposition (H.-Sagerschnig, 2010)**

*The conformal spin structure $\mathcal{C}$ together with its generic twistor spinor $\chi$ is uniquely determined by $\mathcal{D}$.*

This is shown via a Fefferman-type construction which starts from any generic 2-distribution $\mathcal{D} \subset TM$ and associates $(\mathcal{C}_\mathcal{D}, \chi_\mathcal{D})$. In particular, there are non-flat conformal spin structures with generic twistor spinors.

# Generic twistor spinors in low dimension [HS, 2010]

The space of conformal Killing fields of $(\mathcal{C}, \chi)$ decomposes into symmetries of the generic distribution $\mathcal{D}_\chi = \ker \gamma\chi$ and a complement $\mathsf{cKf}^{\perp}_{\chi}(M, \mathcal{C})$. In this situation it can be shown that the space $\mathsf{cKf}^{\perp}_{\chi}(M, \mathcal{C})$ can be identified with the space of almost Einstein scales $\mathsf{aEs}(M, \mathcal{C})$, so

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$$\mathsf{cKf}(M, \mathcal{C}) = \mathsf{sym}(\mathcal{D}_\chi) \oplus \mathsf{aEs}(M, \mathcal{C}).$$

Using the appropriate coupling maps one obtains explicit formulas: for a $g \in \mathcal{C}$, the almost Einstein scale part of a conformal Killing field $\xi \in \mathfrak{X}(M)$ is given by $\sigma = \mathbf{b}(\chi, \frac{4}{5}\xi \cdot \not{D}\chi + (D_{[a}\xi_{b]}) \cdot \chi) \in \mathcal{E}[1]$.

Conversely, an almost Einstein scale $\sigma \in \mathcal{E}[1]$ is mapped to a conformal Killing field $\xi_a = \mathbf{b}(\gamma_a\chi, -\frac{2}{5}\sigma\not{D}\chi + (D\sigma) \cdot \chi) \in \mathcal{E}_a[2] \cong \mathfrak{X}(M)$