Geometric overdetermined systems and the BGG-machinery

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Plan

1. Geometric overdetermined systems
   - Some interesting problems and equations
   - Geometric prolongation
   - Study of singularity sets
2. The BGG-Machinery and applications to invariant prolongation
   - Parabolic geometries and tractor bundles
   - The BGG-Machinery
   - Prolongation of first BGG operators
3. Further applications
   - Geometric construction of solutions and solution coupling

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   - Parabolic geometries and tractor bundles
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Some questions:

Two questions in Riemannian (or conformal) geometry:

- Let g be a pseudo-Riemannian metric on a manifold M. Can we rescale g conformally to $\hat{g} = fg$ with some positive function f such that $\hat{g}$ is Einstein,

$$\mathrm{Ric}(\hat{g}) = \lambda \hat{g} \ ?$$

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- If M is even-dimensional, can one rescale a Riemannian metric g conformally to a Kähler metric?

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- If ∇ is an affine torsion-free connection on M, is it metrizable? I.e, can one describe its geodesics by a Riemannian metric?
- Does the affine connection allow a projectively equivalent Ricci-flat connection?

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- Given a metric $g \in [g]$, one has its Levi-Civita connection D and can form the Riemannian curvature tensor R^g.
- To ask whether one can rescale a given metric g to an Einstein metric amounts to the question whether there is an Einstein metric in a given conformal class; i.e., whether for some $g \in \mathcal{C} = [g]$ the Ricci curvature $\mathrm{Ric}^g := \mathrm{tr}_{(1,3)} \mathrm{R}^g \in \Gamma(S^2 T^* M)$ is a multiple of g.

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This problem is governed by the operator [Bailey-Eastwood-Gover, 1994],

$$\Theta^g : \mathrm{C}^\infty(M) \to \Gamma(S^2_0 T^* M),$$
$$\Theta^g(\sigma) = (DD\sigma + \mathsf{P}^g\,\sigma) + \frac{1}{n}(\triangle\sigma - \operatorname{tr}_{(1,2)} \mathsf{P}^g\,\sigma)g$$

where

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- The operator Θ^g is conformally covariant between $\mathrm{C}^\infty(M)$ and $S^2_0 T^*M$: if one switches to another metric $\hat{g} = e^{2f} g$ in the conformal class, then

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- To define a conformally invariant operator, one introduces conformal density bundles $\mathcal{E}[w]$: these are line bundles which are trivialized by a choice of $g \in [g]$. The trivializations of $\sigma \in \mathcal{E}[w]$ with respect to $\hat{g} = e^{2f} g$ and g are related according to

$$[\sigma]_{\hat{g}} = e^{wf} [\sigma]_g.$$

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- By forming the weighted bundles $\mathbf{H}_0 = \mathcal{E}[1]$ and $\mathbf{H}_1 = S^2_0 T^* M \otimes \mathcal{E}[1]$ one obtains a conformally invariant operator

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- Following [Gover, Jour.Geom.Phys. (2010)] one calls $\ker \Theta \subset \mathcal{E}[1]$ the space of almost Einstein scales.

Example 2: Metrization of projective structures

- Two torsion-free linear connections D and $\hat{D}$ on TM are projectively equivalent iff there exists a one form $\Upsilon \in \Omega^1(M)$ with

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- An interesting question in projective differential geometry is whether a given projective class of connections $[D]$ contains the Levi-Civita connection of some metric, i.e., whether the corresponding set of unparameterized geodesics is metrizable.
- It was observed by [Sinjukov, Nauka (1979)] and [Mikeš, Acta Univ. Palack. Olomuc. (1996)] that this problem is governed by the equation

$$D\sigma - \frac{1}{n+1}\,\mathrm{sym}\big(\mathrm{id} \otimes \mathrm{tr}_{(1,2)}(D\sigma)\big) = 0$$

  for $\sigma \in \Gamma(S^2 TM)$.

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- We look for an equivalent first order system such that all first order derivatives of the dependent variables are given by the dependent variables themselves.
- In classical language, this means that one introduces additional variables for derivatives of $\sigma \in \Gamma(\mathbf{H}_0)$ and derives differential consequences for these variables from the equation $\Theta_0(\sigma) = 0$.

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We then call the tuple $(\mathbf{V}, \Pi, L, \nabla)$ a geometric prolongation of Θ_0.

Immediate applications of a geometric prolongation

If $(\mathbf{V}, \Pi, L, \nabla)$ is a geometric prolongation of Θ_0, then

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Moreover, the curvature of the prolongation connection can be used to obtain obstructions against the existence of parallel sections of $\mathbf{V}$ resp. solutions of $\Theta_0(\sigma) = 0$.

Study of singularities. Example: Einstein rescalings

To rewrite $\Theta(\sigma) = 0$ in closed form we introduce new variables $\mu = D^g\sigma$ and $\rho = \frac{1}{n}(\triangle^g - J^g)\sigma$, where $\triangle^g = -\operatorname{tr}^g \circ D^g \circ D^g$ and $J^g = \operatorname{tr}(\mathsf{P}(g))$.

Then

$$\Theta(\sigma) = 0 \text{ iff } \nabla \begin{pmatrix} \rho \\ \mu \\ \sigma \end{pmatrix} = \begin{pmatrix} D\rho - \mathsf{P}^g(\cdot, \mu) \\ D\mu + \sigma P + \rho g \\ D\sigma - \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

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The prolongation connection ∇ preserves the bilinear-form **h** given by the (quadratic) formula $\begin{pmatrix} \rho \\ \mu \\ \sigma \end{pmatrix} \mapsto 2\sigma\rho + g(\mu, \mu)$.

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In particular, for σ, μ, ρ corresponding to a solution of $\Theta(\sigma) = 0$, the expression $2\sigma\rho + g(\mu, \mu) \in C^\infty(M)$ is necessarily constant equal α, which shall be non-zero for our discussion.

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If $\sigma(x) = 0$, then $g_x(D\sigma(x), D\sigma(x)) \neq 0$ and we see that $D\sigma$ is non-vanishing along M_0, which shows that M_0 is a hypersurface in M.

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Moreover, $T_x M_0 = D\sigma(x)^\perp \subset T_x M$ and depending on whether $\alpha = g(D\sigma(x), D\sigma(x))$ is greater or smaller zero, M_0 inherits a signature $(p-1, q)$ resp. $(p, q-1)$-metric.

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In the case where M is a closed manifold this yields a Poincaré-Einstein manifold: The manifold M is decomposed into an Einstein-manifold M_- and a conformal boundary M_0, the singularity set of σ. Since $M = M_- \cup M_0$ is compact M_- is a conformally compact Einstein manifold.

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- The case of conformal geometry already exemplifies that one immediately encounters great obstacles, since there is no unique (Levi-Civita) connection as in Riemannian geometry.
- Major advances to overcome this obstacle were achieved in the 1920s by Élie Cartan and Tracy Thomas:
- Given a conformal structure of signature (p, q), $p + q = n$, the latter constructed a natural bundle $\mathbf{S}$ of rank $n + 2$ endowed with a canonical connection ∇^S and compatible signature $(p + 1, q + 1)$-metric $\mathbf{h}$. This is now called the conformal standard tractor bundle.

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Definition

A Cartan geometry of type (G, P) on a manifold M is a P-principal bundle $\mathcal{G} \to M$ endowed with a Cartan connection form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. ω is P-equivariant, reproduces fundamental vector fields and provides a trivialization $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$.

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Theorem (Cartan, 1923)

There is an equivalence of categories between conformal structures of signature (p, q) and Cartan geometries of type $(\mathrm{SO}(p+1, q+1), P)$ whose curvature satisfies a normalization condition.

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- For a parabolic geometry $(\mathcal{G}, \omega)$ there is a canonical regularity condition which implies that it induces a geometric structure on the underlying manifold M.
- There is also a natural normalization condition on ω, which yields the class of normal parabolic geometries.
- The equivalent description of geometric structures as parabolic geometries is a powerful tool for natural resp. invariant constructions.

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- Let $\mathbf{C_k} = \Lambda^k T^*M \otimes \mathbf{V}$. Then $\Gamma(\mathbf{C_k}) = \Omega^k(M, \mathbf{V})$ and one can form the twisted de-Rham sequence of the tractor connection ∇^V,

$$\Gamma(\mathbf{C}_0) \stackrel{\nabla^V}{\to} \Gamma(\mathbf{C}_1) \stackrel{d^\nabla}{\to} \Gamma(\mathbf{C}_2) \stackrel{d^\nabla}{\to} \cdots$$

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- For a parabolic geometry there is a canonical Lie algebra differential ∂^* called the Kostant codifferential. It gives rise to a complex

$$\mathbf{C}_0 \stackrel{\partial^*}{\leftarrow} \mathbf{C}_1 \stackrel{\partial^*}{\leftarrow} \mathbf{C}_2 \stackrel{\partial^*}{\leftarrow} \cdots$$

The BGG-sequence

- The differential ∂^* yields bundles $\mathbf{Z}_k = \ker \partial^*$ of cycles, $\mathbf{B}_k = \mathrm{im}\ \partial^*$ borders and homologies $\mathbf{H}_k = \mathbf{Z}_k/\mathbf{H}_k$, and one has the canonical projections $\Pi_k : \mathbf{Z}_k \to \mathbf{H}_k$.

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- Now the BGG-sequence is formed by natural differential operators

$$\Gamma(\mathbf{H}_0) \stackrel{\Theta_0}{\to} \Gamma(\mathbf{H}_1) \stackrel{\Theta_1}{\to} \Gamma(\mathbf{H}_2) \stackrel{\Theta_2}{\to} \cdots$$

It was presented in [Čap-Slovăk-Souček, Ann. of Math. (2001)] and a simplified construction was obtained in [Calderbank-Diemer, Crelle's Journal (2001)]

The BGG-sequence

- The differential ∂^* yields bundles $\mathbf{Z}_k = \ker \partial^*$ of cycles, $\mathbf{B}_k = \mathrm{im}\ \partial^*$ borders and homologies $\mathbf{H}_k = \mathbf{Z}_k / \mathbf{H}_k$, and one has the canonical projections $\Pi_k : \mathbf{Z}_k \to \mathbf{H}_k$.
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  It was presented in [Čap-Slovăk-Souček, Ann. of Math. (2001)] and a simplified construction was obtained in [Calderbank-Diemer, Crelle's Journal (2001)]
- The main technical step in the development of the BGG-machinery is the construction of the canonical BGG-splitting-operators $L_k : \Gamma(\mathbf{H}_k) \to \Gamma(\mathbf{Z}_k)$.

The first BGG-operator

- We are mostly interested in the first BGG-operator $\Theta_0 : \Gamma(\mathbf{H}_0) \to \Gamma(\mathbf{H}_1)$, defined via the composition $\Pi_1 \circ \nabla^V \circ L_0$,

$$\begin{array}{ccc} \mathrm{im}\,(L_0) & \xrightarrow{\nabla^V} & \Gamma(\mathbf{Z}_1) \\ {\scriptstyle L_0}\uparrow & & \downarrow{\scriptstyle \Pi_1} \\ \Gamma(\mathbf{H}_0) & \xrightarrow{\Theta_0} & \Gamma(\mathbf{H}_1) \end{array}.$$

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- $(\mathbf{V}, \Pi_0, L_0, \nabla^V)$ is however not a geometric prolongation for general representations V, since the converse does not hold:
  If $\sigma \in \ker \Theta_0$, then $\nabla^V(L_0(\sigma))$ need not necessarily vanish, but may lie in $\Gamma(\mathbf{B_1}) = \mathrm{im}\, \partial^*$.

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  Solutions of this equation are known as twistor spinors.
- Both cases are very special: parallel sections of the tractor connection are already in 1:1-correspondence with solutions, which reflects the fact that the modelling representations are still very simple.

Examples of first BGG-operators for conformal structures

- For an exterior power $\mathbf{V} = \Lambda^{k+1}\mathbf{S}$, $k \geq 1$ one obtains the operator governing conformal Killing k-forms,

$$\begin{aligned}\Theta_0 : \Omega^k(M) &\to \Gamma(T^*M \otimes \Lambda^k T^*M),\\ \Theta_0(\sigma) &= D\sigma - \mathrm{alt}_{(1,\cdots,k+1)} D\sigma \\ &\quad - \frac{k}{n-k+1}\mathrm{alt}_{(2,\cdots,k+1)}\big(g \otimes (\mathrm{tr}_{(1,2)} D\sigma)\big).\end{aligned}$$

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- Already in this case a solution of $\Theta_0(\sigma)$ need not satisfy that also $\nabla^V(L_0(\sigma)) = 0$. In fact, this imposes additional equations on a conformal Killing form σ, and solutions to this extended system have been termed normal conformal Killing forms by [Leitner, Rend.Circ.Mat.Pal. (2005)].

Theorem (H.-Somberg-Souček-Šilhan, 2010)

Let $\mathbf{V}$ *be a tractor bundle for a regular parabolic geometry. There exists a natural connection* $\tilde{\nabla}$ *on* $\mathbf{V}$ *such that*

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   commutes, and this implies that $(\mathbf{V}, \Pi_0, L_0, \tilde{\nabla})$ *is a natural geometric prolongation of* Θ_0*.*

$\tilde{\nabla}$ *is unique under a natural condition and is called the* **prolongation connection** *of* Θ_0*.*

Natural Prolongation

Corollary

Let V be a G-representation and $(\mathbf{V}, \tilde{\nabla}, \Pi_0, L_0)$ the geometric prolongation of Θ_0.

1. *The space $\ker \Theta_0 \subset \mathcal{H}_0$ has rank $\leq \dim V$.*
2. *Every $\sigma \in \ker \Theta_0$ is determined by its r-jet at some point, with $r \in \mathbb{N}$ only depending on the representation V.*
3. *If $\sigma \in \ker \Theta_0$ is not globally vanishing, its singularity set $\sigma^{-1}(\{0\})$ has an open dense complement.*

Example: Prolongation of the equation governing projective metrizability

- A class of projectively equivalent connections $[D]$ on an n-manifold is equivalently described as a parabolic geometry $(\mathcal{G}, \omega)$ of type $(SL(n+1), P)$ with P the stabilizer of a line in $\mathbb{R}^{n+1}$. The tractor bundle $\mathbf{V} = \mathcal{G} \times_P S^2\mathbb{R}^{n+1}$ yields the first BGG-operator

$$\Theta_0 : \Gamma(S^2 TM) \to \Gamma(T^*M \otimes S^2 TM)$$

$$\Theta_0(\sigma) = D\sigma - \frac{1}{n+1} \operatorname{sym}\big(\mathrm{id} \otimes \mathrm{tr}_{(1,2)}(D\sigma)\big)$$

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which governs the existence of geodesically equivalent metrics.

- After choice of a connection $D \in [D]$ the tractor bundle $\mathbf{V}$ can be written as a direct sum $S^2TM \oplus TM \oplus \mathrm{C}^\infty(M)$, and a section $s \in \Gamma(\mathbf{V})$ will be written

$$[s]_D = \begin{pmatrix} \rho \\ \mu \\ \sigma \end{pmatrix} \in \begin{pmatrix} \mathrm{C}^\infty(M) \\ \Gamma(TM) \\ \Gamma(S^2TM) \end{pmatrix}.$$

Example: Prolongation of the equation governing projective metrizability

- One calculates the splitting operator $L_0 : \Gamma(S^2TM) \to \Gamma(\mathbf{V})$ as

$$\sigma \mapsto \begin{pmatrix} \frac{1}{n(n+1)} \operatorname{tr}_{(1,3)(2,4)} D^2\sigma + \frac{1}{2n} \operatorname{tr}_{(1,3)(2,4)} \mathrm{P} \otimes \sigma \\ -\frac{1}{n+1} \operatorname{tr}_{(1,2)} D\sigma \\ \sigma \end{pmatrix}.$$

- The explicit form of the prolongation connection is

$$\tilde{\nabla} \begin{pmatrix} \rho \\ \mu \\ \sigma \end{pmatrix} = \begin{pmatrix} D\rho - 2\operatorname{tr}_{(2,3)} \mathrm{P} \otimes \mu - \frac{4}{n} \operatorname{tr}_{(1,4)(3,5)} A \otimes \sigma \\ D\mu - 2\operatorname{tr}_{(2,3)} \mathrm{P} \otimes \sigma + \rho \operatorname{id} + \frac{2}{n} \operatorname{tr}_{(2,5)(4,6)} C \otimes \sigma \\ D\sigma + \operatorname{sym}(\operatorname{id} \otimes \mu) \end{pmatrix}.$$

  Here $A \in \Gamma(T^*M \otimes \Lambda^2 T^*M)$ is the Cotton-York tensor of D and $C \in \Gamma(\Lambda^2 T^*M \otimes \operatorname{End}(TM))$ the Weyl-curvature.

- This prolongation agrees with the one found by direct calculation in [Eastwood-Matveev, IMA (2008)]

Outline

Geometric construction of solutions

A useful application of the BGG-machinery is the geometric realization of non-flat structures that admit solutions to certain overdetermined systems. Often one can also characterize the resulting structures.

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A useful application of the BGG-machinery is the geometric realization of non-flat structures that admit solutions to certain overdetermined systems. Often one can also characterize the resulting structures.

This employs the so called Fefferman-type constructions: Starting from a geometry on M, one naturally associates another geometry on a (possibly larger) manifold N. This generalizes the classical Fefferman construction, which takes a CR-structure on M and associates a conformal structure on an $U(1)$-bundle $N \to M$ which admits a light-like conformal Killing field.

Example: Geometric construction of twistor spinors

In recent joint work with K. Sagerschnig [Ann.Glob.Ann.Geom.] we employed a Fefferman-type construction to produce conformal spin structures on 5 and 6 manifolds that carry generic twistor spinors.

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We sketch the process for 5-manifolds:
We start with a maximally non-integrable 2-distribution $\mathcal{D} \subset TM$ of the 5-manifold M. This says that the bundle $[\mathcal{D}, \mathcal{D}]$ spanned by Lie brackets of sections of $\mathcal{D}$ is 3-dimensional and $[\mathcal{D}, [\mathcal{D}, \mathcal{D}]] = TM$. These are very classical structures, also called generic 2-distributions, that are related to the geometry of second order ODEs.

Example: Geometric construction of twistor spinors

In the case where $\mathcal{D}$ is oriented the structure $(M, \mathcal{D})$ can be modelled as a Cartan geometry of type (G_2, P). Here G_2 shall denote the the connected real Lie group with fundamental group $\mathbb{Z}_2$ and Lie algebra the split real form of the exceptional complex Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$. $P \subset G_2$ is a suitable parabolic subgroup.

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It is well known that one can realize $G_2 \subset \mathrm{SO}(3,4)$ as the stabilizer of a $\Phi \in \Lambda^3\mathbb{R}^7$. For our purposes we use the embedding $G_2 \subset \mathrm{Spin}(3,4)$, where G_2 can be realized as the stabilizer of an arbitrary non-isotropic spinor $X \in \Delta_{\mathbb{R}}^{3,4}$, the real 8-dimensional real spin representation of $\mathrm{Spin}(3,4)$.

Example: Geometric construction of twistor spinors

The Fefferman-type construction starts by taking the Cartan geometry $(\mathcal{G}, \omega)$ describing the generic 2-distribution $\mathcal{D} \subset TM$. The parabolic subgroup $P \subset G_2$ naturally embeds into the parabolic subgroup $\tilde{P} \subset \mathrm{Spin}(3,4)$, and this allows one to build the extended principal bundle

$$\tilde{\mathcal{G}} := \mathcal{G} \times_P \tilde{P}, \ \tilde{\mathcal{G}} \to M.$$

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Then one shows that the G_2-Cartan connection form ω on $\mathcal{G}$ canonically extends to a $\mathrm{Spin}(3,4)$-Cartan connection form $\tilde{\omega}$ on $\tilde{\mathcal{G}}$. There is some further technical work necessary to check regularity and normality of the extended $\mathrm{Spin}(3,4)$-Cartan form.

Example: Geometric construction of twistor spinors

The resulting conformal spin structure of signature $(2,3)$ that is then described by $(\tilde{\mathcal{G}}, \tilde{\omega})$ has a spin tractor bundle

$$\Sigma = \tilde{\mathcal{G}} \times_{\tilde{P}} \Delta_{\mathbb{R}}^{3,4},$$

and it follows from the construction that the corresponding spin tractor connection ∇^{Σ} preserves a canonical (non-trivial) section $\mathbf{X} \in \Gamma(\Sigma)$,

$$\nabla^{\Sigma}\mathbf{X} = 0.$$

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The first BGG-projection is a map $L_0 : \Sigma \to \mathcal{S}[\frac{1}{2}]$, taking values in the weighted spin bundle $\mathcal{S}[\frac{1}{2}]$.

Example: Geometric construction of twistor spinors

Since the first BGG-operator

$$\Theta_0 : \Gamma(\mathcal{S}[\frac{1}{2}]) \to \Gamma(T^*M \otimes \mathcal{S}[\frac{1}{2}])$$

is the twistor operator discussed earlier and parallel sections project to solutions we see that we obtain a solution $\chi \in \Gamma(\mathcal{S}[\frac{1}{2}])$ of the twistor equation $D\chi + \frac{1}{5}\gamma \not{D}\chi = 0$.

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With respect to the canonical spinor pairing

$$\mathbf{b}_{2,3} : \mathcal{S}[\frac{1}{2}] \otimes \mathcal{S}[\frac{1}{2}] \to M \times \mathbb{R}$$

this twistor spinor is generic in the sense that $\mathbf{b}_{2,3}(\chi, \not{D}\chi) \neq 0$.

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Theorem

A conformal spin structure $\mathcal{C}$ of signature $(2, 3)$ on a 5-manifold is induced from an oriented generic 2-distribution $\mathcal{D} \subset TM$ if and only if there exists a generic twistor spinor χ on M.

Solution coupling

Another application of the BGG-machinery is to derive coupling formulas between solutions of (possibly different) systems.

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For the Fefferman-type construction $(M, \mathcal{D}) \rightsquigarrow (M, \mathcal{C})$ just discussed such maps appear naturally when one decomposes a given conformal Killing field $\xi \in \mathfrak{X}(M)$ into a part that respects the distribution $\mathcal{D}$ and a canonical *complementary part*, which in this case turns out to be isomorphic to the space of almost Einstein scales on M.

Solution coupling and automorphism decomposition

Proposition

Given a conformal spin structure of signature $(2,3)$ *and a generic twistor spinor* $\chi \in \Gamma(S[\frac{1}{2}])$*, the space of conformal Killing fields decomposes into the space of almost Einstein scales and the space of infinitesimal symmetries of the corresponding rank 2-distribution.*

Solution coupling and automorphism decomposition

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Explicitly, for a $g \in \mathcal{C}$, the almost Einstein scale part of a conformal Killing field $\xi \in \mathfrak{X}(M)$ is given by

$$\sigma = \mathbf{b}_{2,3}(\chi, \frac{4}{5}\xi \cdot \not{D}\chi + (D\xi) \cdot \chi) \in \mathcal{E}[1].$$

Conversely, an almost Einstein scale $\sigma \in \mathcal{E}[1]$ is mapped to a conformal Killing field

$$\xi = \mathbf{b}_{2,3}(\gamma\chi, -\frac{2}{5}\sigma\not{D}\chi + (D\sigma) \cdot \chi) \in \mathfrak{X}(M)$$