# Prolongation of conformally invariant overdetermined operators


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# Plan

The main point of this talk is to show how one gets a description of the kernel of an conformally invariant overdetermined operator as the space of parallel sections of some connection.
We start by illustrating the problem for one particular and very well understood equation. Then we proceed to lay out out other conformally invariant overdetermined systems which can be described in a similar fashion using *BGG-operators*. This description is then used for rewriting the systems in closed form

# Illustration of the Problem: Almost Einstein scales

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be a conformal class of metrics on $M$.

**Definition**

A function $\sigma \in \mathrm{C}^\infty(M)$ is called an *almost Einstein scale* for $g \in [g]$ if the open set $U = M/\sigma^{-1}\{0\}$ is dense and $\sigma^{-2} g$ is Einstein on $U$.

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Observing the transformation behaviour of the Schouten tensor

$$P_{ab} = \frac{1}{n-2}(\mathrm{Ric}_{ab} - \frac{R}{2(n-1)}g_{ab})$$

one obtains that $\sigma^{-2}g$ is Einstein on the complement of the singularity-set iff

$$\big(D_a D_b \sigma + \sigma P_{ab}\big)_0 = 0. \tag{1}$$

Here $D$ is the Levi-Civita connection of $g$ and subscript 0 takes the trace-free part.

Let us encode this via the operator

$$\Theta^g : \mathrm{C}^\infty(M) \to S^2_0 T^* M, \tag{2}$$
$$\sigma \mapsto \left(DD\sigma + \sigma P\right)_0 \tag{3}$$

which maps $\mathrm{C}^\infty(M)$ into symmetric, trace-free bilinear forms on $TM$.

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which maps $\mathrm{C}^\infty(M)$ into symmetric, trace-free bilinear forms on $TM$. $\Theta^g$ is conformally covariant: with $\mathrm{m}(\cdot)$ denoting the multiplication operator by a function,

$$\Theta^{\tilde{g}} = \mathrm{m}(e^f) \circ \Theta^g \circ \mathrm{m}(e^{-f})$$

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for $\tilde{g} = e^{2f} g$.
Equivalently, one can say that one has a conformally invariant operator; for this we need the conformal density bundles $\mathcal{E}[w]$ for $w \in \mathbb{R}$: these are line bundles which are trivialized for every $g \in [g]$ such that for $f \in \mathcal{E}[w]$ the resulting trivializations transform as

$$[f]_{\tilde{g}} = e^{wf}[f]_g$$

for $\tilde{g} = e^{2f} g$.

Then every $\Theta^g$ defines the same operator

$$\Theta : \mathcal{E}[1] \to S^2 T^* M \otimes \mathcal{E}[1],$$

and we say that $\Theta$ is conformally invariant since it operators between natural bundles for the conformal structures and is defined via a universal formula in metric terms.

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Now the equation (1) resp. the operator (2) is overdetermined. There is a well known way to write the system in closed form:
Let

$$[\mathcal{S}]_g := \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1].$$

A section $s \in \Gamma([\mathcal{S}]_g)$ will be written

$$s = \begin{pmatrix} \rho \\ \varphi_a \\ \sigma \end{pmatrix} \in \begin{pmatrix} \mathcal{E}[-1] \\ \mathcal{E}_a[1] \\ \mathcal{E}[1] \end{pmatrix}. \tag{4}$$

We now define the bundle $\mathcal{S}$ as the equivalence class of the $[\mathcal{S}]_g, g \in [g]$ under the relation

$$[s]_{\hat{g}} = \begin{pmatrix} \hat{\rho} \\ \hat{\varphi}_a \\ \hat{\sigma} \end{pmatrix} = \begin{pmatrix} \rho - \Upsilon_a \varphi^a - \frac{1}{2}\sigma \Upsilon^b \Upsilon_b \\ \varphi_a + \sigma \Upsilon_a \\ \sigma \end{pmatrix} \tag{5}$$

where $\Upsilon = df$.

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where $\Upsilon = df$.
The insertion of $\mathcal{E}[-1]$ into $\mathcal{S}$ as the top slot is independent of the choice of $g \in [g]$ and defines a section $\tau_+ \in \mathcal{S}[1]$. The insertion of $\mathcal{E}[1]$ into $\mathcal{S}$ as the bottom slot is well defined only via a choice of $g \in [g]$ and defines a section $\tau_- \in \mathcal{S}[-1]$.
Projection to the bottom slot is well defined and gives a map

$$\Pi_0 : \mathcal{S} \to \mathcal{E}[1], \begin{pmatrix} \rho \\ \mu \\ \sigma \end{pmatrix} \mapsto \sigma.$$

There is a well defined connection on $\mathcal{S}$: one defines

$$[\nabla_c^{\mathcal{S}} s]_g = \nabla_c^{\mathcal{S}} \begin{pmatrix} \rho \\ \varphi_a \\ \sigma \end{pmatrix} = \begin{pmatrix} D_c \rho - P_c{}^b \varphi_b \\ D_c \varphi_a + \sigma P_{ca} + \rho \mathbf{g}_{ca} \\ D_c \sigma - \varphi_c \end{pmatrix}. \tag{6}$$

Here $\mathbf{g} \in S^2 T^* M \otimes \mathcal{E}[2]$ is the conformal metric.

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Here $\mathbf{g} \in S^2 T^* M \otimes \mathcal{E}[2]$ is the conformal metric. One defines the splitting operator

$$L_0^{\mathcal{S}} : \mathcal{E}[1] \to \Gamma(\mathcal{S}), \tag{7}$$

$$\sigma \mapsto \begin{pmatrix} -\frac{1}{n}(\triangle + J)\sigma \\ D\sigma \\ \sigma \end{pmatrix}.$$

and calculates

$$\nabla^{\mathcal{S}} \circ L_0^{\mathcal{S}}(\sigma) = \begin{pmatrix} D_c(\triangle\sigma + J\sigma) - P_c{}^p D_p \sigma \\ (D_a D_b \sigma + P_{ab}\sigma) - \frac{1}{n}(\triangle\sigma + J\sigma)\mathbf{g}_{ab} \\ 0 \end{pmatrix}.$$

The middle slot is just $\Theta(\sigma)$ and it turns out that it is a differential consequence of $\Theta(\sigma) = 0$ that also the top slot vanishes. Thus:

**Proposition**

*$\Pi_0$ and $L_0 : \mathcal{E}[1] \to \mathcal{S}$ restrict to inverse isomorphisms between $\nabla^{\mathcal{S}}$-parallel sections of $\mathcal{S}$ and the kernel of $\Theta$.*

We will say that $(\mathcal{S}, \nabla^{\mathcal{S}}, \Pi_0, L_0)$ is a geometric prolongation of $\Theta$. It is well known that parallel sections are in $1:1$-correspondence with Holonomy-invariant elements of the modelling vector space. We immediately gain some consequences:

### Corollary

*The solution space of equation* (1) *is finite-dimensional and bounded by* $\operatorname{rank} \mathcal{S} = n + 2$*. Equality can only be obtained for a locally conformally flat structure.*

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**Proof** :

The dimensional bound is clear. Let $K$ denote the curvature of $\nabla^S$. Then $R \in \Omega^2(M, \mathfrak{so}(\mathcal{S}))$ is computet to be

$$K_{c_1 c_2} = \begin{pmatrix} 0 & -A_{ac_1c_2} & 0 \\ 0 & C_{c_1c2}{}^a{}_b & A^a{}_{c_1c_2} \\ 0 & 0 & 0 \end{pmatrix} \tag{8}$$

with $C_{c_1c2}{}^a{}_b$ the Weyl-curvature and

$$A_{abc} = D_b P_{ca} - D_c P_{ba}$$

the Cotton-York tensor.

Having the maximal solution space implies trivial holonomy of $\nabla^{\mathcal{S}}$; especially, the curvature $K$ has to vanish, and we see from (8) that then both conformal curvature tensors are trivial. $\square$

Having the maximal solution space implies trivial holonomy of $\nabla^{\mathcal{S}}$; especially, the curvature $K$ has to vanish, and we see from (8) that then both conformal curvature tensors are trivial. □ One more simple consequence is:

### Corollary

*Any solution of $(DD\sigma + P\sigma)_0 = 0$ is determined by its 2-jet at a point. Especially: If $\sigma$ is a non-trivial solution, it is automatically non-vanishing on an open-dense subset.*

**Proof** :
$L_0$ is a second order splitting operator. Especially, if the 2-jet of a $\sigma \in \mathrm{C}^\infty(M)$ vanishes, $s := L_0\sigma = 0$. But a parallel section of $\mathcal{S}$ is determined by its value at one point. □

# Tractor bundles, cohomology and the BGG-sequence

$\mathcal{S}$ is called the *standard tractor bundle* endowed with the standard tractor connection $\nabla^{\mathcal{S}}$ and the tractor metric $\mathbf{h}$. Any subbundle $\mathcal{T}$ of of a tensor power of $\mathcal{S}$ is called a *tractor bundle* and is endowed with the connection $\nabla^{\mathcal{T}}$ induced by $\nabla^{\mathcal{S}}$.

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$$T^*M \hookrightarrow \mathfrak{so}(\mathcal{S}, \mathbf{h}) = \Lambda^2(\mathcal{S})$$

allow one to define an algebraic differential

$$\partial^* : \Omega^{k+1}(M, \mathcal{T}) \to \Omega^k(M, \mathcal{T}),$$
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We will abbreviate $\Omega^k(M, \mathcal{T}) = \mathcal{C}_k$. Then $\partial^*$ provides us with a complex

$$\mathcal{C}_0 \stackrel{\partial^*}{\leftarrow} \mathcal{C}_1 \stackrel{\partial^*}{\leftarrow} \mathcal{C}_2 \stackrel{\partial^*}{\leftarrow} \cdots .$$

This complex gives rise to spaces $\mathcal{Z}_k = \ker \, \partial^*$ of chains, $\mathcal{B}_k = \mathrm{im} \, \partial^*$ of borders and cohomologies $\mathcal{H}_k = \mathcal{Z}_k / \mathcal{H}_k$.

On the other hand, $\nabla$ gives rise to a sequence

$$\mathcal{C}_0 \xrightarrow{\nabla} \mathcal{C}_1 \xrightarrow{\mathrm{d}^\nabla} \mathcal{C}_2 \xrightarrow{\mathrm{d}^\nabla} \cdots$$

and that's all we need for introducing the BGG-machinery as deveoloped by [Čap-Slovak-Souček,2001] and [Calderbank-Diemer,2001]:
The main point is that $\mathrm{d}^\nabla$ and $\partial^*$ give rise to canonical differential splitting operators $L_k : \mathcal{H}_k \to \mathcal{Z}_k$ in following way:

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While for a general section $\varphi \in \Omega^k(M, \mathcal{T})$ with $\partial^*(\varphi) = 0$ one need not have that also $\partial^*(\mathrm{d}^\nabla(\varphi)) = 0$, there is in fact a well defined subspace $\mathcal{L}_k$ for which

$$\mathcal{Z}_k \supset \mathcal{L}_k \xrightarrow{\mathrm{d}^\nabla} \mathcal{Z}_{k+1} \subset \mathcal{C}_{k+1}.$$

On $\mathcal{L}_k$ the natural projections $\Pi_k : \mathcal{Z}_k \to \mathcal{H}_k$ restricts to an isomorphism, whose inverse is a (differential) splitting operator $L_k$.

One can thus form the BGG-operators $\Theta_k$ as the composition $\Pi_{k+1} \circ \mathrm{d}^\nabla \circ L_k$:

$$\begin{array}{ccc} \mathcal{L}_k & \xrightarrow{\mathrm{d}^\nabla} & \mathcal{Z}_{k+1} \\ {\scriptstyle L_k}\uparrow & & \downarrow{\scriptstyle \Pi_{k+1}} \\ \mathcal{H}_k & \xrightarrow{\Theta_k} & \mathcal{H}_{k+1} \end{array}$$

and obtains the *BGG-sequence*

$$\mathcal{C}_0 \xrightarrow{\Theta_0} \mathcal{C}_1 \xrightarrow{\Theta_1} \mathcal{C}_2 \xrightarrow{\Theta_2} \cdots .$$

We remark that only in the conformally flat case will this be a complex.

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We remark that only in the conformally flat case will this be a complex. The first operator in this sequence is always to be overdetermined.
**Examples** :
If $\mathcal{T} = \mathcal{S}$, $\Theta_0$ is the operator governing Einstein scales introduced at the beginning.

If $\mathcal{T} = \Lambda^{k+1}\mathcal{T}$ we will obtain the operator governing **conformal Killing forms**

$$\{\sigma \in \mathcal{E}_{[a_1 \cdots a_k]}[k+1] \ :$$

$$D_c \sigma_{a_1 \cdots a_k} - D_{[a_0} \sigma_{a_1 \cdots a_k]} - \frac{k}{n-k+1} \mathbf{g}^{pq} D_p \sigma_{q a_2 \cdots a_k} = 0\}.$$

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With respect to a metric one has

$$[\Lambda^{k+1}\mathcal{T}]_g = \begin{pmatrix} & \mathcal{E}_{[a_1 \cdots a_k]}[k-1] & \\ \mathcal{E}_{[a_1 \cdots a_{k+1}]}[k+1] & \big| & \mathcal{E}_{[a_1 \cdots a_{k-1}]}[k-1] \\ & \mathcal{E}_{[a_1 \cdots a_k]}[k+1] & \end{pmatrix}.$$

via the identification

$$\begin{pmatrix} & \rho_{a_1 \cdots a_k} & \\ \varphi_{a_0 \cdots a_k} & \big| & \mu_{a_2 \cdots a_k} \\ & \sigma_{a_1 \cdots a_k} & \end{pmatrix} \mapsto \tau_- \wedge \sigma + \varphi + \tau_+ \wedge \tau_- \wedge \mu + \tau_+ \wedge \rho. \qquad (9)$$

The tractor connection on $\Lambda^{k+1}\mathcal{T}$ is given by

$$\nabla_c^{\Lambda^{k+1}\mathcal{T}} \begin{pmatrix} \rho_{a_1\cdots a_k} \\ \varphi_{a_0\cdots a_k} \mid \mu_{a_2\cdots a_k} \\ \sigma_{a_1\cdots a_k} \end{pmatrix} = \tag{10}$$

$$= \begin{pmatrix} D_c\rho_{a_1\cdots a_k} - P_c{}^p\varphi_{pa_1\cdots a_k} - kP_{c[a_1}\mu_{a_2\cdots a_k]} \\ \begin{pmatrix} D_c\varphi_{a_0\cdots a_k} + (k+1)\mathbf{g}_{c[a_0}\rho_{a_1\cdots a_k]} \\ +(k+1)P_{c[a_0}\sigma_{a_1\cdots a_k]} \end{pmatrix} \mid \begin{pmatrix} D_c\mu_{a_2\cdots a_k} \\ -P_c{}^p\sigma_{pa_2\cdots a_k} + \rho_{ca_2\cdots a_k} \end{pmatrix} \\ D_c\sigma_{a_1\cdots a_k} - \varphi_{ca_1\cdots a_k} + k\delta_{c[a_1}\mu_{a_2\cdots a_k]}. \end{pmatrix}.$$

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and the first BGG-splitting operator $L_0^{\mathcal{T}} : \mathcal{E}_{[a_1\cdots a_k]}[k+1] \to \Lambda^{k+1}\mathcal{T}$ is

$$L_0^{\mathcal{T}}(\sigma) = \tag{11}$$

$$= \begin{pmatrix} \begin{pmatrix} -\frac{1}{n(k+1)}D^pD_p\sigma_{a_1\cdots a_k} + \frac{k}{n(k+1}D^pD_{[a_1}\sigma_{|p|a_2\cdots a_k]} \\ +\frac{k}{n(n-k+1)}D_{[a_1}D^p\sigma_{|p|a_2\cdots a_k]} \\ +\frac{2k}{n}P^p{}_{[a_1}\sigma_{|p|a_2\cdots a_k]} - \frac{1}{n}J\sigma_{a_1\cdots a_k} \end{pmatrix} \\ D_{[a_0}\sigma_{a_1\cdots a_k]} \mid -\frac{1}{n-k+1}\mathbf{g}^{pq}D_p\sigma_{qa_2\cdots a_k} \\ \sigma_{a_1\cdots a_k} \end{pmatrix}.$$

Then, via (10) and (11) one computes that for $\sigma \in \mathcal{E}_{[a_1 \cdots a_k]}[k+1]$ the projection of $\nabla^{\Lambda^{k+1}\mathcal{T}} \circ L_0^{\Lambda^{k+1}\mathcal{T}}(\sigma)$ to the lowest slot $\mathcal{E}_{c[a_1 \cdots a_k]}[k+1]$ in $\Omega^1(M, \Lambda^{k+1}\mathcal{T})$ is given by

$$\sigma_{a_1 \cdots a_k} \mapsto D_c \sigma_{a_1 \cdots a_k} - D_{[a_0}\sigma_{a_1 \cdots a_k]} - \frac{k}{n-k+1} \mathbf{g}^{pq} D_p \sigma_{q a_2 \cdots a_k}. \qquad (12)$$

This is the projection of $\sigma_{a_1 \cdots a_k}$ to the highest weight part of $\mathcal{E}_{c[a_1 \cdots a_k]}[k+1]$ which is formed by trace-free elements with trivial alternation, we write

$$\mathcal{E}_{\{c[a_1 \cdots a_k]\}_0}[k+1] := \{\sigma_{a_1 \cdots a_k} : 0 = \sigma_{[c a_1 \cdots a_k]} \text{ and } 0 = \mathbf{g}^{c a_1} \sigma_{c a_1 \cdots a_k}\}.$$

We thus in fact have the first BGG-operator

$$\Theta_0^{\Lambda^{k+1}\mathcal{T}} : \mathcal{E}_{[a_1 \cdots a_k]}[k+1] \to \mathcal{E}_{\{c[a_1 \cdots a_k]\}_0}[k+1],$$
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Forms in the kernel of $\Theta_0^{\Lambda^{k+1}\mathcal{T}}$ are thus the *conformal Killing k-forms*.

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Forms in the kernel of $\Theta_0^{\Lambda^{k+1}\mathcal{T}}$ are thus the *conformal Killing k-forms*.

Unlike the case of $k = 0$ and almost Einstein scales this is no longer a prolongation: While $\Pi_0$ is easily seen to projects $\nabla$-parallel sections of $\mathcal{T}$ into $\ker\,\Theta_0$, one only has $\nabla L_0(\sigma)) \in \mathrm{im}\,\partial^*$ for a $\sigma \in \ker\,\Theta_0$ and not necessarily $\nabla L_0(\sigma) = 0$.

# General approach:

Our approach to prolong the overdetermined equations

$$\Theta_0 \sigma \stackrel{!}{=} 0$$

for some first BGG-operator $\Theta_0$ is to modify the tractor connection in a suitable way:
We will find a natural space of deformations of tractor connections such that the BGG-construction still works and yields the *same* underlying operator $\Theta_0$. (All other splitting- and BGG- operators may change).
The deformed connection will describe the kernel of $\Theta_0$ as parallel sections.
Working in this class of connections we will find a unique one describing the underlying system of equations.

# The deformation of the tractor connection

Let $\Psi \in \Omega^1(M, \mathfrak{gl}(\mathcal{T}))^1$. The change in curvature after a deformation

$$\nabla \to \tilde{\nabla} = \nabla + \Psi$$

is given by

$$R_\Psi = R + \mathrm{d}^\nabla \Psi + [\Psi, \Psi]$$

with $R \in \Omega^2(M, \mathfrak{gl}(\mathcal{T}))$ the original curvature of $\nabla^{\mathcal{T}}$.

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The main deformation result is

**Theorem 1**

There exists a unique $\Psi \in \Omega^1(M, \mathfrak{gl}(\mathcal{T}))^1$ such that

- $\Psi s \in \mathrm{im}\ \partial^*$ and
- $\partial^*(R_\Psi s) = 0$

for all $s \in \mathcal{T}$.

# The prolongation connection $\tilde{\nabla} = \nabla + \Psi$

Before proving Theorem 1, let us show how it solves the prolongation problem:
We can now do the BGG-machinery with $\tilde{\nabla}$. Let us first check that this still yields the same first BGG-operator $\Theta_0$ as with $\nabla$:

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We can now do the BGG-machinery with $\tilde{\nabla}$. Let us first check that this still yields the same first BGG-operator $\Theta_0$ as with $\nabla$:

1. Since
$$\tilde{\nabla} \circ L_0 = \nabla \circ L_0 \mod \mathrm{im}\ \partial^*$$
we see $\partial^* \circ \tilde{\nabla} \circ L_0 = 0$, which implies that $L_0$ is the first BGG-splitting operator of $\tilde{\nabla}$.
2. Again, since $\tilde{\nabla} = \nabla \mod \mathrm{im}\ \partial^*$ and $\Pi_1$ kills $\mathrm{im}\ \partial^*$, we have
$$\tilde{\theta}_0 = \Pi_1 \circ \tilde{\nabla} \circ L_0 = \Theta_0,$$
and thus our deformation doesn't change the first BGG-operator.

We now show that the diagram

$$\begin{array}{ccc} \mathrm{im}\ L_0 & \xrightarrow{\tilde{\nabla}} & \mathcal{Z}_1 \\ {\scriptstyle L_0}\uparrow & & \uparrow{\scriptstyle \tilde{L}_1} \\ \mathcal{H}_0 & \xrightarrow{\Theta_0} & \mathcal{H}_1 \end{array}$$

commutes:

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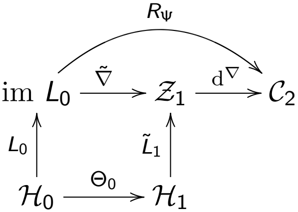
commutes:

1. By definition of $\Theta_0$, $\tilde{\nabla} \circ L_0$ with values in $\mathcal{Z}_1$ lifts $\Theta_0$ over $\Pi_1$. For it to agree with $\tilde{L}_1 \circ \Theta_0$ we thus must have $\partial^* \circ \mathrm{d}^{\tilde{\nabla}} \circ \tilde{\nabla} \circ L_0 = 0$.

We now show that the diagram




commutes:

1. By definition of $\Theta_0$, $\tilde{\nabla} \circ L_0$ with values in $\mathcal{Z}_1$ lifts $\Theta_0$ over $\Pi_1$. For it to agree with $\tilde{L}_1 \circ \Theta_0$ we thus must have $\partial^* \circ \mathrm{d}^{\tilde{\nabla}} \circ \tilde{\nabla} \circ L_0 = 0$.
2. But since $\mathrm{d}^{\tilde{\nabla}} \circ \tilde{\nabla} = R_\Psi$ with $R_\Psi$ the curvature of $\tilde{\nabla}$,

$$\partial^* \circ \mathrm{d}^{\tilde{\nabla}} \circ \tilde{\nabla} \circ L_0 = \partial^* \circ R_\Psi \circ L_0 = 0$$

holds by assumption on $\Psi$.

# The prolongation connection $\tilde{\nabla} = \nabla + \Psi$

Thus we have

**Theorem 2**

There exists a natural connection $\tilde{\nabla}$ on $\mathcal{T}$ such that $\Pi_0$ and $L_0$ restrict to inverse isomorphisms between $\tilde{\nabla}$-parallel sections of $\mathcal{T}$ and the kernel of $\Theta_0$. I.e.: $(\mathcal{T}, \Pi_0, L_0, \tilde{\nabla})$ is a natural geometric prolongation of $\Theta_0$.

To prove Theorem 1 one employs a completely algorithmic inductive procedure: The *failure* of $\nabla + \Psi$ to satisfy the conditions of the theorem is given by

$$\partial^* \circ R_\Psi \in \mathcal{B}_1 \subset \Omega^1(\mathcal{G}, \mathfrak{gl}(\mathcal{T})).$$

Recall that we have a natural filtration of $\mathcal{B}_1$ with $\mathcal{B}^i \supset \mathcal{B}^{i+1}$ and $\mathcal{B}^j = 0$ for some high enough $j$ and assume that we already got a $\Psi \in \Omega^1(M, \mathfrak{gl}(\mathcal{T}))^1$ which achieves that $\partial^* \circ R_\Psi \in \mathcal{B}_1^i$.

Then, for a $\phi$ which also maps $\mathcal{T}$ into $\mathcal{B}_1^i$ we find that

$$\partial^* \circ R_{\Psi+\phi} - \partial^* \circ R_\Psi = \Box \circ \phi \tag{13}$$

modulo terms in $\mathcal{B}_1^{i+1}$. Here $\Box$ denotes the *Kostant Laplacian*: the only important fact for us is that it is **invertible** on $\mathrm{im}\ \partial^* = \mathcal{B}_1 \subset \Omega^1(M, \mathcal{T})$. This tells us to proceed by taking

$$\phi := -\Box \circ \partial^* R_\Psi,$$

then $\partial^* \circ R_{\Psi+\varphi}$ sits in the next higher filtration component, and after finitely many steps we arrive at a solution.

An exposition of this prolongation procedure in the realm of conformal geometry can be found at **math.dg/0811.4122**. There one also finds an explicit step by step calculation for the prolongation connection of conformal Killing forms.

A treatment of the alternative normalization condition $\partial^* \circ R = 0$ for connections on tractor bundles will appear in a joint paper with J. Šilhan, V. Souček and P. Somberg.

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We will now quickly go over some applications:

# Algebraic obstruction tensors for free

Since $\tilde{L}_1 \circ \Theta_0 = \tilde{\nabla} \circ L_0$ one has that the composition of the first two BGG-operators for $\tilde{\nabla}$ is

$$\tilde{\Theta}_1 \circ \Theta_0 = \Pi_2 \circ R_\Psi \circ L_0.$$

Especially, when $\sigma \in \ker \Theta_0$, then necessarily $\Pi_2(R_\Psi(L_0(\sigma))) = 0$.

For instance, for a conformal Killing $k$-form with $k \geq 2$ one obtains without a line of computation that

$$\mathcal{C}_{\{c_1 c_2}{}^{p}{}_{[a_1}\sigma_{|p|a_2 \cdots a_k]\}_0} = 0.$$

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This obstruction has been observed as a side result of calculations done in ad hoc prolongation by Kashiwada (68), Semmelmann (2001) and Gover-Šilhan (2006). This description is completely conceptual.

# Construction of sharp(er) obstructions a la Gover-Nurowski

When one chooses a metric in the conformal class in the conformal case, one obtains the Weyl (resp. Levi-Civita-) connection on $TM$ and $T^*M$ and its tensor powers, and may thus couple these connections with the prolongation connection $\tilde{\nabla}$.

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Then for a $\tilde{\nabla}$-parallel section $s \in \mathcal{T}$ one has $Rs = 0$. Differentiating this one obtains $0 = \tilde{\nabla}(Rs) = (\tilde{\nabla}R)s + R\tilde{\nabla}s = (\tilde{\nabla}R)s$ by parallelity and thus

$$(\tilde{\nabla}^k R)s = 0 \ \forall k \in \mathbb{N}_0.$$

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In the case of the standard tractor bundle of conformal geometry $\tilde{\nabla} = \nabla$ and Gover-Nurowski (2006) obtained sharp obstructions against the existence of Einstein scales under a genericity assumption on the Weyl curvature, using in fact only the equations for $k = 0$ and $k = 1$.