Holonomy reduction over singularity sets in conformal and projective geometry

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Introduction

Given a manifold M carrying a geometric structure we consider a solution σ of an invariant overdetermined system $\Theta(\sigma) = 0$ and assume that it vanishes at some point, $\sigma(x) = 0$.

We then want to understand the zero set $M_0 := \sigma^{-1}(\{0\})$ and the interior $M_- := M \backslash M_0$.

It will be shown for some simple but intersting operators in projective and conformal geometry that the zero set M_0 inherits a geometric structure and M_- is equipped with a canonical connection.

The approach taken here is to regard the holonomy reduction provided by the (prolonged) solution σ, which yields different but related structures on the zero set M_0 and the interor M_-.

Some questions:

Two questions in Riemannian (or conformal) geometry:

- Let g be a pseudo-Riemannian metric on a manifold M. Can we rescale g conformally to $\hat{g} = fg$ for some positive function f such that $\hat{g}$ is Einstein,

$$\mathrm{Ric}(\hat{g}) = \lambda \hat{g} \ ?$$

- If M is even-dimensional, can one rescale a Riemannian metric g conformally to a Kähler metric?

Some questions:

Two questions in affine (or projective) geometry:

- If ∇ is an affine torsion-free connection on M, is it metrizable? I.e, can one describe its geodesics by an Riemannian metric?
- Does the affine connection allow a projectively equivalent Ricci-flat connection?

Prolongation

The above questions yield overdetermined PDEs $\Theta(\sigma) = 0$ on a function or field σ.

An overdetermined system is said to be brought into closed form, or it is said to be prolonged, if it is written in an equivalent form $\nabla s = 0$, where $s = (\sigma, \mu, \cdots, \rho)$ is an extension of σ by new variables which encode partial finite jet information of σ, and ∇ is a linear connection that incorporates differential consequences of the original equation $\Theta(\sigma) = 0$.

The equivalent encoding as a closed system allows us to globalize (jet-)information on the solution, which will be used to study its behaviour and its singularities.

Some typical easy consequences of the prolonged form:

- If σ is non-trivial, s is nowhere vanishing. Sometimes already this is enough for regularity of the singularity set.
- Since s is determined by a finite jet of σ at a point x, any non-trivial solution σ of $\Theta(\sigma) = 0$ must already be non-vanishing on an open-dense subset.
- Sometimes we have a priori knowledge on $\mathrm{Hol}(\nabla)$, like that it preserves a metric. This yields global invariants of σ and its derivatives which only depend on these data at one point.
- In some cases the prolongation connection ∇ is directly induced from the structural data of the underlying geometry. Then a solution of $\Theta(\sigma) = 0$ in fact yields a holonomy reduction of the geometry.

Example 1: Einstein rescalings of a (pseudo)-Riem. metric

The Schouten tensor $\mathrm{P}(g)$ of a metric g is a linear combination of $\mathrm{Ric}(g)$ and g - so g is Einstein iff $\mathrm{Ric}(g)$ or equivalently $\mathrm{P}(g)$ is trace-free.
If one rewrites the rescaled metric as $\hat{g} = \sigma^{-2} g$ for a $\sigma \in \mathrm{C}^\infty(M, \mathbb{R}_+)$ the explicit transformation law $\mathrm{P}(g) \rightsquigarrow \mathrm{P}(\hat{g})$ yields that $\mathrm{P}(\hat{g})$ is trace-free iff

$$\Theta(\sigma) := \mathbf{tf}(D^g D^g \sigma + \mathrm{P}(g)) = 0.$$

To rewrite $\Theta(\sigma) = 0$ in closed form we introduce new variables $\mu = D^g \sigma$ and $\rho = \frac{1}{n}(\triangle^g - J^g)\sigma$, where $\triangle^g = -\mathrm{tr}^g \circ D^g \circ D^g$ and $J^g = \mathrm{tr}(\mathrm{P}(g))$.

Then

$$\Theta(\sigma) = 0 \text{ iff } \nabla \begin{pmatrix} \rho \\ \mu \\ \sigma \end{pmatrix} = \begin{pmatrix} D\rho - \mathrm{P}^g(\cdot, \mu) \\ D\mu + \sigma P + \rho g \\ D\sigma - \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Example 1: Einstein rescalings of a (pseudo)-Riem. metric

The prolongation connection ∇ preserves the bilinear-form **h** given by the (quadratic) formula $\begin{pmatrix}\rho\\ \mu\\ \sigma\end{pmatrix} \mapsto 2\sigma\rho + g(\mu,\mu)$.

In particular, for σ, μ, ρ corresponding to a solution of $\Theta(\sigma) = 0$, the expression $2\sigma\rho + g(\mu,\mu) \in C^\infty(M)$ is necessarily constant equal α, which shall be non-zero for our discussion.

If $\sigma(x) = 0$, then $g_x(D\sigma(x), D\sigma(x)) \neq 0$ and we see that $D\sigma$ is non-vanishing along M_0, which shows that M_0 is a hypersurface in M.

Moreover, $T_xM_0 = D\sigma(x)^\perp \subset T_xM$ and depending on whether $\alpha = g(D\sigma(x), D\sigma(x))$ is greater or smaller zero, M_0 inherits a signature $(p-1, q)$ resp. $(p, q-1)$-metric.

Example 2: Projectively equivalent Ricci-flat connections

Let D be a torsion-free affine connection on the n-manifold M and assume that D induces the flat connection on $\Lambda^n T^*M$, which is equivalent to $\mathrm{Ric}(D)$ being symmetric.

We ask whether there is a projectively equivalent connection $\hat{D}$,

$$\hat{D}_a\varphi_b = D_a\varphi_b - \Upsilon_a\varphi_b - \Upsilon_b\phi_a$$

for some $\Upsilon \in \Omega^1(M)$ with

$$\mathrm{Ric}(\hat{D}) = 0.$$

If we restrict to exact $\Upsilon = D\log(\frac{1}{|\sigma|})$ one obtains that $\mathrm{Ric}(\hat{D}) = 0$ iff

$$\Theta(\sigma) = DD\sigma + \frac{1}{n-1}\mathrm{Ric}(D)\sigma = 0.$$

Example 2: Projectively equivalent Ricci-flat connections

The prolongation of this equation introduces a new variable $\varphi = D\sigma$, and then

$$\Theta(\sigma) = 0 \text{ iff } \nabla \begin{pmatrix} \varphi \\ \sigma \end{pmatrix} = \begin{pmatrix} D\varphi + \frac{1}{n-1}\operatorname{Ric}(D)\sigma \\ D\sigma - \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If we allow a non-trivial solution σ of $\Theta(\sigma) = 0$ to have a zero at $x \in M$, then necessarily $D\sigma(x) \neq 0$. In particular $M_0 = \sigma^{-1}(\{0\})$ is a hypersurface in M with $T_x M_0 = (D\sigma(x))^o \subset T_x M$.

Example 2: Projectively equivalent Ricci-flat connections

If $\xi, \eta \in \mathfrak{X}(M_0)$ are vector fields along M_0, then $0 = \phi(\eta)$ along M_0, and then also

$$0 = (D_\xi \phi)(\eta(x)) + \phi(D_\xi(\eta)).$$

But $D_\xi(\phi)(\eta(x)) = -\frac{1}{n-1} \operatorname{Ric}(D)(\xi, \eta)\sigma(x) = 0$, and therefore $0 = \phi(D_\xi(\eta)) = (D_\xi \eta) \cdot \sigma$ and $D_\xi \eta \in \mathfrak{X}(M_0)$, which says that D restricts to a connection on M_0.

Let now $[D]$ be a projective class of connections, i.e., a class of torsion-free affine connections with the same geodesics as unparameterized curves. Then we see that a solution of $\Theta(\sigma) = 0$ provides a canonical Ricci-flat connection on M_- and the singularity hypersurface M_0 is totally geodesic in $(M, [D])$.

Example 3: The twistor spinor equation in split signature

Let g be a split signature (n, n) metric on the $2n$-manifold M endowed with a spin structure and corresponding positive and negative (real 2^{n-1}-dimensional) Spin bundles S_+, S_-.

A positive twistor spinor is a solution $\chi \in \Gamma(S_+)$ of

$$\Theta(\chi) = D\chi + \frac{1}{2n}\gamma \not{D}\chi = 0,$$

where γ is Clifford multiplication and $\not{D}$ is the Dirac operator.

The prolongation connection ∇ of this system lives on $S_+ \oplus S_-$, acts on $(\chi, \not{D}\chi)$ and has

$$\mathrm{Hol}(\nabla) \subset \mathrm{Spin}(n+1, n+1).$$

Example 3: The twistor spinor equation in split signature

Assume that $\chi(x) = 0$ for some $x \in M$ and that $\not{D}\chi(x)$ is pure, i.e., the kernel of $\not{D}\chi(x)$ is maximal isotropic in T_xM.

Then, since $\mathrm{Hol}(\nabla) \subset \mathrm{Spin}(n+1, n+1)$, $\not{D}\chi(x)$ is also pure at any other $x \in M_0 = \chi^{-1}(\{0\})$.
And since $D\chi(x) = -\frac{1}{2n}\gamma\not{D}\chi$ this implies that $D\chi$ has constant rank along M_0, which is therefore a submanifold.

A more detailed analysis of the $\mathrm{Spin}(n+1, n+1)$-orbit of $(0, \tau)$, pure τ and some more work gives:

Proposition

1. *The zero set $M_0 = \chi^{-1}(\{0\})$ is a totally geodesic maximally isotropic submanifold of M.*
2. *χ is pure on the open dense complement $M_- = M \backslash M_0$ and determines a foliation of M_- by maximally isotropic leafs.*

We now summarize what happened in these examples and then switch to a new viewpoint:

- We had an overdetermined system $\sigma, \Theta(\sigma) = 0$.
- Introducing new variables for some derivatives of σ one can write down an equivalent closed system $s, \nabla s = 0$.
- Parellelicity of s implies that information on σ and its derivatives globalizes in some way.

Another interpretation: Holonomy reduction for Cartan geometries

Projective, conformal and conformal spin structures can be equivalently encoded as Cartan geometries.

A Cartan geometry has a type (G, P), with G a Lie group and $P \subset G$ a closed subgroup.

A Cartan geometry of type (G, P) on a manifold M is given by a P-principal bundle $\mathcal{G}$ over M endowed with a 1-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ satisfying some properties making it into a Cartan connection form.

We are interested in cases where the overdetermined system $\Theta(\sigma) = 0$ is canonically associated to an irreducible G-representation V via the BGG-construction of [Čap-Slovak-Souček, 2001].

Holonomy reduction for Cartan geometries

It is not easy to describe the operator Θ induced by the G-representation V quickly.

Also, for every system $\Theta(\sigma) = 0$ of BGG-type the prolongation connection ∇ that rewrites this system into $\nabla s = 0$ exists [H.-Šilhan-Somberg-Souček], but in most cases it is a complicated object.

We restrict our discussion here to some very special cases, which include the three examples above, where it is very easy to describe the prolongation connection ∇ of this system: It is a linear connection induced by the Cartan connection form:

Holonomy reduction for Cartan geometries

The Cartan connection form is not a principal connection form, but we can form the extended bundle $\hat{\mathcal{G}} := \mathcal{G} \times_P G$ and extend ω canonically to a G-principal connection form $\hat{\omega}$.

Then, given a G-representation V, we form the associated bundle $\mathbf{V} = \mathcal{G} \times_P V$, which carries the linear connection ∇ induced by $\hat{\omega}$ since we can also write $\mathbf{V} = \hat{\mathcal{G}} \times_G V$.

Then for the examples discussed above parallel sections $s \in \Gamma(\mathbf{V}), \nabla s = 0$ correspond to solutions of the overdetermined system $\Theta(\sigma) = 0$, which therefore reduce the holonomy of $\hat{\omega}$.

Holonomy reduction for Cartan geometries

While it is clear that one has an holonomy reduction of the extended bundle, this doesn't give a global reduction of the underlying Cartan geometry.

In fact, we cannot expect to obtain a global reduction over all of M, since we saw in the behaviour exhibited above that we should obtain different Cartan geometries over different subsets of M.

In fact, it turns out that the holonomy reduction does yield Cartan reductions over the correct submanifolds of M.

Holonomy reduction for Cartan geometries

Let $\hat{s} : \hat{\mathcal{G}} \to V$ be the G-equivariant map corresponding to the section $s \in \Gamma(\mathbf{V})$ and take $u \in \mathcal{G} \in \hat{\mathcal{G}}$ with $\hat{s}(u) = v$. Then one has the the holonomy reduction $\mathcal{H} \subset \hat{\mathcal{G}}$ through u: $\hat{s}$ restricts to the constant function $v \in V$ on $\mathcal{H}$.

Then $\hat{\mathcal{G}} = \mathcal{H} \cdot G$, which implies $\hat{s}(\hat{\mathcal{G}}) = G \cdot v$. We say that σ resp. s has G-orbit type $G \cdot v$.

The different behaviour of σ on different subsets of M stems from the fact that the single G-orbit $G \cdot v$ decomposes into a disjoint union on P-orbits $G \cdot v = \cup_i P \cdot v_i$ for some $v_i \in G \cdot v$.

For $x \in M$ and $\mathcal{G}_x$ the fiber in P-principal bundle $\mathcal{G}$ over x one has $\hat{s}(\mathcal{G}_x) = P \cdot v_i$ for some i and we say that σ has P-orbit type $P \cdot v_i$ at x.

Holonomy reduction for Cartan geometries

We define $M_i \subset M$ as the set of all x with $\sigma(x)$ having orbit type $P \cdot v_i$.

The reduction now proceeds in four steps:

1. Show that M_i are submanifolds of M.
2. With $M_i \overset{j}{\hookrightarrow} M$, show that $j^*\mathcal{G}$ and $j^*\mathcal{H}$ intersect in a P_{v_i}-principal bundle $\mathcal{G}_i$ over M_i.
3. Show that that pullback of the Cartan connection form $j^*\omega$ restricts to a Cartan connection form on $\mathcal{G}_i$ of type (G_{v_i}, P_{v_i}).
4. Show that the reduced connection satisfies special curvature properties due to the normalization condition on the original curvature.

- In [Example 1, Conformal Einstein] from above this method yields
  - M_0 inherits normal parabolic geometry describing an induced conformal structure.
  - M_- inherits a torsion-free reductive Cartan geometry describing the distinguished (pseudo)-Riemannian metric. The normalization condition implies that this is an Einstein metric.
- In [Example 2, Projective Ricci-flat] one obtains that
  - M_0 is a totally geodesic hypersurface and the restricted Cartan connection form is normal.
  - M_- inherits a Ricci-flat torsion-free connection.
- In [Example 3, Twistor spinor with $\chi(x) = 0$, $\not{D}\chi(x)$ pure] one has
  - M_0 is a maximally isotropic totally geodesic submanifold and the restricted Cartan connection form is normal.
  - On M_- one has that χ is pure and induces a foliation by integrable maximally isotropic distributions.

Reductions of the homogenous model

A Cartan geometry has its homogeneous model G/P with maximal symmetry group G. We can always use this model case to understand what kind of reductions a parallel section $s \in \Gamma(\mathbf{V})$ provides.

The prolongation connection ∇ of the system $\Theta(\sigma) = 0$ lives on the bundle $\mathbf{V} = G \times_P V$: it is the flat connection with respect to the trivialization $(gP, v) \mapsto [g, g^{-1}v] \in \mathbf{V}$. In particular, the homogeneous model allows also the maximal number of solutions to $\Theta(\sigma) = 0$.

Given an element $v \in V$, we now study the corresponding parallel section of type $G \cdot v$, which is given by the equivariant map $g \mapsto g^{-1}v$.

We have that

$$M_i := \{x = gP \in G/P : \; g^{-1} \cdot v \in P \cdot v_i\}$$

is G_{v_i}-homogeneous and $M_i = G_{v_i}/P_{v_i}$.

Example: The reduction of $\mathbb{RP}^n$

The homogeneous model of projective structures in dimension n is

$$M = G/P = \mathrm{SL}(n+1)/P = \mathbb{RP}^n,$$

with P the stabilizer of some line $\mathbb{R}e_+$ in $\mathbb{R}^{n+1}$.

The overdetermined system $\sigma, \in \mathrm{C}^\infty(M)$,

$$DD\sigma + P\sigma = 0$$

is associated to the dual standard representation $(\mathbb{R}^{n+1})^*$.

Example: The reduction of $\mathbb{RP}^n$

There is only one G-orbit type $G \cdot v$, $0 \neq v \in (\mathbb{R}^{n+1})^*$, but

$$G \cdot v = P \cdot v_0 \cup P \cdot v_-,$$

with $(e_+, v_0) = 0$, $(e_+, v_-) = 1$.

Then $G_{v_-} \cong G_{v_0} \cong \mathrm{SL}(n) \ltimes \mathbb{R}^n$. We have $P_{v_-} = \mathrm{SL}(n)$, and thus the geometry over M_- is reduced to a (torsion-free) geometry of type $(\mathrm{SL}(n) \ltimes \mathbb{R}^n, \mathrm{SL}(n))$. $M_- \hookrightarrow \mathbb{RP}^n$ as flat affine space $\mathbb{R}^n$.

Let be $\bar{P}$ the stabilizer of the line through $e_+ \in v_0^o \cong \mathbb{R}^n$. in $\mathrm{SL}(n) \subset \mathcal{G}_{v_0}$. Then $P_{v_0} = \bar{P} \ltimes \mathbb{R}^n$, and the geometry over M_0 is reduced to $(\mathrm{SL}(n) \ltimes \mathbb{R}^n, \bar{P} \ltimes \mathbb{R}^n)$, which factorizes to a geometry of type $(\mathrm{SL}(n), \bar{P})$: We have $M_0 \hookrightarrow \mathbb{RP}^n$ as $\mathbb{RP}^{n-1}$.