Geometric overdetermined differential equations and holonomy

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Introduction

Given a manifold M carrying a geometric structure we consider a *solution* σ *of an invariant overdetermined system* $\Theta(\sigma) = 0$ and assume that it vanishes at some point, $\sigma(x) = 0$.

We then want to understand the *zero set* $M_0 := \sigma^{-1}(\{0\})$ and the *interior* $M_- := M \backslash M_0$.

It will be shown for some simple but interesting operators in projective and conformal geometry that the zero set M_0 inherits a geometric structure and M_- is equipped with a canonical connection.

The approach taken here is to regard the *holonomy reduction provided by the (prolonged) solution* σ, which yields different but related structures on the zero set M_0 and the interor M_-.

Specific examples of overdetermined problems:

Conformally invariant overdetermined problems:

- Let g be a Riemannian metric on a manifold M. Can we rescale g conformally to $\hat{g} = fg$ for some positive function f such that $\hat{g}$ is *Einstein*,

$$\mathrm{Ric}(\hat{g}) = \lambda \hat{g} \ ?$$

- If M is even-dimensional, can one rescale a Riemannian metric g conformally to a *Kähler* metric?

Projectively invariant overdetermined problems:

Two torsion-free affine connections ∇ and $\hat{\nabla}$ on TM are *projectively equivalent* if they have the *same unparameterized geodesics*. For given affine connection ∇ one thus obtains a *projective structure* $[\nabla]$.

- If ∇ is an affine torsion-free connection on M, is it *metrizable*? I.e, can one describe its geodesics by an Riemannian metric?
- Does the affine connection allow a projectively equivalent *Ricci-flat* connection?

Prolongation

The above questions yield overdetermined PDEs $\Theta(\sigma) = 0$ on a function or field σ.

An overdetermined system is said to be brought into *closed* form, or it is said to be *prolonged*, if it is written in an equivalent form $\nabla s = 0$, where $s = (\sigma, \mu, \cdots, \rho)$ is an extension of σ by new variables which encode partial finite jet information of σ, and ∇ is a linear connection that incorporates *differential consequences* of the original equation $\Theta(\sigma) = 0$.

The equivalent encoding as a closed system allows us to *globalize (jet-)information* on the solution, which will be used to study its behaviour and its singularities.

Some typical easy consequences of the prolonged form:

- If σ is non-trivial, *s is nowhere vanishing*. Sometimes already this is enough for regularity of the singularity set.
- Since s is determined by a finite jet of σ at a point x, any non-trivial solution σ of $\Theta(\sigma) = 0$ must already be *non-vanishing on an open-dense subset*.
- Sometimes we have a priori knowledge on $\mathrm{Hol}(\nabla)$, like that it preserves a metric. This yields *global invariants of σ and its derivatives* which only depend on these data at one point.
- In some cases the prolongation connection ∇ is directly induced from the structural data of the underlying geometry. Then a solution of $\Theta(\sigma) = 0$ in fact yields a *holonomy reduction of the geometry*.

Example: Einstein rescalings of a (pseudo)-Riem. metric

The *Schouten tensor* $\mathsf{P}(g)$ of a metric g is a linear combination of $\mathrm{Ric}(g)$ and g - so g is *Einstein iff* $\mathrm{Ric}(g)$ or equivalently $\mathsf{P}(g)$ *is trace-free*.
If one rewrites the rescaled metric as $\hat{g} = \sigma^{-2} g$ for a $\sigma \in \mathrm{C}^\infty(M, \mathbb{R}_+)$ the explicit transformation law $\mathsf{P}(g) \rightsquigarrow \mathsf{P}(\hat{g})$ yields that $\mathsf{P}(\hat{g})$ is trace-free iff

$$\Theta(\sigma) := \mathbf{tf}(D^g D^g \sigma + \mathsf{P}(g)) = 0.$$

To rewrite $\Theta(\sigma) = 0$ in closed form we introduce *new variables* $\mu = D^g \sigma$ and $\rho = \frac{1}{n}(\triangle^g - J^g)\sigma$, where $\triangle^g = -\mathrm{tr}^g \circ D^g \circ D^g$ and $J^g = \mathrm{tr}(\mathsf{P}(g))$.

Then

$$\Theta(\sigma) = 0 \text{ iff } \nabla \begin{pmatrix} \rho \\ \mu \\ \sigma \end{pmatrix} = \begin{pmatrix} D\rho - \mathsf{P}^g(\cdot, \mu) \\ D\mu + \sigma P + \rho g \\ D\sigma - \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Example: Einstein rescalings of a (pseudo)-Riem. metric

The prolongation connection ∇ *preserves* the *bilinear-form* **h** given by the (quadratic) formula $\begin{pmatrix} \rho \\ \mu \\ \sigma \end{pmatrix} \mapsto 2\sigma\rho + g(\mu, \mu)$.

In particular, for σ, μ, ρ corresponding to a solution of $\Theta(\sigma) = 0$, the expression $2\sigma\rho + g(\mu, \mu) \in \mathrm{C}^\infty(M)$ *is necessarily constant equal* α, which shall be non-zero for our discussion.

If $\sigma(x) = 0$, then $g_x(D\sigma(x), D\sigma(x)) \neq 0$ and we see that $D\sigma$ is non-vanishing along M_0, which shows that M_0 *is a hypersurface in* M.

Moreover, $T_x M_0 = D\sigma(x)^\perp \subset T_x M$ and depending on whether $\alpha = g(D\sigma(x), D\sigma(x))$ is greater or smaller zero, M_0 *inherits a signature* $(p-1, q)$ *resp.* $(p, q-1)$*-metric*.

Let us observe what happened in the example and then switch to a new viewpoint:

- We had an overdetermined system $\sigma, \Theta(\sigma) = 0$.
- Introducing new variables for some derivatives of σ one can write down an equivalent closed system $s, \nabla s = 0$.
- Parellelicity of s implies that information on σ and its derivatives globalizes in some way.

A different viewpoint: Cartan geometries

Projective, conformal and conformal spin structures can be *equivalently encoded as Cartan geometries*.

The model for a Cartan geoometry is a *Klein geometry*, which is a pair (G, P) with G a Lie group and $P \subset G$ a closed subgroup. G is then regarded as the *automorphism group* of the homogeneous space G/P.

A general Cartan geometry on a manifold M is modelled on a homogeneous space G/P but is 'curved' and has 'less/no' symmetries:

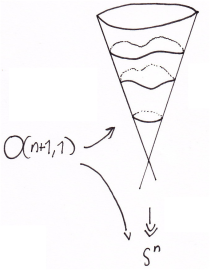
Klein geometry	**Cartan geometry**
$G \longrightarrow G$	$\mathrm{Aut} \longrightarrow \mathcal{G}$
$\downarrow P$	$\downarrow P$
G/P	M

Klein-model-spaces and associated Cartan geometries

• Euclidean geometry can be understood as the study of the invariants under the action of the Euclidean group $\mathrm{Euc}(n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$ on Euclidean space $\mathbb{E}^n \cong \mathbb{R}^n \cong \mathrm{Euc}(n)/\mathrm{O}(n)$.
$\rightsquigarrow$ n-dimensional Riemannian structure

• Projective space $\mathbb{P}^n$ can be described as $\mathrm{SL}(n+1)/P$, with $G = \mathrm{SL}(n+1)$ acting transitively on the space of lines in $\mathbb{R}^{n+1}$ and $P \subset \mathrm{SL}(n+1)$ the stabilizer of a line.
$\rightsquigarrow$ n-dimensional projective structure

• The *conformal n-sphere* is realized by regarding the (transitive) action of the Lorentz-group $G = \mathrm{SO}(n+1,1)$ on the light-cone $\mathcal{C} \subset \mathbb{R}^{n+1,1}$ and having P the stabilizer of some $\mathbb{R}_+$-ray in this cone.
$\rightsquigarrow$ n-dimensional conformal



Definition of a Cartan geometry

Definition

A Cartan geometry of type (G, P) on a manifold M is a P-principal bundle $\mathcal{G} \to M$ endowed with a Cartan connection form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$.

- ω is P-equivariant
- ω reproduces fundamental vector fields
- ω trivialization $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$.

The Cartan geometry is *reductive* if $\mathfrak{g}$ decomposes into $\mathfrak{p} \ltimes \mathfrak{n}$ as a P-module, i.e., $\mathfrak{p}$ has a P-invariant complement in $\mathfrak{g}$. In that case the Cartan connection form ω decomposes into

- a *Soldering form* $\theta \in \Omega^1(\mathcal{G}, \mathfrak{n})$ giving an identification $TM = \mathcal{G} \times_P \mathfrak{n}$,
- a *P-principal connection for* $\gamma \in \Omega^1(\mathcal{G}, \mathfrak{p})$, providing in particular a linear connection on TM.

Tractor bundles and holonomy for Cartan connections

A general (non-reductive) Cartan geometry $(\mathcal{G}, \omega)$ of type (G, P) does not induce connections on P-associated bundles, but the Cartan bundle naturally extends to a G-principal bundle $(\hat{G}, \hat{\omega})$:

$$\begin{array}{ccc} \mathcal{G} & \hookrightarrow & \hat{\mathcal{G}} \\ \downarrow P & & \downarrow G \\ M & \xrightarrow{id} & M \end{array}$$

This allows us to define the holonomy of the Cartan connection form ω via the extended principal connection form:

$$\mathrm{Hol}(\omega) := \mathrm{Hol}(\hat{\omega}) \subset G.$$

For any G-representation V we can form the associated *tractor bundle* $\mathcal{V} = \mathcal{G} \times_P V$ endowed with its induced tractor connection ∇^V.

Orbit decompositions and holonomy reductions

Flat situation	**Curved situation**
$G \longrightarrow G$	$\mathrm{Aut} \longrightarrow \mathcal{G}$
$\downarrow P$	$\downarrow P$
G/P	M

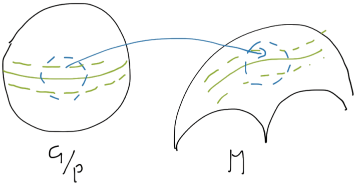
- While G acts transitively on G/P, one may regard the H-orbit decomposition of G/P for any given subgroup $H \subseteq G$.
- Does there exist a generalisation of such H-orbits in the curved situation?

Theorem (A. Čap, A.R. Gover and M. H., 2014)

For a given curved structure M modelled on G/P with holonomy group reduced to $H \subseteq G$ there exists a natural $1:1$ correspondence

$$\{H\text{−orbits in } G/P\} \leftrightarrow \{\text{curved orbits in } M\}.$$

This yields the curved orbit decomposition of M, and each orbit carries itself a natural Cartan geometry.



Applications:

- Classification of conformal holonomy (J. Alt, A. Di Scala and T. Leistner, 2014)
- Differential geometric compactifications
- Close relationships with overdetermined systems of PDEs

Relationship with BGG-solutions

For every G-representation V one has a naturally associated overdetermined differential operator, the *first BGG-operator* $\Theta_0 : \mathcal{H}_0 \to \mathcal{H}_1$, which defines the *first BGG-equation* $\Theta_0(\sigma) = 0$ [Čap-Slovak-Souček 2001, Calderbank-Diemer 2000].

$$\begin{array}{ccc} \nabla^{\mathcal{V}} - \text{parallel sections} \subset \Gamma(\mathcal{V}) & & \textit{prolonged system} \\ \Pi_0 \downarrow \quad \uparrow L_0 & & \\ \text{normal solutions} \subset \ker \Theta_0 & & \textit{overdetermined system} \end{array}$$

- General solutions of $\Theta_0(\sigma) = 0$ correspond to sections of $\mathcal{V}$ which are parallel with respect to a (modified) *prolongation connection* [H.-Somberg-Souček-Šilhan 2012], and therefore don't induce holonomy reductions of the parabolic structure.
- *Normal solutions* (following [Leitner, 2005]) of $\Theta_0(\sigma) = 0$ are those which correspond to parallel sections of $\mathcal{V}$. In particular, normal solutions are equivalent to holonomy reductions.

Examples of first BGG-operators for conformal structures

- If one takes the standard tractor bundle $\mathbf{S}$ of a conformal structure $(M, [g])$ one obtains the operator governing Einstein rescalings discussed in the first example.
- If $(M, [g])$ is a conformal spin structure with spin bundle $\mathbf{\Delta}$ and Clifford symbol $\gamma \in \Gamma(T^*M \otimes \mathrm{End}(\mathbf{\Delta}))$, one also has a spin tractor bundle $\mathbf{\Sigma}$. Let $\not{D} : \Gamma(\mathbf{\Delta}) \to \Gamma(\mathbf{\Delta})$ be the Dirac operator.
  The first BGG-operator of $\mathbf{\Sigma}$ is the *twistor operator*

$$\Gamma(\mathbf{\Delta}) \to \Gamma(T^*M \otimes \mathbf{\Delta}),$$
$$\chi \mapsto D\chi + \frac{1}{n}\gamma \otimes \not{D}\chi.$$

  Solutions of this equation are known as *twistor spinors*.
- Both cases are very special: parallel sections of the tractor connection are already in 1:1-correspondence with solutions, which reflects the fact that the modelling representations are still very simple.

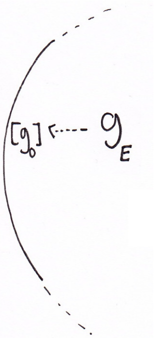
Conformal Holonomy Characterizations

- $\mathrm{SU}(p+1, q+1) \hookrightarrow \mathrm{SO}(2p+2, 2q+2)$:
  CR-structure $\rightsquigarrow$ signature $(2p+1, 2q+1)$-conformal structure on S^1-bundle
  $+$ lightlike conformal Killing field
  [Fefferman 1976, Graham 1987, Čap-Gover 2010]
- $G_2 \hookrightarrow \mathrm{Spin}(3, 4)$:
  generic rank 2-distribution on 5-manifold $\rightsquigarrow$
  signature $(2, 3)$-conformal spin structure
  $+$ generic twistor spinor
  [H.-Sagerschnig 2011]
- $\mathrm{SL}(3) \hookrightarrow \mathrm{Spin}(3, 3)$:
  projective 2-dimensional structure $\rightsquigarrow$
  split signature $(2, 2)$-conformal spin structure
  $+$ 2 compatible twistor spinors
  [H.-Tagavi–Chabert-Žádník-Šilhan-Sagerschnig, 2016]

Poincare-Einstein manifolds

Let $(M, [g])$ with $[g] = \{fg \mid f \in \mathrm{C}^\infty(M, \mathbb{R}_{>0})\}$ be an n-dimensional Riemannian signature conformal structure.

- For a reduction to $\mathrm{SO}(n+1) \hookrightarrow \mathrm{SO}(n+1,1)$ we obtain a single curved orbit of type $\mathrm{SO}(n+1)/\mathrm{SO}(n)$, which carries an Einstein metric with positive scalar curvature.
- For a reduction to $\mathrm{SO}(n,1) \hookrightarrow \mathrm{SO}(n+1,1)$ we obtain an open curved orbit of type $\mathrm{SO}(n,1)/\mathrm{SO}(n)$, which carries an Einstein metric with negative scalar curvature and a closed curved orbit of type $\mathrm{SO}(n,1)/\bar{P}$ which carries a conformal structure.



We recover a description of **almost Einstein structures** [Gover, 2010]: The reduction $\mathrm{SO}(n,1) \hookrightarrow \mathrm{SO}(n+1,1)$ yields examples of *Poincaré-Einstein* manifolds with an Einstein metric on an interior part of a manifold and a *conformal structure at infinity*.

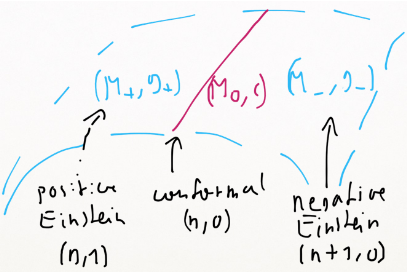
Klein-Einstein manifolds

Let M be an $n+1$-dimensional manifold endowed with a projective structure $\mathbf{p}$ whose *normal projective tractor connection* $\nabla^{\mathcal{T}}$ preserves a signature $(n+1,1)$ *tractor metric* $\mathbf{g}$. Equivalently, the *projective holonomy* is reduced to

$$\mathrm{Hol}(\mathbf{p}) \subseteq \mathrm{SO}(n+1,1) \subseteq \mathrm{SL}(n+2).$$

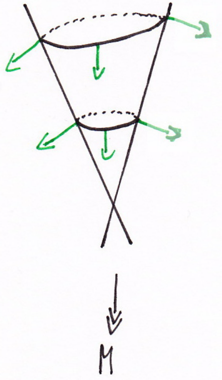
One obtains a decomposition of M as a *Klein-Einstein* manifold:

- $(M_0, \mathbf{c})$ is a conformal n-dimensional space.
- (M_+, g_+) an Einstein metric with $\mathrm{Ric}(g_+) = n g_+$.
- (M_-, g_-) an Einstein metric with $\mathrm{Ric}(g_-) = -n g_-$.
- $(M_0, \mathbf{c})$ is the *projective infinity* of (M_-, g_-).



Another viewpoint: Fefferman-Graham ambient metrics

A conformal structure
$[g]$ can be understood as the ray-subbundle $\mathcal{C} \subset S^2 T^* M$
which consists of all metrics in the given conformal class.

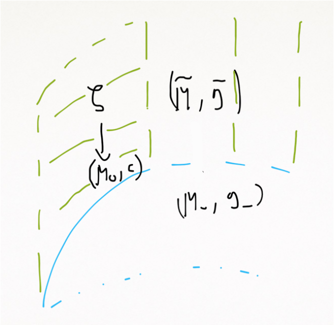


The *Fefferman-Graham ambient metric*
$\tilde{g}$ is a *signature* $(n+1,1)$ *metric*
on $n+2$*-dimensional ambient space* $\tilde{M} = \mathcal{C} \times (-1,1)$
and extends a tautological (degenerate) form $\mathbf{g}_0$ on $\mathcal{C}$.

- $n = p + q$ odd:
  $\tilde{g}$ is uniquely determined as an infinite order jet along $\mathcal{C}$ by the normalization condition that $\mathrm{Ric}(\tilde{g})$ vanishes to infinite order along $\mathcal{C}$.
- $n = p + q$ even: $\tilde{g}$ is unique up addition of terms of order higher than $\frac{n}{2}$ under the normalization condition that $\mathrm{Ric}(\tilde{g})$ vanishes to order $\frac{n}{2} - 2$ along $\mathcal{C}$ and to order $\frac{n}{2} - 1$ in tangential directions along $\mathcal{C}$.

BGG-solutions and parallel ambient fields

According to recent work (Čap-Gover-Graham-H. 2016) ambient holonomy equals conformal holonomy: $\mathrm{Hol}(\mathbf{c}) = \mathrm{Hol}(\widetilde{\nabla})$.



To be precise, this holds for infinitesimal holonomy, and literally in the simply-connected, real-analytic, odd dimensional situation.

In particular, one gets correspondences between solutions to overdetermined equations and parallel ambient objects:

$\bar{V}$	$\widetilde{M}$	M_0	M_-
$\mathbb{R}^{n+2}$	Parallel field	Einstein metric $\overline{g}_E$ in $\mathbf{c}$	...
$\Delta^{n+1,1}$	Parallel spinor	twistor spinor $\overline{\chi}$	Killing spinor χ