Invariant Prolongation of Overdetermined Systems arising for Parabolic Geometries

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Parabolic geometries give rise to many interesting overdetermined geometric equations and in this talk we are going to describe a general prolongation procedure yielding a natural connection whose parallel sections are in 1:1-correspondence with the solutions of the prolonged system.

We will begin by giving one example in conformal geometry and another one for projective structures.

Example in conformal geometry: (almost) Einstein scales

A conformal structure on a manifold M is an equivalence class $[g]$ of pseudo-Riemannian metrics, where two metrics g and $\hat{g}$ are equivalent iff there is a function $f \in \mathrm{C}^\infty(M)$ such that $\hat{g} = e^{2f} g$.

A nice example of a conformally invariant operator is

$$\Theta^g : \mathrm{C}^\infty(M) \to S^2_0 T^* M, \tag{1}$$

$$\sigma \mapsto \textbf{trace-free part}\big(DD\sigma + \sigma P\big). \tag{2}$$

Here D is the Levi-Civita connection of a metric g in the conformal class and $P = P_{ab}$ is the Schouten-tensor, which is a trace-modification of the Ricci tensor. $S^2_0 T^* M$ denotes symmetric, trace-free bilinear forms on TM.

Θ^g describes the equation governing Einstein scales: for $\sigma \in \mathrm{C}^\infty(M), \sigma > 0$ one has $\Theta^g \sigma = 0$ iff $\sigma^{-2} g$ is Einstein.

The operator Θ^g is conformally covariant between $\mathrm{C}^\infty(M)$ and $S^2_0 T^* M$: if one switches to another metric $\hat{g} = e^{2f} g$ in the conformal class, then

$$\Theta^{\hat{g}} \circ m(e^f) = m(e^f) \circ \Theta^g,$$

where $m(e^f)$ is simply the multiplication operator with e^f. This yields a conformally invariant operator between the weighted bundles $\mathcal{H}_0 = \mathcal{E}[1]$ and $\mathcal{H}_1 = S^2_0 T^* M \otimes \mathcal{E}[1]$.

Metrization of projective structures (Eastwood-Matveev, Bryant-Dunajski-Eastwood)

An example for an interesting equation appearing for a projective geometry $(M, [\nabla])$ is the equation **trace-free part**$(\nabla\sigma) = 0$, for $\sigma \in S^2TM$.

This gives a projectively invariant operator

$$\Theta : \mathcal{E}^{(ab)}[-2] \to \mathcal{E}_c{}^{(ab)}[-2]$$

between (projectively) weighted spaces.

Assume that there is a ∇ in the projective class with symmetric Ricci tensor. Then:
Solutions σ of this equation which are positive definite correspond to metrics whose Levi-Civita connection sits in the projective class of metrics $[\nabla]$.

The above equations can be described by first BGG-operators, which will be introduced/recalled below. So one could hope that this uniform description of these overdetermined systems can be used to obtain a general prolongation method.

Given an overdetermined operator $\Theta : \mathcal{H}_0 \to \mathcal{H}_1$, what we are looking for is a *geometric prolongation*: We want an extension

$$\Pi : \mathcal{V} \to \mathcal{H}_0$$

of $\mathcal{H}_0$, a (differential) splitting

$$L : \mathcal{H}_0 \to \mathcal{V}$$

of Π and a linear connection ∇ on $\mathcal{V}$ such that Π and L restrict to inverse isomorphisms between ∇-parallel sections of $\mathcal{V}$ and the kernel of Θ.

We will also call the tuple $(\mathcal{V}, \Pi, L, \nabla)$ a geometric prolongation of Θ.

A general method yielding a natural prolongation connection. The Setting.

We will work with a *parabolic geometry* of type (G, P), G a semisimple Lie group and P a parabolic subgroup:

The geometric structure on the manifold M is encoded in a P-principal bundle $\mathcal{G}$ over M endowed with a Cartan connection form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$.

The curvature $\kappa \in \Omega^2(\mathcal{G}, \mathfrak{g})$, given by $\kappa(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)]$, satisfies *regularity* and and *normality* conditions for achieving equivalence with an underlying geometric structure on M.

The solution to the prolongation problem of BGG-operators which we are going to describe will work for arbitrary regular parabolic geometries and will be natural resp. invariant.

First, some background: basic structure on tractor bundles

For a G-representation V the associated bundle $\mathcal{V} = G \times_P V$ is called a *tractor bundle*.
Tractor bundles make themselves so welcome by carrying, in contrast to mere assocated P-representations, canonical linear connections:

The Cartan connection form ω can be extended G-equivariantly to a principal connection on the extended bundle $\mathcal{G}' = \mathcal{G} \times_P G$ and thus the associated bundle $\mathcal{V} = G' \times_G V = G \times_P V$ is endowed with a linear connection, which is denoted by ∇ and called the *tractor connection* on $\mathcal{V}$.
∇ gives rise to a sequence

$$\mathcal{C}_0 \xrightarrow{\nabla} \mathcal{C}_1 \xrightarrow{\mathrm{d}^\nabla} \mathcal{C}_2 \xrightarrow{\mathrm{d}^\nabla} \cdots$$

on the chain spaces $\mathcal{C}_k = \Omega^k(M, \mathcal{V})$.

On the other hand, one has the (algebraic) Kostant co-differential $\partial^* : \mathcal{C}_{k+1} \to \mathcal{C}_k$ which gives the complex

$$\mathcal{C}_0 \stackrel{\partial^*}{\leftarrow} \mathcal{C}_1 \stackrel{\partial^*}{\leftarrow} \mathcal{C}_2 \stackrel{\partial^*}{\leftarrow} \cdots .$$

This complex gives rise to spaces $\mathcal{Z}_k = \ker \partial^*$ of chains, $\mathcal{B}_k = \mathrm{im}\ \partial^*$ of borders and homologies $\mathcal{H}_k = \mathcal{Z}_k / \mathcal{H}_k$.

d^∇ and ∂^* are strongly related: tractor bundles $\mathcal{V}$ and their chain spaces $\mathcal{C}_k$ carry natural filtrations $\cdots \mathcal{C}_k^i \supset \mathcal{C}_k^{i+1} \cdots$. Regularity of the parabolic geometry is then equivalent to the fact that d^∇ always induces a certain canonical Lie algebra differential

$$\partial : \mathrm{gr}(\mathcal{C}_k) \to \mathrm{gr}(\mathcal{C}_{k+1})$$

on the associated graded spaces. We will also simply say that d^∇ and ∂ coincide in lowest homogeneity.

That's all one needs for doing BGG:

Now the BGG-machinery starts by observing that d^∇ and ∂^* give rise to canonical splittings $L_k : \mathcal{H}_k \to \mathcal{Z}_k$: While $\mathcal{Z}_k$ is mapped by d^∇ into $\mathcal{C}_k$ one has a well defined subspace $\mathcal{L}_k$ for which

$$\mathcal{Z}_k \supset \mathcal{L}_k \overset{\mathrm{d}^\nabla}{\to} \mathcal{Z}_{k+1} \subset \mathcal{C}_{k+1}.$$

On $\mathcal{L}_k$ the natural projections $\Pi_k : \mathcal{Z}_k \to \mathcal{H}_k$ restricts to an isomorphism, whose inverse is a (differential) splitting operator L_k. One can thus form the BGG-operators Θ_k as the composition $\Pi_{k+1} \circ \mathrm{d}^\nabla \circ L_k$:

$$\begin{array}{ccc} \mathcal{L}_k & \xrightarrow{\mathrm{d}^\nabla} & \mathcal{Z}_{k+1} \\ {\scriptstyle L_k}\uparrow & & \downarrow{\scriptstyle \Pi_{k+1}} \\ \mathcal{H}_k & \xrightarrow{\Theta_k} & \mathcal{H}_{k+1} \end{array}$$

Example: (almost) Einstein cales:

Conformal structures of signature (p, q) are modelled on Cartan geometries of type $(SO(p+1, q+1), P)$ and the standard representation of $SO(p+1, q+1)$ on $\mathbb{R}^{p+q+2}$ this gives rise to the standard tractor bundle $\mathcal{S} := \mathcal{G} \times_P \mathbb{R}^{p+q+2}$ of conformal geometry.

With respect to a metric g in the conformal class, which corresponds to a Weyl structure, the tractor bundle decomposes into

$$[\mathcal{S}]_g = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$$

and one writes elements $[s]_g = \sigma \oplus \varphi_a \oplus \rho \in [\mathcal{S}]_g$ as

$$[s]_g = \begin{pmatrix} \rho \\ \varphi_a \\ \sigma \end{pmatrix}.$$

The tractor connection is explicitly given by

$$\nabla_c s = \nabla_c \begin{pmatrix} \rho \\ \varphi_a \\ \sigma \end{pmatrix} = \begin{pmatrix} D_c\rho - P_c{}^b\varphi_b \\ D_c\varphi_a + \sigma P_{ca} + \rho g_{ca} \\ D_c\sigma - \varphi_c \end{pmatrix},$$

and the splitting operator L_0 is

$$\sigma \in \mathcal{E}[1] \mapsto \begin{pmatrix} -\frac{1}{n}(\triangle\sigma + P_a{}^a\sigma) \\ \nabla\sigma \\ \sigma \end{pmatrix}.$$

Composition of ∇ with L_0 and computing the first homology $\mathcal{H}_1$ to be $\mathcal{S}_0^2 T^*M[1]$ yields

$$\begin{aligned} \Theta_0 : \mathcal{E}[1] &\to S_0^2 T^*M[1], \\ \sigma &\mapsto \textbf{trace-free part}\big(DD\sigma + \sigma P\big), \end{aligned}$$

whose kernel consists of (almost) Einstein scales of $[g]$.

Commutativity of the first BGG-diagram

In this case the *first BGG-diagram*

$$\begin{array}{ccc} \mathrm{im}\ L_0 & \xrightarrow{\nabla} & \mathcal{Z}_1 \\ {\scriptstyle L_0}\uparrow & & \uparrow{\scriptstyle L_1} \\ \mathcal{H}_0 & \xrightarrow{\Theta_0} & \mathcal{H}_1 \end{array}$$

commutes, which says that L_0 and Π_0 restrict to inverse isomorphisms between almost Einstein scales and ∇-parallel sections of $\mathcal{S}$. Otherwise put: $(\mathcal{S}, \Pi_0, L_0, \nabla)$ is a geometric prolongation of Θ_0.

However, for a general, more complicated representation V of G and its tractor bundle $\mathcal{V}$, the associated first BGG-diagram will fail to commute. The problem is that while Π_0 is easily seen to projects ∇-parallel sections of $\mathcal{V}$ into $\ker\, \Theta_0$, one only has $\nabla L_0(\sigma)) \in \mathrm{im}\, \partial^*$ for a $\sigma \in \ker\, \Theta_0$ and not necessarily $\nabla L_0(\sigma) = 0$.

The solution:

We are going to deform ∇ to a connection $\tilde{\nabla} = \nabla + \Psi$ which will satisfy a natural condition that will imply commutativity of the first BGG-diagram and solve the prolongation problem.

The deformation of the tractor connection

We will consider deformations $\Psi \in \Omega^1(M, \mathfrak{gl}(\mathcal{V}))^1$. The homogeneity 1-condition exactly means that $\tilde{\nabla} = \nabla + \Psi$ still coincides with ∂ in lowest homogeneity, which is necessary and sufficient for doing the BGG-constructions with $\tilde{\nabla}$.

Let R_Ψ be the curvature of $\tilde{\nabla} = \nabla + \Psi$. The main result is

Theorem 1

There exists a unique $\Psi \in \Omega^1(M, \mathfrak{gl}(\mathcal{V}))^1$ such that

- $\Psi s \in \mathrm{im}\ \partial^*$ and
- $\partial^*(R_\Psi s) = 0$

for all $s \in \mathcal{V}$.

A completely algorithmic inductive procedure:

The *failure* of $\nabla + \Psi$ to satisfy the conditions of the theorem is given by

$$\partial^* \circ R_\Psi \in \mathcal{B}_1 \subset \Omega^1(\mathcal{G}, \mathfrak{gl}(\mathcal{V})).$$

Recall that we have a natural filtration of $\mathcal{B}_1$ with $\mathcal{B}^i \supset \mathcal{B}^{i+1}$ and $\mathcal{B}^j = 0$ for some high enough j and assume that we already got a $\Psi \in \Omega^1(M, \mathfrak{gl}(\mathcal{V}))^1$ which achieves that $\partial^* \circ R_\Psi \in \mathcal{B}_1^i$.

Then, for a ϕ which also maps $\mathcal{V}$ into $\mathcal{B}_1^i$ we find that

$$\partial^* \circ R_{\Psi+\phi} - \partial^* \circ R_\Psi = \Box \circ \phi \tag{3}$$

modulo terms in $\mathcal{B}_1^{i+1}$. Here $\Box$ denotes the *Kostant Laplacian*: the only important fact for us is that it is **invertible** on $\mathrm{im}\,\partial^* = \mathcal{B}_1 \subset \Omega^1(M, \mathcal{V})$.

This tells us to proceed by taking

$$\phi := -\Box \circ \partial^* R_\Psi,$$

then $\partial^* \circ R_{\Psi+\varphi}$ sits in the next higher filtration component, and after finitely many steps we arrive at a solution.

An exposition of this prolongation procedure in the realm of conformal geometry can be found at **math.dg/0811.4122**. There one also finds an explicit step by step calculation for the prolongation connection of conformal Killing forms.

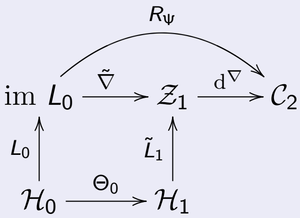
A general treatment of the normalization condition $\partial^* \circ R = 0$ will appear in a joint paper with J. Šilhan, V. Souček and P. Somberg.

The prolongation connection $\tilde{\nabla} = \nabla + \Psi$

We can now do the BGG-machinery with $\tilde{\nabla}$. Let us first check that this still yields the same first BGG-operator Θ_0 as with ∇:

1. Since
$$\tilde{\nabla} \circ L_0 = \nabla \circ L_0 \mod \mathrm{im}\, \partial^*$$
we see $\partial^* \circ \tilde{\nabla} \circ L_0 = 0$, which implies that L_0 is the first BGG-splitting operator of $\tilde{\nabla}$.
2. Again, since $\tilde{\nabla} = \nabla \mod \mathrm{im}\, \partial^*$ and Π_1 kills $\mathrm{im}\, \partial^*$, we have
$$\tilde{\theta}_0 = \Pi_1 \circ \tilde{\nabla} \circ L_0 = \Theta_0,$$
and thus our deformation doesn't change the first BGG-operator.

We now show that the diagram



commutes:

1. By definition of Θ_0, $\tilde\nabla \circ L_0$ with values in $\mathcal{Z}_1$ lifts Θ_0 over Π_1. For it to agree with $\tilde L_1 \circ \Theta_0$ we thus must have $\partial^* \circ \mathrm{d}^{\tilde\nabla} \circ \tilde\nabla \circ L_0 = 0$.
2. But since $\mathrm{d}^{\tilde\nabla} \circ \tilde\nabla = R_\Psi$ with R_Ψ the curvature of $\tilde\nabla$,

$$\partial^* \circ \mathrm{d}^{\tilde\nabla} \circ \tilde\nabla \circ L_0 = \partial^* \circ R_\Psi \circ L_0 = 0$$

holds by assumption on Ψ.

The prolongation connection $\tilde{\nabla} = \nabla + \Psi$

Thus we have

Theorem 2

There exists a natural connection $\tilde{\nabla}$ on $\mathcal{V}$ such that Π_0 and L_0 restrict to inverse isomorphisms between $\tilde{\nabla}$-parallel sections of $\mathcal{V}$ and the kernel of Θ_0. I.e.: $(\mathcal{V}, \Pi_0, L_0, \tilde{\nabla})$ is a natural geometric prolongation of Θ_0.

We will now quickly go over some applications:

Some algebraic obstruction tensors for free

Since $\tilde{L}_1 \circ \Theta_0 = \tilde{\nabla} \circ L_0$ one has that the composition of the first two BGG-operators for $\tilde{\nabla}$ is

$$\tilde{\Theta}_1 \circ \Theta_0 = \Pi_2 \circ R_\Psi \circ L_0.$$

Especially, when $\sigma \in \ker \Theta_0$, then necessarily $\Pi_2(R_\Psi(L_0(\sigma))) = 0$.

If the geometry is 1-graded and one knows that $\mathcal{H}_2$ is concentrated in lowest homogeneity, then this latter term turns out to be the projection of the action of the (generalized) Weyl curvature on σ to the highest weight part.

For instance, for a conformal Killing k-form with $k \geq 2$ one obtains without a line of computation that

$$\mathcal{C}_{\{c_1 c_2}{}^{p}{}_{[a_1}\sigma_{|p|a_2 \cdots a_k]\}_0} = 0.$$

This obstruction has been observed as a side result of calculations done in ad hoc prolongation by Kashiwada (68), Semmelmann (2001) and Gover-Šilhan (2006). This description is completely conceptual.

Similarly one sees that twistor spinors are killed by the Weyl curvature.

For the projective example above one gets that

$$(W_{c_1 c_2}{}^{a}{}_{p}\sigma^{bp} + W_{c_1 c_2}{}^{b}{}_{p}\sigma^{ap})_0$$

must vanish for a solution σ^{ab} describing a metrization of the projective class.

Construction of sharp(er) obstructions a la Gover-Nurowski

When one chooses a Weyl structure, for instance when one chooses a metric in the conformal class in the conformal case, one obtains the Weyl (resp. Levi-Civita-) connection on TM and T^*M and its tensor powers, and may thus couple these connections with the prolongation connection $\tilde{\nabla}$.

Then for a $\tilde{\nabla}$-parallel section $s \in \mathcal{V}$ one has $Rs = 0$. Differentiating this one obtains $0 = \tilde{\nabla}(Rs) = (\tilde{\nabla} R)s + R\tilde{\nabla} s = (\tilde{\nabla} R)s$ by parallelity and thus

$$(\tilde{\nabla}^k R)s = 0 \; \forall k \in \mathbb{N}_0.$$

In the case of the standard tractor bundle of conformal geometry $\tilde{\nabla} = \nabla$ and Gover-Nurowski (2006) obtained sharp obstructions against the existence of Einstein scales under a genericity assumption on the Weyl curvature, using in fact only the equations for $k = 0$ and $k = 1$.