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A non-normal Fefferman-type construction of split-signature conformal structures admitting twistor spinors

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The Fefferman-construction

The original Fefferman construction [Fefferman,'76] canonically associated a conformal structure on a circle bundle over a CR-structure. It was shown by Sparling and discussed by [Graham, '87] that a conformal structure is the Fefferman-space of some CR-structure if and only if it admits a light-like conformal Killing field which also satisfies additional (conformally invariant) properties.

The characterizing property can alternatively be understood as a *holonomy reduction* of the conformal structure: It was shown in [Čap-Gover, '10] that a conformal structure $(M, \mathcal{C})$ is locally the Fefferman-space of a CR-structure if and only if its conformal holonomy satisfies
$\mathrm{Hol}(\mathcal{C}) \subset \mathrm{SU}(p+1, q+1) \subset \mathrm{SO}(2p+2, 2q+2)$.

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Generalization: Fefferman-type constructions

A generalization of the original Fefferman-construction was described in [Čap, '05], and in recent years a number of constructions have been discussed in that framework:

- The original construction was treated by [Čap-Gover, '10]
- A construction of [Biquard, '00] of conformal structures from quaternionic contact structures was treated by [Alt, '10]
- Nurowski's conformal structures that are associated to generic rank 2 distributions on 5-manifolds and Bryant's [Bryant, '06] conformal structures associated to generic rank 3 distributions on 6-manifolds were discussed in [H.-Sagerschnig, '10, '11]

In all cited cases the Fefferman-type construction is *normal*, which allows one to derive a holonomy-based characterization of the induced structures.

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A non-normal construction

Here we discuss a (generically) *non-normal Fefferman-type construction*. We associate a split signature (n, n) conformal spin structure to a projective structure of dimension n.

The original motivation for this Fefferman-type construction was work by [Dunajski-Tod, '10]:

Extending a construction due to [Walker, '54], which associates a pseudo-Riemannian split signature (n, n)-metric to an affine torsion-free connection on an *n*-manifold, they associate a conformal split signature (n, n)-metric to a projective class of torsion-free affine connections on an *n*-manifold. Using a normal form for the induced metrics it is also shown that they admit a twistor spinor. For $n = 2$ this construction was also observed in work by [Nurowski-Sparling, '03].

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Parabolic geometries and Fefferman-type constructions

Parabolic geometries are Cartan geometries of type (G, P), with P a parabolic subgroup of a Lie group G: A parabolic geometry of the given type an a manifold M is described by a principal P-bundle $\mathcal{G} \to M$ that is endowed with a Cartan connection form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{p})$.

Parabolic geometries allow uniform regularity and normality conditions, and if these conditions are satisfied, the parabolic structure is an equivalently description of an underlying geometric structure, like projective, conformal or CR-structures.

A Fefferman-type construction [Čap, '05] is a natural procedure that starts with a parabolic geometry of a type (G, P) on a manifold M and associates a parabolic geometry of another type $(\tilde{G}, \tilde{P})$ on a (possibly larger) manifold $\tilde{M}$.

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Parabolic geometries and Fefferman-type constructions

The algebraic premise for this construction is an inclusion of Lie groups $i : G \hookrightarrow \tilde{G}$ with the property that $Q := G \cap \tilde{P} \subset P$. The Fefferman-type construction $(G, P) \rightsquigarrow (\tilde{G}, \tilde{P})$ is then possible if G acts locally transitive on the homogeneous space $\tilde{G}/\tilde{P}$.

The first step is to form the correspondence space $\tilde{M} := \mathcal{G}/Q = \mathcal{G} \times_P P/Q$. $\mathcal{G} \to \tilde{M}$ is then a Q-principal bundle endowed with the Cartan connection form ω of type (G, Q).

The second step is to form the extended Cartan bundle $\tilde{\mathcal{G}} = \mathcal{G} \times_Q \tilde{P}$ and canonically extend ω to a Cartan connection form $\tilde{\omega}$ on $\tilde{\mathcal{G}}$. Then $(\tilde{\mathcal{G}}, \tilde{\omega})$ is a Cartan geometry of type $(\tilde{G}, \tilde{P})$ on $\tilde{M}$.

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Normality and holonomy

The Fefferman-type construction $(G, P) \rightsquigarrow (\tilde{G}, \tilde{P})$ is called *normal* if normality of ω automatically implies normality of $\tilde{\omega}$. This has immediate strong consequences, which are best visible by holonomy methods:

To form the *holonomy* $\mathrm{Hol}(\omega)$ of a parabolic geometry $(\mathcal{G}, \omega)$, one extends $\mathcal{G}$ to a principal G-bundle $\hat{G} = \mathcal{G} \times_P G$ and canonically extends ω to the a principal connection form $\hat{\omega}$ on $\hat{G}$. Then $\mathrm{Hol}(\omega) := \mathrm{Hol}(\hat{\omega})$.

It immediately follows from the Fefferman-type construction that $\mathrm{Hol}(\tilde{\omega}) = \mathrm{Hol}(\omega)$, and if the construction is normal, $\mathrm{Hol}(\tilde{\omega})$ is the well-defined holonomy of the parabolic geometry on $\tilde{M}$.

In particular, this implies that if ω is non-flat, $(\tilde{\mathcal{G}}, \tilde{\omega})$ is a non-flat parabolic geometry on $\tilde{M}$ with holonomy contained in $G \subset \tilde{G}$.

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Induced solutions of BGG-equations

In many cases the inclusion $G \hookrightarrow \tilde{G}$ is realized as the stabilizer of an element in a $\tilde{G}$-representation V. It is well known that the *tractor bundle* $\mathcal{V} = \tilde{\mathcal{G}} \times_{\tilde{P}} V$ carries the *tractor connection* ∇ that is naturally induced from the Cartan connection form $\tilde{\omega}$. Then $\text{Hol}(\tilde{\omega}) \subset G$ is equivalent to the existence of a parallel section $s \in \Gamma(\mathcal{V})$ of a suitable type.

By the general theory of BGG-operators on parabolic geometries as developed by [Čap-Slovak-Souček, '01], such a parallel section s is equivalent to a normal solution of the first BGG-operator $\Theta_0 : \mathcal{H}_0 \to \mathcal{H}_1$ associated to $\mathcal{V}$.

This $1 : 1$-correspondence is realized by a natural (tensorial) projection $\Pi_0 : \mathcal{V} \to \mathcal{H}_0$ and the *first BGG-splitting operator* $L_0 : \Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{V})$ of $\mathcal{V}$.

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Examples of normal Fefferman-type constructions of conformal structures

- $\mathrm{SU}(p+1, q+1) \hookrightarrow \mathrm{SO}(2p+2, 2q+2)$:
  CR-structure $\rightsquigarrow$
  signature $(2p+1, 2q+1)$-conformal structure on S^1-bundle
  + lightlike conformal Killing field (with additional properties)

- $\mathrm{Sp}(n+1, 1) \hookrightarrow \mathrm{SO}(4n+4, 4)$:
  quaternionic contact structure $\rightsquigarrow$
  signature $(4n+3, 3)$ conformal structure
  + 2 orthogonal lightlike conformal Killing fields

- $G_2 \hookrightarrow \mathrm{Spin}(3, 4)$:
  generic rank 2-distribution on 5-manifold $\rightsquigarrow$
  signature $(2, 3)$-conformal spin structure + generic twistor
  spinor

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$\mathrm{SL}(n+1) \hookrightarrow \mathsf{Spin}(n+1, n+1)$

This Fefferman-type construction is based on an inclusion
$\mathrm{SL}(n+1) \hookrightarrow \mathsf{Spin}(n+1, n+1)$:

Denote by $\Delta = \Delta_+^{n+1,n+1} \oplus \Delta_-^{n+1,n+1}$ the real 2^{n+1}-dimensional
spin representation of $\tilde{G} = \mathsf{Spin}(n+1, n+1)$. Then we fix two
pure spinors $s_F \in \Delta_-^{n+1,n+1}$, $s_E \in \Delta_\pm^{n+1,n+1}$ with non-trivial pairing
- here s_E lies in $\Delta_+^{n+1,n+1}$ if n is even or $\Delta_-^{n+1,n+1}$ if n is odd.

These assumptions guarantee that the kernels $E, F \subset \mathbb{R}^{n+1,n+1}$ of
s_E, s_F with respect to Clifford multiplication are complementary
maximally isotropic subspaces.

Then
$G := \{g \in \mathsf{Spin}(n+1, n+1) : g \cdot s_E = s_E, g \cdot s_F = s_F\} \cong \mathrm{SL}(n+1)$,
defines an embedding $G = \mathrm{SL}(n+1) \overset{i}{\hookrightarrow} \mathsf{Spin}(n+1, n+1)$.

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Fefferman-space $\tilde{M}$ and induced structure

One computes $\tilde{M} = \mathcal{G} \times_Q P/Q \cong (T^*M \otimes \mathcal{E}[2])/\{0\}$. Here we use the notation $\mathcal{E}[w]$ for suitably weighted (projective) version of the density bundle.

The invariant spinors s_E and s_F give rise to pure spin tractors:

The spin tractor bundle of $(M, \mathcal{C})$ is $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$, where $\mathcal{S}_\pm = \tilde{\mathcal{G}} \times_{\tilde{P}} \Delta_\pm^{n+1,n+1} = \mathcal{G} \times_Q \Delta_\pm^{n+1,n+1}$. Since $s_E \in \Delta_\pm^{n+1,n+1}$ and $s_F \in \Delta_-^{n+1,n+1}$ are Q-invariant, they induce canonical sections $\mathbf{s}_E \in \Gamma(\mathcal{S}_\pm)$ and $\mathbf{s}_F \in \Gamma(\mathcal{S}_-)$.

The conformal Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \mathfrak{g})$ induces a tractor connection ∇ on each conformal tractor bundle; the spin tractors $\mathbf{s}_E, \mathbf{s}_F$ are parallel with respect to the induced tractor connections on the respective spin tractor bundles. But these are not necessarily the normal conformal tractor connection!

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Normality of the induced conformal Cartan connection

Proposition

For $n = 2$ the Fefferman-type construction $\mathrm{SL}(3) \hookrightarrow \mathrm{Spin}(3,3)$ is normal. For $n \geq 3$ The conformal Cartan connection form $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ induced by the normal projective Cartan connection form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is normal if and only ω is flat, in which case also $\tilde{\omega}$ is flat.

Outline of the argument:

The normalization condition on a conformal structure automatically implies that it is also torsion-free, i.e., that $\tilde{\kappa} : \tilde{\mathcal{G}} \to \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \mathfrak{p}$ has values in $\tilde{\mathfrak{p}} \subset \tilde{\mathfrak{g}}$. If the Fefferman-type construction $\mathrm{SL}(n+1) \hookrightarrow \mathrm{Spin}(n+1, n+1)$ is normal, this forces the curvature of the projective structure κ to have values in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{q}$, since $\mathfrak{q} = \mathfrak{g} \cap \tilde{\mathfrak{p}}$. Now one has to treat two separate cases:

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Normality

In the case where $n = 2$ the projective curvature consists only of the projective Cotton tensor, and has values in $\mathfrak{p}_+ \subset \mathfrak{q}$. Then a detailed discussion of the normalization condition indeed implies normality of the Fefferman-type construction.

However when $n \geq 3$, the projective curvature is uniquely determined by the projective Weyl tensor, and this has values in a P-module larger than $\mathfrak{q}$. But then, if the curvature κ doesn't vanish, it immediately follows from equivariancy-properties that κ has values outside of $\mathfrak{q} \subset \mathfrak{p}$.

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The normal case $n = 2$:

Proposition

- The *conformal holonomy* $\mathrm{Hol}(\tilde{\omega})$ *is contained in* $\mathrm{SL}(3)$.

- The normal conformal tractor connection $\nabla^{\tilde{\mathcal{T}}, nor}$ preserves the decomposition $\tilde{\mathcal{T}} = \tilde{\mathcal{E}} \oplus \tilde{\mathcal{F}}$.

- The adjoint tractor **K** is parallel with respect to the normal tractor connection, i.e. $\nabla \mathbf{K} = 0$. Thus **K** corresponds to a *normal conformal Killing field* $k \in \mathfrak{X}(\tilde{M})$.

- The spin tractor bundle has two sections $\mathbf{s}_E$ and $\mathbf{s}_F$ with non-trivial pairing that are parallel with respect to the normal tractor connection, i.e. $\nabla^{\mathcal{S}_+, nor} \mathbf{s}_E = 0$ and $\nabla^{\mathcal{S}_-, nor} \mathbf{s}_F = 0$. Thus they correspond to *two pure twistor spinors* $\chi_E \in \Gamma(\mathbf{S}_+[\frac{1}{2}])$ *and* $\chi_F \in \Gamma(\mathbf{S}_-[\frac{1}{2}])$.

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The non-normal case $n \geq 3$:

Since for $n \geq 3$ the induced Cartan connection form $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ is not already the normal conformal connection form, one needs a modification $\Psi \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{p}})$ with the property that $\tilde{\omega}^{nor} = \tilde{\omega} + \Psi$ is normal, i.e., the curvature function $\tilde{\kappa}^{nor}$ of the modified Cartan connection form has to lie in the kernel of the Kostant co-differential $\tilde{\partial}^* : \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$.

The inductive normalization procedure that is necessary for a full computation of the modification Ψ makes it difficult to obtain an explicit formula for this map. It turns out however that certain properties of Ψ can be obtained without an explicit form. In particular, the normalized connection can be shown to still preserve one of the pure tractor spinors:

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The non-normal case $n \geq 3$:

Theorem

$\mathbf{s}_F \in \Gamma(\mathcal{S}_-)$ is parallel with respect to the normal conformal spin tractor connection $\nabla^{\mathcal{S}_-,nor}\mathbf{s}_F = 0$. In particular, the conformal spin structure $(M, \mathcal{C})$ carries a canonical *pure twistor spinor* $\chi_F \in \Gamma(\mathbf{S}_-[\frac{1}{2}])$.

In a addition, the proof shows that while the adjoint tractor $\mathbf{K} \in \mathcal{A}\tilde{M}$ is no longer parallel with respect to the normal adjoint tractor connection, it is still the BGG-splitting of a conformal Killing field $k \in \mathfrak{X}(\tilde{M})$. Moreover, $\mathbf{K}^2 = \mathbb{I}$, and the -1–Eigenspace of $\mathbf{K}$ is precisely the kernel of $\mathbf{s}_F$.

The pair (χ_F, k) characterizes the Fefferman-type spaces of $(\mathrm{SL}(n+1) \hookrightarrow \mathrm{Spin}(n+1, n+1)$ algebraically.

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Complete characterization

Theorem

A split signature (n, n) conformal structure $\mathcal{C}$ on a manifold $\tilde{M}$ is induced by an n-dimensional projective structure if and only if $\mathcal{C}$ admits a pair of a pure twistor spinor χ_F and a conformal Killing field k with the compatibilty properties stated above and satisfies the integrability condition

$$W_{abcd}\xi^a\eta^c = 0 \text{ for all vertical fields } \xi, \eta \in \mathfrak{X}(\tilde{M}). \qquad (I)$$

If integrability condition (I) is satisfied, one can see directly how to factorize $\mathcal{C}$ to a projective structure: Let $g \in \mathcal{C}$ be some metric in the conformal class and D its Levi-Civita covariant derivative. Denote by $M = \tilde{M}/\ker\chi_F$ the leaf space of the totally isotropic (and integrable) n-dimensional kernel of χ_F. Then D induces a canonical projective class of affine torsion-free connections on M.