A non-normal Fefferman-type construction: Holonomy-methods and characterization

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The Fefferman-construction

The original Fefferman construction [Fefferman,'76] canonically associated a conformal structure on a circle bundle over a CR-structure. It was shown by Sparling and discussed by [Graham, '87] that a conformal structure is the Fefferman-space of some CR-structure if and only if it admits a light-like conformal Killing field which also satisfies additional (conformally invariant) properties.

The characterizing property can alternatively be understood as a *holonomy reduction* of the conformal structure: It was shown in [Čap-Gover, '10] that a conformal structure (M, c) is locally the Fefferman-space of a CR-structure if and only if its conformal holonomy satisfies $\mathrm{Hol}(c) \subset \mathrm{SU}(p+1, q+1) \subset \mathrm{SO}(2p+2, 2q+2)$.

Generalization: Fefferman-type constructions

A generalization of the original Fefferman-construction was described in [Čap, '05], and in recent years a number of constructions have been discussed in that framework:

- The original construction was treated by [Čap-Gover, '10]
- A construction of [Biquard, '00] of conformal structures from quaternionic contact structures was treated by [Alt, '10]
- Nurowski's conformal structures that are associated to generic rank 2 distributions on 5-manifolds and Bryant's [Bryant, '06] conformal structures associated to generic rank 3 distributions on 6-manifolds were discussed as Fefferman-type constructions in [H.-Sagerschnig, '10, '11]

In all cited cases the Fefferman-type construction is *normal*, which allows one to derive a holonomy-based characterization of the induced structures.

A non-normal construction

In this talk we discuss a (generically) *non-normal Fefferman-type construction*. We associate a split signature (n, n) conformal spin structure to a projective structure of dimension n.

The original motivation for this Fefferman-type construction was work by [Dunajski-Tod, '10]:

Extending a construction due to [Walker, '54], which associates a pseudo-Riemannian split signature (n, n)-metric to an affine torsion-free connection on an n-manifold, they associate a conformal split signature (n, n)-metric to a projective class of torsion-free affine connections on an n-manifold. Using a normal form for the induced metrics it is also shown that they admit a twistor spinor. For $n = 2$ this construction was also observed in work by [Nurowski-Sparling, '03].

Parabolic geometries

Parabolic geometries are Cartan geometries of type (G, P), with P a parabolic subgroup of a Lie group G: A parabolic geometry of the given type an a manifold M is described by a principal P-bundle $\mathcal{G} \to M$ that is endowed with a Cartan connection form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{p})$.

$$\begin{array}{ccc} (\mathcal{G}, \omega) & \longleftarrow & P \\ \downarrow & & \\ M & & \end{array}$$

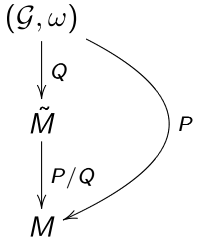
Parabolic geometries allow uniform regularity and normality conditions, and if these conditions are satisfied, the parabolic structure is an equivalently description of an underlying geometric structure, like projective, conformal or CR-structures.

Different subgroups of G: Correspondence spaces

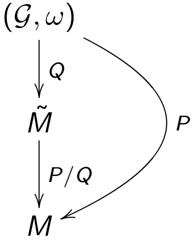
When $Q \subset P$ is a closed subgroup, one may also regard a given Cartan geometry $(\mathcal{G}, \omega)$ of type (G, P) on M as a Cartan geometry of type (G, Q) on

$$\tilde{M} := \mathcal{G}/Q = \mathcal{G} \times_P P/Q :$$

$\mathcal{G} \to \tilde{M}$ is then a Q-principal bundle endowed with the Cartan connection form ω of type (G, Q).



Conversely, for a given geometry $\mathcal{G} \to \tilde{M}$ of type (G, Q) one may ask whether it is in fact the correspondence space of a geometry of type (G, P).



Answer: [Čap, 2005]: Denote with $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ the curvature form.

- When $i_v i'_v \tilde{\Omega} = 0$ for all fields $v, v' \in \omega^{-1}(\mathfrak{p})$, then $\mathcal{G}$ is (locally) a P-bundle over a manifold M.
- When $i_v \tilde{\Omega} = 0$ for all fields $v \in \omega^{-1}(\mathfrak{p})$, then ω defines a Cartan connection form of type (G, P) on $\mathcal{G} \to M$.

Fefferman-type constructions over the same manifold

- Let G be some Lie group that acts transitively on $\tilde{G}/\tilde{P}$, i.e.,

$$G/Q \cong \tilde{G}/\tilde{P}.$$

- Now let $(\mathcal{G}, \omega)$ be Cartan geometry of type (G, Q). We form the *extended principal bundle*

$$\tilde{\mathcal{G}} = \mathcal{G} \times_Q \tilde{P}$$

  and canonically extend ω to a Cartan connection form $\tilde{\omega}$ on $\tilde{\mathcal{G}}$. This gives a canonical inclusion (as bundles over $\tilde{M}$)

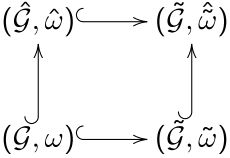
$$(\mathcal{G}, \omega) \hookrightarrow (\tilde{\mathcal{G}}, \tilde{\omega})$$

- $(\tilde{\mathcal{G}}, \tilde{\omega})$ is a Cartan geometry of type $(\tilde{G}, \tilde{P})$ on $\tilde{M}$.

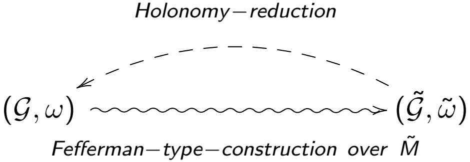
Question: How does one recognize whether a given Cartan geometry of type $(\tilde{G}, \tilde{P})$ is in fact induced by a Cartan geometry of type (G, P)?
Answer: One forms the holonomy $\mathrm{Hol}(\tilde{\omega})$:

Holonomy reduction

To form the *holonomy* $\mathrm{Hol}(\omega)$ of a parabolic geometry $(\mathcal{G}, \omega)$, one extends $\mathcal{G}$ to a principal G-bundle $\hat{G} = \mathcal{G} \times_P G$ and canonically extends ω to the a principal connection form $\hat{\omega}$ on $\hat{G}$. Then $\mathrm{Hol}(\omega) := \mathrm{Hol}(\hat{\omega})$.



For a Fefferman-type construction over the same manifold one obtains a commutative diagram and obtains a geometry of type $(\tilde{G}, \tilde{P})$ with reduced holonomy G with a 'single curved orbit' of type G/P.



Characterization in terms of underlying objects

In many cases the inclusion $G \hookrightarrow \tilde{G}$ is realized as the stabilizer of an element in a $\tilde{G}$-representation V, and in that case one can obtain an equivalent description of the holonomy reduction via BGG-solutions:

- The tractor bundle $\mathcal{V} = \tilde{\mathcal{G}} \times_{\tilde{P}} V$ carries the tractor connection $\nabla^{\mathcal{V}}$ that is naturally induced from the Cartan connection form $\tilde{\omega}$.
- Then $\mathrm{Hol}(\tilde{\omega}) \subset G$ is equivalent to the existence of a parallel section $s \in \Gamma(\mathcal{V})$ of a suitable type.
- By the general theory of BGG-operators on parabolic geometries [Čap-Slovak-Souček '01, Calderbank-Diemer '00] , such a parallel section s is equivalent to a normal solution of the first BGG-operator $\Theta_0 : \mathcal{H}_0 \to \mathcal{H}_1$ associated to $\mathcal{V}$:

$$\begin{array}{ccc} \Gamma(\mathcal{V}) & \quad & \nabla^{\mathcal{V}} - \text{parallel sections} \\ \Pi_0 \downarrow \;\; \uparrow L_0 & \quad & \Pi_0 \downarrow \;\; \uparrow L_0 \\ \Gamma(\mathcal{H}_0) & \quad & \ker \Theta_0^{\mathcal{V}} \end{array}$$

General (combined) Fefferman-type construction

A general Fefferman-type construction has both a 'correspondence space step' and an 'group extension step':

$$
\begin{array}{ccccccc}
(\mathcal{G},\omega) & \overset{\textit{Correspondence}}{\rightsquigarrow} & (\mathcal{G},\omega) & \overset{\textit{Extension}}{\rightsquigarrow} & (\tilde{\mathcal{G}},\tilde{\omega}) & \overset{\textit{Normalization}}{\rightsquigarrow} & (\tilde{\mathcal{G}},\tilde{\omega}^{nor}) \\
\downarrow P & & \downarrow Q & & \downarrow \tilde{P} & & \downarrow \tilde{P} \\
M & & \tilde{M} & & \tilde{M} & & \tilde{M}
\end{array}
$$

In general normality conditions on ω won't automatically imply normality conditions for $\tilde{\omega}$, and one needs one more 'normalization step'.

- If normality of ω automatically implies normality of $\tilde{\omega}$, the Fefferman-type construction $(G, P) \rightsquigarrow (\tilde{G}, \tilde{P})$ is called *normal*.
- It immediately follows in this case that $\mathrm{Hol}(\tilde{\omega}) = \mathrm{Hol}(\omega)$, and $\mathrm{Hol}(\tilde{\omega})$ is the well-defined holonomy of the parabolic geometry on $\tilde{M}$.

Examples of normal Fefferman-type constructions of conformal structures

- $\mathrm{SU}(p+1, q+1) \hookrightarrow \mathrm{SO}(2p+2, 2q+2)$:
  CR-structure $\rightsquigarrow$ signature $(2p+1, 2q+1)$-conformal structure on S^1-bundle + lightlike conformal Killing field (with additional properties) which inserts trivially into Weyl and Cotton tensors.
- $\mathrm{Sp}(n+1, 1) \hookrightarrow \mathrm{SO}(4n+4, 4)$:
  quaternionic contact structure $\rightsquigarrow$ signature $(4n+3, 3)$ conformal structure + 2 orthogonal lightlike conformal Killing fields inserting trivially into Weyl and Cotton tensors.
- $G_2 \hookrightarrow \mathrm{Spin}(3, 4)$:
  generic rank 2-distribution on 5-manifold $\rightsquigarrow$ signature $(2, 3)$-conformal spin structure + generic twistor spinor (same manifold, so no integrability conditions)

$\mathrm{SL}(n+1) \hookrightarrow \mathsf{Spin}(n+1,n+1)$

This Fefferman-type construction is based on an inclusion $\mathrm{SL}(n+1) \hookrightarrow \mathsf{Spin}(n+1,n+1)$:

Denote by $\Delta = \Delta_+^{n+1,n+1} \oplus \Delta_-^{n+1,n+1}$ the real 2^{n+1}-dimensional spin representation of $\tilde{G} = \mathsf{Spin}(n+1,n+1)$. Then we fix two pure spinors $s_F \in \Delta_-^{n+1,n+1}, s_E \in \Delta_\pm^{n+1,n+1}$ with non-trivial pairing - here s_E lies in $\Delta_+^{n+1,n+1}$ if n is even or $\Delta_-^{n+1,n+1}$ if n is odd.

These assumptions guarantee that the kernels $E, F \subset \mathbb{R}^{n+1,n+1}$ of s_E, s_F with respect to Clifford multiplication are complementary maximally isotropic subspaces.

Then $G := \{g \in \mathsf{Spin}(n+1,n+1) : g \cdot s_E = s_E, g \cdot s_F = s_F\} \cong \mathrm{SL}(n+1)$, defines an embedding $G = \mathrm{SL}(n+1) \overset{i}{\hookrightarrow} \mathsf{Spin}(n+1,n+1)$.

Fefferman-space $\tilde{M}$ and induced structure

One computes $\tilde{M} = \mathcal{G} \times_Q P/Q \cong (T^*M \otimes \mathcal{E}[2])/\{0\}$. Here we use the notation $\mathcal{E}[w]$ for suitably weighted (projective) version of the density bundle.

The invariant spinors s_E and s_F give rise to pure spin tractors:

The spin tractor bundle of (M, c) is $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$, where $\mathcal{S}_\pm = \tilde{\mathcal{G}} \times_{\tilde{P}} \Delta_\pm^{n+1,n+1} = \mathcal{G} \times_Q \Delta_\pm^{n+1,n+1}$. Since $s_E \in \Delta_\pm^{n+1,n+1}$ and $s_F \in \Delta_-^{n+1,n+1}$ are Q-invariant, they induce canonical sections $\mathbf{s}_E \in \Gamma(\mathcal{S}_\pm)$ and $\mathbf{s}_F \in \Gamma(\mathcal{S}_-)$.

The conformal Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \mathfrak{g})$ induces a tractor connection ∇ on each conformal tractor bundle; the spin tractors $\mathbf{s}_E, \mathbf{s}_F$ are parallel with respect to the induced tractor connections on the respective spin tractor bundles. However:
these are not necessarily the normal conformal tractor connections.

Normality of the induced conformal Cartan connection

Proposition

- *The Fefferman-type construction*

$$(\mathrm{SL}(3), P) \rightsquigarrow (\mathsf{Spin}(3,3), \tilde{P})$$

 is normal.

- *For $n \geq 3$ the Fefferman-type construction*

$$(\mathrm{SL}(n+1), P) \rightsquigarrow (\mathsf{Spin}(n+1, n+1), \tilde{P})$$

 is not normal:
 The conformal Cartan connection form $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ induced by the normal projective Cartan connection form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is normal if and only ω is flat, in which case also the projective Cartan connection form $\tilde{\omega}$ is flat.

The normal case $n = 2$:

Proposition

- *The conformal holonomy* $\mathrm{Hol}(\tilde{\omega})$ *is contained in* $\mathrm{SL}(3)$.
- *The normal conformal tractor connection* $\nabla^{\tilde{\mathcal{T}},nor}$ *preserves the decomposition* $\tilde{\mathcal{T}} = \tilde{\mathcal{E}} \oplus \tilde{\mathcal{F}}$.
- *The adjoint tractor* $\mathbf{K}$ *is parallel with respect to the normal tractor connection, i.e.* $\nabla \mathbf{K} = 0$*. Thus* $\mathbf{K}$ *corresponds to a normal conformal Killing field* $k \in \mathfrak{X}(\tilde{M})$.
- *The spin tractor bundle has two sections* $\mathbf{s}_E$ *and* $\mathbf{s}_F$ *with non-trivial pairing that are parallel with respect to the normal tractor connection, i.e.* $\nabla^{\mathcal{S}_+,nor}\mathbf{s}_E = 0$ *and* $\nabla^{\mathcal{S}_-,nor}\mathbf{s}_F = 0$*. Thus they correspond to two pure twistor spinors* $\chi_E \in \Gamma(\mathbf{S}_+[\frac{1}{2}])$ *and* $\chi_F \in \Gamma(\mathbf{S}_-[\frac{1}{2}])$.

Relating BGG-solutions in the normal case

Proposition

Suppose $(\tilde{M}, \mathbf{c})$ *is a conformal structure of signature* $(2,2)$ *associated to a 2-dimensional projective structure* $(M, \mathbf{p})$ *via the Fefferman-type construction.*

- *Then* $\mathbf{aEs}(\mathbf{c}) = \mathbf{aRs}(\mathbf{p}) \oplus \ker \Theta_0^{\mathcal{T}}$.
- *Let* $g \in \mathbf{aEs}(\mathbf{c})$ *be defined on the open-dense subset* $U \subset M$ *and let* $\widetilde{D}^g$ *be the Levi-Civita connection of* g*. Then* g *corresponds to an almost Ricci-flat structure of* $\mathbf{p}$ *if and only if* $\widetilde{D}^g \chi_f = 0$ *and* g *corresponds to an element of* $\ker \Theta_0^{\mathcal{T}}$ *if and only if* $\widetilde{D}^g \chi_e = 0$*. In both cases it automatically follows that* $\mathrm{Ric}(g) = 0$*.*

Remark: If $\mathbf{aRs}(\mathbf{p}) \neq \{0\}$ then $\mathbf{p}$ is locally projectively flat, and therefore also $\mathbf{c}$ is locally conformally flat.

Relating infinitesimal symmetries

Another application of our construction concerns the conformal Killing fields on $\tilde{M}$. Note that under $\mathfrak{sl}(3)$ the Lie algebra $\mathfrak{so}(3,3)$ decomposes into the following irreducible pieces

$$\mathbb{R}^3 \oplus \mathbb{R}^{3^*} \oplus \mathfrak{sl}(3) \oplus \mathbb{R}. \tag{1}$$

Analogously to [Čap-Gover '08, H.-Sagerschnig '09] one can prove that:

Proposition

The space of conformal Killing fields on $(\tilde{M}, c)$ decomposes as

$$\mathbf{aEs(c)} \oplus \mathbf{inf(p)} \oplus \mathbb{R}k. \tag{2}$$

The non-normal case $n \geq 3$:

Since for $n \geq 3$ the induced Cartan connection form $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ is not already the normal conformal connection form, one needs to find the unique modification

$$\Psi \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{p}})$$

such that

$$\tilde{\omega}^{nor} = \tilde{\omega} + \Psi$$

is normal:

One must find Ψ such that the curvature function $\tilde{\Omega}^{nor}$ of the modified Cartan connection form lies in the kernel of the Kostant co-differential $\tilde{\partial}^* : \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$:

$$\tilde{\partial}^*(\tilde{\Omega}^{nor}) = 0.$$

The inductive normalization procedure that is necessary for a full computation of the modification Ψ makes it difficult to obtain an explicit formula for this map. It turns out however that certain properties of Ψ can be obtained without an explicit form:

The non-normal case $n \geq 3$:

Theorem

$\mathbf{s}_F \in \Gamma(\mathcal{S}_-)$ is parallel with respect to the normal conformal spin tractor connection $\nabla^{\mathcal{S}_-,nor}\mathbf{s}_F = 0$. In particular, the conformal spin structure (M, c) carries a canonical pure twistor spinor $\chi_F \in \Gamma(\mathbf{S}_-[\frac{1}{2}])$.

In a addition, the proof shows that while the adjoint tractor $\mathbf{K} \in \mathcal{A}\tilde{M}$ is no longer parallel with respect to the normal adjoint tractor connection, it is still the BGG-splitting of a conformal Killing field $k \in \mathfrak{X}(\tilde{M})$.

Theorem

Let $\chi_e \in \Gamma(S_\pm[\frac{1}{2}])$ and $k \in \Gamma(T\widetilde{M})$ be as above. Then

- $\mathbf{s}_E = L_0^{\widetilde{\mathcal{S}}_+}(\chi_e)$,
- $\mathbf{K} = L_0^{\Lambda^2\widetilde{\mathcal{T}}}(k)$.

Collecting induced algebraic and integral properties

- The underlying objects (χ_f, χ_e, k) fully characterizes the Fefferman-type spaces of $(\mathrm{SL}(n+1) \hookrightarrow \mathrm{Spin}(n+1, n+1)$ *algebraically*:
  On the level of tractors, $\mathbf{s}_F = L_0(\chi_f)$ and $\mathbf{s}_E = L_0(\chi_e)$ are pure tractor spinors with non-trivial pairing $(\mathbf{s}_E, \mathbf{s}_F) \neq 0$ and thus define a decomposition $\mathcal{T} = \mathcal{E} \oplus \mathcal{F}$ via their Clifford-kernels. The endomorphism $\mathbf{K} = L_0(k) \in \mathfrak{so}(\mathcal{T})$ is an involution that acts by $\mathbb{I}$ on $\mathcal{E}$ and $-\mathbb{I}$ on $\mathcal{F}$.
- Since $\tilde{\nabla}^{nor}\mathbf{s}_F = 0$, it obviously follows that $\widetilde{\Omega}^{nor}$ has values in $(\widetilde{\mathcal{E}} \otimes \widetilde{\mathcal{F}})_0 \oplus (\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{F}})$.
- We also have an automatically implied *integrability conditon* for the structure $(\tilde{\mathcal{G}}, \tilde{\omega}^{nor})$:

$$i_v \widetilde{\Omega}^{nor} \in \widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{F}} \text{ for } v \in \ker \chi_f .$$

Remark: Factorization without Characterization

Under the 'algebraic' conditions on (χ_f, χ_e, k), it is seen that integrability condition

$$i_v\widetilde{\Omega}^{nor} \in \widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{F}} \text{ for } v \in \ker \chi_f$$

is in fact equivalent to the condition on the Cartan curvature

$$W_{abcd}\xi^a\eta^c = 0 \text{ for all vertical fields } \xi, \eta \in \ker \gamma_{\chi_f} \subset \mathfrak{X}(\tilde{M}) \qquad \text{(I)}$$

on the Weyl curvature tensor.

If integrability condition (I) is satisfied, one can also see directly how to factorize c to a projective structure: Let $g \in c$ be some metric in the conformal class and D its Levi-Civita covariant derivative. Denote by $M = \tilde{M}/\ker \chi_F$ the leaf space of the totally isotropic (and integrable) n-dimensional kernel of χ_f. Then D induces a canonical projective class of affine torsion-free connections on M.

Converse direction

Let now $(\tilde{\mathcal{G}}, \tilde{\omega}^{nor})$ describe a split signature (n, n) conformal spin structure with the additional data and properties collected above, summarized as $(\tilde{M}, c, \chi_f, \chi_e, k)$.

Basic problem at this point:

- We would like to show that $(M, c, \chi_f, \chi_e, k)$ is induced by some projective structure (M, p) via the Fefferman-type construction.
- However: while it is relatively easy to show that under the given condition $(\tilde{M}, c)$ induces a projective structure on a leaf space, (M, p), this doesn't help us to show that in fact all of $(\tilde{M}, c)$ is completely determined by this quotient.

“De-Normalizing: Going back to projective”

We define a modified Cartan connection

$$\widetilde{\omega}' := \widetilde{\omega}^{nor} - \frac{1}{2} i_k \widetilde{\Omega}^{nor},$$

Proposition

The Cartan connection $\widetilde{\omega}'$ has the following properties:

- $\widetilde{\omega}'$ *is an* $\mathrm{SL}(n+1)$*-connection, i.e.* $\widetilde{\nabla}' \mathbf{s}_E = \widetilde{\nabla}' \mathbf{s}_F = 0$.
- *The curvature* $\widetilde{\Omega}'$ *satisfies* $\widetilde{\Omega}' = \mathrm{Proj}_{(E \otimes F)_0} \widetilde{\Omega}^{nor}$.
- $\mathrm{Proj}_{\mathfrak{g}_0} \widetilde{\omega'} = \mathrm{Proj}_{\mathfrak{g}_0} \widetilde{\omega}^{nor}$

Proof: One computes

$$\tilde{\Omega}' = [...] = \tilde{\Omega}^{nor} + \frac{1}{2} \mathbf{K} \bullet \tilde{\Omega}^{nor} = (\tilde{\Omega}^{nor})_{E \otimes F},$$

since**K** acts by multiplication with -2 on $\widetilde{\mathcal{F}} \wedge \widetilde{\mathcal{F}}$.

Observations and Consequences

Since $\tilde{\omega}'$ is thus a Cartan connection with $\mathrm{SL}(n+1)$-holonomy, we can naturally regard it as Cartan connection on a $Q = \mathrm{SL}(n+1) \cap P$-subbundle $\mathcal{G} \subset \tilde{\mathcal{G}}$.

It follows from the last property of the proposition above, that when $(\tilde{\mathcal{G}}, \tilde{\omega}^{nor})$ is induced by a projective structure $(\mathcal{G}, \omega)$, then $\tilde{\omega}'$ equals ω modulo $\mathfrak{p}_+$: That's good, because it then follows that the process

$$\tilde{\omega} \rightsquigarrow \tilde{\omega}^{nor} \rightsquigarrow \tilde{\omega}'$$

takes on $\mathrm{SL}(n+1)$-connection into another $\mathrm{SL}(n+1)$-connection which only differs in $\mathfrak{p}_+$: thus, any information on the projective structure (which resides in $\mathfrak{g}_0$) is preserved.

However, even in the case where we would 'know already' that $(\tilde{G}, \tilde{\omega}^{nor})$ was induced by a $(\mathcal{G}, \omega)$ the new connection $\tilde{\omega}'$ need not be P-equivariant, and can thus not be factorized immediately.

Factorization

Proposition

Let $(\mathcal{G} \to \tilde{M}, \omega)$ be a Cartan geometry of type $(SL(n+1), Q)$ with curvature Ω. Suppose that Ω has values in $\mathfrak{p}$, $i_w i_v \Omega \in \mathfrak{p}_+$ for all $v \in \omega^{-1}(\mathfrak{p}) \subset T\mathcal{G}$ and $w \in T\mathcal{G}$, and $i_{v_1} i_{v_2} \Omega = 0$ and for all $v_1, v_2 \in \omega^{-1}(\mathfrak{p})$. Then $\mathcal{G}$ is locally a P-bundle over $M = \mathcal{G}/P$ and it defines a canonical projective structure on M.

Main points of the proof: (trivially adapted proof from [Čap, 2005])

- The fact that $i_{v_1} i_{v_2} \Omega = 0$ for all $v_1, v_2 \omega^{-1}(\mathfrak{p})$ implies by usual Cartan-methods that $\mathcal{G}$ is locally a P-bundle $\mathcal{G} \to M$. We will restrict $\mathcal{G}$ to assume this globally. We define $M = \mathcal{G}/P$ and $\mathcal{G}_0 = \mathcal{G}/P_+$.
- Let $\sigma : \mathcal{G}_0 \to \mathcal{G}$ be an $\mathrm{SL}(n)$-equivariant splitting. It follows from $i_v \Omega \in \mathfrak{p}_+$, for all $v \in \omega^{-1}(\mathfrak{p})$, that

$$\mathcal{L}_{\zeta_X} \omega = -\mathrm{ad}(X) \circ \omega \mod \mathfrak{p}_+,$$

for all $X \in \mathfrak{p}$.

Complete characterization

Theorem

A split signature (n, n) conformal structure c on a manifold $\tilde{M}$ is induced by an n-dimensional projective structure if and only if c admits (χ_f, χ_e, k) such that

- χ_f *is a pure twistor spinor with (parallel) pure tractor spinor* $\mathbf{s}_f = L_0(\chi_f)$
- χ_e *is a pure spinors with pure tractor spinor* $\mathbf{s}_e = L_0(\chi_e)$
- k *is a conformal Killing field with tractor endomorphism* $\mathbf{K} = L_0(K)$ *which acts by* $\mathbb{I}$ *on* $\ker \mathbf{s}_E$ *and* $-\mathbb{I}$ *on* $\ker \mathbf{s}_F$

and in addition the following integrability condition is satisfied:

$$W_{abcd}\xi^a\eta^c = 0 \text{ for all vertical fields } \xi, \eta \in \ker \gamma_{\chi_f} \subset \mathfrak{X}(\tilde{M}).$$