Natural Prolongations of BGG-Operators

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Example 1: Einstein metrics in a conformal class

- Let g and $\hat{g}$ be pseudo-Riemannian metrics of signature $(p, q), p + q = n$ on an n-manifold M.
- We say that g and $\hat{g}$ are conformally related iff there is a function $f \in \mathrm{C}^\infty(M, \mathbb{R}_+)$ such that $\hat{g} = fg$.
- This defines an equivalence relation for pseudo-Riemannian metrics; the equivalence class of a metric g is denoted by $[g]$ and defines a *conformal structure* on M.
- Given a metric $g \in [g]$, one has its Levi-Civita connection D and can form the Riemannian curvature tensor R^g.
- It is a natural question whether there is an Einstein metric in a given conformal class; i.e., whether for some $g \in [g]$ the Ricci curvature $\mathrm{Ric}^g := \mathrm{tr}_{(1,3)} \mathrm{R}^g \in \Gamma(S^2 T^* M)$ is a multiple of g.

Example 1: Einstein metrics in a conformal class

- This question is governed by the operator

$$\Theta^g : \mathrm{C}^\infty(M) \to \Gamma(S_0^2 T^* M),$$
$$\Theta^g(\sigma) = (DD\sigma + \mathsf{P}^g\,\sigma) + \frac{1}{n}(\triangle\sigma - \mathrm{tr}_{(1,2)}\,\mathsf{P}^g\,\sigma)g.$$

Here

$$\mathsf{P}^g := \frac{1}{n-2}\big(\mathrm{Ric}^g - \frac{\mathrm{Sc}^g}{2(n-1)}g\big)$$

is the Schouten-tensor; $S_0^2 T^* M$ denotes symmetric, trace-free bilinear forms on TM. The convention for the Laplace operator is $\triangle := -\,\mathrm{tr}_{(1,2)} \circ D^2$.

- For $\sigma \in \mathrm{C}^\infty(M, \mathbb{R}_+)$ one has $\Theta^g(\sigma) = 0$ iff $\sigma^{-2}g$ is Einstein.

Example 1: Einstein metrics in a conformal class

- The operator Θ^g is conformally covariant between $\mathrm{C}^\infty(M)$ and $S^2_0 T^*M$: if one switches to another metric $\hat{g} = e^{2f} g$ in the conformal class, then

$$\Theta^{\hat{g}} \circ m(e^f) = m(e^f) \circ \Theta^g,$$

  where $m(e^f)$ is simply the multiplication operator with e^f.
- To define a conformally invariant operator, one introduces *conformal density bundles* $\mathcal{E}[w]$: these are line bundles which are trivialized by a choice of $g \in [g]$. The trivializations of $\sigma \in \mathcal{E}[w]$ with respect to $\hat{g} = e^{2f} g$ and g are related according to $[\sigma]_{\hat{g}} = e^{wf}[\sigma]_g$.
- By forming the *weighted bundles* $\mathbf{H}_0 = \mathcal{E}[1]$ and $\mathbf{H}_1 = S^2_0 T^*M \otimes \mathcal{E}[1]$ one obtains a *conformally invariant* operator

$$\Theta : \Gamma(\mathbf{H}_0) \to \Gamma(\mathbf{H}_1) :$$

  the definition of Θ does not depend on the choice of $g \in [g]$.

Example 2: Metrization of projective structures

- Two torsion-free linear connections D and $\hat{D}$ on TM are projectively equivalent iff there exists a one form $\Upsilon \in \Omega^1(M)$ with
$$\hat{D}\omega = D\omega + \Upsilon \otimes \omega + \omega \otimes \Upsilon$$
for all $\omega \in \Omega^1(M)$. Projectively equivalent connections have the same unparameterized geodesics.
- An interesting question in projective differential geometry is whether a given projective class of connections $[D]$ contains the Levi-Civita connection of some metric.
- It was observed by [Sinjukov, Nauka (1979)] and [Mikeš, Acta Univ. Palack. Olomuc. (1996)] that this problem is governed by the equation
$$D\sigma - \frac{1}{n+1}\operatorname{sym}\big(\mathrm{id} \otimes \mathrm{tr}_{(1,2)}(D\sigma)\big) = 0$$
for $\sigma \in \Gamma(S^2 TM)$.
- This yields a projectively invariant operator between suitably weighted bundles.

Prolongations of overdetermined systems

- Given an overdetermined system of equations described by an operator $\Theta : \Gamma(\mathbf{H}_0) \to \Gamma(\mathbf{H}_1)$ we want to rewrite the system in *closed form*:
- We look for an equivalent first order system such that all first order derivatives of the dependent variables are given by the dependent variables themselves.
- In classical language, this means that one introduces additional variables for derivatives of $\sigma \in \Gamma(\mathbf{H}_0)$ and derives differential consequences for these variables from the equation $\Theta_0(\sigma) = 0$.

Prolongations of overdetermined systems

We will employ the following notation:

- The 'additional variables' are encoded in an extension of the bundle $\mathbf{H}_0$ to a bundle $\mathbf{V}$ which has a projection $\mathbf{V} \xrightarrow{\Pi} \mathbf{H}_0$.
- The expression of derivatives of $\sigma \in \Gamma(\mathbf{H}_0)$ in terms of the 'new variables' is done via a linear differential operator $L : \Gamma(\mathbf{H}_0) \to \Gamma(\mathbf{V})$ which splits Π, i.e., $\Pi \circ L = \mathrm{id}_{\Gamma(\mathbf{H}_0)}$.
- The resulting closed system is encoded in a linear connection $\nabla : \Gamma(\mathbf{V}) \to \Gamma(T^*M \otimes \mathbf{V})$.
- Equivalence of the closed system with the equation $\Theta(\sigma) = 0$ then says that the projection Π and the splitting L restrict to inverse isomorphisms between the space of parallel sections of ∇ and the kernel of Θ_0.

We then call the tuple $(\mathbf{V}, \Pi, L, \nabla)$ a *geometric prolongation* of Θ_0.

Immediate applications of a geometric prolongation

If $(\mathbf{V}, \Pi, L, \nabla)$ is a geometric prolongation of Θ_0, then

- the solution space $\ker(\Theta_0)$ is finite-dimensional and bounded by $\mathrm{rank}\ \mathbf{V}$,
- if L_0 is a differential operator of order r, then every solution is determined by its r^{th}-order jet in a point,
- if $\Theta_0(\sigma) = 0$ and σ is not trivial, then σ is non-vanishing on an open-dense set.

Moreover, the curvature of the prolongation connection can be used to obtain obstructions for the existence of parallel sections of $\mathbf{V}$ resp. solutions of $\Theta_0(\sigma) = 0$.

The problem of naturality/invariance

- We want to construct prolongations which respect the underlying geometric structure.
- For instance, in conformal geometry we don't want our constructions to depend on a choice of metric in the conformal class.
- This case already exemplifies that one immediately encounters great obstacles, since there is no unique Levi-Civita connection as in Riemannian geometry.
- Major advances to overcome this obstacle were achieved in the 1920s by Élie Cartan and Tracy Thomas:
- Given a conformal structure of signature $(p, q), p + q = n$, the latter constructed a natural bundle $\mathbf{S}$ of rank $n + 2$ endowed with a canonical connection ∇^S and compatible signature $(p + 1, q + 1)$-metric $\mathbf{h}$. This is now called the *conformal standard tractor* bundle.

Cartan's description of conformal structures

A few years earlier, Élie Cartan had worked with what would now be considered the structure bundle $\mathcal{G}$ of **S**: Let $G := \mathrm{SO}(p+1, q+1), \mathfrak{g} = \mathfrak{so}(p+1, q+1)$ and define $P \subset G$ as the stabilizer of an isotropic ray in $\mathbb{R}^{p+1,q+1}$.

Definition

A Cartan geometry of type (G, P) on a manifold M is a P-principal bundle $\mathcal{G} \to M$ endowed with a Cartan connection form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. ω is P-equivariant, reproduces fundamental vector fields, and provides a trivialization $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$.

Theorem (Cartan, 1923)

There is an equivalence of categories between conformal structures of signature (p, q) and Cartan geometries of type $(\mathrm{SO}(p+1, q+1), P)$ whose curvature satisfies a normalization condition.

Parabolic geometries and underlying structures

- The definition of a Cartan geometry makes sense for arbitrary Lie groups G with closed subgroup P, and in the case where P is a parabolic subgroup of a semi-simple Lie group one calls $(\mathcal{G}, \omega)$ a *parabolic geometry*.
- For a parabolic geometry $(\mathcal{G}, \omega)$ there is a canonical *regularity* condition which implies that it induces a geometric structure on the underlying manifold M.
- There is also a natural normalization condition on ω, which yields the class of *normal* parabolic geometries.
- The equivalent description of geometric structures as parabolic geometries is a powerful tool for natural resp. invariant constructions.

Parabolic geometries and tractor bundles

- Given an arbitrary parabolic geometry $(\mathcal{G}, \omega)$ of type (G, P) and a G-representation V, one can build the associated *tractor bundle* $\mathbf{V} := \mathcal{G} \times_P V$.
- The Cartan connection form ω can be extended to a G-principal connection form ω' on an extended bundle and then endows $\mathbf{V}$ with its *tractor connection* ∇^V.
- Let $\mathbf{C_k} = \Lambda^k T^*M \otimes \mathbf{V}$. Then $\Gamma(\mathbf{C_k}) = \Omega^k(M, \mathbf{V})$ and one can form the *twisted de-Rham sequence* of the tractor connection ∇^V,

$$\Gamma(\mathbf{C}_0) \stackrel{\nabla^V}{\to} \Gamma(\mathbf{C}_1) \stackrel{\mathrm{d}^\nabla}{\to} \Gamma(\mathbf{C}_2) \stackrel{\mathrm{d}^\nabla}{\to} \cdots$$

- For a parabolic geometry there is a canonical Lie algebra differential ∂^* called the *Kostant codifferential*. It gives rise to a complex

$$\mathbf{C}_0 \stackrel{\partial^*}{\leftarrow} \mathbf{C}_1 \stackrel{\partial^*}{\leftarrow} \mathbf{C}_2 \stackrel{\partial^*}{\leftarrow} \cdots$$

The BGG-sequence

- The differential ∂^* yields bundles $\mathbf{Z}_k = \ker \partial^*$ of cycles, $\mathbf{B}_k = \mathrm{im}\ \partial^*$ borders and homologies $\mathbf{H}_k = \mathbf{Z}_k/\mathbf{H}_k$, and one has the canonical projections $\Pi_k : \mathbf{Z}_k \to \mathbf{H}_k$.
- Now the *BGG-sequence* is formed by natural differential operators

$$\Gamma(\mathbf{H}_0) \stackrel{\Theta_0}{\to} \Gamma(\mathbf{H}_1) \stackrel{\Theta_1}{\to} \Gamma(\mathbf{H}_2) \stackrel{\Theta_2}{\to} \cdots$$

  It was presented in [Čap-Slovăk-Souček, Ann. of Math. (2001)] and a simplified construction was obtained in [Calderbank-Diemer, (J. Reine u. Angew. Math.) (2001)]
- The main technical step in the development of the BGG-machinery is the construction of the canonical *BGG-splitting-operators* $L_k : \Gamma(\mathbf{H}_k) \to \Gamma(\mathbf{Z}_k)$.

The first BGG-operator

- We are mostly interested in the first BGG-operator $\Theta_0 : \Gamma(\mathbf{H}_0) \to \Gamma(\mathbf{H}_1)$, defined via the composition $\Pi_1 \circ \nabla^V \circ L_0$,

$$\begin{array}{ccc} \mathrm{im}\,(L_0) & \xrightarrow{\nabla^V} & \Gamma(\mathbf{Z}_1)\ . \\ {\scriptstyle L_0}\uparrow & & \downarrow{\scriptstyle \Pi_1} \\ \Gamma(\mathbf{H}_0) & \xrightarrow{\Theta_0} & \Gamma(\mathbf{H}_1) \end{array}$$

- If $s \in \Gamma(\mathbf{V})$ is ∇^V-parallel, then automatically $\Theta_0(\Pi_0(s)) = 0$. Thus, parallel sections project into $\ker \Theta_0$.
- $(\mathbf{V}, \Pi_0, L_0, \nabla^V)$ is however not a geometric prolongation for general representations V, since the converse does not hold: If $\sigma \in \ker \Theta_0$, then $\nabla^V(L_0(\sigma))$ need not necessarily vanish, but may lie in $\Gamma(\mathbf{B_1}) = \mathrm{im}\ \partial^*$.

Examples of first BGG-operators for conformal structures

- If one takes the standard tractor bundle $\mathbf{S}$ of a conformal structure $(M, [g])$ one obtains the operator governing Einstein rescalings discussed in the first example.
- If $(M, [g])$ is a conformal spin structure with spin bundle $\mathbf{\Delta}$ and Clifford symbol $\gamma \in \Gamma(T^*M \otimes \mathrm{End}(\mathbf{\Delta}))$, one also has a spin tractor bundle $\mathbf{\Sigma}$. Let $\not{D} : \Gamma(\mathbf{\Delta}) \to \Gamma(\mathbf{\Delta})$ be the Dirac operator.
  The first BGG-operator of $\mathbf{\Sigma}$ is the *twistor operator*

$$\Gamma(\mathbf{\Delta}) \to \Gamma(T^*M \otimes \mathbf{\Delta}),$$
$$\chi \mapsto D\chi + \frac{1}{n}\gamma \otimes \not{D}\chi.$$

  Solutions of this equation are known as *twistor spinors*.
- Both cases are very special: parallel sections of the tractor connection are already in 1:1-correspondence with solutions, which reflects the fact that the modelling representations are still very simple.

Examples of first BGG-operators for conformal structures

- For an exterior power $\mathbf{V} = \Lambda^{k+1}\mathbf{S}$, $k \geq 1$ one obtains the operator governing conformal Killing k-forms,

$$\Theta_0 : \Omega^k(M) \to \Gamma(T^*M \otimes \Lambda^k T^*M),$$
$$\Theta_0(\sigma) = D\sigma - \mathrm{alt}_{(1,\cdots,k+1)} D\sigma - \frac{k}{n-k+1}\mathrm{alt}_{(2,\cdots,k+1)}\big(g \otimes (\mathrm{tr}_{(1,2)}\, D\sigma)\big).$$

- Already in this case a solution of $\Theta_0(\sigma)$ need not satisfy that also $\nabla^V(L_0(\sigma)) = 0$. In fact, this imposes additional equations on a conformal Killing form σ, and solutions to this extended system have been termed *normal* conformal Killing forms by [Leitner, Rend.Circ.Mat.Pal. (2005)].

Prolongation of first BGG-operators

- For BGG-operators appearing for parabolic geometries whose structure group G has 1-graded Lie algebra a prolongation was constructed by [Branson-Čap-Eastwood-Gover, Int. Journ. Mat. (2006)]. This works also for semilinear equations with prescribed symbol; however, it doesn't respect invariance of the original equation. A generalization of this approach to higher gradings is current work of K. Neusser.
- Another result was obtained for the prolongation of the equations for infinitesimal automorphisms of parabolic geometries, [Čap, JEMS (2008)]: The *adjoint tractor bundle* $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$ together with an explicit modification of the adjoint tractor connection by curvature is shown to describe the infinitesimal automorphisms of a parabolic geometry $(\mathcal{G}, \omega)$

Prolongation of first BGG-operators

- For specific equations invariant prolongations have been found by direct calculations; For the equation governing conformal Killing forms above, this was done by [Gover-Šilhan, Diff. Geom. Appl. (2008)]. This approach soon becomes computationally impossible.
- The solution to the prolongation problem for BGG-operators which we are going to describe will work for arbitrary regular parabolic geometries and will be natural. Explicit calculations can be done by an algorithm:

Theorem (Natural Prolongation)

Let $\mathbf{V}$ *be a tractor bundle for a regular parabolic geometry. There exists a natural connection* $\tilde{\nabla}$ *on* $\mathbf{V}$ *such that*

1. *The BGG-construction can still be carried out for* $\tilde{\nabla}$ *and yields BGG-splitting operators* $\tilde{L}_k$ *and BGG-operators* $\tilde{\Theta}_k$*.*
2. *The first BGG-splitting operator and first BGG-operator for* $\tilde{\nabla}$ *coincide with the corresponding objects for* ∇^V*.*
3. *The diagram*

$$\begin{array}{ccc} \mathrm{im}\ L_0 & \xrightarrow{\tilde{\nabla}} & \Gamma(\mathbf{Z_1}) \\ {\scriptstyle L_0}\uparrow & & \uparrow{\scriptstyle \tilde{L}_1} \\ \Gamma(\mathbf{H}_0) & \xrightarrow{\Theta_0} & \Gamma(\mathbf{H}_1) \end{array}$$

   commutes, and this implies that $(\mathbf{V}, \Pi_0, L_0, \tilde{\nabla})$ *is a natural geometric prolongation of* Θ_0*.*

$\tilde{\nabla}$ *is unique under a natural condition and is called the* **prolongation connection** *of* Θ_0*.*

Natural prolongation

- This is shown via a modification procedure for tractor connections, which will yield the required connection in the form $\tilde{\nabla} = \nabla^V + \Psi$, $\Psi \in \Omega^1(M, \mathrm{End}(\mathbf{V}))$.
- One first imposes conditions on the modification map Ψ which yield a class of modified connections for which the BGG-construction can still be carried out and produces the same first BGG-(splitting)-operator as the tractor connection on $\mathbf{V}$.
- Next, one imposes a natural condition on the curvature $R_\Psi \in \Omega^2(M, \mathrm{End}(\mathbf{V}))$ of $\tilde{\nabla} = \nabla^V + \Psi$ which is seen to imply commutativity of the first BGG-diagram.
- Now there is an inductive algorithm which yields Ψ in terms of geometric data of the underlying structure after an 'unnatural' choice. For instance, in conformal geometry, this produces the desired Ψ in terms of Riemannian data of a metric g in the conformal class.
- Finally, one observes that this Ψ is actually unique, and thus the result doesn't depend on any special choices during the construction.

Natural Prolongation

One immediately obtains:

Corollary

Let V be a G-representation and $(\mathbf{V}, \tilde{\nabla}, \Pi_0, L_0)$ the geometric prolongation of Θ_0.

1. *The space* $\ker \Theta_0 \subset \mathcal{H}_0$ *has rank* $\leq \dim\ V$.
2. *Every* $\sigma \in \ker \Theta_0$ *is determined by its r-jet at some point, with* $r \in \mathbb{N}$ *only depending on the representation* V.
3. *If* $\sigma \in \ker \Theta_0$ *is not globally vanishing, its singularity set* $\sigma^{-1}(\{0\})$ *has an open dense complement.*

Example: Prolongation of the equation governing projective metrizability

- A class of projectively equivalent connections $[D]$ on an n-manifold is equivalently described as a parabolic geometry $(\mathcal{G}, \omega)$ of type $(SL(n+1), P)$ with P the stabilizer of a line in $\mathbb{R}^{n+1}$. The tractor bundle $\mathbf{V} = \mathcal{G} \times_P S^2\mathbb{R}^{n+1}$ yields the first BGG-operator

$$\Theta_0 : \Gamma(S^2 TM) \to \Gamma(T^*M \otimes S^2 TM)$$

$$\Theta_0(\sigma) = D\sigma - \frac{1}{n+1} \operatorname{sym}\big(\mathrm{id} \otimes \mathrm{tr}_{(1,2)}(D\sigma)\big)$$

  which governs the existence of geodesically equivalent metrics.
- After choice of a connection $D \in [D]$ the tractor bundle $\mathbf{V}$ can be written as a direct sum $S^2 TM \oplus TM \oplus \mathrm{C}^\infty(M)$, and a section $s \in \Gamma(\mathbf{V})$ will be written

$$[s]_D = \begin{pmatrix} \rho \\ \mu \\ \sigma \end{pmatrix} \in \begin{pmatrix} \mathrm{C}^\infty(M) \\ \Gamma(TM) \\ \Gamma(S^2 TM) \end{pmatrix}.$$

Example: Prolongation of the equation governing projective metrizability

- One calculates the splitting operator $L_0 : \Gamma(S^2 TM) \to \Gamma(\mathbf{V})$ as

$$\sigma \mapsto \begin{pmatrix} \frac{1}{n(n+1)} \operatorname{tr}_{(1,3)(2,4)} D^2\sigma + \frac{1}{2n} \operatorname{tr}_{(1,3)(2,4)} \mathsf{P} \otimes \sigma \\ -\frac{1}{n+1} \operatorname{tr}_{(1,2)} D\sigma \\ \sigma \end{pmatrix}.$$

- The explicit form of the prolongation connection is

$$\tilde{\nabla} \begin{pmatrix} \rho \\ \mu \\ \sigma \end{pmatrix} = \begin{pmatrix} D\rho - 2 \operatorname{tr}_{(2,3)} \mathsf{P} \otimes \mu - \frac{4}{n} \operatorname{tr}_{(1,4)(3,5)} A \otimes \sigma \\ D\mu - 2 \operatorname{tr}_{(2,3)} \mathsf{P} \otimes \sigma + \rho \operatorname{id} + \frac{2}{n} \operatorname{tr}_{(2,5)(4,6)} C \otimes \sigma \\ D\sigma + \operatorname{sym}(\operatorname{id} \otimes \mu) \end{pmatrix}.$$

  Here $A \in \Gamma(T^*M \otimes \Lambda^2 T^*M)$ is the *Cotton-York* tensor of D and $C \in \Gamma(\Lambda^2 T^*M \otimes \operatorname{End}(TM))$ the *Weyl-curvature*.
- This prolongation agrees with the one found by direct calculation in [Eastwood-Matveev, IMA (2008)]

Conformal structures with G_2-holonomy

- Let $(M,[g])$ be a conformal structure of signature $(2,3)$.
- $(M,[g])$ can be described as a parabolic geometry $(\tilde{\mathcal{G}},\tilde{\omega})$ of type $(\mathrm{SO}(3,4),\tilde{P})$, with $\tilde{P}$ the stabilizer of an isotropic ray in $\mathbb{R}^{3,4}$. The standard tractor bundle of $[g]$ is given by $\mathbf{S}=\tilde{\mathcal{G}}\times_{\tilde{P}}\mathbb{R}^7$ and is endowed with a canonical $(3,4)$-metric $\mathbf{h}$.
- We define the conformal holonomy $\mathrm{Hol}([g]):=\mathrm{Hol}(\nabla^S)\subset\mathrm{SO}(3,4)$ and are interested in the case where $\mathrm{Hol}([g])\subset G_2$:
- G_2 shall be the the real Lie group with fundamental group $\mathbb{Z}_2$ and Lie algebra the split real form of the exceptional complex Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$. It is well known that one can realize $G_2\subset\mathrm{SO}(3,4)$ as the stabilizer of a $\Phi\in\Lambda^3\mathbb{R}^7$.

G_2-holonomy and generic distributions

- Conformal structures with G_2-holonomy are strongly related with *generic distributions* on a 5-manifold M:
- Let $\mathbf{D} \subset TM$ be a subbundle of constant rank 2. We say that $\mathbf{D}$ is a generic rank 2-distribution if Lie brackets $[\xi, \eta]$ of sections $\xi, \eta \in \Gamma(\mathbf{D})$ span a subbundle of constant rank 3 and TM is spanned by Lie brackets of the form $[\xi, [\eta, \zeta]]$, $\xi, \eta, \zeta \in \Gamma(\mathbf{D})$.
- It is a classical result by [Cartan, 1910] that a generic distributions $\mathbf{D} \subset TM$ can be described as a parabolic geometry $(\mathcal{G}, \omega)$ of type (G_2, P), with $P = G_2 \cap \tilde{P}$.
- Surprisingly, it was only observed recently, [Nurowski, J.Geom.Phys. (2005)], that the inclusion $G_2 \hookrightarrow \mathrm{SO}(3,4)$ can be used to canonically associate a conformal class of $(2,3)$-metrics $[g]_{\mathbf{D}}$ to a generic distribution $\mathbf{D}$.

G_2-holonomy and generic distributions

- This is a *generalized Fefferman concstruction*: If $(\mathcal{G}, \omega)$ is the Cartan geometry describing the distribution $\mathbf{D}$, one obtains a conformal geometry $(\tilde{\mathcal{G}}, \tilde{\omega})$, where $\tilde{\mathcal{G}} := \mathcal{G} \times_P \tilde{P}$ and $\tilde{\omega}$ is the canonically extended form.
- The construction $\mathbf{D} \rightsquigarrow [g]_{\mathbf{D}}$ is functorial. To obtain strong relations between $[\mathbf{D}]$ and $[g]_{\mathbf{D}}$ one needs in addition that the normalization conditions on ω imply the corresponding conditions on $\tilde{\omega}$. This was done by [Sagerschnig, 2008 (Thesis)].
- This implies that the induced conformal structures have holonomy contained in G_2, and in joint work [M.H.-Sagerschnig, SIGMA (2009)] the converse construction was presented:

Theorem

Let $(M, [g])$ *be a conformal structure of signature* $(2, 3)$ *with* $\mathrm{Hol}([g]) \subset G_2$. *Then* $[g]$ *is canonically associated to a generic rank two distribution* $\mathbf{D}$.

Characterization of G_2-holonomy in terms of normal conformal Killing 2 forms

- Recall that G_2 was realized in $\mathrm{SO}(3,4)$ as the stabilizer of a $\Phi \in \Lambda^3\mathbb{R}^7$. This is easily seen to provide a parallel section $\mathbf{\Phi}$ of $\Lambda^3\mathbf{S}$, which then yields a normal conformal Killing 2-form $\phi \in \Gamma(\Lambda^2 T^*M \otimes \mathcal{E}[3])$ for $(M,[g])$.
- The characterization of G_2-holonomy in terms of a normal conformal Killing 2-form ϕ is a bit technical, since one has to find good conditions on ϕ which imply that its corresponding tractor object is of the right orbit type. It turns out that one has to demand that ϕ is locally decomposable and satisfies the genericity condition $\phi \wedge \mu \wedge \rho \neq 0$ for

$$
\begin{aligned}
\mu &:= \mathrm{tr}_{(1,2)}\, D\phi \in T^*M \\
\rho &:= +2\triangle\phi + 4\,\mathrm{alt}(\mathrm{tr}_{(1,3)}\, DD\phi) + 3\,\mathrm{alt}(\mathrm{tr}_{(2,3)}\, DD\phi) \\
&\quad + 24\,\mathrm{alt}(\mathrm{tr}_{(1,3)}\, P\otimes\phi) - 6\,\mathrm{tr}_{(1,2)}\, P\otimes\phi \in \Lambda^2 T^*M.
\end{aligned}
$$

Characterization of G_2-holonomy in terms of twistor spinors

- Another way to obtain a conformal object from **D** is to use the realization of G_2 in the orthogonal spin group of signature $(3,4)$: Let $\Delta_{\mathbb{R}}^{3,4}$ be the real 8-dimensional spin representation of $\mathrm{Spin}(3,4)$.
- There is a unique signature $(4,4)$ symmetric bilinear form on $\Delta_{\mathbb{R}}^{3,4}$, and it is shown by [Kath, 1999 (Habil.)] that one can present G_2 as the isotropy group of an arbitrary non-null $\mathrm{X} \in \Delta_{\mathbb{R}}^{3,4}$.
- This implies in particular that the induced conformal structure of an (orientable) generic distribution always carries a canonical spin structure.
- We can thus form a parallel spin tractor $\mathbf{X} \in \Gamma(\mathbf{\Sigma})$ over M; and with $\mathbf{\Delta}_{\mathbb{R}}$ the real spin bundle of $(M,[g])$, we can project $\mathbf{X}$ to a twistor spinor $\chi \in \Gamma(\mathbf{\Delta}_{\mathbb{R}} \otimes \mathcal{E}[\frac{1}{2}])$.

Characterization of G_2-holonomy in terms of twistor spinors

- It turns out that characterization via a twistor spinor is very simple: The real 4-dimensional spin representation $\Delta_{\mathbb{R}}^{2,3}$ carries a non-degenerate skew-symmetric bilinear form which can be related to the symmetric $(4,4)$-form on $\Delta_{\mathbb{R}}^{3,4}$.
- Now via the first BGG-splitting operator a twistor spinor $\chi \in \Gamma(\mathbf{\Delta}_{\mathbb{R}} \otimes \mathcal{E}[\frac{1}{2}])$ splits to a parallel spin tractor $\mathbf{X} \in \Gamma(\Sigma)$ and the condition of $\mathbf{X}$ being non-null can be related to a condition on χ:
- Let $\not{D} : \Gamma(\mathbf{\Delta}_{\mathbb{R}}) \to \Gamma(\mathbf{\Delta}_{\mathbb{R}})$ be the Dirac operator, then

Theorem

Let $(M, [g])$ be a conformal spin manifold of signature $(2,3)$ and ω the skew-symmetric form on the 4-dimensional real spin bundle $\mathbf{\Delta}_{\mathbb{R}}$. Then $[g]$ is induced from a generic rank 2-distribution iff there is a twistor spinor $\chi \in \Gamma(\mathbf{\Delta}_{\mathbb{R}} \otimes \mathcal{E}[\frac{1}{2}])$ with non-vanishing $\omega(\chi, \not{D}\chi)$.

Decomposition of infinitesimal automorphisms

- We can relate the symmetries of a generic distribution with those of the induced conformal structure:
- A vector field $\xi \in \mathfrak{X}(M)$ is a symmetry of $\mathbf{D}$ if $\mathcal{L}_\xi(\eta) \in \Gamma(\mathbf{D})$ for all $\eta \in \Gamma(\mathbf{D})$.
- A vector field $\xi \in \mathfrak{X}(M)$ is said to be a *conformal Killing field* if it preserves the conformal structure $[g]_{\mathbf{D}}$: for every representative metric g there is an $f \in \mathrm{C}^\infty(M)$ with $\mathcal{L}_\xi g = fg$.
- Since the construction $\mathbf{D} \rightsquigarrow [g]_{\mathbf{D}}$ is functorial, one has an inclusion of symmetries of $\mathbf{D}$ into the conformal Killing fields, we write

$$\mathrm{sym}(\mathbf{D}) \hookrightarrow \mathrm{cKf}([g]_{\mathbf{D}}).$$

Decomposition of infinitesimal automorphisms

- It follows from the description of infinitesimal automorphisms of parabolic geometries [Čap, JEMS (2008)] that the first BGG-operators of the adjoint tractor bundles $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}_2$ and $\tilde{\mathcal{A}}M := \tilde{\mathcal{G}} \times_{\tilde{P}} \mathfrak{so}(3,4)$ describe the symmetries of **D** and the conformal Killing fields of $[g]$.
- Now as a G_2-module, $\mathfrak{so}(3,4)$ decomposes into $\mathbb{R}^{3,4} \oplus \mathfrak{g}_2$. This implies a decomposition of the conformal adjoint tractor bundle $\tilde{A}M$ into **S** and $\mathcal{A}M$.
- This decomposition is compatible with the prolongation connections on the respective bundles. Via explicit formulas for BGG-splitting operators this yields the following decomposition theorem:

Proposition (Decomposition of conf. Killing fields via a twistor spinor)

Let $[g]_{\mathbf{D}}$ be the conformal $(2,3)$-structure induced by a generic 2-distribution $\mathbf{D} \subset TM$. Every conformal Killing field decomposes into a symmetry of the distribution $\mathbf{D}$ and another part corresponding to an Einstein scale (which may have a singularity set). Via the canonical twistor spinor $\chi \in \Gamma(\mathbf{\Delta}_{\mathbb{R}} \otimes \mathcal{E}[\frac{1}{2}])$ this decomposition can be made explicit:

- *An Einstein scale $\sigma \in \mathrm{C}^\infty(M)$ corresponds to the Killing field $\xi \in \mathfrak{X}(M)$ defined by the relation*

$$g(\xi, \eta) = \omega(\frac{2}{5}\sigma \not{D}\chi + \gamma(D\sigma)\chi, \gamma(\eta)\chi)$$

  for all $\eta \in \mathfrak{X}(M)$.

- *The Einstein scale part $\sigma \in \mathrm{C}^\infty(M)$ of a Killing field $\xi \in \mathfrak{X}(M)$ is given by*

$$\sigma = \omega(\frac{4}{5}\gamma(\xi)\not{D}\chi + \gamma(D\xi)\chi, \chi).$$