Holonomy-reductions of Cartan connections

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Plan

Holonomy of Cartan connections

Let $(\mathcal{G}, \omega)$ be a Cartan geometry of type (G, P) on M: G is a Lie group, and $P \subset G$ a closed subgroup; $\mathcal{G} \to M$ is a P-principal bundle and $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is a Cartan connection form: ω is P-equivariant, reproduces fundamental vector fields and provides an isomorphism $T\mathcal{G} = \mathcal{G} \times \mathfrak{g}$.

To define the holonomy of the Cartan geometry $(\mathcal{G}, \omega)$ we need an auxiliary principal connection form:

For this we extend $\hat{\mathcal{G}} = \mathcal{G} \times_P G$, which is now a G- principal bundle over M and canonically extends ω to a G-principal connection form $\hat{\omega} \in \Omega^1(\hat{\mathcal{G}}, \mathfrak{g})$.

Holonomy of Cartan connections

For any given $u \in \hat{\mathcal{G}}$ the holonomy group of $\hat{\omega}$ is $\mathrm{Hol}_u(\hat{\omega}) \subset G$, and if we forget about the base-point $u \in \hat{\mathcal{G}}$ we obtain the holonomy group of $\hat{\omega}$ up to G-conjugacy, denoted by $\mathrm{Hol}(\hat{\omega})$.We define this to be the holonomy of the original Cartan connection form:

$$\mathrm{Hol}(\omega) := \mathrm{Hol}(\hat{\omega}).$$

If $\mathrm{Hol}(\omega) \neq G$ we have reduced holonomy. Interesting situations with reduced holonomy are often those due to the existence of some parallel section:

Parallel sections of tractor bundles

Since $(\hat{\mathcal{G}}, \hat{\omega})$ is a G-principal bundle together with a principal connection form, for every G-representation V one obtains an associated bundle $\mathcal{V}$ together with an induced linear connection ∇^V.
One has $\mathcal{V} = \hat{\mathcal{G}} \times_G V = \mathcal{G} \times_P V$ and says that $\mathcal{V}$ is a tractor bundle together with its canonical tractor connection ∇^V.

A section of $\mathcal{V}$ is equivalent to a G-equivariant map $s : \hat{\mathcal{G}} \to V$, and this section is parallel if and only if for every parallel curve $c : [0, 1] \to \hat{\mathcal{G}}$ one has that $s \circ c : [0, 1] \to V$ is constant.

G- and P-types of parallel sections

If we assume the manifold M to be connected any two fibers of the G-bundle $\hat{\mathcal{G}} \to M$ can be joined by a parallel curve. Thus for parallel s the full image $s(\hat{\mathcal{G}}) \subset V$ is the same as the image $s(\hat{\mathcal{G}}_x)$ of an arbitrary fiber, and by G-equivariancy of s it follows that this is some G-orbit $\mathcal{O} \subset V$. We call $\mathcal{O}$ the G-type of s.

Since the Cartan bundle $\mathcal{G}$ canonically includes into the extended bundle $\hat{\mathcal{G}}$ there is also an additional point-wise data: For a given $x \in M$ we have the P-fiber $\mathcal{G}_x \subset \hat{\mathcal{G}}_x$ and can also form $s(\mathcal{G}_x) \subset \mathcal{O} \subset V$. This is a well-defined P-orbit in $\mathcal{O}$ that depends on x. We say that $s(\mathcal{G}_x) \in P\backslash\mathcal{O}$ is the P-type of s at $x \in M$.

G- and P-types of parallel sections

This yields a decomposition of the manifold M: For given $\bar{\alpha} = P \cdot \alpha \in P \backslash \mathcal{O}$ we define $M_{\bar{\alpha}}$ the set of all points in M of given P-type $\bar{\alpha}$, and then

$$M = \bigcup_{\bar{\alpha} \in P \backslash \mathcal{O}} M_{\bar{\alpha}}.$$

We now discuss this decomposition in a simple case in projective geometry:

Example: A parallel metric on the projective standard tractor bundle

Let M be a connected smooth n-manifold and $[D]$ a projective class of torsion-free affine connections on M, i.e., a projective structure. This is described as a Cartan geometry $(\mathcal{G}, \omega)$ of type $(\mathrm{SL}(n+1), P)$, with P the stabilizer of a line in $\mathbb{R}^{n+1}$.

The standard tractor bundle of $(M, [D])$ is $\mathcal{T} = \mathcal{G} \times_P \mathbb{R}^{n+1}$, and the preserved line in $\mathbb{R}^{n+1}$ gives a well-defined (projectively weighted) section $\mathbf{X}$ of $\mathcal{T}[1]$

Assume that $\mathbf{h} \in \Gamma(S^2\mathcal{T}^*)$ is a parallel non-degenerate symmetric bilinear form on $\mathcal{T}$ of signature (p, q). Then the G-type of $\mathbf{h}$ is an $\mathrm{SL}(n+1)$-orbit $\mathcal{O} \subset S^2 T^*$, namely the homogeneous space of all (p, q)-signature inner products on $\mathbb{R}^{n+1}$.

Example: A parallel metric on the projective standard tractor bundle

As a P-space, $\mathcal{O}$ decomposes into three pieces: $P\backslash\mathcal{O} = P\backslash\mathcal{O}_+ \cup P\backslash\mathcal{O}_0 \cup P\backslash\mathcal{O}_-$: $P\backslash\mathcal{O}_+$ consists of those inner products in $\mathcal{O}$ which are positive on the P-preserved line in $\mathbb{R}^{n+1}$, $P\backslash\mathcal{O}_-$ are those which are negative and with respect to the inner products in $P\backslash\mathcal{O}_0$ the preserved line is isotropic.

The corresponding P-type decomposition $M = M_+ \cup M_0 \cup M_-$ is determined by the sign of the function $\sigma := \mathbf{h}(\mathbf{X}, \mathbf{X})$.

Holonomy-reductions of type $\mathcal{O}$

Back in the general situation, where $(\mathcal{G}, \omega)$ is an arbitrary Cartan geometry of type (G, P), let $\mathcal{O}$ be a G-homogeneous space, which need not necessarily sit inside some G-representation. The non-linear tractor bundle $\mathcal{G} \times_P \mathcal{O}$ inherits a natural connection from ω and it is useful to define a holonomy reduction of ω in this setting:

Definition

A holonomy reduction of $(\mathcal{G}, \omega)$ of type $\mathcal{O}$ is a G-equivariant map $s : \hat{\mathcal{G}} \to \mathcal{O}$ which is parallel with respect to the induced connection on $\mathcal{G} \times_P \mathcal{O}$.

P-type decomposition on the homogeneous model

We now ask what a holonomy reduction of type $\mathcal{O}$ and its induced decomposition look like on the homogeneous model $M = G/P$.

The extended Cartan bundle $\hat{\mathcal{G}} = G \times_P G$ is canonically trivialized to $G/P \times G$ via the map

$$G/P \times G \to G \times_P G$$
$$(gP, g') \mapsto [g, g^{-1}g']_P.$$

Then, if $s : \hat{\mathcal{G}} \to \mathcal{O}$ is a G-equivariant map, this trivializes to a map $s : G/P \times G \to \mathcal{O}$, and it is easy to see that s is parallel if and only if this map is constant in G/P. In particular, by G-equivariancy, s is completely determined by $\alpha = s(eP, e) \in \mathcal{O}$.

P-type decomposition on the homogeneous model

We denote $H = G_\alpha \subset G$ the isotropy group of $\alpha \in \mathcal{O}$. Then $\mathcal{O} = G/H$ and s is given by

$$s : G/P \times G \to G/H, (gP, g') \mapsto (g')^{-1}H.$$

For a point $gP \in G/P$ we have $\hat{\mathcal{G}}_{gP} \supset \mathcal{G}_{gP} = (gP, gP)$ and thus

$$s(\mathcal{G}_{gP}) = Pg^{-1}H \in P\backslash\mathcal{O} = P\backslash G/H.$$

P-type decomposition on the homogeneous model

The map

$$\begin{aligned} G/P &\to P\backslash\mathcal{O}, \\ gP &\mapsto P-\text{type at } gP \end{aligned}$$

thus factorizes to the isomorphism

$$\begin{aligned} H\backslash G/P &\to P\backslash\mathcal{O} = P\backslash G/H, \\ HgP &\mapsto Pg^{-1}H \end{aligned}$$

between double co-set spaces.

This shows that $M_{Pg^{-1}H} = HgP/P = H \cdot (gP/P) \subset G/P$. So the points in G/P of type $\bar{\alpha} = Pg^{-1}H \in P\backslash\mathcal{O}$ are exactly those in the H-orbit of $gP/P \in G/P$.

Comparison theorem

Theorem

Let $(\mathcal{G} \to M, \omega)$ and $(\mathcal{G}' \to M', \omega')$ be Cartan geometries of type (G, P) which have given holonomy reductions of type $\mathcal{O}$. Assume, for some $\bar{\alpha} \in P \backslash \mathcal{O}$, that there are points $x \in M_{\bar{\alpha}}$, $x' \in M'_{\bar{\alpha}}$. Then there exists a local diffeomorphism $\varphi : N \to N'$ between neighborhoods N of x and N' of x' with $\varphi(x) = x'$ which maps $M_{\bar{\beta}} \cap N'$ to $M'_{\bar{\beta}} \cap N'$ for all $\bar{\beta} \in P \backslash \mathcal{O}$.

This says that the P-type of s is locally determined by its P-type at one point.

Sketch of proof of comparison theorem: adapted normal coordinates

The proof is based on adapted normal coordinates. Take some point $u \in \mathcal{G}_x \subset \hat{G}_x$ with $s(u) = \alpha \in \mathcal{O}$. To form normal coordinates for the Cartan geometry we choose some complement $\mathfrak{g}_- \subset \mathfrak{g}$ to $\mathfrak{p} \subset \mathfrak{g}$. Denote, for $X \in \mathfrak{g}_-$, by $\zeta^X \in \mathfrak{X}(\mathcal{G})$ the vector field that is defined by $\zeta^X_u = \omega_u^{-1}(X)$. Let $\pi : \hat{\mathcal{G}} \to M$ denote the surjective submersion of the principal bundle.

Let $\Psi : \mathfrak{g}_- \to \mathcal{G},\ \Psi(X) := \mathrm{Fl}_1^{\zeta_X}(u)$, i.e., we follow the flow of the vector field ζ_X, starting at u to time 1. Then it is easy to see that the map $\psi = \pi \circ \Psi$ defines a local diffeomorphism between suitable neighborhoods $W \subset \mathfrak{g}_-$ and $N \subset M$ of $0 \in \mathfrak{g}_-$ resp. $x \in M$.

Sketch of proof of comparison theorem: adapted normal coordinates

Now we define the local section $\tau : N \to \hat{\mathcal{G}}$, $\tau(\psi(X)) := \Psi(X) \cdot \exp(-X)$. Then the radial curves

$$c : \mathbb{R} \to \hat{\mathcal{G}},$$
$$t \mapsto \tau(tX) = \mathrm{Fl}^X_t(u) \cdot \exp(-tX)$$

are parallel with respect to $\hat{\omega}$.

In particular, since $s : \hat{\mathcal{G}} \to \mathcal{O}$ is constant on parallel curves, it follows that for all $X \in W \subset \mathfrak{g}_-$ one has $s(\tau(X)) = s(\tau(0)) = s(u) = \alpha \in \mathcal{O}$. Then by equivariancy $s(\Psi(X)) = s(\tau(X) \cdot \exp(X)) = \exp(-X) \cdot \alpha$, and therefore the P-type of s at $\psi(X)$ equals $P \cdot \exp(-X) \cdot \alpha$.

Consequences of the comparison theorem

The comparison theorem tells us that the local structure of the P-type decomposition of type $\mathcal{O}$ can already be seen on the homogeneous model:

Choosing the model space $M' = G/P$ and the map $s' : G/P \times G \to \mathcal{O}$, $s'((gP, g')) = g'^{-1} \cdot \alpha$ we have computed that the P-type decomposition of G/P is then simply the $H = G_\alpha$-orbit decomposition of G/P.

We therefore obtain:

Corollary

For all $\bar{\alpha} \in P \backslash \mathcal{O}$, $M_{\bar{\alpha}}$ is either empty or an initial submanifold of M that is locally diffeomorphic to $G_\alpha/(G_\alpha \cap P)$.

The reduced Cartan geometries on curved orbits

$G_\alpha / G_\alpha \cap P$ is the homogeneous model of Cartan geometries of type $(G_\alpha, G_\alpha \cap P)$. This Cartan geometric structure carries over to the curved orbit $M_{\bar{\alpha}}$:

Theorem

Let $\alpha \in P \backslash \mathcal{O}$ be such that $M_{\bar{\alpha}} \neq \emptyset$. Then $M_{\bar{\alpha}} \subset M$ carries in a natural way a Cartan geometry of type $(G_\alpha, G_\alpha \cap P)$ that is induced from the holonomy reduced Cartan geometry $(\mathcal{G} \to M, \omega)$.

The remaining work in specific cases is to see what the normalization conditions that were employed for the original Cartan connection form, respectively its curvature, imply for the reduced structure.

Projective structure with a tractor metric

In our example of a projective structure $(M,[D])$ endowed with a parallel tractor metric $\mathbf{h} \in \Gamma^2(S^2 T^* M)$ we have a global G-type $\mathcal{O} \subset S^2 \mathbb{R}^{n+1*}$ which is the space of all signature (p,q) inner products on $\mathbb{R}^{n+1}$. In particular, for all $h \in \mathcal{O}$, we have $G_h = \mathrm{SO}(p,q) \subset \mathrm{SL}(n+1)$.

Denote the line in $\mathbb{R}^{n+1}$ that is stabilized by the parabolic subgroup $P \subset G$ by $\mathbb{R}X$. For $h \in \mathcal{O}$ with $h(X,X) > 0$ we then have $G_h \cap P = P_h = \mathrm{SO}(p-1,q)$.

It follows that $M_+ = \{x \in M : \sigma(x) > 0\}$ (with $\sigma = \mathbf{h}(\mathbf{X},\mathbf{X})$) is endowed with a Cartan geometry of type $(\mathrm{SO}(p,q), \mathrm{SO}(p-1,q))$. This describes a signature $(p-1,q)$ metric on M_+, and the normalization condition of the projective Cartan connection ω respectively its curvature imply that this metric is Einstein.

Projective structure with a tractor metric

Analogously, $M_- = \{x \in M : \sigma(x) < 0\}$ carries an Einstein metric of signature $(p, q-1)$.

For $h \in \mathcal{O}$ with $h(X, X) = 0$ we have that $G_h \cap P = P_h$ is the parabolic subgroup of $\mathrm{SO}(p, q)$ that preserves an isotropic line. Therefore the hypersurface $M_0 = \sigma^{-1}(\{0\}$ carries a conformal structure of signature $(p-1, q-1)$.

Since the projective structure $(M, [D])$ is an instance of a parabolic geometry, parallel sections of the tractor bundle $S^2\mathcal{T}^* = \mathcal{G} \times_P S^2(\mathbb{R}^{n+1})^*$ are equivalent to normal solutions of the corresponding first BGG-equation on σ.

Zero-sets of natural quotients

When s is a parallel section of a tractor bundle $\mathcal{V} = \mathcal{G} \times_P V$ and $W \subset V$ is some P-subbundle, we can form $\mathcal{W} = \mathcal{G} \times_P V$ and take the quotient $\sigma = s/\mathcal{W}$.

Let $\mathcal{O} \subset V$ be the G-type of s. Now since W is P-invariant we see that the possible P-types of s, which are $P \backslash \mathcal{O}$, decompose into

$$P \backslash \mathcal{O}_0 := P \backslash (W \cap \mathcal{O})$$

and some complement.

It is easy to see that $\sigma = s/\mathcal{W}$ vanishes at some $x \in M$ if and only if the P-type of s at x is contained in $P \backslash \mathcal{O}_0$. In particular it follows from the comparison theorem that the local structure of the zero set of σ is already visible on the homogeneous model G/P.

Relation to normal BGG-solutions

In the case where G is a semi-simple Lie group and $P \subset G$ is a parabolic subgroup, the BGG-machinery of [Čap-Slovak-Souček] relates sections of a tractor bundle $\mathcal{V} = \mathcal{G} \times_P V$ with solutions of an overdetermined system $\Theta_0(\sigma) = 0$ on a natural quotient $\sigma = s/\mathcal{V}^1$ of s. The normal solutions of $\Theta_0(\sigma) = 0$ are in $1:1$-correspondence with parallel sections $s \in \Gamma(\mathcal{V})$.

It follows that

1. The structure of the zero set $\sigma^{-1}(\{0\})$ of normal solutions of $\Theta_0(\sigma) = 0$ is already completely visible on the homogeneous model G/P.
2. The zero set decomposes into a union of curved orbits, each of which carries a canonical (reduced) Cartan geometry.

Global holonomy reductions for $H \subset G$ acting transitively

Some interesting special cases of holonomy reductions of type $\mathcal{O}$ occur when $P \backslash \mathcal{O}$ only consists of one point. If $H = G_\alpha, \alpha \in \mathcal{O}$ is an isotropy group of that reduction, the duality $P \backslash \mathcal{O} = P \backslash G/H \cong H \backslash G/P$ gives the equivalent condition that H acts transitively on G/P. In this case there is a global reduction from the Cartan geometry (G, P) on M to a Cartan geometry of type $(H, H \cap P)$.

Holonomy reductions of Fefferman-type spaces

Let $\mathcal{O}$ be the set of all orthogonal complex structures on $\mathbb{R}^{2p+2,2q+2}$. With $\mathbb{J} \in \mathcal{O}$ one has $H = G_{\mathbb{J}} = \mathrm{U}(p+1, q+1)$, and this acts transitively on $\mathrm{SO}(2p+2, 2q+2)/P$. It was shown by [Čap-Gover, Leitner], that conformal holonomy $\mathrm{Hol}(\omega) \subset \mathrm{U}(p+1, q+1)$ already implies locally that $\mathrm{Hol}(\omega) \subset \mathrm{SU}(p+1, q+1)$.

Given the parallel orthogonal complex structure, the corresponding normal BGG-solution is a light-like conformal Killing field on $(M, [g])$. The resulting reduced Cartan geometry locally factorizes to a CR-structure, and the conformal geometry is completely determined by that CR-structure via the classical Fefferman-construction.

Holonomy reductions of Fefferman-type spaces

Let $\mathcal{O}$ be the set of all non-isotropic spinors in the 8-dimensional real spin representation $\Delta_{\mathbb{R}}^{3,4}$ of $\mathrm{Spin}(3,4)$. The stabilizer of such a spinor provides an embedding of G_2 into $\mathrm{Spin}(3,4)$, and since G_2 is seen to act transitively on $\mathrm{Spin}(3,4)/P$ there is again only a singe P-type.

Given the non-isotropic parallel spinor on the conformal manifold $(M,[g])$, the corresponding first BGG-solution is a twistor spinor χ. The resulting holonomy reduction describes the geometry of a generic rank 2-distribution, which is formed by $\ker \chi \cap \ker g$.

This is again an instance of a Fefferman-type construction [H.-Sagerschnig] and the original conformal structure is completely determined by this rank 2-distribution.