universität wien

DISSERTATION

Natural Prolongations of BGG-Operators

Verfasser

Mag. Matthias Hammerl

angestrebter akademischer Grad:
Doktor der Naturwissenschaften (Dr. rer. nat.)

Wien, Juli 2009

Studienkennzahl lt. Studienblatt:	A 091 405
Dissertationsgebiet lt. Studienblatt:	Mathematik
Betreuet von:	a.o. Univ.-Prof. Dr. Andreas Čap.

Contents

CHAPTER 1

Introduction

The main result of this work is a natural prolongation procedure for certain overdetermined systems arising for parabolic geometries. The overdetermined linear differential operators which describe the systems in question are the first BGG-operators, i.e., the first operators appearing in BGG-sequences. These operators are natural with respect to the underlying geometric structure on the manifold, and the same is true for the resulting equations. A main feature of our method is that also the prolongation connection we construct is natural.

A simple and interesting case in which one can illustrate both problem and solution comes from conformal geometry:

1.1. An example from conformal geometry: the prolongation of the conformally invariant operator governing (almost) Einstein scales

A conformal structure on a manifold M is an equivalence class $[g]$ of pseudo-Riemannian metrics, where two metrics g and $\hat{g}$ are equivalent if and only if there is a function $f \in \mathrm{C}^\infty(M)$ such that $\hat{g} = e^{2f}g$. The simplest way to explain what a conformally invariant operator is, is to give an example: The formula for the operator will at first depend on the choice of metric $g \in [g]$, resp. its Levi-Civita connection D, and maps smooth functions on M into trace-free symmetric bilinear forms on M:

$$\Theta_0^g : \mathrm{C}^\infty(M) \to \Gamma(S_0^2 T^*M), \tag{1}$$

$$\sigma \mapsto \big(DD\sigma + \sigma \mathrm{P}\big)_0. \tag{2}$$

Here

$$\mathrm{P} = \mathrm{P}(g) = \frac{1}{n-2}(\mathrm{Ric}(g) - \frac{\mathrm{Sc}(g)}{2(n-1)}g) \in \Gamma(S^2T^*M)$$

is the Schouten tensor of g, which is a trace modification of Ricci curvature $\mathrm{Ric} = \mathrm{Ric}(g)$ by scalar curvature $\mathrm{Sc} = \mathrm{Sc}(g)$. Subscript 0 takes the trace-free part.

Now it is well known ([**BEG94**]), and we will check this in chapter 6, that Θ_0^g describes the equation governing Einstein scales: for $\sigma \in \mathrm{C}^\infty(M)$ nowhere vanishing one has $\Theta_0^g\sigma = 0$ if and only if $\sigma^{-2}g$ is Einstein, i.e., iff $\mathrm{Ric}(s^{-2}g)$ is purely trace. Moreover, the operator Θ_0^g is *conformally covariant* between $\mathrm{C}^\infty(M)$ and $\Gamma(S_0^2T^*M)$: if one switches to another metric $\hat{g} = e^{2f}g$ in the conformal class,

$$\Theta_0^{\hat{g}} \circ m(e^f) = m(e^f) \circ \Theta_0^g, \tag{3}$$

where $m(e^f)$ is simply the multiplication operator with e^f.

If we want an operator which doesn't depend on a choice of metric we need to introduce the conformal density bundles: For every $w \in \mathbb{R}$, $\mathbf{E}[w]$ is a line bundle which is trivialized canonically after a choice of a $g \in [g]$. Let $\hat{g} = e^{2f} g$. If $[\sigma]_g$ is the trivialization of a section $\sigma \in \mathcal{E}[w] := \Gamma(\mathbf{E}[w])$, then $[\sigma]_{\hat{g}} = e^{wf}[\sigma]_g$. From the definition it is clear that one can use $\mathbf{E}[2]$ to define a *conformal metric*

$$\boldsymbol{g} \in \Gamma(S^2 T^*M \otimes \mathbf{E}[2])$$

which trivializes to g for $g \in [g]$. Its inverse is denoted by $\boldsymbol{g}^{-1} \in \Gamma(S^2 TM \otimes \mathbf{E}[-2])$. We will also regard $\boldsymbol{g}$ and $\boldsymbol{g}^{-1}$ as isomorphisms; e.g., for $\varphi \in \Omega^1(M), \boldsymbol{g}^{-1}(\varphi) \in \Gamma(TM \otimes \mathbf{E}[-2])$

To obtain a conformally invariant operator from Θ_0^g we will tensor the source and target spaces of Θ_0^g by suitable density bundles:

$$\mathbf{H}_0 := \mathbf{E}[1],$$
$$\mathbf{H}_1 := S_0^2 T^*M \otimes \mathbf{E}[1].$$

Then (3) is seen to be equivalent to Θ_0^g defining the same operator

$$\Theta_0 : \mathcal{H}_0 := \Gamma(\mathbf{H_0}) \to \Gamma(\mathbf{H}_1) =: \mathcal{H}_1$$

for every $g \in [g]$, and we say that Θ_0 is a conformally invariant operator.

In general, a conformally invariant operator is obtained by a universal formula in the Levi-Civita connection, the metric, the curvature, and the volume form, possibly followed by contractions, such that one obtains a well-defined operator between natural bundles for the conformal structure.

The example of the operator for Einstein scales above has another interesting property: it is overdetermined, and thus one can wish to have a prolongation of the corresponding system of equations: in classical terms, this means that one wants to introduce more dependent variables and derive differential consequences of the overdetermined system, such that one can write down a closed system of equations; i.e., a system of first order PDEs in which all (first order) derivatives of the dependent variables are expressed in the dependent variables themselves.

1.1.1. The standard tractor bundle of conformal geometry and the prolongation of the equation governing Einstein scales. The prolongation of $\Theta_0(\sigma) = \left(DD\sigma + \sigma \mathrm{P}\right)_0 \stackrel{!}{=}$ is well known and conformally invariant. With respect to a metric g in the conformal class the *standard tractor bundle* $\mathbf{S}$ of a conformal geometry is given by

$$[\mathbf{S}]_g = \mathbf{E}[1] \oplus T^*M \otimes \mathbf{E}[1] \oplus \mathbf{E}[-1] \tag{4}$$

and one writes elements $[s]_g = \sigma \oplus \varphi \oplus \rho \in [\mathcal{S}]_g$ as

$$[s]_g = \begin{pmatrix} \rho \\ \varphi \\ \sigma \end{pmatrix}. \tag{5}$$

We will say more about the standard tractor bundle than would be strictly necessary at this place to write down the prolongation, since many the structures introduced below for $\mathbf{S}$ will also occur later for general tractor bundles.

For $\hat{g} = e^{2f} g$ one has the transformation

$$[s]_{\hat{g}} = \begin{pmatrix} \hat{\rho} \\ \hat{\varphi} \\ \hat{\sigma} \end{pmatrix} = \begin{pmatrix} \rho - \boldsymbol{g}^{-1}(\Upsilon, \varphi) - \frac{1}{2}\sigma \boldsymbol{g}^{-1}(\Upsilon, \Upsilon) \\ \varphi + \sigma \Upsilon \\ \sigma \end{pmatrix} \tag{6}$$

where $\Upsilon = df \in \Omega^1(M)$. $\mathbf{S}$ is defined as the equivalence class of $[\mathbf{S}]_g$ for $g \in [g]$ with respect to this transformation (see also [**BEG94**]). The space of sections $\Gamma(\mathbf{S})$ is denoted $\mathcal{S}$.

The filtration

$$\begin{aligned} [\mathbf{S}]_g^{-1} &= [\mathbf{S}]_g = \mathbf{E}[1] \oplus T^*M \otimes \mathbf{E}[1] \oplus \mathbf{E}[-1], \\ [\mathbf{S}]_g^0 &= T^*M \otimes \mathbf{E}[1] \oplus \mathbf{E}[-1], \\ [\mathbf{S}]_g^1 &= \mathbf{E}[-1] \end{aligned}$$

is compatible with (6) and thus defines a filtration

$$\mathbf{S} = \mathbf{S}^{-1} \supset \mathbf{S}^0 \supset \mathbf{S}^1$$

of $\mathbf{S}$. By construction this filtration splits according to (4) with respect to a metric g in the conformal class

We will denote the filtration of $\mathbf{S}$ as a composition series in the form

$$\mathbf{S} = \mathbf{E}[1] \oplus\!\!\!\!+ T^*M \otimes \mathbf{E}[1] \oplus\!\!\!\!+ \mathbf{E}[-1], \tag{7}$$

since

$$\begin{aligned} \mathbf{S}^{-1}/\mathbf{S}^0 &= \mathbf{E}[1], \\ \mathbf{S}^0/\mathbf{S}^1 &= T^*M \otimes \mathbf{E}[1], \\ \mathbf{S}^1 &= \mathbf{E}[-1]. \end{aligned}$$

Thus (7) says that the *associated graded*

$$\mathrm{gr}(\mathbf{S}) := \mathrm{gr}_{-1}(\mathbf{S}) \oplus \mathrm{gr}_0(\mathbf{S}) \oplus \mathrm{gr}_1(\mathbf{S}) := \mathbf{S}^{-1}/\mathbf{S}^0 \oplus \mathbf{S}^0/\mathbf{S} \oplus \mathbf{S}^1$$

is given by (4).

The bundle $[\mathbf{S}]_g$ is endowed with the connection

$$\nabla_\xi [s]_g = \nabla_\xi \begin{pmatrix} \rho \\ \varphi \\ \sigma \end{pmatrix} = \begin{pmatrix} D_\xi \rho - \mathrm{P}(\xi, \boldsymbol{g}^{-1}(\varphi)) \\ D_\xi \varphi + \sigma i_\xi \mathrm{P} + \rho i_\xi \boldsymbol{g} \\ D_\xi \sigma - \varphi(\xi) \end{pmatrix}, \tag{8}$$

which is invariant with respect to the transformation (6) and thus gives a well defined connection on $\mathcal{S}$, called the standard tractor connection.

We furthermore see from (6) that one has a well-defined projection Π to the 'lowest slot' $\mathbf{H}_0 = \mathbf{E}[1]$ of $\mathbf{S}$. This projection splits via the differential operator $L_0 : \Gamma(\mathbf{H}_0) = \mathcal{H}_0 \to \mathcal{S}$, which is again defined via a metric g:

$$\sigma \in \mathcal{E}[1] \mapsto \begin{pmatrix} \frac{1}{n}(\triangle \sigma - J\sigma) \\ D\sigma \\ \sigma \end{pmatrix},$$

with $\triangle\sigma = -\boldsymbol{g}^{-1}(DD\sigma)$ and $J \in \mathcal{E}[-2]$ the trace of $\mathrm{P} \in S^2T^*M$ with respect to $\boldsymbol{g}^{-1} \in \Gamma(S^2TM \otimes \mathbf{E}[-2])$. We calculate

$$\nabla(L_0(\sigma)) = \begin{pmatrix} \frac{1}{n}D(\triangle\sigma - J\sigma) - \mathrm{P}(\cdot, \boldsymbol{g}^{-1}(D\sigma)) \\ (DD\sigma + \mathrm{P}\sigma) + \frac{1}{n}(\triangle\sigma - J\sigma)\boldsymbol{g} \\ 0 \end{pmatrix}.$$

Since $\frac{1}{n}(\triangle\sigma - J\sigma)\boldsymbol{g}$ is minus the trace part of $(DD\sigma + \mathrm{P}\sigma)$, we see that $\Theta_0 : \mathcal{H}_0 \to \mathcal{H}_1$ comes about as the composition of $\nabla \circ L_0$ with the projection to the middle slot of $\mathcal{S}$. This is an example of the calculation of a first BGG-operator.

Let now $\sigma \in \mathrm{C}^\infty(M)$ satisfy $(DD\sigma + P\sigma) + \frac{1}{n}(\triangle\sigma - J\sigma)\boldsymbol{g} = 0$. Then also

$$D^3\sigma + (DP)\sigma + PD\sigma + \frac{1}{n}D(\triangle\sigma - J\sigma)\boldsymbol{g} = 0,$$

and an alternation in the first two slots followed by trace yields

$$-\mathrm{Ric}(\cdot, \boldsymbol{g}^{-1}(D\sigma)) + JD\sigma - P(\cdot, \boldsymbol{g}^{-1}D\sigma) + \frac{1}{n}(n-1)D(\triangle\sigma - J\sigma) = 0.$$

Then, using the definition of P, this is immediately seen to be equivalent to vanishing of $\frac{1}{n}D(\triangle\sigma - J\sigma) - \mathrm{P}(\cdot, \boldsymbol{g}^{-1}(D\sigma))$. Therefore $\Theta_0\sigma = 0$, which is the middle equation of $\nabla(L_0(\sigma)) = 0$, already implies that also the top slot of $\nabla(L_0(\sigma))$ vanishes, and we see that $\Theta_0(\sigma) = 0$ is equivalent to $\nabla(L_0(\sigma)) = 0$. We therefore say that $(\mathbf{S}, \nabla, \Pi, L_0)$ is a *geometric prolongation* of $\Theta_0 : \mathcal{H}_0 \to \mathcal{H}_1$: The maps Π and L_0 restrict to inverse isomorphisms between the space of parallel sections of $\mathbf{S}$ with respect to ∇ and the space of Einstein scales in $\mathcal{H}_0$. This is of course well known ([**BEG94**]), and will be obtained in a far more conceptual way in 6.1. The approach taken here to get a prolongation of the equation $\Theta_0(\sigma) \overset{!}{=} 0$ can be stated in the classical language: we introduced more dependent variables via the extension of $\mathbf{H}_0$ by $\mathbf{S}$ and found the differential consequences of $\Theta_0(\sigma) = 0$, namely $\nabla(L_0(\sigma)) = 0$.

1.2. The general procedure: prolongation of first BGG-operators

The general situation will be the following: We consider a parabolic geometry

$$\begin{array}{ccc} (\mathcal{G}, \omega) & \longleftarrow & P \\ \downarrow & & \\ M & & \end{array}$$

of type (G, P) on a manifold M. In the example above this would encode a conformal structure $[g]$ on M. The necessary parts of the, by now quite extensive, general theory of parabolic geometries will be recalled in chapter 2.

Any G-representation V gives rise to the associated *tractor bundle*

$$\mathbf{V} = \mathcal{G} \times_P V$$

which shares basic properties with the standard tractor bundle of conformal geometry discussed above: It is endowed with a canonical linear connection $\nabla : \Gamma(\mathbf{V}) = \mathcal{V} \to \Omega^1(M, \mathbf{V})$ and there is again a a canonical quotient $\mathbf{H}_0$

via a surjection $\Pi : \mathbf{V} \to \mathbf{H}_0$. The crucial fact is, as we will discuss in some detail in the next chapter, that this projection admits a differential splitting

$$L_0 : \Gamma(\mathbf{H}_0) = \mathcal{H}_0 \to \mathcal{V} = \Gamma(\mathcal{V}).$$

This splitting is the central part in the construction of the first BGG-operator

$$\Theta_0 : \mathcal{H}_0 \to \Omega^1(M, \mathbf{H}_0)),$$

which will be described in section 2.4 of chapter 2.

Then we will prove in chapter 4:

THEOREM. *There exists a natural linear connection $\tilde{\nabla}$ on $\mathcal{V}$ such that the projection $\Pi : \mathcal{V} \to \mathcal{H}_0$ and the differential splitting operator $L_0 : \mathcal{H}_0 \to \mathcal{V}$ restrict to inverse isomorphisms between the space of parallel sections of $(\mathbf{V}, \tilde{\nabla})$ and the kernel of Θ_0.*

I.e., every first BGG-operator has a geometric prolongation $(\mathbf{V}, \tilde{\nabla}, \Pi, L_0)$.

Inconveniently, this natural connection coincides with the associated tractor connection only in a few special cases.

For 1-graded parabolic geometries a prolongation procedure for first BGG-operators has already been constructed in [**BČEG06**]. The result of this thesis has two advantages: One advantage of the procedure presented here is that it works for arbitrary regular parabolic geometries. The main point however is that the prolongation connection we construct is natural.

1.3. Overview of this text

In chapter 2 we will present the necessary background for parabolic geometries. The general facts on Cartan geometries will be recalled succinctly, mostly to fix the notation. The main focus will lie on tractor bundles, their cohomology and the construction of the BGG-sequence.

In chapter 3 we will construct a natural modification/adjustment of the tractor connection such that the first BGG-diagram commutes. This is the main technical part in the solution of the prolongation problem, which will then be discussed in chapter 4.

Next we come to examples: in chapter 5 we start by treating the simplest BGG-operators in projective geometry, namely those which arise for the standard- resp. the dual standard tractor bundle and then proceed to two more elaborate examples.

In chapter 6 we treat conformally invariant equations. After reviewing the standard tractor bundle in our framework we proceed to the prolongation of the operator governing conformal Killing k-forms in 6.2. In section 6.3 we give a convenient description of the spin-tractor bundle and use this to study the twistor-spinor equation.

Finally, in chapter 7, we use BGG-techniques and prolongation connections to study a generalized Fefferman construction associating conformal structures of signature $(2, 3)$ to generic rank two distributions in dimension five. We obtain a characterizations of those conformal structures which are induced by generic distributions via conformal Killing two-forms and another characterization via twistor-spinors. Moreover, we give a decomposition theorem for conformal Killing fields.

Acknowledgements

Special thanks go to Andreas Čap for his support and encouragement throughout this thesis project. It was his work, which started my interest in representation-theoretic approaches towards geometry, and I have undoubtedly benefited greatly from his explanations, suggestions, and general approach.

I also thank Prof. Vladimir Souček for an interesting time in Prague early in 2008, when parts of this thesis were discussed and presented. Talking with Josef Šilhan often provided for more insight. I also thank Thomas Leistner, who pointed me to the beautiful example [**EM07**]. Several conversations with Felipe Leitner have been very valuable to me; in particular, he introduced me to the notion of orbit types of parallel tractors.

Without question, this thesis has benefited greatly from many discussions with Katja Sagerschnig, who often enjoyed the doubtful privilege of hearing the 'first version' of many of the arguments presented here. Due to her work, I also became interested in generic rank two distributions and G_2, which led to the joint characterization project [**HS09**], parts of which are contained in chapter 7.

I also thank the referees of [**Ham08**] and [**HS09**] for their careful reports with corrections and useful suggestions, which concern parts of chapters 6 and 7 of this thesis. Last but not least, I thank Clemens Hanel, who often provided support with his extensive knowledge on LaTeX-related questions.

CHAPTER 2

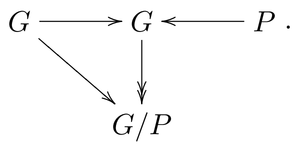
Preliminaries on parabolic geometries, tractor bundles and the BGG-machinery

In this chapter we will recall basic notions of Cartan- and parabolic geometries and then treat tractor bundles and the BGG-sequence. For an extensive treatment of parabolic geometries we refer to [**ČS09**]. The basic references for tractor calculus are [**ČG02**] and [**ČG00**].

2.1. Cartan geometries

Let us first discuss Cartan geometries of type (G, P), with G some (real) Lie group and $P < G$ a closed subgroup. We will always assume that G/P is connected. The Lie algebras of G and P will be denoted by $\mathfrak{g}$ resp. $\mathfrak{p}$.

The first step is to view the homogeneous space $M = G/P$ as a geometric structure, whose automorphism group is exactly G, acting upon $M = G/P$ from the left. Diagrammatically:



One can intrinsically describe this geometric data via the *Maurer-Cartan* form $\omega^{MC} \in \Omega^1(G, \mathfrak{g})$ which is just left-trivialization of $TG = G \times \mathfrak{g}$. It satisfies the *Maurer-Cartan* equation

$$d\omega^{MC}(X, Y) + [\omega(X), \omega(Y)] = 0 \text{ for all } X, Y \in \mathfrak{g}. \tag{9}$$

We say that the automorphisms of (G, ω^{MC}) are the (right-)P-equivariant diffeomorphisms of G preserving ω^{MC}:

Proposition. $\mathbf{Aut}(G, \omega^{MC}) = \{\Psi : G \to G : \Psi(gp) = \Psi(g)p$ *for all* $p \in P$ *and* $\Psi^*\omega^{MC} = \omega^{MC}\}$.

Then it is well known that $\mathbf{Aut}(G, \omega^{MC}) = G$.

Now a *Cartan geometry* of type (G, P) is a P-principal bundle $\Pi : \mathcal{G} \to M$ endowed with a *Cartan connection* form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, which satisfies the possible generalizations of ω^{MC} to a so called 'curved' setting: Denote the P-right action of $g \in P$ by r^g, i.e., $r^g(u) = u \cdot p$, and the fundamental vector fields for this action by ζ_Y for $Y \in \mathfrak{p}$, i.e., $\zeta_Y(u) = \frac{d}{dt}_{|t=0}(u \cdot \exp(tY))$. Then one demands that

(C.1) $\omega_{u \cdot p}(T_u r^p \xi) = \mathrm{Ad}(p^{-1})\omega_u(\xi)$ for all $p \in P$, $u \in \mathcal{G}$, and $\xi \in T_u\mathcal{G}$.
(C.2) $\omega(\zeta_Y) = Y$ for all $Y \in \mathfrak{p}$.
(C.3) $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$.

We say that ω is right-equivariant, reproduces fundamental vector fields and is an absolute parallelism $\omega : T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$.

The automorphisms of $(\mathcal{G}, \omega)$ are

$$\mathbf{Aut}(\mathcal{G}, \omega) := \{\Psi \in \mathrm{Diff}(\mathcal{G})^P : \Psi^*\omega = \omega\}, \tag{10}$$

which is a Lie group (see for instance [**ČS09**]) with Lie algebra a subalgebra of the P-invariant vector fields $\mathfrak{X}(\mathcal{G})$:

$$\mathbf{aut}(\mathcal{G}, \omega) := \{\xi \in \mathfrak{X}(\mathcal{G})^P : \xi \text{ is complete and } \mathcal{L}_\xi \omega = 0\}. \tag{11}$$

One has that $\dim \mathbf{Aut}(\mathcal{G}, \omega) \leq \dim G = \dim \mathbf{Aut}(G, \omega^{MC})$, with equality implying that $(\mathcal{G}, \omega)$ and (G, ω^{MC}) are locally isomorphic as Cartan geometries.

Let $\underline{\mathrm{Ad}} : P \to \mathrm{GL}(\mathfrak{g}/\mathfrak{p})$ be the representation on $\mathfrak{g}/\mathfrak{p}$ induced by Ad, which makes sense since $\mathfrak{p} \subset \mathfrak{g}$ is $\mathrm{Ad}(P)$-invariant. It is a direct consequence of (C.1)-(C.3) that

$$TM = \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}. \tag{12}$$

In particular, $\dim M = \dim \mathfrak{g}/\mathfrak{p}$. For a general Cartan geometry $(\mathcal{G}, \omega)$ of type (G, P) the failure of ω to satisfy the Maurer-Cartan equation (9) is measured by the *curvature* form $\Omega \in \Omega^2(\mathcal{G}, \mathfrak{g})$,

$$\Omega(\xi, \eta) := d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)] \tag{13}$$

for $\xi, \eta \in \mathfrak{X}(\mathcal{G})$. One can show that Ω vanishes, i.e., ω is *locally flat*, if and only if $(\mathcal{G}, \omega)$ is locally isomorphic to (G, ω^{MC}).

Since the Cartan connection defines an absolute parallelism

$$\omega : T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$$

its curvature can be equivalently encoded in the curvature function $\kappa \in \mathrm{C}^\infty(\mathcal{G}, \Lambda^2(\mathfrak{g}^*) \otimes \mathfrak{g})$,

$$\kappa(u)(X, Y) := \Omega(\omega_u^{-1}(X), \omega_u^{-1}(Y)).$$

One verifies that Ω vanishes upon insertion of a vertical field ζ_Y for $Y \in \mathfrak{p}$, i.e. it is horizontal. Moreover, Ω is P-equivariant, and we will write $\Omega \in \Omega^2_{\mathrm{hor}}(\mathcal{G}, \mathfrak{g})^P$. Thus we can view κ as a P-equivariant function $\mathcal{G} \mapsto \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$.

We denote by

$$\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g} \tag{14}$$

the associated bundle corresponding to the restriction of the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ to P, called the *adjoint tractor bundle* (general tractor bundles will be introduced later).

Since the curvature-form $\Omega \in \Omega^2_{\mathrm{hor}}(\mathcal{G}, \mathfrak{g})^P$ is horizontal and P-equivariant, it factorizes to a $\mathcal{A}M$-valued 2-form $K \in \Omega^2(M, \mathcal{A}M)$ on M. Thus $\Omega \in \Omega^2_{\mathrm{hor}}(\mathcal{G}, \mathfrak{g})^P, K \in \Omega^2(M, \mathcal{A}M)$ and $\kappa \in \mathrm{C}^\infty(\mathcal{G}, \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g})^P$ all encode essentially the same object, namely the curvature of the Cartan connection form ω. Technical reasons will determine which representation should be used at a given point.

2.2. Parabolic geometries

2.2.1. $|k|$-graded Lie algebras. Let $\mathfrak{g}$ be a real semisimple Lie algebra. We say that $\mathfrak{g}$ is $|k|$-graded if

$$\mathfrak{g} = \underbrace{\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}}_{\mathfrak{g}_-} \oplus \mathfrak{g}_0 \oplus \underbrace{\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k}_{\mathfrak{p}_+} \tag{15}$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, with $\mathfrak{g}_i = \{0\}$ for $|i| > k$.

The grading of $\mathfrak{g}$ induces a filtration

$$\mathfrak{g} = \mathfrak{g}^{-k} \supset \mathfrak{g}^{-k+1} \supset \cdots \supset \mathfrak{g}^k = \mathfrak{g}_k \supset \{\}$$

via

$$\mathfrak{g}^i := \bigoplus_{j=i}^{k} \mathfrak{g}_j.$$

$\mathfrak{p} := \mathfrak{g}^0 = \mathfrak{g}_0 \ltimes \mathfrak{p}_+$ is then a *parabolic* subalgebra of $\mathfrak{g}$. $\mathfrak{g}_0$ is reductive and decomposes into the semisimple part $\mathfrak{g}_0^{ss}$ and its center $\mathfrak{g}_0^c$. There is a unique element E in $\mathfrak{g}_0^c$, called the *grading element* of $\mathfrak{g}$, which has the property that $\mathrm{ad}(E)_{|\mathfrak{g}_j} = j\mathrm{id}_{\mathfrak{g}_j}$.

2.2.2. Group level. For a given semisimple $|k|$-graded $\mathfrak{g}$ will fix a Lie group G with Lie algebra $\mathfrak{g}$ and define

$$\begin{aligned} G_0 &:= \{g \in G : \mathrm{Ad}(g)\mathfrak{g}_i = \mathfrak{g}_i \ \forall i\}, \\ P_+ &:= \{g \in G : (\mathrm{Ad}(g) - \mathrm{id})\mathfrak{g}^i \subset \mathfrak{g}^{i+1} \ \forall i\} \text{ and} \\ P &:= \{g \in G : \mathrm{Ad}(g)\mathfrak{g}^i \subset \mathfrak{g}^i \ \forall i\}. \end{aligned} \tag{16}$$

These are closed subgroups of G with Lie algebras $\mathfrak{g}_0$, $\mathfrak{p}_+$ and $\mathfrak{p}$. One has that $P = G_0 \ltimes P_+$ with G_0 reductive and P_+ a contractible nilpotent group isomorphic via the exponential map to $\mathfrak{p}_+$. In particular, we can write $P = G_0 \ltimes \mathfrak{p}_+$.

Definition 2.2.1. A *parabolic geometry of type* (G, P) is a Cartan geometry $(\mathcal{G}, \omega)$ on a manifold M of type (G, P) with $\mathfrak{g}$ a $|k|$-graded semisimple Lie algebra and P defined as above.

Diagrammatically, we will say that

$$\begin{array}{ccc} (\mathcal{G}, \omega) & \longleftarrow & P \\ \big\downarrow & & \\ M & & \end{array}$$

is a parabolic geometry of type (G, P) over M

We will often use that the Killing form on $\mathfrak{g}$ induces a P-equivariant duality between $(\mathfrak{g}/\mathfrak{p})$ and $\mathfrak{p}_+$. I.e.: $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$. Then, since $TM = \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$, one has $T^*M = \mathcal{G} \times_P \mathfrak{p}_+$. In particular, T^*M is canonically endowed with the structure of a (pointwise) nilpotent Lie algebra.

2.2.3. Gradings and filtrations of representations. Let V be a P- or G-representation. The action of $p \in P$ resp $p \in G$ on $v \in V$ written $p \cdot v$, and likewise the infinitesimal action of $X \in \mathfrak{p}$ resp. $X \in \mathfrak{g}$ is written $X \cdot v$. E.g., for $V = \mathfrak{g}$ the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ and $X, Y \in \mathfrak{g}$ one has $X \cdot Y = -Y \cdot X = [X, Y]$.

2.2.3.1. *Let V be a G-representation.* Then the semisimple element E acts diagonalizable on V since $\mathfrak{g}$ is a semisimple Lie algebra. If moreover V is irreducible, the eigenvalues of E will form a (necessarily finite) chain in $\alpha+\mathbb{N}$ for some $\alpha \in \mathbb{R}$ and E thus induces a G_0-invariant grading $V = V_0 \oplus \cdots \oplus V_r$ of V. It is easy to see that $\mathfrak{g}_i V_j \subset V_{i+j}$. The induced filtration

$$V = V^0 \supset \cdots \supset V^r \supset V^{r+1} = \{0\}$$

then satisfies $\mathfrak{g}^i V^j \subset V^{i+j}$. In particular $\mathfrak{p}_+ V^j \subset V^{j+1}$ and the filtration is $P = G_0 \ltimes \mathfrak{p}_+$-invariant.

2.2.3.2. *If V is just a P-representation,* there is still a natural P-invariant filtration

$$V = V^0 \supset V^1 \cdots \supset V^r \supset V^{r+1} = \{0\} \tag{17}$$

such that $\mathfrak{p}_+ V^i \subset V^{i+1}$. This comes about by defining V^r as the annihilator of the $\mathfrak{p}_+$-action on V; then one sets recursively

$$V^i := \{v \in V : Y \cdot v \in V^{i+1} \ \forall\ Y \in \mathfrak{p}_+\}.$$

Here r is chosen in hindsight such that all inclusions in (17) are proper. The *associated graded* $\mathrm{gr}(V)$ of V is then defined by

$$\mathrm{gr}(V) = \mathrm{gr}_0(V) \oplus \cdots \oplus \mathrm{gr}_r(V) = V/V^1 \oplus \cdots \oplus V^{r-1}/V^r \oplus V^r,$$

and this grading is evidently P-invariant. In fact, $\mathfrak{p}_+$ acts trivially on $\mathrm{gr}(V)$ by construction.

Remark 2.2.2. A map of homogeneity $\geq i$ between filtered spaces descends to a canonical map of homogeneity i between the associated graded spaces; however, a lift of a homogeneous map between graded spaces to a map between the corresponding filtered spaces depends on a choice of isomorphism of filtered spaces between $\mathrm{gr}(V)$ and V.

2.2.3.3. *Associated graded spaces of natural bundles:* Consider the bundle $\mathcal{G}_0 := \mathcal{G}/P_+$: since P_+ is a normal subgroup in P and $P/P_+ = G_0$, $\mathcal{G}_0$ is a G_0-principal bundle over M.

Let V be a P-representation and let $\mathbf{V} = \mathcal{G} \times_P V$ be the corresponding P-associated bundle.

Lemma 2.2.3. *There is a canonical isomorphism* $\mathrm{gr}(\mathbf{V}) \cong \mathcal{G}_0 \times_{G_0} V$

Proof. Since $\mathfrak{p}_+$ acts trivially on $\mathrm{gr}(V)$ and $\mathfrak{p}_+ \lhd P$ the quotient of $\mathcal{G} \times \mathrm{gr}(V)$ by the P-right action

$$(u, v) \cdot p = (u \cdot p, p^{-1} \cdot v), u \in \mathcal{G}, v \in V, p \in P$$

is the same as the quotient of $\mathcal{G}_0 \times V$ by

$$(u, v) \cdot p = (u \cdot g, g^{-1} \cdot v), u \in \mathcal{G}_0, v \in V, p \in G_0.$$

□

2.2.4. Lie algebra (co)homology. Let now V be an irreducible G-representation. We can act on V by both $\mathfrak{g}_-$ and $\mathfrak{p}_+$, and this will yield two Lie algebra differentials ∂ and ∂^*. We begin by introducing the P-representations

$$C_l := \Lambda^l(\mathfrak{g}/p)^* \otimes V = \Lambda^l \mathfrak{p}_+ \otimes V. \tag{18}$$

which will be the chain spaces of algebraic differentials. $C_l = \Lambda^l \mathfrak{p}_+ \otimes V$ inherits the canonical G_0-invariant gradings from $\mathfrak{p}_+$ and V and the induced P-invariant filtrations. For instance, $C_0^i = V^i$ and $C_1^i = \sum_{j=1}^i \mathfrak{g}_j \otimes V_{i-j}$. We will say that an element of C_l^i is of *homogeneity* $\geq i$. This refers to the natural concept of homogeneity of maps between filtered spaces: take for instance a $\varphi \in C_2 = \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes V$. Then φ is of homogeneity $\geq i$, i.e., $\varphi \in C_2^i$, if for all $X_s \in \mathfrak{g}_s$ and $X_t \in \mathfrak{g}_t$ one has $\varphi(X_s, X_t) \in V^{i+s+t}$.

Both ∂ and ∂^* below will make

$$C_* := C_0 \oplus \cdots \oplus C_n$$

into a complex.

2.2.4.1. *The differentials ∂ and ∂^*:* The Lie algebra differential ∂ of $\mathfrak{g}_-$ with values in V is defined by

$$\partial : \Lambda^l \mathfrak{g}_-^* \otimes V \to \Lambda^{l+1}\mathfrak{g}_-^* \otimes v; \text{for } \varphi \in \Lambda^l \mathfrak{g}_-^* \otimes V, \text{and } X_0, \ldots, X_l \in \mathfrak{g}_- \tag{19}$$
$$\partial\varphi(X_0, \cdots, X_l) := \sum_{i=0}^{l} (-1)^i X_i \cdot \varphi(X_0, \cdots, \widehat{X_i}, \cdots, X_l)$$
$$+ \sum_{0 \leq i < j \leq l} (-1)^{i+j} \varphi([X_i, X_j], X_0, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_l).$$

One has the G_0-equivariant isomorphism $\mathfrak{g}_- \to \mathfrak{g}/\mathfrak{p}$; likewise for the dual spaces $\mathfrak{g}_-^* \cong (\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$. Thus one can view ∂ a as map $\partial : C_l \to C_{l+1}$. It is straightforward to check that $\partial \circ \partial = 0$, and one obtains the complex

$$V = C_0 \xrightarrow{\partial} C_1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_n. \tag{20}$$

Since the differential (19) and the identification $\mathfrak{g}_-^* \cong (\mathfrak{g}/\mathfrak{p})^*$ are immediately seen to be G_0-equivariant, so is $\partial : C_l \to C_{l+1}$. It is however easy to see that ∂ cannot be $\mathfrak{p}_+$-equivariant, Take for instance $v \in V, X \in \mathfrak{g}_{-1}$ and $Y \in \mathfrak{g}_1$ and compute that $(Y \cdot (\partial v))(X) - (\partial(Y \cdot v))(X) = [Y, X] \cdot v$, which need not vanish.

Dually to ∂ one has the *Kostant codifferential*

$$\partial^* : \Lambda^{l+1}\mathfrak{p}_+ \otimes V \to \Lambda^l \mathfrak{p}_+ \otimes V; \text{ for } Z_0, \cdots, Z_l \in \mathfrak{p}_+ \text{ and } v \in V, \tag{21}$$
$$\partial^*(Z_0 \wedge \cdots \wedge Z_l \otimes v) := \sum_{i=0}^{l} (-1)^{i+1} Z_0 \wedge \cdots \widehat{Z_i} \cdots \wedge Z_l \otimes Z_i \cdot v$$
$$+ \sum_{0 \leq i < j \leq l} (-1)^{i+j} [Z_i, Z_j] \wedge Z_0 \cdots \wedge \widehat{Z_i} \wedge \cdots \wedge \widehat{Z_j} \wedge \cdots \wedge Z_l \otimes v.$$

Since ∂^* as defined by (21) is easily seen to be P-equivariant, and since the the identification $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$ is so too, the codifferential $\partial^* : C_{l+1} \to C_l$ is also P-equivariant; moreover, one has that $\partial^* \circ \partial^* = 0$. Thus it provides the complex

$$V \overset{\partial^*}{\leftarrow} C_1 \overset{\partial^*}{\leftarrow} C_2 \cdots \overset{\partial^*}{\leftarrow} C_n.$$

We will denote the space of cycles of ∂^* by $Z_l = \ker \partial^* \subset C_l$ and the space of borders of ∂^* by $B_l = \mathrm{im}\ \partial^* \subset C_l$. The homologies are $H_l = H_l(\mathfrak{p}_+, V) = Z_l/B_l$.

Remark 2.2.4. For complex semisimple $|k|$-graded Lie algebras [**Kos61**] provides an algorithmic computation of the homologies H_i. The results have been transfered to the case of real semisimple $|k|$-graded Lie algebras in [**Šil04**].

In this text we will mostly be interested in H_0 and H_1, since first BGG-operators operate between the corresponding associated bundles. For H_0, it follows from irreducibilty of V that $H_0 = V/V^1$. H_1 is obtained automatically when one computes the first BGG-operator, as is shown in the examples in chapters 5 and 6. H_2 will appear when constructing certain tensorial obstruction maps (4.2, 6.2.6.)

We don't give the details of the computations here, which involve the computation of part of the Hasse diagram of $\mathfrak{g}$ and have been written down in complete detail in [**ČS09**]. One can also employ the implementation [**Šil**]. For the purposes of computation one employs the algorithm of [**BE89**], which works with the highest weight of the dual representation V^*, written as a sum of fundamental weights. For the explicit examples of chapters 5 and 6 we will therefore provide this via the Dynkin diagram of the (complexified) representation and the coefficient of each fundamental weight in the representation V^* presented by an integer over the corresponding simple root.

2.2.4.2. *The Hodge decomposition via the Kostant Laplacian $\Box$:* It is due to Kostant [**Kos61**] that ∂ and ∂^* are adjoint with respect to a natural inner product on the complex $C_* = C_0 \oplus \cdots \oplus C_n$. This is used to produce a Hodge decomposition via the *Kostant Laplacian*

$$\Box = \partial \circ \partial^* + \partial^* \circ \partial.$$

One has

$$C_i = \mathrm{im}\ \partial \oplus \ker \Box \oplus \mathrm{im}\ \partial^*, \tag{22}$$

and then necessarily $\ker(\partial) = \mathrm{im}\ \partial \oplus \ker \Box$ and $\ker(\partial^*) = \mathrm{im}\ \partial^* \oplus \ker \Box$.

This implies that $H_i(\mathfrak{p}_+, V) = H_i = Z_i/B_i$ includes into C_i as $\ker \Box$ as a G_0-module, but since $\Box$ is not $\mathfrak{p}$-invariant, $\ker \Box$ is not P-invariant either, and thus the identification $H_i = \ker \Box \subset C_i$ only makes sense as G_0-modules.

Since ∂^* vanishes on $\mathrm{im}\ \partial^*$, one has that $\partial^* \circ \partial : \mathrm{im}\ \partial^* \to \mathrm{im}\ \partial^*$ agrees with the restriction of $\Box$ to $\mathrm{im}\ \partial^*$. Thus

$$\Box_{|\mathrm{im}\ \partial^*} = \partial^* \circ \partial : \mathrm{im}\ \partial^* \to \mathrm{im}\ \partial^*$$

has trivial kernel by (22) and is therefore an isomorphism,where

$$((\partial^* \circ \partial)_{|\mathrm{im}\ \partial^*})^{-1} = (\Box_{|\mathrm{im}\ \partial^*})^{-1}. \tag{23}$$

2.2.4.3. *Tractor bundles.* A natural bundle $\mathbf{V} = \mathcal{G} \times_P V$ associated to a G-representation is called a *tractor bundle.* Any G-representation V is also a $\mathfrak{g}$-module, and thus one has a P-equivariant action of $\mathfrak{g}$ on V, i.e.: for all $g \in P$,$X \in \mathfrak{g}$ and $v \in V$,

$$\mathrm{Ad}(g)(X) \cdot v = g \cdot (X \cdot (g^{-1} \cdot v)). \tag{24}$$

Thus, G-representations are special cases of $(\mathfrak{g}, P)$-representations, i.e., representations V of both $\mathfrak{g}$ and P which are compatible via (24). A bit more generally, we will say that P-associated bundles to $(\mathfrak{g}, P)$-representations are also tractor bundles. (24) is just P-equivariancy of the Lie algebra action $\mathfrak{g} \times V \to V$, and thus yields an action of $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$ on $\mathbf{V}$, which we will denote by $\bullet$.

With

$$\mathbf{C}_i := \mathcal{G} \times_P C_i = \Lambda^i T^*M \otimes \mathbf{V}$$

and the P-equivariant Kostant codifferential as defined in (21) we get the complex

$$\mathbf{V} \overset{\partial^*}{\leftarrow} \mathbf{C}_1 \overset{\partial^*}{\leftarrow} \mathbf{C}_2 \cdots \overset{\partial^*}{\leftarrow} \mathbf{C}_n. \tag{25}$$

The vector bundles of cycles, borders and homologies are

$$\begin{aligned} \mathbf{Z}_i &= \mathcal{G} \times_P Z_i = \ker \partial^* \subset \mathbf{C}_i, \\ \mathbf{B}_i &= \mathcal{G} \times_P B_i = \mathrm{im}\ \partial^* \subset \mathbf{C}_i \text{ and} \\ \mathbf{H}_i &= \mathbf{Z}_i/\mathbf{B}_i = \mathcal{G} \times_P H_i. \end{aligned}$$

We have the natural projections

$$\Pi_i : \mathbf{Z}_i \to \mathbf{H}_i. \tag{26}$$

The spaces of sections of $\mathbf{C}_i, \mathbf{Z}_i, \mathbf{B}_i$ and $\mathbf{H}_i$ will be denoted by $\mathcal{C}_i, \mathcal{Z}_i, \mathcal{B}_i$ and $\mathcal{H}_i$.

Since $\partial : C_l \to C_{l+1}$ is only G_0-equivariant, one has to switch to the associated graded spaces: Since $P = G_0 \ltimes \mathfrak{p}_+$ and $\mathfrak{p}_+$ acts trivially on the spaces $\mathrm{gr}(C_l)$, we obtain a P-equivariant map $\partial : \mathrm{gr}(C_l) \to \mathrm{gr}(C_{l+1})$, and on the level of associated bundles this yields the complex

$$\mathrm{gr}(\mathbf{V}) = \mathrm{gr}(\mathbf{C}_0) \overset{\partial}{\to} \mathrm{gr}(\mathbf{C}_1) \overset{\partial}{\to} \cdots \overset{\partial}{\to} \mathrm{gr}(\mathbf{C}_n).$$

Likewise, the G_0-equivariant Kostant-Laplacian $\Box : C_l \to C_l$ gives rise to maps

$$\Box : \mathrm{gr}(\mathbf{C}_l) \to \mathrm{gr}(\mathbf{C}_l).$$

We remark that the maps ∂, ∂^* and $\Box$ are all homogeneous of degree 0 when viewed as maps on the associated graded spaces. E.g., $\Box$ preserves $\mathrm{gr}_i(\mathcal{C}_l)$ for all i, l and ∂ maps $\mathrm{gr}_i(\mathcal{C}_l)$ into $\mathrm{gr}_i(\mathcal{C}_{l+1})$.

2.2.5. Regular and normal parabolic geometries. Recall that property (C.3) of the Cartan connection form ω yields the trivialization $T\mathcal{G} = \mathcal{G} \times \mathfrak{g}$. In particular, the filtration of $\mathfrak{g}$ gives rise to a filtration

$$T^{-k}\mathcal{G} \supset \cdots \supset T^k\mathcal{G} \supset \{0\}.$$

Since TM is the associated bundle to the P-representation $\mathfrak{g}/\mathfrak{p}$, which carries the $\underline{\mathrm{Ad}}(P)$-invariant filtration

$$\mathfrak{g}^{-k}/\mathfrak{p} \supset \cdots \supset \mathfrak{g}^{-1}/\mathfrak{p}$$

, one has the filtration of the tangent bundle

$$T^iM = \mathcal{G} \times_P \mathfrak{g}^i/\mathfrak{p}, \tag{27}$$

$$TM = T^{-k}M \supset T^{-k+1}M \supset \cdots \supset T^{-1}M \supset \{0\}. \tag{28}$$

It is easy to see that the filtration of TM is induced directly by the filtration of $T\mathcal{G}$: for $u \in \mathcal{G}$ and $x = \Pi(u)$ one has $T^i_xM = T_u\Pi(T^i_u\mathcal{G})$ for all $i \in \mathbb{Z}$.

We say that M is a *filtered manifold* if the filtration of TM is compatible with the Lie bracket of vector fields: one has

$$[\xi, \eta] \in \Gamma(T^{i+j}M) \text{ for } \xi \in \Gamma(T^iM), \eta \in \Gamma(T^jM). \tag{29}$$

In this case it is easy to see that the Lie bracket

$$[\cdot,\cdot] : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

induces an algebraic Lie bracket on all fibers of $\mathrm{gr}(TM)$. This is encoded in the *Levi bracket*

$$\mathcal{L} : \mathrm{gr}(TM) \times \mathrm{gr}(TM) \to \mathrm{gr}(TM). \tag{30}$$

But $\mathrm{gr}(TM)$ already carries a natural pointwise (nilpotent) Lie algebra structure $\{\cdot,\cdot\}$ coming from $\mathfrak{g}_-$: By Lemma 2.2.3, $\mathrm{gr}(TM) = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_-$. Thus the G_0-equivariant Lie-bracket on $\mathfrak{g}_-$ carries over to an *algebraic bracket*

$$\{\cdot,\cdot\} : \mathrm{gr}(TM) \otimes \mathrm{gr}(TM) \to \mathrm{gr}(TM). \tag{31}$$

Definition 2.2.5. A parabolic geometry $(\mathcal{G}, \omega)$ on a filtered manifold M is called *regular* if the algebraic- and the Levi bracket on $\mathrm{gr}(TM)$ coincide. I.e., if $\{\cdot,\cdot\} = \mathcal{L}(\cdot,\cdot)$.

One can translate this into a simple condition on the curvature;

Proposition 2.2.6. *Let M be endowed with the filtration* (27) *induced by $(\mathcal{G}, \omega)$. Then M is a filtered manifold and ω is regular if and only if $K \in \Omega^2(M, \mathcal{A}M)$ is homogeneous of degree ≥ 1, i.e., $K \in \Omega^2(M, \mathcal{A}M)^1$.*

2.2.5.1. *Additionally, parabolic geometries allow a uniform normalization condition:* The curvature form K of ω lies in $\Omega^2(M, \mathcal{A}M)$, with $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$ the adjoint tractor bundle of (14). In notation of 2.2.4.3, $\Omega^2(M, \mathcal{A}M)$ is the chain space $\mathbf{C}_2$ for the G-representation $\mathfrak{g}$. Thus one can use the Kostant codifferential $\partial^* : \mathbf{C}_2 \to \mathbf{C}_1$ to define a normalization condition:

Definition 2.2.7. A Cartan connection form ω is called *normal* if

$$\partial^*(K) = 0.$$

In this case one has the *harmonic curvature* $K_H = \Pi_2(K) \in \mathcal{H}_2(\mathcal{A}M)$.

In the picture of P-equivariant functions on $\mathcal{G}$ the harmonic curvature corresponds to the composition of the curvature function κ with the projection $\Pi_2 : Z_2(\mathfrak{g}) \to H_2(\mathfrak{g})$, i.e., to $\kappa_H = \Pi_2 \circ \kappa$.

2.2.6. Weyl structures. A *Weyl structure* for $(\mathcal{G}, \omega)$ is a G_0-equivariant section $\sigma : \mathcal{G}_0 \to \mathcal{G}$. There always exist global Weyl structures and they form an affine space modelled on

$$\mathrm{C}^\infty_{G_0}(\mathcal{G}_0, \mathfrak{p}_+) = \Gamma(\mathrm{gr}(T^*M))$$

(cf. [**ČS03**] or [**ČS09**]).

2.2.6.1. *Reductions of structure group from P to G_0:*

LEMMA 2.2.8. *Given a natural bundle $\mathbf{V} = \mathcal{G} \times_P V$ for a P-representation V, a choice of Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ yields an isomorphism*

$$\mathrm{gr}(\mathbf{V}) \cong \mathbf{V}.$$

PROOF. For $u \in \mathcal{G}_0, v \in V$, $[u, v] \in \mathcal{G}_0 \times_{G_0} V$ is the orbit of (u, v) under the G_0-action

$$(u, v) \cdot g = (u \cdot g, g^{-1} \cdot v),\ \ g \in G_0,$$

and analogously

$$[u, v] = (u, v) \cdot P, \text{with}$$
$$(u, v) \cdot p = (u \cdot p, \mathrm{Ad}(p^{-1})v) \text{ for } u \in \mathcal{G}, v \in V, p \in P.$$

This immediately gives that

$$\begin{aligned} \mathcal{G}_0 \times V &\cong \mathcal{G} \times_P V, \\ [u, v] &\mapsto [\sigma(u), v] \end{aligned} \tag{32}$$

is well defined by G_0-equivariance of σ. This yields an isomorphism $\mathcal{G}_0 \times_{G_0} V \cong \mathcal{G} \times_P V = \mathbf{V}$, but by Lemma 2.2.3 we have $\mathcal{G}_0 \times_{G_0} V = \mathrm{gr}(\mathbf{V})$. □

We say that a bundle which associated to $\mathcal{G}$ via a P-representation is a natural bundle for $(\mathcal{G}, \omega)$; and as we have just seen, every natural bundle can be written as a G_0-associated bundle to $\mathcal{G}_0$ after a choice of Weyl structure.

In particular, one gets a reduction of structure group from P to G_0 of the chain spaces $\mathbf{C_l}$ via $\mathbf{C_l} \cong \mathcal{G}_0 \times_{G_0} C_l = \mathrm{gr}(\mathbf{C_l})$ and ∂ and $\Box$ lift to maps on $\mathbf{C}_* = \oplus \mathbf{C}_l$.

We now discuss transformation rules for $\mathfrak{g}$, where $\mathrm{gr}(TM) = TM$. Moreover, we will need the transformation law only for the case where the $(\mathfrak{g}, P)$-representation V has just 3-grading components, i.e., where $V = V_0 \oplus V_1 \oplus V_2$.

We will write $[v]_\sigma \in [\mathbf{V}]_\sigma = \mathrm{gr}(\mathbf{V})$ for the the element in $\mathrm{gr}(\mathbf{V})$ corresponding to a $v \in \mathbf{V}$ via the isomorphism of Lemma 2.2.8.

Let $\hat{\sigma} = \sigma \cdot \exp(\Upsilon)$ be another Weyl structure for a $\Upsilon \in \Omega^1(\mathcal{G}_0, T^*M)$.

Then it is shown in [**ČS09**], following Proposition 5.1.5, that for

$$[v]_\sigma = \begin{pmatrix} v_2 \\ v_1 \\ v_0 \end{pmatrix},$$

one has

$$[v]_{\hat{\sigma}} = \begin{pmatrix} v_2 + \Upsilon \bullet v_1 + \frac{1}{2} \Upsilon \bullet (\Upsilon \bullet v_0) \\ v_1 + \Upsilon \bullet v_0 \\ v_0 \end{pmatrix}. \tag{33}$$

Here we employ the canonical action $\bullet$ of T^*M on $\mathrm{gr}(\mathbf{V})$, which is in fact the restriction of the algebraic action of $\mathcal{A}M$ discussed in 2.2.4.3.

2.2.6.2. *Soldering form, Weyl connections and Schouten tensors:* Given a Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ one can consider the pullback $\sigma^*\omega \in \Omega^1(\mathcal{G}_0, \mathfrak{g})$, which will be G_0-equivariant. Since the decomposition of $\mathfrak{g}$ into $\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$ is G_0-equivariant, we can decompose $\sigma^*(\omega)$ correspondingly into

$$\sigma^*(\omega) = \theta \oplus \gamma \oplus \mathrm{P}. \tag{34}$$

Then θ is easily seen to be a *soldering form*: this means that $\theta \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_-)$ is G_0-equivariant, and with $\Pi^{\mathcal{G}_0} : \mathcal{G}_0 \to M$ the canonical surjection one has that for every $u \in \mathcal{G}_0$,

$$\ker T_u \Pi^{\mathcal{G}_0} = \ker \theta_u. \tag{35}$$

It is a standard statement that θ therefore yields an isomorphism

$$\mathcal{G}_0 \times_{G_0} \mathfrak{g}_- \cong TM. \tag{36}$$

This also follows from Lemma 2.2.8 since $\mathrm{gr}(\mathfrak{g}/\mathfrak{p}) = \mathfrak{g}_-$ as a G_0-module.

Let us treat the other two components in the decomposition (34): Since $\gamma \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0)$ reproduces fundamental G_0-vector fields and is G_0-equivariant it is a principal connection form on $\mathcal{G}_0$. In particular, after a choice of Weyl structure, every P-associated bundle $\mathbf{V} = \mathcal{G} \times_P V$, is endowed with a linear connection, denoted by D and called the *Weyl connection*.

The $\mathfrak{p}_+$-component $\mathrm{P} \in \Omega^1(\mathcal{G}_0, \mathfrak{p}_+)$ of $\sigma^*\omega$ is called the *Rho-* or generalized Schouten tensor. Since P is a horizontal, P-equivariant, $\mathfrak{p}+$-valued 1-form on $\mathcal{G}$, and since the P-associated bundle to $\mathfrak{p}_+$ is T^*M, P factorizes to a section of $T^*M \otimes T^*M$

Under a achange of the Weyl structure to $\hat{\sigma} = \sigma \cdot \exp(\Upsilon)$ for a $\Upsilon \in \Omega^1(\mathcal{G}_0, T^*M)$ it is shown in [**ČS09**], that for $s \in \Gamma(V)$,

$$\hat{D}_\xi s = D_\xi s - \{\xi, \Upsilon\} \bullet s \tag{37}$$

and

$$(i_\xi \hat{\mathrm{P}})_a = (i_\xi \mathrm{P})_a - (D_\xi \Upsilon)_a + \frac{1}{2}\{\Upsilon, \{\Upsilon, \xi\}\}_a. \tag{38}$$

2.2.7. The adjoint tractor bundle. The *adjoint tractor bundle* $\mathcal{A}M$ of $(\mathcal{G}, \omega)$ comes about as the associated bundle to the adjoint representation of G on $\mathfrak{g}$, as already defined in (14): $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$. The filtration of $\mathfrak{g}$ carries over to a filtration $\mathcal{A}M = \mathcal{A}^{-k}M \supset \cdots \supset \mathcal{A}^k M \supset \{0\}$ of $\mathcal{A}M$. The geometry $(\mathcal{G}, \omega)$ is said to be *torsion-free* if $K \in \Omega^2(M, \mathcal{A}^0 M)$.

The Lie bracket of $\mathfrak{g}$ carries over to an *algebraic bracket* $\{\cdot, \cdot\}$ on $\mathcal{A}M$. Since $TM = \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$, one has a natural projection $\Pi^{\mathcal{A}} : \mathcal{A}M \to TM$. Moreover, since the Killing form on $\mathfrak{g}$ provides a P-equivariant duality between $\mathfrak{p}_+$ and $\mathfrak{g}/p$, one has that $T^*M = \mathcal{G} \times_P \mathfrak{p}_+$. But $\mathfrak{p}_+ \subset \mathfrak{g}$ as a P-module, which yields the canonical embedding

$$T^*M \hookrightarrow \mathcal{A}M. \tag{39}$$

The algebraic brackets on $\mathrm{gr}(TM)$ and T^*M introduced above are induced by the algebraic bracket on $\mathcal{A}M$.

Since $\mathfrak{g}_-$ includes into $\mathfrak{g}$ as a G_0 module and $\mathcal{A}M$ is reduced by a Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ to $\mathcal{G}_0 \times_{\mathcal{G}_0} \mathfrak{g}$, a choice of Weyl structure yields the embedding

$$TM = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_- \hookrightarrow \mathcal{G}_0 \times_{G_0} \mathfrak{g} = \mathcal{A}M.$$

That T^*M always includes canonically into $\mathcal{A}M$ was already observed in (39). In particular, since $\mathcal{A}M$ acts on $\mathbf{V}$ by

$$\bullet : \mathcal{A}M \otimes \mathbf{V} \to \mathbf{V},$$

we obtain actions

$$\begin{aligned} \bullet &: TM \otimes \mathbf{V} \to \mathbf{V} \text{ and} \\ \bullet &: T^*M \otimes \mathbf{V} \to \mathbf{V}, \end{aligned} \tag{40}$$

the first of which depends on the choice of Weyl structure.

2.2.7.1. *Formulas for ∂ and ∂^*.* The reduction of structure group of the chain space $\mathbf{C}_l = \Lambda^l T^*M \otimes \mathbf{V}$ from P to G_0 provided by a Weyl structure was seen in 2.2.6 to provide an isomorphism $\mathbf{C}_l \cong \mathrm{gr}(\mathbf{C}_l)$, and one can then view ∂ as a map $\partial : \mathbf{C}_l \to \mathbf{C}_{l+1}$.

On the first chain space $\mathrm{gr}(\mathbf{V}) = \mathrm{gr}(\mathbf{C}_0)$ it is easy to see that one has for $x \in M$, $X \in T_xM$ and $v \in \mathbf{V}_x$,

$$\partial(v)(X) = X\bullet v, \text{ for } v \in \mathbf{V}_x, X \in T_xM \tag{41}$$

In fact, one has more generally that for $x \in M$, $\phi \in (\Lambda^l T^*_x M \otimes \mathbf{V})_x$ and $X_0, \dots X_l \in T_xM$

$$\begin{aligned} \partial\phi(X_0, \dots, X_l) = &\sum_{k=0}^{l} (-1)^k X_k \bullet \phi(X_0, \dots, \widehat{X_k}, \dots, X_l) \\ &+ \sum_{0 \le i < j \le l} (-1)^{i+j} \varphi(\{X_i, X_j\}, X_0, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_l), \end{aligned} \tag{42}$$

where the bracket $\{\cdot,\cdot\}$ is just the bracket of $\mathcal{A}M$ restricted to $TM \subset \mathcal{A}M$.

The action $\bullet$ of T^*M on $\mathbf{V}$ provides a map $T^*M \otimes \mathbf{V} \to \mathbf{V}$: for $\varphi \in T^*M \otimes \mathbf{V}$ we write $\bullet(\varphi)$ for the corresponding element in $\mathbf{V}$. Take some $x \in M$. Then every $\varphi \in \mathbf{C}_{1x}$ is a sum of terms of the form $Y \otimes v$ for $Y \in T^*_x M$ and $v \in \mathbf{V}_x$. Since $\partial^*(Y_i \otimes v) = -Y_i \bullet v$ we see $\partial^*(\varphi) = -\bullet(\varphi)$.

One has similar formulas for all chain spaces, but we will only need one for the second chain space $\mathbf{C}_2$, and this only in the special situation when $\mathfrak{g}$ is 1-graded. Since then $\mathfrak{g}_-$ and $\mathfrak{g}_+$ are abelian the last terms in formulas (19) and (21) disappear. An element $\varphi \in \mathbf{C}_{2x} = \Lambda^2 T^*_x M \otimes \mathbf{V}_x$ can be written as a sum of terms $Y_1 \wedge Y_2 \otimes v$ for $Y_1, Y_2 \in T^*_x M$ and $v \in \mathbf{V}_x$. Now

$$Y_1 \wedge Y_2 \otimes v = Y_1 \otimes Y_2 \otimes v - Y_2 \otimes Y_1 \otimes v.$$

Since T^*M acts trivially on itself via $\bullet$, we can view $\bullet$ as a map $T^*M \otimes \big(T^*M \otimes \mathbf{V}\big) \to T^*M \otimes \mathbf{V}$ where the first term T^*M acts on the other terms tensorially. Again, this map is denoted by $\bullet(\varphi)$ for $\varphi \in \mathbf{C}_{2x}$. Then

$$\bullet(Y_1 \wedge Y_2 \otimes v) = Y_2 \otimes Y_1 \bullet v - Y_1 \otimes Y_2 \bullet v = -\partial^*(Y_1 \wedge Y_2 \otimes v).$$

Thus, for $\varphi \in \mathbf{C}_2$ we have $\partial^*(\varphi) = -\bullet(\varphi)$.

More generally, one can check that $\partial^* : \mathbf{C}_{l+1} \to \mathbf{C}_l$ coincides with (the negative of)

$$\Lambda^{l+1} T^*M \otimes \mathbf{V} \subset T^*M \otimes \Lambda^l T^*M \otimes \mathbf{V} \xrightarrow{\bullet} \Lambda^l T^*M \otimes \mathbf{V}, \tag{43}$$

where we have T^*M acting on $\Lambda^l T^*M \otimes \mathbf{V}$ via $\bullet$.

2.2.7.2. *The fundamental derivative.* A section s of of $\mathcal{A}M$ corresponds to a P-equivariant function f from $\mathcal{G}$ to $\mathfrak{g}$. But taking $\xi_u := \omega_u^{-1}(f(u))$, one obtains a P-invariant vector fields on $\mathcal{G}$, and this provides a 1:1 correspondence between $\Gamma(\mathcal{A}M)$ and $\mathfrak{X}(\mathcal{G})^P$. Let now v be a section of a P-associated bundle $\mathbf{V}$. Then we define $D^\omega_s v \in \Gamma(\mathbf{V})$ for $s \in \Gamma(\mathcal{A}M)$ by differentiating the P-equivariant function g corresponding to v with the P-invariant field ξ corresponding to s; so the result will be P-equivariant again. This operation is tensorial in $\xi \in \mathfrak{X}(\mathcal{G})^P$ resp. $s \in \Gamma(\mathcal{A}M)$ and thus yields a map

$$D^\omega : \Gamma(\mathbf{V}) \to \Gamma(\mathcal{A}M^* \otimes \mathbf{V}), \tag{44}$$

called the *fundamental derivative.* Since $\mathfrak{g} = \mathfrak{g}^*$ via the Killing-form, one can also view it as an operator $D^\omega : \Gamma(\mathbf{V}) \to \Gamma(\mathcal{A}M \otimes \mathbf{V})$.

2.3. The tractor connection

Given a representation of G on V, we called the associated bundle $\mathbf{V} = \mathcal{G} \times_P V$ a *tractor bundle* in 2.2.4.3. If we form the *extended bundle* $\mathcal{G}' := \mathcal{G} \times_P G$, which is now a G-principal bundle over M, we can equivariantly extend ω to a $\mathfrak{g}$-valued 1-form on $\mathcal{G}'$, which turns out to be a principal connection form on $\mathcal{G}'$. In particular, this yields a linear connection on every tractor bundle $\mathbf{V}$ since we have $\mathbf{V} = \mathcal{G} \times_P V = \mathcal{G}' \times_G V$. This is the induced *tractor connection* $\nabla = \nabla^V$ on $\mathbf{V}$.

For a tractor bundle $\mathbf{V}$ associated to a $(\mathfrak{g}, P)$-representation, which need not necessarily extend to a G-representation, one defines the tractor connection as follows: Let $\xi \in \mathfrak{X}(M)$ and $s \in \Gamma(\mathcal{A}M)$ be some lift of ξ, i.e., $\Pi^{\mathcal{A}}(s) = \xi$. Let v be a section of $\mathbf{V}$. Then one checks that

$$\nabla_\xi v := D^\omega_s v + s \bullet v \tag{45}$$

is in fact independent of the lift s of ξ and defines a connection ∇ on $\mathbf{V}$. In the case where V is in fact a G-representation, this construction of the tractor connection coincides with the construction via the extended bundle $\mathcal{G}'$.

If $f : \mathrm{C}^\infty(\mathcal{G}, V)^P$ is the equivariant function corresponding to v and $\hat{\xi} \in \mathfrak{X}(M)$ is a P-invariant lift of ξ, then (45) translates to $\nabla_\xi v$ being the section corresponding to the equivariant function

$$\hat{\xi} \cdot f + \omega(\hat{\xi}) \cdot f. \tag{46}$$

If $R \in \Omega^2(M, \mathrm{End}(\mathbf{V}))$ denotes the curvature of the tractor connection ∇, then one has in fact by construction that

$$R = K \bullet. \tag{47}$$

Thus, for $\xi_1, \xi_2 \in \mathfrak{X}(M)$ and $v \in \Gamma(\mathbf{V})$, one has that $R(\xi_1, \xi_2)v = K(\xi_1, \xi_2) \bullet v$. In other words: the curvature of the induced tractor connection is the action of $K \in \Omega^2(M, \mathcal{A}M)$ on $\mathbf{V}$.

The tractor connection ∇ on $\mathbf{V}$ induces the covariant exterior derivatives

$$\mathrm{d}^\nabla : \Omega^l(M,\mathbf{V}) \to \Omega^{l+1}(M,\mathbf{V}),$$

$$(\mathrm{d}^\nabla\varphi)(\xi_0,\cdots,\xi_l) = \sum_{k=0}^{l}(-1)^k\nabla_{\xi_k}\varphi(\xi_0,\cdots,\widehat{\xi_k},\cdots,\xi_l) + \sum_{0\le i<j\le l}(-1)^{i+j}\varphi([\xi_i,\xi_j],\xi_0,\cdots,\widehat{\xi_i},\cdots,\widehat{\xi_j},\cdots,\xi_l).$$

With $\mathcal{C}_l = \Omega^l(M,\mathbf{V})$ the space of sections of the chain spaces $\mathbf{C}_l = \mathcal{G}\times_P C_l$ already introduced in 2.2.4.3 we obtain the sequence

$$\Gamma(\mathbf{V}) =: \mathcal{V} \xrightarrow{\nabla} \mathcal{C}_1 \xrightarrow{\mathrm{d}^\nabla} \cdots \xrightarrow{\mathrm{d}^\nabla} \mathcal{C}_n. \tag{48}$$

This however is a complex if and only if the geometry $(\mathcal{G},\omega)$ is locally flat: $\mathrm{d}^\nabla\circ\mathrm{d}^\nabla : \mathcal{C}^l \to \mathcal{C}^{l+2}$ is well known (cf. for example [**Mic08**], 19.13) to be the (algebraic) action of the curvature $R = K\bullet$; i.e.,

$$\mathrm{d}^\nabla\mathrm{d}^\nabla\varphi = \mathrm{alternation}(K\bullet\varphi).$$

To be precise, i.e., to fix the factor of this action, for $\xi_1,\cdots,\xi_{l+2}\in\mathfrak{X}(M)$ this is

$$\frac{1}{2(l!)}\sum_{\sigma\in\mathfrak{S}_{l+2}}\mathrm{sgn}(\sigma)K(\xi_{\sigma(1)},\xi_{\sigma(2)})\bullet\varphi(\xi_{\sigma(3)},\ldots,\xi_{\sigma(l+2)}).$$

with $\mathfrak{S}_{l+2}$ the permutation group of $1,\cdots,l+2$.

As we have seen in 2.2.6, a choice of Weyl structure $\sigma:\mathcal{G}_0\to\mathcal{G}$ yields a Weyl connection $D:\mathcal{V}\to\Omega^1(M,\mathbf{V})$, and we ask how it relates to the tractor connection ∇. First recall that $\sigma^*\omega\in\Omega^1(\mathcal{G}_0,\mathfrak{g})$ decomposes into $\eta\oplus\gamma\oplus\mathrm{P}$ (see (34)), with η being a soldering form yielding an isomorphism $TM = \mathcal{G}_0\times_{G_0}\mathfrak{g}_-$, $\gamma\in\Omega^1(\mathcal{G}_0,\mathfrak{g}_0)$ a principal connection form and $\mathrm{P}\in\Gamma(T^*M\otimes T^*M)$ the Schouten tensor.

One can view $\partial:\mathbf{V}\to\mathbf{C}_1 = T^*M\otimes\mathbf{V}$, which depends on σ, also as an element of

$$T^*M\otimes\mathrm{End}(\mathbf{V}). \tag{49}$$

Similarly, also the algebraic action $\bullet : T^*M\otimes\mathbf{V}\to\mathbf{V}$ can be viewed as an element of $TM\otimes\mathbf{V}^*\otimes\mathbf{V} = TM\otimes\mathrm{End}(\mathbf{V})$. Since $\mathrm{P}\in\Gamma(T^*M\otimes T^*M)$ we can compose P with the tensor product of the identity on T^*M and $\bullet\in TM\otimes\mathrm{End}(\mathbf{V})$: We will write

$$\mathrm{P}\bullet\in T^*M\otimes\mathrm{End}(\mathbf{V}). \tag{50}$$

Now according to (46) (to be precise, we restrict this formula to the G_0-subbundle $\sigma(\mathcal{G}_0)\subset\mathcal{G}$), one calculates $\nabla_\xi v\in\Gamma(\mathbf{V})$ for $\xi\in\mathfrak{X}(M)$ and $s\in\Gamma(\mathbf{V})$ as follows: let $\hat\xi\in\mathfrak{X}(\mathcal{G}_0)$ be the horizontal lift of $\xi\in\mathfrak{X}(M)$ with respect to the principal connection form γ; then ξ is automatically G_0-equivariant. Let $f\in\mathrm{C}^\infty(\mathcal{G}_0,V)^{G_0}$ be the G_0-equivariant function corresponding to v. Then $\nabla_\xi v$ corresponds to the (again G_0-equivariant) function

$$\hat\xi\cdot f + \omega(\hat\xi)\cdot f.$$

The first term is just the function corresponding to $D_\xi s$. For the second term, the decomposition $\omega = \eta \oplus \gamma \oplus \mathrm{P}$ and $\gamma(\hat{\xi}) = 0$ yields that

$$\omega(\hat{\xi}) \cdot f = \eta(\hat{\xi}) \cdot f + P(\hat{\xi}) \cdot f.$$

Now $\eta(\hat{\xi}) \in \mathfrak{g}_-$ and $P(\hat{\xi}) \in \mathfrak{p}_+$ correspond to $\xi \in \Gamma(TM)$ resp. $P(\xi) \in \Gamma(T^*M)$, where we view P as a section of $T^*M \otimes T^*M$, as discussed in 2.2.6. Then, via (49) and (50), we obtain

$$\nabla = \partial + D + \mathrm{P}\bullet. \tag{51}$$

Now $D : \mathcal{V} \to \mathcal{C}_1$ is homogeneous of degree ≥ 1, and so is $\mathrm{P}\bullet : \mathcal{V} \to \mathcal{C}_1$: For $\mathrm{P}\bullet$ this means that for every $v \in \mathcal{V}^i$ and $\xi \in \mathfrak{X}(M)^j$ one has $\mathrm{P}(\xi)\bullet v \in \mathcal{C}_1^{i+j+1}$, and analogously for D. Since, as was already remarked in 2.2.4, $\partial : \mathbf{C}_l \to \mathbf{C}_{l+1}$ has homogeneity 0, we see that ∇ and ∂ coincide in homogeneity 0. If the curvature $K \in \Omega^2(M, \mathcal{A}M)$ is of homogeneity ≥ 1, i.e., if ω is regular, one can show that also $\mathrm{d}^\nabla - \partial$ is always of homogeneity ≥ 1. In fact, this is a special case of Lemma 3.1.1 in the next chapter. We will then say that d^∇ and ∂ agree in lowest homogeneity, and this will be an important fact for several constructions, in particular for the BGG-machinery below in 2.4.

For a precise formulation of ∂ and d^∇ coinciding in lowest homogeneity for a regular parabolic geometry, we introduce the canonical surjections

$$\pi_{l,i} : \mathcal{C}_l^i \to \mathcal{C}_l^i / \mathcal{C}_l^{i+1} = \mathrm{gr}_i(\mathcal{C}_l), \tag{52}$$

which project elements of homogeneity $\geq i$ in $\mathcal{C}_l$ to $\mathrm{gr}_i(\mathcal{C}_l)$. Then we have, for $v \in \mathcal{C}_l^i$,

$$\pi_{l+1,i}(\mathrm{d}^\nabla v) = \partial(\pi_{l,i}(v)) \in \mathrm{gr}_i(\mathcal{C}_{l+1}). \tag{53}$$

This says that one can consider d^∇ as a natural lift of the algebraic map $\partial : \mathrm{gr}(\mathbf{C}_l) \to \mathrm{gr}(\mathbf{C}_{l+1})$ to the differential operator $\mathrm{d}^\nabla : \mathbf{C}_l \to \mathbf{C}_{l+1}$.

Once one has chosen a Weyl structure one gets a reduction of structure group of $\mathbf{C}_0 \oplus \cdots \oplus \mathbf{C}_n$ to G_0. Recall that the inclusion of H_i into Z_i as $\ker \Box \subset C_i$ is G_0-equivariant and that the Hodge decomposition on the algebraic level (22) implies

$$\mathbf{C}_i = \mathrm{im}\ \partial \oplus \ker \Box \oplus \mathrm{im}\ \partial^*. \tag{54}$$

Thus one can also view $\mathbf{H}_i$ as included into $\mathbf{C}_i$ as $\ker \Box$. This gives the decomposition

$$\mathbf{Z}_i = \mathbf{H}_i \oplus \mathbf{B}_i, \tag{55}$$

which depends on the choice of Weyl structure.

2.4. The BGG-sequence

We now introduce the BGG-machinery. As above, V will be a $(\mathfrak{g}, P)$-representation. The main object here is a differential splitting operator $L_i = L_i^V : \mathcal{H}_i \to \mathcal{C}_i$ of the natural projection $\Pi_i : \mathcal{Z}_i \to \mathcal{H}_i$. This was first done in general in [**ČSS01**], and with a simplified construction in [**CD01**].

2.4.1. Unique lifts. We will show that there is a unique maximal subspace $\mathcal{L}_i \subset \mathcal{Z}_i$ such that

$$\mathrm{d}^\nabla(\mathcal{L}_i) \subset \mathcal{Z}_{i+1} \subset \mathcal{C}_{i+1}. \tag{56}$$

Note that for a general section $s \in \mathcal{Z}_i \subset \mathcal{C}_i$ one need not have that also $\partial^*(\mathrm{d}^\nabla s) = 0$. But this will hold on the subspace $\mathcal{L}_i$.

LEMMA 2.4.1. *(*[**Čap05**]*) $\partial^* \circ \mathrm{d}^\nabla : \mathcal{B}_l \to \mathcal{B}_l$ is invertible. The inverse is a differential operator, which will be denoted by $\mathcal{I}_l$.*

PROOF. The inverse of the operator $\partial^* \circ \mathrm{d}^\nabla : \mathcal{B}_l \to \mathcal{B}_l$ will be constructed via a choice of Weyl structure, the choice of which doesn't matter since the inverse, if it is shown to exist, is necessarily unique.

Having chosen a Weyl structure, we have $\Box$ acting on $\mathcal{B}_l$, and can therefore form

$$\mathcal{N}_l := (\Box_{|\mathcal{B}_l})^{-1} \circ \partial^* \circ \mathrm{d}^\nabla - \mathrm{id}_{\mathcal{B}_l} : \mathcal{B}_l \to \mathcal{B}_l.$$

Assume $s \in \mathcal{B}_l$ is of homogeneity $\geq i$, i.e., $s \in \mathcal{B}_l^i$. Recall from (53) that $\pi_{l,i} \circ \partial^* \circ \mathrm{d}^\nabla = \partial^* \circ \partial \circ \pi_{l,i}$ on $\mathcal{C}_l^i$. Thus $\pi_{l,i} \circ (\Box^{-1} \circ \partial^* \circ \partial - \mathrm{id}_{\mathcal{B}_l})$ vanishes on $\mathcal{B}_l^i$ for any homogeneity i. This means that in fact $\mathcal{N}_l s \in \mathcal{B}_l^{i+1}$. Thus, if the highest nontrivial homogeneity of $\mathcal{B}_l$ is k_0, we have that $\mathcal{N}_l^{k_0+1} = 0$. I.e, $\mathcal{N}_l : \mathcal{B}_l \to \mathcal{B}_l$ is nilpotent. Therefore $(\Box_{|\mathcal{B}_l})^{-1} \circ \partial^* \circ \mathrm{d}^\nabla = \mathrm{id}_{\Gamma(\mathcal{B}_l)} + \mathcal{N}_l$ is invertible on $\mathcal{B}_l$ and so is $\partial^* \circ \mathrm{d}^\nabla$. $\Box$

2.4.2. BGG-splitting-operators. It is now easy to show

LEMMA 2.4.2. *For every $\sigma \in \mathcal{H}_l$ there is a unique $s \in \mathcal{Z}_l$ such that $\Pi_l(s) = \sigma$ and $\partial^*(\mathrm{d}^\nabla s) = 0$.*

PROOF. First take an $s \in \mathcal{Z}_l$ such that $\partial^*(\mathrm{d}^\nabla s) = 0$ and assume that $\Pi_l(s) = 0$. This means that $s \in \mathcal{B}_l$. However, $\partial^*(\mathrm{d}^\nabla s) = 0$ already implies $s = 0$ for $s \in \mathcal{B}_l$ by Lemma 2.4.1. This shows uniqueness.

For existence, let $s \in \mathcal{Z}_l$ be an arbitrary lift of σ. Since $\partial^*(\mathrm{d}^\nabla s) \in \mathcal{B}_l$, Lemma 2.4.1 shows that there exists a $b \in \mathcal{B}_l$ such that $\partial^*(\mathrm{d}^\nabla b) = \partial^*(\mathrm{d}^\nabla s)$. But then $s - b \in \mathcal{Z}_l$ projects onto σ as well but additionally satisfies $\partial^*(\mathrm{d}^\nabla(s-b)) = 0$. $\Box$

By 2.4.2 we can define a differential splitting operator $L_l : \mathcal{H}_l \to \mathcal{Z}_l$ by associating to $\sigma \in \mathcal{H}_l$ the unique lift s in $\mathcal{Z}_l$ which satisfies $\mathrm{d}^\nabla s \in \mathcal{Z}_{l+1}$. Now we can define $\mathcal{L}_l = \mathrm{im}\ L_l(\mathcal{H}_l)$. The operators

$$L_l : \mathcal{H}_l \to \mathcal{L}_l \subset \mathcal{Z}_l$$

are called the *BGG-splitting-operators.*

As a surprisingly direct consequence one has ([**Čap05**])

COROLLARY 2.4.3. *If the geometry $(\mathcal{G}, \omega)$ is regular and normal, i.e, $K \in \Omega^2(M, \mathcal{A}M)^1$ and $\partial^*(K) = 0$, the full curvature can be recovered from the harmonic curvature $K_H = \Pi_2(K) \in \mathcal{H}_2$; in particular, K_H is the full obstruction to flatness.*

PROOF. We have already observed that we can project K to $\mathcal{H}_2$ since K lies in the kernel of ∂^*. Now $K \in \Omega^2(M, \mathcal{A}M)$ is the curvature of the tractor connection ∇ on $\mathcal{A}M$, but every linear connection satisfies the differential

Bianchi identity $d^\nabla K = 0$. Thus $K = L_2(K_H)$. I.e: the full curvature can be recovered from K_H. $\square$

In chapter 7 we will often only need the following consequence of the definition of $L_0 : \mathcal{H}_0 \to \mathcal{V}$: if $s \in \mathcal{V}$ is parallel, one trivially has $\partial^*(\nabla s) = 0$, and thus $s = L_0(\Pi_0(s))$. This is important enough to merit a

LEMMA 2.4.4. *On the space of parallel sections of a tractor bundle* $\mathbb{V}$, $L_0 \circ \Pi_0$ *is the identity. I.e., if* $s \in \mathcal{V}$ *with* $\nabla^{\mathbf{V}} s = 0$, *then*

$$s = L_0(\Pi_0(s)). \tag{57}$$

In particular, if the projection of a parallel $s \in \mathcal{V}$ *to its lowest homogeneity part in* $\Gamma(\mathcal{H}_0) = \Gamma(\mathbf{V}/\mathbf{V}^1)$ *vanishes, then* $s = 0$.

2.4.3. BGG-operators. We can now form the BGG-operators $\theta_l = \theta_l^V$,

$$\theta_l : \mathcal{H}_l \to \mathcal{H}_{l+1}, \tag{58}$$

$$\theta_l := \Pi_{l+1} \circ d^\nabla \circ L_l. \tag{59}$$

Diagrammatically,

$$\begin{array}{ccc} \mathcal{L}_l & \xrightarrow{d^\nabla} & \mathcal{Z}_{l+1} \\ {\scriptstyle L_l}\uparrow & & \downarrow{\scriptstyle \Pi_{l+1}} \\ \mathcal{H}_l & \xrightarrow{\theta_l} & \mathcal{H}_{l+1}. \end{array}$$

Note that this construction makes sense since for $\sigma \in \mathcal{H}_l$ one has by construction $d^\nabla(L_l\sigma) \in \mathcal{Z}_{l+1} = \ker \partial^* \subset \mathcal{C}_{l+1}$ and this element is projectable into $\mathcal{H}_{l+1}$.

The BGG-operators thus form a sequence

$$\Gamma(\mathbf{V}/\mathbf{V}^1) = \mathcal{H}_0 \xrightarrow{\theta_0} \mathcal{H}_1 \xrightarrow{\theta_1} \cdots \xrightarrow{\theta_{n-1}} \mathcal{H}_n.$$

This sequence won't be a complex in the general curved situation. In the case where $(\mathcal{G}, \omega)$ is a flat parabolic geometry, i.e.,$\Omega \in \Omega^2(\mathcal{G}, \mathfrak{g})$ as defined in (13) vanishes, it does form a complex: For this, we first note that in the flat case the diagram

$$\begin{array}{ccc} \mathcal{L}_l & \xrightarrow{d^\nabla} & \mathcal{Z}_{l+1} \\ {\scriptstyle L_l}\uparrow & & \uparrow{\scriptstyle L_{l+1}} \\ \mathcal{H}_l & \xrightarrow{\theta_l} & \mathcal{H}_{l+1} \end{array} \tag{60}$$

commutes, i.e., $L_{l+1} \circ \theta_l : \mathcal{H}_l \to \mathcal{L}_{l+1}$ agrees with the composition $d^\nabla \circ L_l : \mathcal{H}_l \to \mathcal{C}_l$: this commutativity is evidently equivalent to $d^\nabla(\mathcal{L}_l) \subset \mathcal{L}_{l+1} \subset \mathcal{C}_{l+1}$. But if $K \in \Omega^2(M, \mathcal{A}M)$ vanishes, one has $\partial^* \circ d^\nabla \circ d^\nabla = 0$ trivially since

$$d^\nabla \circ d^\nabla = \text{alternation} \circ K\bullet = 0.$$

Thus, having commutativity of (60), we see that

$$\theta_{l+1} \circ \theta_l = \Pi_{l+2} \circ d^\nabla \circ d^\nabla \circ L_l = 0,$$

and the BGG-sequence does indeed form a complex in flat case.

In the curved case, one can in some cases restrict to certain *subcomplexes*, see [**ČS05**].

Commutativity of (60) will play a major role in the next two sections.

CHAPTER 3

A natural adjustment procedure for tractor connections

In all of the following, $(\mathcal{G}, \omega)$ will be a regular parabolic geometry of type (G, P).

3.1. A natural space of modifications

In this section we will mainly be concerned with mending the non-commutativity of the diagram

$$\begin{array}{ccc} \mathcal{L}_0 & \xrightarrow{\nabla} & \mathcal{Z}_1 \\ {\scriptstyle L_0}\uparrow & & \uparrow{\scriptstyle L_1} \\ \mathcal{H}_0 & \xrightarrow{\Theta_0} & \mathcal{H}_1 \end{array} \tag{61}$$

This will be done by changing the tractor connection ∇ on $\mathbf{V}$ by a modification $\Psi \in \Omega^1(M, \mathrm{End}(\mathbf{V}))$. For such a Ψ, $\tilde{\nabla} = \nabla + \Psi$ is the new connection $\tilde{\nabla} s = \nabla s + \Psi s$ defined by

$$\tilde{\nabla}_\xi s := \nabla_\xi s + \Psi(\xi) s$$

for $\xi \in \mathfrak{X}(M)$ and $s \in \mathcal{V} = \Gamma(\mathbf{V})$.

The basic step in the construction of the BGG-splitting operators in section 2.4 was Lemma 2.4.1, where we constructed an inverse of

$$\partial^* \circ \mathrm{d}^\nabla : \mathcal{B}_l \to \mathcal{B}_l. \tag{62}$$

The important point here was that we knew what $\partial^* \circ \mathrm{d}^\nabla$ did in lowest homogeneity: by regularity of $(\mathcal{G}, \omega)$, we have, via the canonical surjections

$$\pi_{l,i} : \mathcal{C}_l^i \to \mathcal{C}_l^i / \mathcal{C}_l^{i+1} = \mathrm{gr}_i(\mathcal{C}_l),$$

that on $\mathcal{B}_l^i$,

$$\pi_{l,i} \circ \partial^* \circ \mathrm{d}^\nabla{}_{|\mathcal{B}_l^i} = \Box \ \circ \pi_{l,i|\mathcal{B}_l^i}. \tag{63}$$

This was used to invert $\partial^* \circ \mathrm{d}^\nabla$ on $\mathcal{B}_l$. If we want to obtain a BGG-sequence for a modified connection $\tilde{\nabla} = \nabla + \Psi$ it is therefore natural to demand that also $\tilde{\nabla}$ equals ∂ in lowest homogeneity, which just says that $\Psi \in \Omega^1(M, \mathrm{End}(\mathbf{V}))$ should be of homogeneity ≥ 1, or $\Psi \in \Omega^1(M, \mathrm{End}(\mathbf{V}))^1$. Explicitly, this means that for all $\xi \in \Gamma(TM)^i = \mathfrak{X}(M)^i$ and $s \in \mathcal{V}^j$ one has that $\Psi(\xi) s \in \mathcal{V}^{i+j+1}$. In fact, we have to following simple lemma.

Lemma 3.1.1. *If $\Psi \in \Omega^1(M, \mathrm{End}(\mathbf{V}))^1$, then one has for the modified connection $\tilde{\nabla} = \nabla + \Psi$ that, for all homogeneities $j \in \mathbb{Z}$,*

$$\pi_{l+1,j} \circ \mathrm{d}^{\tilde{\nabla}}{}_{|\mathcal{C}_l^j} = \partial \ \circ \pi_{l,j}. \tag{64}$$

Proof. Choose a Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$. This yields the decomposition (51), $\nabla = \partial + D + \mathrm{P}\bullet$ with D the Weyl-connection on $\mathbf{V}$ and P the Schouten tensor; and we have observed that $\partial : \mathbf{V} \to \mathbf{C}_1$ is of homogeneity 0 while D and $\mathrm{P}\bullet$ are homogeneous of degree ≥ 1 as maps $\mathcal{V} \to \mathcal{C}_1$.

For $\xi \in \mathfrak{X}(M)^i$ and $s \in \mathcal{V}^j$ one has by assumption on Ψ, $\Psi(\xi)s \in \mathcal{V}^{i+j+1}$. Therefore also Ψ is homogeneous of degree ≥ 1 as a map $\mathbf{V} \to \mathbf{C}_1$. This gives $\tilde{\nabla} - \partial = D + \mathrm{P}\bullet + \Psi$, and thus

$$(\tilde{\nabla}_\xi s - \partial s(\xi)) = D_\xi s + \mathrm{P}(\xi)\bullet s + \Psi(\xi)s \in \mathcal{V}^{i+j+1};$$

In particular,

$$\pi_{0,i+j}(\tilde{\nabla}_\xi s) = \pi_{0,i+j}(\partial s(\xi)) = \partial(\pi_{0,j}(s))(\xi),$$

which holds for all homogeneities i, j, and therefore yields (64) for $l = 0$.

Take now $s \in \mathcal{C}_l^j$ and $\xi_i \in \mathfrak{X}(M)^{j_i}$ for $0 \leq i \leq l$ and homogeneities $j_i \in \mathbb{Z}$. Then, by definition,

$$(\mathrm{d}^{\tilde{\nabla}} s)(\xi_0, \cdots, \xi_l) = \sum_{k=0}^{l} (-1)^k \tilde{\nabla}_{\xi_k} \varphi(\xi_0, \cdots, \widehat{\xi_k}, \cdots, \xi_l) \qquad (65)$$
$$+ \sum_{0 \leq i < k \leq l} (-1)^{i+k} \varphi([\xi_i, \xi_k], \xi_0, \cdots, \widehat{\xi_i}, \cdots, \widehat{\xi_k}, \cdots, \xi_l).$$

And by regularity of the geometry,

$$[\xi_i, \xi_k] - \{\xi_i, \xi_k\} \in \mathcal{V}^{j_i + j_k + 1}$$

for all $x \in M, 0 \leq i < k \leq l$. Together with (64) for $l = 0$, (65) implies that

$$\pi_{0,j+\sum_{k=0}^l j_k}((\mathrm{d}^{\tilde{\nabla}} s)(\xi_0, \ldots, \xi_l)) =$$
$$\pi_{0,j+\sum_{k=0}^l j_k}\Big(\sum_{k=0}^{l} (-1)^k \xi_k \bullet \varphi(\xi_0, \cdots, \widehat{\xi_k}, \cdots, \xi_l)$$
$$+ \sum_{0 \leq i < k \leq l} (-1)^{i+k} \varphi(\{\xi_i, \xi_k\}, \xi_0, \cdots, \widehat{\xi_i}, \cdots, \widehat{\xi_k}, \cdots, \xi_l)\Big)$$

This is the formula for ∂ by (42). □

Since $\Box = \partial^* \circ \partial$ on $\mathrm{gr}(\mathbf{B}_l) \subset \mathrm{gr}(\mathbf{C}_l)$, this implies that (63) also holds with $\mathrm{d}^{\tilde{\nabla}}$ instead of d^{∇}: we have $\pi_{l,i} \circ \partial^* \circ \mathrm{d}^{\nabla}{}_{|\mathcal{B}_l^i} = \Box \circ \pi_{l,i|\mathcal{B}_l^i}$ for all l, i. The resulting seqeunce of BGG-(splitting)-operators for $\tilde{\nabla} = \nabla + \Psi$ is denoted $\tilde{L}_0, \cdots, \tilde{\Theta}_0, \cdots$.

In view of our prolongation problem, apart from this necessary homogeneity condition, another natural condition on the modification Ψ is that the first BGG-operator

$$\tilde{\Theta}_0 : \mathcal{H}_0 \to \mathcal{H}_1$$

of $\tilde{\nabla}$ coincides with the original first BGG-operator Θ_0. I.e.: we want to achieve commutativity of diagram (61) without changing the bottom operator. For this, one makes the following simple observation:

Lemma 3.1.2. *Let $\Psi \in \Omega^1(M, \mathrm{End}(\mathbf{V}))$ be such that for all $s \in \mathcal{V}$ one has that $\Psi s \in \mathcal{B}_1 = \mathrm{im}\ \partial^*$. Then the first BGG-splitting operator $\tilde{L}_0 : \mathcal{H}_0 \to \mathcal{V}$ coincides with L_0, and also $\tilde{\Theta}_0 = \Theta_0 : \mathcal{H}_0 \to \mathcal{H}_1$.*

Proof. Take $s \in \mathcal{L}_0$. Then, by Lemma 2.4.2, $s = L_0(\Pi_0(s))$. Thus $\tilde{\nabla} s = \nabla s + \Psi s$. Since $s \in \mathcal{L}_0$, we have that $\nabla s \in \ker \partial^*$. But by assumption on Ψ, $\Psi s \in \mathrm{im}\ \partial^* \subset \ker \partial^*$. Thus $\tilde{\nabla} s = \nabla s + \Psi s \in \ker \partial^*$. I.e.: $s \in \mathcal{V}$ is a lift of $\Pi_0(s) \in \mathcal{H}_0$ into $\mathcal{V}$ such that $\partial^*(\tilde{\nabla} s) = 0$. By Lemma 2.4.2 this split is unique, and by the definition of splitting operators it is $s = \tilde{L}_0(\Pi_0(s))$. Thus indeed $\tilde{L}_0 = L_0$.

To see that $\tilde{\Theta}_0 = \Theta_0$ we just consider the definition

$$\tilde{\Theta}_0 = \Pi_1 \circ \tilde{\nabla} \circ \tilde{L}_0 = \Pi_1 \circ (\nabla + \Psi) \circ L_0 :$$

Since Ψ is a map $\mathcal{V} \to \mathcal{B}_1 = \Gamma(\mathrm{im}\ \partial^*) = \Gamma(\ker \Pi_1)$, we have in fact

$$\tilde{\Theta}_0 = \Pi_1 \circ \nabla \circ L_0 = \Theta_0.$$

□

Thus the natural space of modifications of ∇ we will allow is

Definition 3.1.3.

$$\mathcal{D}_{\mathcal{B}} := \{\Psi \in \Omega^1(M, \mathrm{End}(\mathbf{V}))^1 : \Psi s \in \mathcal{B}_1 \ \forall s \in \mathcal{V}\}.$$

We endow the space $\mathcal{D}_{\mathcal{B}} \subset \Omega^1(M, \mathrm{End}(\mathbf{V}))$ with the filtration

$$\mathcal{D}^i_{\mathcal{B}} := \{\Psi \in \Omega^1(M, \mathrm{End}(\mathbf{V}))^1 : \Psi s \in \mathcal{B}^i_1 \ \forall s \in \mathcal{V}\}.$$

If the the tractor bundle is filtered

$$\mathbf{V} = \mathbf{V}^0 \supset \mathbf{V}^1 \supset \cdots \supset \mathbf{V}^r \supset \{0\},$$

the filtration of $\mathcal{D}_{\mathcal{B}}$ is

$$\mathcal{D}_{\mathcal{B}} = \mathcal{D}^2_{\mathcal{B}} \supset \cdots \supset \mathcal{D}^{r+k}_{\mathcal{B}} \supset \mathcal{D}^{r+k+1}_{\mathcal{B}} = \{0\}.$$

3.2. The normalization condition, existence and uniqueness

The basic observation is now

Proposition 3.2.1. *Let $\Psi \in \mathcal{D}_{\mathcal{B}}$ and R_Ψ be the curvature of the modified connection $\tilde{\nabla} = \nabla + \Psi$. Assume that one has for all $s \in \mathcal{V}$ that $\partial^*(R_\Psi s) = 0$. For this we will also write*

$$\partial^* \circ R_\Psi = 0. \tag{66}$$

Then the diagram

$$\begin{array}{ccc} \mathcal{L}_0 & \xrightarrow{\tilde{\nabla}} & \mathcal{Z}_1 \\ {\scriptstyle L_0}\uparrow & & \uparrow{\scriptstyle \tilde{L}_1} \\ \mathcal{H}_0 & \xrightarrow{\Theta_0} & \mathcal{H}_1 \end{array} \tag{67}$$

commutes.

Proof. Take $\sigma \in \mathcal{H}_0$. Then $\tilde{\nabla}(L_0(\sigma)) \in \mathcal{Z}_1$ and by construction of Θ_0,

$$\Theta_0 = \Pi_1 \circ \tilde{\nabla} \circ L_0,$$

so $\tilde{\nabla}(L_0(\sigma)) \in \mathcal{Z}_1$ is a lift of $\Theta_0 \sigma \in \mathcal{H}_1$. By assumption on Ψ resp. R_Ψ,

$$\mathrm{d}^{\tilde{\nabla}}(\tilde{\nabla}(L_0(\sigma))) = R_\Psi L_0(\sigma) \in \mathcal{Z}_2.$$

Thus $\mathrm{d}^{\tilde{\nabla}}(\tilde{\nabla}(L_0(\sigma))) \in \mathcal{L}_1$ is the unique lift of $\Pi_1(\tilde{\nabla}(L_0(\sigma))) = \Theta_0(\sigma) \in \mathcal{H}_1$, and by construction this is $\tilde{L}_1(\Theta_0(\sigma))$. I.e, we have

$$\tilde{\nabla}(L_0(\sigma)) = L_1(\Theta_0(\sigma))$$

for all $\sigma \in \mathcal{H}_0$ and thus commutativity of (67). □

The next two Lemmas lie at the heart of our adjustment procedure employed afterwards in Proposition 3.2.4, which obtains existence and uniqueness of $\Psi \in \mathcal{D}_{\mathcal{B}}$ with $\partial^* \circ R_\Psi = 0$. The goal is to control the change in curvature which occurs when one changes ∇ to $\nabla + \Psi$: first recall that with R the curvature of ∇ and R_Ψ the curvature of $\nabla + \Psi$ one has the basic formula

$$R_\Psi = R + \mathrm{d}^\nabla \Psi + \Psi \wedge \Psi, \tag{68}$$

cf. for instance [**Ram05**], Chapter 5, Proposition 5.3. Here

$$\begin{aligned} &\wedge : \big(T^*M \otimes \mathrm{End}(\mathbf{V})\big) \otimes \big(T^*M \otimes \mathrm{End}(\mathbf{V})\big) \to \Lambda^2 T^*M \otimes \mathrm{End}(\mathbf{V}), \\ &\Psi \wedge \Psi'(\xi,\eta)s = \Psi(\xi)\Psi'(\eta)s - \Psi(\eta)\Psi'(\xi) \\ &\text{for } x \in M, \Psi, \Psi' \in T^*_xM \otimes \mathrm{End}(\mathbf{V}), \xi, \eta \in T_xM, s \in \mathbf{V}_x. \end{aligned} \tag{69}$$

Written out, (68) is given by

$$\begin{aligned} &R_\Psi(\xi,\eta)s = R(\xi,\eta)s \\ &+ \nabla_\xi(\Psi(\eta)s) - \Psi(\eta)\nabla_\xi s - \nabla_\eta(\Psi(\xi)s) + \Psi(\xi)\nabla_\eta s - \Psi([\xi,\eta])s \\ &+ \Psi(\xi)\Psi(\eta)s - \Psi(\eta)\Psi(\xi)s. \end{aligned}$$

LEMMA 3.2.2. *Let $\Psi \in \Omega^1(M, \mathrm{End}(\mathbf{V}))^1$ and $\varphi \in \mathcal{D}^i_{\mathcal{B}}$ for some $2 \le i \le r+k$. Define*

$$\Delta(\Psi,\varphi) := \mathrm{d}^\nabla\varphi + \varphi \wedge \Psi + \Psi \wedge \varphi + \varphi \wedge \varphi \in \Omega^2(M, \mathrm{End}(\mathbf{V})). \tag{70}$$

Then

$$\pi_{1,i}\big(\partial^*(\Delta(\Psi,\varphi)s)\big) = \Box\pi_{1,i}(\varphi s).$$

PROOF. We will work with some Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$. By definition

$$\mathrm{d}^\nabla\varphi(\xi,\eta)s = \nabla_\xi(\varphi(\eta)s) - \varphi(\eta)\nabla_\xi s - \nabla_\eta(\varphi(\xi)s) + \varphi(\xi)\nabla_\eta s - \varphi([\xi,\eta])s.$$

With D the Weyl connection on $\mathbf{V}$ and P the Schouten tensor, $\nabla_\xi = \partial_\xi + D_\xi + (i_\xi P)\bullet$. Now

$$D_\xi + (i_\xi P)\bullet : \mathcal{V} \to \mathcal{V}$$

is homogeneous of degree ≥ 0 for all $\xi \in \mathfrak{X}(M)$. By assumption, $\varphi(\eta)s \in \mathcal{B}^{i+j}$ by for all $s \in \mathcal{V}$ and $\eta \in \mathfrak{X}(M)^j$, Furthermore, regularity of ω implies that for $\xi \in \mathfrak{X}(M)^r$ and $\eta \in \mathfrak{X}(M)^s$ the Lie-bracket $[\xi,\eta]$, coincides with the algebraic bracket $\{\xi,\eta\}$ in homogeneity $r+s$. Taking this together, we see that in homogeneity i the map

$$(\xi,\eta) \mapsto \nabla_\xi(\varphi(\eta)s) - \nabla_\eta(\varphi(\xi)s) - \varphi([\xi,\eta])$$

agrees with

$$(\xi,\eta) \mapsto \partial_\xi(\varphi(\eta)s) - \partial_\eta(\varphi(\xi)s) - \varphi(\{\xi,\eta\})s = \partial(\varphi s)(\xi,\eta).$$

The $\mathcal{V}$-valued 2-forms $[\varphi, \Psi]s$, $[\varphi, \varphi]s$ and $(\xi, \eta) \mapsto \varphi(\xi)\partial_\eta s - \varphi(\eta)\partial_\xi s$ are all of the form $(\xi, \eta) \mapsto \varphi(\xi)(\tau(\eta)) - \varphi(\eta)(\tau(\xi))$, with $\tau \in \Omega^1(M, \mathbf{V})$. But by assumption on φ, this is seen to be homogeneous of degree $\geq i+1$.

The only term left in $\Delta(\Psi, \varphi)$ is $(\xi, \eta) \mapsto \Psi(\xi)(\varphi(\eta)s) - \Psi(\eta)(\varphi(\xi)s)$. But this too is homogeneous of degree $\geq i+1$.

Thus we see that $(\Delta(\Psi, \varphi)s$ agrees with $\partial(\varphi s)$ in homogeneity i for all $s \in \mathcal{V}$. But this immediately implies the result. $\square$

LEMMA 3.2.3. *Let $\Psi \in \Omega^1(M, \mathrm{End}(\mathbf{V}))^1$ and $\varphi \in \mathcal{D}^i_{\mathcal{B}}$ for some $2 \leq i \leq r+k$. Denote by R, R' the curvature of $\nabla + \Psi$, resp. $\nabla + \Psi + \varphi$. If $\partial^* \circ R \in \mathcal{D}^i_{\mathcal{B}}$ then*

(1) $\partial^* \circ R' \in \mathcal{D}^i_{\mathcal{B}}$.

(2) $\pi_{1,i} \circ \partial^* \circ R' = \pi_{1,i} \circ \partial^* \circ R + \square \circ \pi_{1,i} \circ \varphi$.

PROOF. We have

$$R' = R + \mathrm{d}^\nabla \varphi + \varphi \wedge \Psi + \Psi \wedge \varphi + \varphi \wedge \varphi = R + \Delta(\Psi, \varphi).$$

We have seen in the proof of Lemma 3.2.2 that $\partial^*(\Delta(\Psi, \varphi)s)$ lies in $\mathcal{B}^i_1$, and so does $\partial^*(Rs)$ by assumption. Also by assumption, $\partial^* \circ R \in \Omega^1(M, \mathrm{End}(\mathbf{V}))^1$, were the focus is on the homogeneity ≥ 1. Thus to see that $\partial^* \circ R' \in \mathcal{D}^i_{\mathcal{B}}$ it remains to check that $\partial^* \circ \Delta(\Psi, \varphi) \in \Omega^1(M, \mathrm{End}(\mathbf{V}))^1$. However, it is evident that $\mathrm{d}^\nabla\varphi + [\varphi, \Psi] + [\Psi, \varphi] + [\varphi, \varphi]$ is homogeneous of degree ≥ 1 as a 2-form on M with values in $\mathrm{End}(\mathbf{V})$, and so is $\partial^* \circ \Delta(\Psi, \varphi)$. Thus we have (1). (2) follows immediately from Lemma 3.2.2. $\square$

Having these technical Lemmas, we can now prove the main result of this chapter:

THEOREM 3.2.4. *There is a unique $\Psi \in \mathcal{D}_{\mathcal{B}}$ with $\partial^* \circ R_\Psi = 0$.*

PROOF. Again we work with some Weyl structure. We will show the following by induction in i:

(*) There is a $\Psi_i \in \mathcal{D}_{\mathcal{B}}$ such that the curvature R of $\tilde{\nabla} = \nabla + \Psi_i$ has the property that $\partial^* \circ R \in \mathcal{D}^{i+1}_{\mathcal{B}}$. I.e.: $\partial^* \circ R \in \Omega^1(M, \mathrm{End}(\mathbf{V}))^1$ and for every $s \in \mathcal{V}$ one has $\partial^*(Rs) \in \mathcal{B}^{i+1}_1$.

(**) If also $\Psi'_i \in \mathcal{D}_{\mathcal{B}}$ has this property, then $(\Psi_i - \Psi'_i) \in \mathcal{D}^{i+1}_{\mathcal{B}}$.

$\Psi_1 = 0$ satisfies the conditions for $i = 1$, because $\mathcal{B}^1_1 = \{0\}$ and $K\bullet$ is homogeneous of degree ≥ 1 by regularity of the geometry. At the end of this induction Ψ_{r+k} will be the solution we sought for and will be unique.

Now assume that for $1 \leq i < r+k$ we have constructed Ψ_i. Define

$$\varphi := -\square^{-1} \circ \partial^* \circ R. \tag{71}$$

By inductive assumption $\varphi \in \mathcal{D}^{i+1}_{\mathcal{B}}$. Let $\Psi_{i+1} = \Psi_i + \varphi$ and consider $\nabla' = \nabla + \Psi_{i+1}$ and its curvature R'. One has

$$R' = R + \mathrm{d}^\nabla \varphi + \varphi \wedge \Psi_i + \Psi_i \wedge \varphi + \varphi \wedge \varphi = R + \Delta(\Psi_i, \varphi).$$

Now Lemma 3.2.3 tells us that

$$\begin{aligned} \pi_{1,i+1}(\partial^*(R's)) &= \pi_{1,i+1}(\partial^*(Rs)) + \pi_{1,i+1}(\partial^*(\Delta(\Psi_i, \varphi)s)) = \\ &= \pi_{1,i+1}(\partial^*(Rs)) + \square\pi_{1,i+1}(\varphi) = \\ &= \pi_{1,i+1}(\partial^*(Rs)) - \square\pi_{1,i+1}(\square^{-1}\partial^*(Rs)) = 0. \end{aligned}$$

Since, as seen in Lemma 3.2.3, $\partial^* \circ R' \in \Omega^1(M, \mathrm{End}(\mathbf{V}))^1$, we therefore have $\partial^* \circ R' \in \mathcal{D}_{\mathcal{B}}^{i+2}$, which shows that Ψ_{i+1} solves (*) for $i+1$.

For the uniqueness part, assume that for some $2 \leq l \leq i+1$ an $\eta \in \Omega^1(M, \mathrm{End}(\mathbf{V}))^1$ has the following properties: for every $s \in \mathcal{V}$, $\eta s \in \mathcal{B}_1^l$ and the curvature R'' of $\nabla + \Psi_{i+1} + \eta$ also satisfies $\partial^*(R'' s) \in \mathcal{B}_1^{i+2}$.

Then $R''s = R' + \Delta(\Psi_{i+1}, \eta)$ and

$$0 = \pi_{1,l}(\partial^*(R''s - R's)) = \pi_{1,l}(\partial^*(\Delta(\Psi_{i+1}, \eta)s)) = \Box \pi_{1,l}(\eta);$$

Invertibility of $\Box$ on $\mathcal{B}^l$ and induction in l yields $\eta s \in \mathcal{B}_1^{i+2}$ for every $s \in \mathcal{V}$. $\Box$

Since $\Psi \in \mathcal{D}_{\mathcal{B}}$ is uniquely determined by the natural condition $\partial^* \circ R_\Psi = 0$, it is natural.

Remark 3.2.5. When one works with a torsion-free, 1-graded parabolic geometry, the proof shows that the modification $\Psi \in \Omega^1(M, \mathrm{End}(\mathbf{V}))^1$ actually lies in $\Omega^1(M, \mathrm{End}(\mathbf{V})^1)$: Since the filtration $TM = T^{-1}M$ is trivial for $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ 1-graded, $\Psi \in \Omega^1(M, \mathrm{End}(\mathbf{V}))^1$ means that for every $\xi \in \mathfrak{X}(M) = \mathfrak{X}(M)^{-1}$ and $s \in \mathcal{V}^i$ one has $\Psi(\xi)s \in \mathcal{V}^i$. I.e.: $\Psi(\xi) \in \mathrm{End}(\mathbf{V})^0$ for all $\xi \in \mathfrak{X}(M)$.

Having additionally that ω is torsion-free means that its curvature K sits in $\Omega^2(M, \mathcal{A}^0 M) \subset \Omega^2(M, \mathcal{A}M)$. Then $K\bullet$ is homogeneous of degree ≥ 2 as a map from $\mathcal{V}$ to $\mathcal{C}_2$, and so is $\partial^* \circ K\bullet$. But again it follows immediately from $TM = T^{-1}M$ that $\Omega^1(M, \mathrm{End}(\mathbf{V}))^2 = \Omega^1(M, \mathrm{End}(\mathbf{V})^1)$. The first adjustment step in the proof above (which employs a Weyl structure) is given by

$$\nabla \rightsquigarrow \nabla + \varphi$$

with

$$\varphi = -\Box^{-1} \circ \partial^* \circ K\bullet.$$

Since $\Box$ preserves homogeneities, we have $\varphi \in \Omega^1(M, \mathrm{End}(\mathbf{V})^1)$. That then also prolongation connection Ψ lies in $\Omega^1(M, \mathrm{End}(\mathbf{V})^1)$ follows inductively: the inductive step in the adjustment procedure employs the change in curvature (70), and this is again seen to be of homogeneity ≥ 2.

3.3. A natural formula

In the context of Theorem 3.2.4 it is even possible to arrive at a natural formula for Ψ. This depends on the simple nature of the action of the Kostant Laplacian $\Box$: for an $0 \leq i \leq n$ consider $\mathrm{gr}_i(B_1) = \mathrm{gr}_i(C_1) \cap \mathrm{im}\, \partial^*$. Let F be a G_0-isotypical component of $\mathrm{gr}_i(B_1)$. Then $\Box$, which is G_0-equivariant, preserves F; in fact, the proof of Kostant's version of the Bott-Borel-Weyl Theorem even shows that $\Box$ acts by scalars on isotypical components, and since $\Box$ is symmetric with respect to a canonical G_0-invariant inner product on C_1 and $\mathrm{im}\, \partial^* \cap \ker \partial^* = \{0\}$, we have that $\Box$ acts by a non-zero real number on F.

The construction of a natural formula proceeds similarly as Theorem 3.2.4, but now the inductive procedure will go on finer than before, namely via the isotypical components of B_1. Let us say that for $1 \leq i \leq r+k$ the space $\mathrm{gr}_i(B_1)$ has $\tau_i \in \mathbb{N}_0$ isotypical G_0-components $F_{i,1}, \dots F_{i,\tau_i}$. By our conventions on the filtration, $\tau_1 = \tau_{r+k+1} = 0$. The associated bundles are denoted $\mathbf{F}_{i,l} = \mathcal{G} \times_P F_{i,l}$ and the space of sections of $\mathbf{F}_{i,l}$ is $\mathcal{F}_{i,l}$.

For $0 \le j \le \tau_i$ we will denote by $\mathcal{D}^i_{\mathcal{B},j}$ the space of all elements φ of $\mathcal{D}^i_{\mathcal{B}}$ which have the property that for every $s \in \mathcal{V}$ the section $\pi_{1,i}(\varphi s)$ of $\mathrm{gr}_i(\mathcal{B}_1)$ has trivial projections to $\mathcal{F}_{i,j+1}, \dots \mathcal{F}_{i,\tau_i}$.

THEOREM 3.3.1. *There is a natural formula for the adjustment* $\tilde{\nabla} = \nabla + \Psi, \Psi \in \mathcal{D}_{\mathcal{B}}$ *constructed in Theorem 3.2.4.*

PROOF. We will proceed by a nested induction upward in i and downward in j via the following statement:

(*) For $1 \le i \le r+k$ and $0 \le j \le \tau_i$ there is an $\Psi_{i,j} \in \mathcal{D}_{\mathcal{B}}$ such that the curvature R of $\tilde{\nabla} = \nabla + \Psi_{i,j}$ has the property that $\partial^* \circ R \in \mathcal{D}^i_{\mathcal{B},j}$. $\Psi_{i,j}$ is natural; more precisely, it is given by a formula only involving multiplications by real numbers and compositions of the operators ∂^*, d^∇ and the algebraic commutator-bracket $[\cdot,\cdot]$ of (69).

We know that (*) is satisfied for $i = 1, j = 0$ for the trivial reason that $\mathrm{gr}_1(\mathcal{B}_1) = \{0\}$ and by regularity of the geometry. Having shown (*) for $(i, 0)$, we have in fact (*) for $(i+1, \tau_{i+1})$. The solution we sought for will be obtained at $(r+k, 0)$ respectively $(r+k+1, 0)$.

So lets assume now that we have shown (*) for (i, j) with $1 < i \le r+k$ and $0 < j \le \tau_i$. We form $\varphi = \partial^* \circ R$, with R the curvature of $\nabla + \Psi_{i,j}$. We have the natural projections $\pi_{\mathcal{F}_{i,l}} : \mathcal{B}^i_1 \to \mathcal{F}_{i,l}$. By assumption, for every $s \in \mathcal{V}$ one has that $\pi_{\mathcal{F}_l}(\varphi s) = 0$ for $l > j$.

Assume that $\Box$ acts by $c \in \mathbb{R}\backslash\{0\}$ on $\mathbf{F}_{i,j}$. Take $\Psi_{i,j-1} := \Psi_{ij} - \frac{1}{c}\varphi$ and form

$$\tilde{\nabla}' := \tilde{\nabla} - \frac{1}{c}\varphi =: \nabla + \Psi_{i,j-1}.$$

Note that by assumption $\varphi \in \mathcal{D}^i_{\mathcal{B}}$, and in particular, $\Psi_{i,j-1} \in \mathcal{D}_{\mathcal{B}}$.

The curvature R' of $\tilde{\nabla}'$ is

$$R' = R + \Delta(\Psi_{i,j}, -\frac{1}{c}\varphi).$$

By Lemma 3.2.3, we have

$$\pi_{\mathcal{F}_l} \circ \partial^* \circ R' = (1 - \frac{c}{c})\pi_{\mathcal{F}_l} \circ \partial^* \circ R.$$

Thus it is immediately clear that $\partial^* \circ R'$ has only $j-1$ nontrivial projections.

Also by Lemma 3.2.3, we know that $\partial^* \circ R'$ sits again in $\mathcal{D}^i_{\mathcal{B}}$, and thus $\Psi_{i,j-1}$ solves (*) for $(i, j-1)$. $\Box$

REMARK 3.3.2. The formula constructed in Theorem 3.3.1 is natural but not canonical: it depends on an an order on the isotypical-typical components of $\mathcal{B}_1$. The problematic terms here are the commutators $[,]$, which result in higher order dependence on the eigenvalue of the component one started with. We note that in the case of a 1-graded, torsion-free geometry these terms vanish if $V = V_0 \oplus V_1 \oplus V_2$. That all formulas which can be arrived at by this procedure are equal also in the general case follows only by uniqueness of the solution shown in Theorem 3.2.4.

CHAPTER 4

Natural prolongation of first BGG-operators via the adjusted tractor connection

4.1. Geometric prolongation of Θ_0

Given a tractor bundle $\mathbf{V}$ for a G or $(\mathfrak{g}, P)$-representation V, we have the first BGG-operator $\Theta_0 : \mathcal{H}_0 \to \mathcal{H}_1$. In chapter 3 we found a natural adjustment $\tilde{\nabla} = \nabla + \Psi$, $\Psi \in \Omega^1(M, \mathrm{End}(\mathbf{V}))^1$, such that we have commutativity of the first BGG-diagram (67); i.e., with the splitting operators $L_0 : \mathcal{H}_0 \to \mathcal{V}$ and $\tilde{L}_1 : \mathcal{H}_1 \to \mathcal{Z}_1$:

$$\tilde{L}_1 \circ \Theta_0 = \tilde{\nabla} \circ L_0. \tag{72}$$

Theorem 4.1.1. *Let V be a $(\mathfrak{g}, P)$-representation, $\mathbf{V}$ the associated tractor bundle and $\mathcal{V}$ its space of sections. Let $\Psi \in \Omega^1(M, \mathrm{End}(\mathbf{V}))^1$ be the unique modification of the standard tractor connection ∇ on $\mathbf{V}$ with the properties*

(1) $\Psi s \in \mathrm{im}\ \partial^* \ \forall s \in \mathcal{V}$,
(2) $\partial^* \circ R_\Psi = 0$ for R_Ψ the curvature of $\nabla + \Psi$.

constructed in Theorem 3.2.4. Then, with $\tilde{\nabla} = \nabla + \Psi$, $(\mathbf{V}, \tilde{\nabla}, \Pi_0, L_0)$ is a geometric prolongation of Θ_0: The maps $\Pi_0 : \mathcal{V} \to \mathcal{H}_0$ and $L_0 : \mathcal{H}_0 \to \mathcal{V}$ restrict to inverse isomorphisms between the space of $\tilde{\nabla}$-parallel sections of $\mathbf{V}$ and the kernel of Θ_0. Since Ψ is uniquely determined by natural conditions (1) and (2) it is natural, and so is $\tilde{\nabla}$.

Proof. Let $s \in \mathcal{V}$ be such that $\tilde{\nabla} s = 0$. For the parallel section s one has $s = \tilde{L}_0(\Pi_0(s))$ by construction of the first splitting operator. But by Lemma 3.1.2 $\tilde{L}_0 = L_0$ and thus $s = L_0(\Pi_0(s))$. Then, by definition of Θ_0 (see (58)),

$$\Theta_0(\Pi_0(s)) = \Pi_1(\tilde{\nabla}(L_0(\Pi_0(s)))) = \Pi_1(\tilde{\nabla} s)) = 0.$$

Conversely, let $\sigma \in \ker(\Theta_0) \subset \mathcal{H}_0$. Then, by (72),

$$\tilde{\nabla}(L_0(\sigma)) = \tilde{L}_1(\Theta_0(\sigma)) = \tilde{L}_1(0) = 0,$$

which proves the claim. □

The connection $\tilde{\nabla}$ on $\mathbf{V}$ will occasionally be referred to as the *prolongation connection* on $\mathbf{V}$ or the prolongation connection of Θ_0.

One immediately has the following corollary:

Corollary 4.1.2. *Let V be a $(\mathfrak{g}, P)$-representation and $(\mathbf{V}, \tilde{\nabla}, \Pi_0, L_0)$ the geometric prolongation of Theorem 4.1.1. The grading of V is assumed to be $V = V_0 \oplus \cdots \oplus V_r$ for $r \in \mathbb{N}$. (See 2.2.3). Then*

(1) The space $\ker \Theta_0 \subset \mathcal{H}_0$ has rank $\leq \dim V$.

(2) *Every $\sigma \in \ker \Theta_0$ is determined by its r-jet at some point.*
(3) *If $\sigma \in \ker \Theta_0$ is nontrivial, its singularity set $\sigma^{-1}(\{0\})$ has an open dense complement.*

Proof. Let $x \in M$ and let $\tilde{\nabla}$ be the prolongation connection of Theorem 4.1.1. Take some identification of V with $\mathbf{V}_x$ and denote by $\mathrm{Hol}_x(\tilde{\nabla}) \subset \mathrm{GL}(V)$ the holonomy group of $\tilde{\nabla}$ at x. Then it is well known that the space of parallel sections of $\mathcal{V}$ with respect to a $\tilde{\nabla}$ is in 1:1- correspondence with $\mathrm{Hol}_x(\tilde{\nabla})$-invariant elements in V; in particular, $\tilde{\nabla}$-parallel sections are determined by their value at x. This immediately shows (1). For (2), one observes inductively that the first BGG-splitting operator $L_0 : \mathcal{H}_0 \to \mathcal{V}$ is a differential operator of order r; thus the r-jet of $\sigma \in \ker \Theta_0$ at x determines $(L_0\sigma)(x) \in \mathbf{V}_x$, which again determines the $\tilde{\nabla}$-parallel section $L_0\sigma$ as we have just observed. To see (3), assume that for some $x \in M$ there is a neighborhood on which σ vanishes. Then evidently $(L_0\sigma)(x) = 0$, since L_0 is a differential operator. Alas, since $s = L_0(\sigma)$ is $\tilde{\nabla}$-parallel, it is determined by its value at x and therefore vanishes globally. To be precise, we assume here that M is connected. □

We remark that the notion of r-jets used in Corollary 4.1.2 is the usual one and ignores the filtration of TM.

Proposition 4.1.3. *Let $s \in \mathcal{V}$ such that $\nabla s = 0$. Then $\Psi s = 0$. Otherwise put: ∇-parallel sections of $\mathcal{V}$ are also $\tilde{\nabla}$-parallel.*

Proof. We will use Theorem 3.3.1, respectively its proof. Let

$$\varphi := \partial^* \circ (K\bullet).$$

Then, since s is ∇-parallel and $K\bullet$ is the curvature of ∇, one has $\varphi s = 0$. But then also $(\mathrm{d}^\nabla \varphi)s = 0$ and $[\varphi, \varphi]s = 0$. Since Ψ is generated from φ by these operations we thus have $\Psi s = 0$. □

Remark 4.1.4. Of course, in general, for $\Psi \neq 0$, the converse does not hold. Another way to see Proposition 4.1.3 is (see also [**Čap08**], Corollary 3.5): Let $s \in \mathcal{V}$ be ∇-parallel, then $\Theta_0(\Pi_0(s)) = 0$, and thus $\tilde{\nabla}(L_0(\Pi_0(s))) = \tilde{\nabla} s = 0$ by Theorem 4.1.1.

Definition 4.1.5. Generalizing [**Lei05**] we say that the *normal* solutions of

$$\Theta_0\sigma \stackrel{!}{=} 0, \ \sigma \in \mathcal{H}_0$$

are those which split to ∇-parallel sections of $\mathbf{V}$, i.e., we define $\ker_{\mathrm{nor}}\Theta_0 \subset \ker \Theta_0$ by

$$\ker_{\mathrm{nor}}\Theta_0 := \{\sigma \in \mathcal{H}_0 \ : \ \nabla L_0(\sigma) = 0\}.$$

This yields additional equations on σ. For specific examples of this see 6.2 and 7.3.3.

4.2. Curvature and obstructions

4.2.1. The Curvature of the prolongation connection and tensorial obstructions.

LEMMA 4.2.1. *Let R be the curvature of the adjusted connection $\tilde{\nabla} = \nabla + \Psi$. Assume that $K \in \Omega^2(M, \mathcal{A}M)^l$. Then, for s a section of $\mathbf{V}$, one has*

$$\pi_{2,l}(Rs) = \mathrm{proj}_{\mathrm{gr}_l(\mathcal{Z}_2)}\big(\pi_{2,l}(K\bullet s)\big). \tag{73}$$

PROOF. In the proof of Theorem 3.2.4 we started out with a regular parabolic geometry: By Proposition 2.2.6, regularity of the Cartan geometry $(\mathcal{G}, \omega)$ is equivalent to $K \in \Omega^2(M, \mathcal{A}M)^1$; i.e., the curvature form K satisfies that for all $\xi \in \mathfrak{X}(M)^i$ and $\eta \in \mathfrak{X}(M)^j$ one has $K(\xi, \eta) \in \mathcal{A}^{i+j+1}M$. This condition is exactly what is needed for the map $\varphi \in \Omega^1(M, \mathrm{End}(\mathbf{V})$ defined (after choice of a Weyl structure) by $\varphi s = -\Box^{-1}(\partial^*(K\bullet s))$ to be homogeneous of degree ≥ 1, i.e., for φ to lie in $\Omega^1(M, \mathrm{End}(\mathbf{V}))^1$. Likewise, when one starts with a curvature form K which is homogeneous of degree $\geq l$, $l \geq 2$, one sees that $\varphi \in \Omega^1(M, \mathrm{End}(\mathbf{V}))^l$, and also the final adjustment Ψ of Theorem 3.2.4 will be homogeneous of degree $\geq l$. In this case the induction process in the proof of Theorem 3.2.4 starts at $i = l$, since for $s \in \mathcal{V} = \mathcal{V}^0 \supset \cdots \supset \mathcal{V}^r$ arbitrary, φs is automatically homogeneous of degree $\geq l$.

Now the resulting $\Psi \in \Omega^1(M, \mathrm{End}(\mathbf{V}))^l$ can be written as $\Psi = \varphi + \varphi'$ with $\varphi' \in \mathcal{D}_B^{l+1}$; in particular, for every $s \in \mathcal{V}$, $\varphi' s \in \mathcal{B}_1^{l+1}$. If we apply Lemma 3.2.2, we see that in homogeneity l, $Rs = R(\nabla + \varphi + \varphi')s$ equals $R(\nabla + \varphi)s$; but the last line of the proof of this Lemma tells us thus that with $\varphi = -\Box^{-1} \circ \partial^* \circ (K\bullet)$

$$\begin{aligned}\pi_{2,l}(Rs) &= \pi_{2,l}(K\bullet s) + \pi_{2,l}(\partial(\varphi s))_l = \\ &= \pi_{2,l}(K\bullet s) - \pi_{2,l}(\partial(\Box^{-1}(\partial^*(K\bullet s)))),\end{aligned}$$

which is the projection of $\pi_{2,l}(K\bullet s) \in \mathrm{gr}_l(\mathcal{C}_2)$ to $\mathrm{gr}_l(\mathcal{Z}_2)$. □

COROLLARY 4.2.2. *Let $(\mathcal{G}, \omega)$ be a 1-graded, torsion-free parabolic geometry, V a $(\mathfrak{g}, P)$-representation. Let R be the curvature of the adjusted connection $\tilde{\nabla} = \nabla + \Psi$ of Theorem 3.2.4. With the first two BGG-operators $\tilde{\Theta}_1$ and Θ_0 of $\tilde{\nabla}$ we define the natural map*

$$\begin{aligned}&\Phi : \mathcal{H}_0 \to \mathrm{gr}_2(\mathcal{H}_2),\\ &\Phi := \mathrm{proj}_{\mathrm{gr}_2(\mathcal{H})} \circ \tilde{\Theta}_1 \circ \Theta_0.\end{aligned}$$

Then Φ is tensorial and

$$\Phi(\sigma) = \pi_{2,2}(RL_0(\sigma)) = \mathrm{proj}_{\mathrm{gr}_2(\mathcal{H}_2)}(K\bullet \pi_{0,0}(\sigma)) \tag{74}$$

for all $\sigma \in \mathcal{H}_0$.

PROOF. Let $s = L_0\sigma$, then (73) for $l = 2$ gives us

$$\pi_{2,2}(Rs) = \mathrm{proj}_{\mathrm{gr}_2(\mathcal{H}_2)}(\pi_{2,2}((K\bullet s)))$$

since $\mathrm{gr}_2(\mathcal{Z}_2) = \mathrm{gr}_2(\mathcal{H}_2)$.

By commutativity of (67), we have $R\bullet s = \mathrm{d}^{\tilde{\nabla}}(\tilde{\nabla}(s)) = \tilde{L}_2(\tilde{\Theta}_1(\Theta_0(\sigma)))$, which gives (74). Since $\pi_{2,2}(K\bullet s)$ only depends on $\sigma = \pi_{0,0}(s)$ the map Φ is indeed tensorial. □

By construction, Φ is in tensorial obstruction map in the sense that $\ker \Theta_0 \subset \ker \Phi$. Of course, this is only interesting when $\mathrm{gr}_2(\mathcal{H}_2) \neq \{\}$.

In the next two chapters 5 and 6 we will treat examples from projective and conformal geometry and calculate this map explicitly.

Remark 4.2.3. One can obtain stronger obstructions when one works on the tractor bundle; this is analogous to the constructions of Gover-Nurowski in [**GN06**]. Fix some Weyl structure. Then we can couple the the prolongation connection $\tilde{\nabla}$ on the tractor bundle $\mathbf{V}$ with the Weyl connection D on T^*M; we will denote the coupled connection on tensor products of T^*M with $\mathbf{V}$ by $\hat{\nabla}$. If $s \in \mathcal{V}$ is a parallel with respect to $\tilde{\nabla}$ then necessarily $Rs = 0$ with R the curvature of the prolongation connection $\tilde{\nabla}$. For $\xi \in \mathfrak{X}(M)$ we then have $0 = \hat{\nabla}_\xi(Rs) = (\hat{\nabla}_\xi R)s + R(\tilde{\nabla}_\xi s)$. But the last summand vanishes by parallelicity of s, and thus we see inductively that $(\hat{\nabla}^k R)s = 0$ for all $k \geq 0$. Therefore this system of equations is a necessary condition for a section $s \in \mathcal{V}$ to be a parallel section. This system of obstructions is invariant in the following sense: if the first $l-1$ expressions vanish, then the l-th expression doesn't depend on the choice of Weyl structure resp. Weyl connection. Again, analogous to the constructions in [**GN06**] one can form determinant-like expressions which are then conformally invariant sequences of functions which determine injectivity of the maps $(\hat{\nabla}^k R)$ and thus obstruct existence of $\tilde{\nabla}$-parallel tractors. While it is clear that one can compute these expressions, this would, a priori, yield quite unmanageable formulas in the general case.

4.3. Infinitesimal automorphisms of parabolic geometries.

In this section we relate the prolongation connections one obtains by the method presented above to those constructed in [**Čap08**] for the special case of adjoint tractor bundles. Nothing new is contained here, but it is useful to recall this case here for later applications in chapter 7.

Since $\mathcal{G} \to M$ is a P-principal bundle over M and the geometric structure is encoded in the Cartan connection form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, one defines an *automorphism* of $(\mathcal{G}, \omega)$ as a P-equivariant diffeomorphism Ψ of $\mathcal{G}$ preserving ω, as was done in (10). The set of *infinitesimal automorphisms* is defined by

$$\mathbf{inf.aut.}(\mathcal{G}, \omega) := \{\xi \in \mathfrak{X}(M)^P : \ \mathcal{L}_\xi \omega = 0\}.$$

Thus the Lie algebra $\mathbf{aut}(\mathcal{G}, \omega)$ (see (11)) of the automorphism group of $(\mathcal{G}, \omega)$ is formed by the complete vector fields in $\mathbf{inf.aut.}(\mathcal{G}, \omega)$.

It was shown in [**Čap08**] that infinitesimal automorphisms of ω are in 1:1-correspondence with adjoint tractors $s \in \Gamma(\mathcal{A}M)$ which are parallel with respect to the modified connection

$$\tilde{\nabla} s = \nabla s + K(\Pi^{\mathcal{A}}(s), \cdot). \tag{75}$$

Here one employs the natural projection

$$\Pi^{\mathcal{A}} : \mathcal{A}M = \mathcal{G} \times_P \mathfrak{g} \to \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p} = TM$$

which projects $s \in \Gamma(\mathcal{A}M)$ to a vector field $\Pi^{\mathcal{A}}(s) \in \mathfrak{X}(M)$ on M and thus allows for the insertions of adjoint tractors into the curvature form $K \in \Omega^2(M, \mathcal{A}M)$.

Recall that $\omega : T\mathcal{G} \to \mathfrak{g}$ is a P-equivariant trivialization of $T\mathcal{G}$, and thus, for a $\xi \in \mathbf{inf.aut}.(\mathcal{G}, \omega)$, the function $f = \omega \circ \xi : \mathcal{G} \to \mathfrak{g}$ is P-equivariant. f therefore defines a section of the adjoint tractor bundle $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$.

We have an explicit formula for the tractor connection $\nabla^{\mathcal{A}}$ on $\mathcal{A}M$, see (45): To compute $\nabla^{\mathcal{A}}_\eta s$ for $\eta \in \mathfrak{X}(M)$ and $s \in \Gamma(\mathcal{A}M)$ we take a P-invariant lift $\eta' \in \mathfrak{X}(\mathcal{G})$ of η and let $s \in \Gamma(\mathcal{A}M)$ be the adjoint tractor corresponding to $f \in \mathrm{C}^\infty(\mathcal{G}, \mathfrak{g})^P$. Then $\nabla^{\mathcal{A}}_\eta s$ corresponds to the P-equivariant map

$$u \mapsto \eta'_u \cdot \omega(\xi) + [\omega_u(\eta'), \omega_u(\xi)]. \tag{76}$$

Proposition 4.3.1 ([**Čap08**]). *Let $s \in \Gamma(\mathcal{A}M)$ be the adjoint tractor corresponding to the P-invariant vector field $\xi \in \mathfrak{X}(\mathcal{G})^P$. Then*

$$\mathcal{L}_\xi \omega = 0 \text{ iff } \nabla^{\mathcal{A}}_\eta s + K(\Pi^{\mathcal{A}}(s), \eta) = 0 \ \forall \eta \in \mathfrak{X}(M).$$

I.e., $\xi \in \mathbf{inf.aut}.(\mathcal{G}, \omega)$ if and only if the corresponding adjoint tractor $s = s_\xi \in \Gamma(\mathcal{A}M)$ is parallel with respect to the connection

$$\tilde{\nabla}^{\mathcal{A}} s = \nabla^{\mathcal{A}} s + K(\Pi^{\mathcal{A}}(s), \cdot). \tag{77}$$

Proof. It is a standard formula that $\mathcal{L}_\xi = i_\xi \circ d + d \circ i_\xi$ as a differential operator on forms. Thus, with $\eta' \in \mathfrak{X}(\mathcal{G})^P$ a P-equivariant vector field which then factorizes to a field $\eta \in \mathfrak{X}(M)$,

$$(\mathcal{L}_\xi \omega)(\eta') = (i_\xi d\omega)(\eta') + d\omega(\xi)(\eta') = d\omega(\xi, \eta') + \eta' \cdot \omega(\xi).$$

Now use that $K \in \Omega^2(M, \mathcal{A}M)$ was defined as the quotient of the horizontal, P-equivariant form $\Omega \in \Omega^2_{\mathrm{hor}}(\mathcal{G}, \mathfrak{g})^P$ defined in (13), which is:

$$\Omega(\xi, \eta') = d\omega(\xi, \eta') + [\omega(\xi), \omega(\eta')].$$

We can thus rewrite $d\omega(\xi, \eta')$ as $\Omega(\xi, \eta') + [\omega(\eta'), \omega(\xi)]$. Therefore

$$\mathcal{L}_\xi \omega(\eta') = \eta' \cdot \omega(\xi) + [\omega(\eta'), \omega(\xi)] + \Omega(\xi, \eta').$$

But now $\Omega(\xi, \eta')$ corresponds to $K(\Pi^{\mathcal{A}}(s), \eta)$ and according to (76) the P-equivariant function $\eta' \cdot \omega(\xi) + [\omega(\eta'), \omega(\xi)]$ corresponds to $\nabla^{\mathcal{A}}_\eta s$. □

Let us now compare this modification to the one obtained by our procedure: Let R be the curvature of the modified connection $\tilde{\nabla}$ of (75). Then it is shown in Lemma 3.3, [**Čap08**] that for an adjoint tractor $s \in \mathcal{A}M$ one has $Rs = D^\omega_s K \in \Omega^2(M, \mathcal{A}M)$, with D^ω the fundamental derivative of 2.2.7.2. Thus, if $(\mathcal{G}, \omega)$ is normal, i.e, $\partial^* K = 0$, by naturality of D^ω_s one has

$$\partial^*(Rs) = \partial^*(D^\omega_s K) = D^\omega_s(\partial^*(K)) = D^\omega_s 0 = 0.$$

I.e, $\Psi s = i_{\Pi^{\mathcal{A}}(s)} K$ satisfies condition (66), namely $\partial^* \circ R_\Psi = \partial^* \circ R = 0$.

We have that $\mathcal{H}_0 = \mathcal{A}M/(\mathcal{A}M)^{-k+1} = TM/TM^{-k+1}$. With the first BGG-splitting operator $\tilde{L}_0 : \mathcal{H}_0 \to \mathcal{A}M$ of $\tilde{\nabla}$ and the first BGG-operator $\tilde{\Theta}_0 : \mathcal{H}_0 \to \mathcal{H}_1$ the proof of Proposition 3.2.1 shows that $(\mathcal{A}M, \tilde{\nabla}, \Pi_0, \tilde{L}_0)$ is a geometric prolongation of $\tilde{\Theta}_0$. In particular, the operator $\tilde{\Theta}_0$ constructed via $\tilde{\nabla}$ has as its kernel the space of infinitesimal automorphisms of ω.

Let us assume that $(\mathcal{G}, \omega)$ is torsion-free, i.e., $K \in \Omega^2(M, \mathcal{A}^0 M)$. Moreover, we demand that the $\mathcal{H}_1$ resp. $H_1 = H_1(\mathfrak{p}_+, \mathfrak{g})$ is *concentrated in* *non-positive homogeneity*: The condition on $H_1 = H_1(\mathfrak{p}_+, \mathfrak{g})$ means that $\Pi_2(Z^1_2(\mathfrak{p}_+, \mathfrak{g})) = \{0\}$. It is shown in Theorem 3.4 of [**Čap08**], that

$$\Psi = K(\Pi^{\mathcal{A}}(\cdot), \cdot) : \mathcal{V} \to \Omega^1(M, \mathcal{A}M) = \mathcal{C}_1$$

has values in $\mathcal{B}_1 = \text{im } \partial^*$. By uniqueness of Ψ with $\partial^* \circ R_\Psi = 0$ and $\Psi s \in \text{im } \partial^* \ \forall s \in \mathcal{V}$, we see that in this case $\Psi s = i_{\Pi(s)} K$ coincides with the solution provided by Theorem 3.2.4 and $\tilde{\nabla}^{\mathcal{A}M} = \nabla^{\mathcal{A}M} + iK$ is the geometric prolongation of the (usual) first BGG-operator $\Theta_0 : \mathcal{H}_0 \to \mathcal{H}_1$. That $H_1(\mathfrak{p}_+, \mathfrak{g})$ is concentrated in non-positive homogeneity is satisfied for parabolic geometries other than projective and contact projective structures (cf. [**ČS09**]).

In chapter 7 we will employ the following corollary (Lemma 4.10 in [**Čap08**]),

Corollary 4.3.2. *If $(\mathcal{G}, \omega)$ is regular, normal, torsion-free and H_1 is concentrated in non-positive homogeneity, every $\nabla^{\mathcal{A}M}$-parallel section of $\mathcal{A}M$ inserts trivially into the curvature form $K \in \Omega^2(M, \mathcal{A}M)$.*

Proof. We just observed that under the given conditions the modification $K(\Pi(\cdot), \cdot)$ agrees with the modification Ψ constructed in Theorem 3.2.4. Thus the claim follows immediately from Proposition 4.1.3. □

Recall that $\nabla^{\mathcal{A}M}$-parallel sections of $\mathcal{A}M$ are in 1:1-correspondence with $\ker_{\text{nor}}(\Theta_0)$ be Definition 4.1.5. These are the *normal* infinitesimal automorphisms.

CHAPTER 5

Explicit examples of prolongations in projective geometry

5.0.1. Index notation. Before we start with examples in projective geometry we introduce some general notation which will be very useful for all explicit calculations occurring in chapters 5, 6 and 7. This index notation is derived from [**GŠ08**], which is itself a version of the so called Penrose abstract index notation (cf. [**PR87**]):

We denote the tangent bundle by $\mathbf{E}^a = TM$ and the cotangent bundle by $\mathbf{E}_a = T^*M$. Tensor products are written by sequences of indices, e.g.,

$$\mathbf{E}^{a_1\cdots a_i}{}_{b_1\cdots b_j} = \otimes^i TM \bigotimes \otimes^j T^*M$$

or

$$T^*M \otimes TM \otimes T^*M = \mathbf{E}_a{}^b{}_c.$$

Symmetric powers will be denoted by round brackets,

$$S^2T^*M = \mathbf{E}_{(ab)},$$

exterior powers by square brackets, e.g,

$$T^*M \otimes \Lambda^k T^*M = \mathbf{E}_{c[a_1\cdots a_k]}.$$

The space of sections of $\mathbf{E}^*_*$ (with $*$ being a placeholder for some indices) is $\mathcal{E}^*_* := \Gamma(\mathbf{E}^*_*)$ and the modelling vector space will be denoted E^*_*. The space of vector fields is thus $\mathcal{E}^a = \mathfrak{X}(M) = \Gamma(TM)$ and one forms are $\mathcal{E}_a = \Omega^1(M) = \Gamma(T^*M)$. For an n-dimensional manifold $E^a \cong \mathbb{R}^n$, $E^a \cong (\mathbb{R}^n)^*$.

Taking traces is done by using the same symbol for an upper and a lower index, resembling the Einstein sum convention for usual indices: for instance, when $\varphi \in \mathcal{E}_a{}^b$, then $\varphi_p{}^p \in \mathrm{C}^\infty(M)$ is the trace of φ. A subscript 0 will denote complete trace-freeness, e.g.:

$$\mathcal{E}_{0}{}_{[ab]}^{\;c} = \{\phi \in \Gamma(\Lambda^2(T^*M) \otimes TM)) : \phi_{ap}{}^p = 0\}.$$

When TM is endowed with a metric $g \in \mathcal{E}_{(ab)}$ we can also contract two lower or two upper indices, e.g., with $g^{-1} \in \mathcal{E}^{(ab)}$ the inverse of g, we have $g_{pq}g^{pq} = \dim(M)$.

A subscript $\odot$ will denote the highest weight parts (with respect to the structure group of the geometry), e.g.:

$$\mathcal{E}_{\odot[ab]c} = \{\phi \in \Gamma(\Lambda^2(T^*M) \otimes T^*M) : g^{pq}\phi_{apq} = 0 \text{ and } \phi_{[apq]} = 0\}.$$

Brackets around indices of a section will denote projections to the corresponding spaces: for instance, if $\varphi_{abc} = \varphi \in \mathcal{E}_{abc} = \otimes^3 T^*M$, then $\varphi_{[ab]c} := \frac{1}{2}(\varphi_{abc} - \varphi_{bac}) \in \mathcal{E}_{[ab]c}$.

Both for projective and conformal structures there is an adapted notion of densities, and the corresponding rank 1-line bundles will be denoted by $\mathbf{E}[w]$ for $w \in \mathbb{R}$; its space of sections is again $\mathcal{E}[w]$ and the modelling 1-dimensional representations are denoted $\mathbb{R}[w]$. Instead of $\mathbf{E}_{ab} \otimes \mathbf{E}[w]$ we will simply write $\mathbf{E}_{ab}[w]$.

5.1. Projective Structures

Let M be a manifold of dimension $n \geq 2$ endowed with a projective class of linear, torsion-free connections $[D]$; here D and $\hat{D}$ are projectively equivalent if there is a $\Upsilon_a \in \mathcal{E}_a$ such that for $\omega_b \in \mathcal{E}_b$ (cf. e.g. [**BEG94**] or [**Eas08**]))

$$\hat{D}_a \omega_b = D_a \omega_b - \Upsilon_a \omega_b - \Upsilon_b \omega_a \tag{78}$$

resp. for $\xi^a \in \mathcal{E}^a$

$$\hat{D}_a \xi^b = D_a \xi^b + \Upsilon_a \xi^b + \Upsilon_p \xi^p \delta_a^{\ b};$$

here $\delta = \mathrm{id}_{TM}$ is the Kronecker-symbol for the identity on TM. We remark that this transformation is such that projectively equivalent connections have the same geodesics up to reparametrization. As in [**EM07**] and [**BEG94**] we will restrict ourselves to projective equivalence classes of affine, torsion-free connections D which induce the flat connection on the bundle of n-forms $\Omega^n(M) = \mathcal{E}_{[a_1 \cdots a_n]}$. There is a simple proof in [**EM07**] which shows that for a torsion-free connection D this is equivalent to symmetry of the Schouten tensor of D defined below in (82).

There is a unique normal Cartan geometry $(\mathcal{G}, \omega)$ on M of type $(G, P) = (\mathrm{SL}(n+1), P)$ which induces the given projective equivalence class of connections on (the oriented bundle) TM. P can be realized as the stabilizer of the ray through the first canonical basis vector of $\mathbb{R}^{n+1}$, explicitly,

$$P = \{ \begin{pmatrix} a & v \\ 0 & A \end{pmatrix} : a \in \mathbb{R} \backslash \{0\}, A \in \mathrm{GL}(n), v \in \mathbb{R}^{n*} \text{ with } a \mathrm{Det} A = 1 \}.$$

G/P is then seen to be projective n-space $\mathbb{R}P^n$.

The Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+1)$ is 1-graded

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathbb{R}^n \oplus \mathfrak{gl}(n) \oplus (\mathbb{R}^n)^*,$$

where an element $X \oplus A \oplus \varphi \in \mathfrak{g}$ for $A \in \mathfrak{sl}(n)$ corresponds to the matrix

$$\begin{pmatrix} 0 & -\varphi \\ X & A \end{pmatrix} \tag{79}$$

and $\mathbb{I}_n \in \mathfrak{gl}(n)$ embeds in $\mathfrak{g}$ as

$$-E := \begin{pmatrix} -\frac{n}{n+1} & 0 \\ 0 & \frac{1}{n+1} \mathbb{I}_n \end{pmatrix}; \tag{80}$$

E is the grading element of $\mathfrak{g}$ introduced in 2.2.1: it is uniquely determined by $\mathrm{ad}(E)_{|\mathfrak{g}_i} = i \, \mathrm{id}_{\mathfrak{g}_i}$.

With this convention the adjoint actions of $G_0 = \mathrm{GL}(n) \subset \mathrm{SL}(n+1)$ on $\mathfrak{g}_{-1} = \mathbb{R}^n$ and $\mathfrak{g}_1 = (\mathbb{R}^n)^*$ are the standard representation and its dual. As an abelian Lie group the subgroup P_+ of P formed by all unipotent matrices is seen to be isomorphic to $\mathfrak{g}_1 = (\mathbb{R}^n)^*$, and since P_+ is a normal subgroup of P one gets the semidirect decomposition $P = \mathrm{SL}(n) \ltimes (\mathbb{R}^n)^*$.

The P-associated bundle to the 1-dimensional representation which infinitesimally maps $\mathbb{I}_n$ to multiplication with $w\frac{n}{n+1}$ is denoted by $E[w]$-its space of sections $\mathcal{E}[w]$ is the space of projective w-densities.

5.1.1. The adjoint tractor bundle and curvature. The curvature of the Cartan connection form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{sl}(n+1))$ can be viewed as a section K of $\mathbf{E}_{c_1c_2} \otimes \mathcal{A}M$, with $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$ the adjoint tractor bundle. Via a connection $D \in [D]$ which induces the flat connection on $\mathcal{E}_{[a_1\cdots a_n]}$, one has according to (79) and (80) an isomorphism of $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{sl}(n+1)$ with

$$
\begin{aligned}
[\mathcal{A}M]_D = \{ &\begin{pmatrix} 0 & -\varphi \\ \xi & A \end{pmatrix} \text{ for } \varphi \in \mathbf{E}_a, \xi \in \mathbf{E}^a, A \in \mathbf{E_0}^a{}_b \} \\
&\oplus \mathbb{R} \begin{pmatrix} -\frac{n}{n+1} & 0 \\ 0 & \frac{1}{n+1}\delta^a_b \end{pmatrix}.
\end{aligned}
$$

Then the curvature is given by

$$
K = \begin{pmatrix} 0 & -A_{ac_1c_2} \\ 0 & C_{c_1c_2}{}^a{}_b \end{pmatrix} \tag{81}
$$

with A the Cotton-York tensor and C the (projectively invariant) Weyl curvature.

Let R be the curvature of D. With the Schouten tensor $\mathrm{P} \in \mathcal{E}_{(ab)}$,

$$
\mathrm{P}_{ab} = \frac{1}{n-1} R_{pa}{}^p{}_b \tag{82}
$$

one has

$$
\begin{aligned}
C_{c_1c_2}{}^a{}_p &= R_{c_1c_2}{}^a{}_p + \mathrm{P}_{c_1p}\delta^a_{c_2} - \mathrm{P}_{c_2p}\delta^a_{c_1}, \\
A_{ac_1c_2} &= 2D_{[c_1}\mathrm{P}_{c_2]a}.
\end{aligned} \tag{83}
$$

Using (83) and the differential Bianchi identity $D_{[p}C_{c_1c_2]}{}^a{}_b = 0$ one obtains

$$
D_p C_{c_1c_2}{}^p{}_b = (n-2)A_{bc_1c_2}.
$$

We will later use that the Schouten tensor transforms as

$$
\hat{\mathrm{P}}_{ab} = \mathrm{P}_{ab} - D_a\Upsilon_b + \Upsilon_a\Upsilon_b \tag{84}
$$

under a transformation $D \rightsquigarrow \hat{D}$ corresponding to Υ_a.

Recall that if $\mathbf{V} = \mathcal{G} \times_P V$ is some tractor bundle corresponding to a $(\mathfrak{g}, P)$-representation on V, the map

$$
\begin{aligned}
\mathfrak{g} \otimes V &\to V, \\
(Z, v) &\mapsto Z \cdot v
\end{aligned}
$$

is P-equivariant and defines an algebraic action of the adjoint tractor bundle

$$
\begin{aligned}
\mathcal{A}M \otimes V &\to V, \\
(s, v) &\mapsto s \bullet v.
\end{aligned}
$$

Via the inclusions $\mathbf{E}^a \hookrightarrow \mathcal{A}M$ and $\mathbf{E}_a \hookrightarrow \mathcal{A}M$ we can also act on $\mathbf{V}$ by TM and T^*M.

5.1.2. A brief summary of the necessary steps involved in some explicit computations of tractor connections, BGG-splitting operators and first BGG-operators.

REMARK 5.1.1. It turned out in chapter 2, section 2.2.6, that one proceeds as follows: first, one chooses a Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$, i.e., a G_0-equivariant splitting of $\Pi : \mathcal{G} \to \mathcal{G}_0$. In the case of projective structures every choice of $D \in [D]$ yields such a reduction. This provides a reduction of structure group of every P-associated bundle to G_0, recall (32). In particular, one obtains an isomorphism between $\mathrm{gr}(\mathbf{C}_l) = \mathrm{gr}(\Lambda^l T^*M \otimes \mathbf{V})$ and $\mathbf{C}_l$, and therewith the algebraic differential $\partial : \mathbf{C}_l \to \mathbf{C}_{l+1}$. After inclusion of TM and T^*M into $\mathcal{A}M$ one obtains actions $\bullet$ of these spaces on $\mathbf{V}$. After having calculated these, one immediately obtains a formula for the tractor connection ∇, since $\nabla = \partial + D + P\bullet$. Moreover, according to (42) and (43) we can express $\partial^* : \mathbf{C}_{l+1} \to \mathbf{C}_l$ via the algebraic action $\bullet$ of T^*M. Having ∂ and ∂^*, we can compute $\Box = \partial \circ \partial^* + \partial^* \circ \partial$ and its inverse. This will allow us to compute the first splitting operator L_0 and the first BGG-operator Θ_0. In the examples below.

5.1.3. The Lie algebra differentials. Let now V be an arbitrary $(\mathfrak{sl}(n+1), P)$-representation. In 2.2.4 we introduced Lie algebra differentials ∂ and ∂^* on the chain spaces $C_k := E_{[c_1 \cdots c_k]} \otimes V$. Since $\mathfrak{g}$ is 1-graded, $\mathfrak{g}_-$ and $\mathfrak{g}_+$ are pointwise abelian Lie algebras the formulas for Lie algebra differentials ∂ and ∂^* introduced simplify:

For $\varphi \in C_k \cong \Lambda^k \mathfrak{g}_-^* \otimes V$ and $X_0, \ldots, X_k \in \mathfrak{g}_-$ we have (after choice of $D \in [D]$,

$$\partial\varphi(X_0, \cdots, X_k) = \sum_{j=0}^{k} (-1)^j X_j \cdot \varphi(X_0, \cdots, \widehat{X_j}, \cdots, X_k). \tag{85}$$

The Kostant codifferential $\partial^* : C_{k+1} \to C_k$ is given by

$$\partial^*(Y_0 \wedge \cdots \wedge Y_k \otimes v) = \sum_{j=0}^{k} (-1)^{j+1} Y_0 \wedge \cdots \wedge \widehat{Y_j} \wedge \cdots \wedge Y_k \otimes (Y_j \cdot v). \tag{86}$$

for $v \in V, Y_0, \ldots, Y_k \in \Omega^1(M)$.

One has $\partial \circ \partial = \partial^* \circ \partial^* = 0$. We recall that it is a consequence of a general result by Kostant ([**Kos61**]), that ∂ and ∂^* are naturally adjoint with respect to a (pointwise) inner product on the chain spaces $\mathcal{C}_k$. This gives a Hodge decomposition

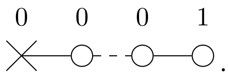
$$C_k = \mathrm{im}\ \partial \oplus \ker \Box \oplus \mathrm{im}\ \partial^* \tag{87}$$

with $\Box = \partial \circ \partial^* + \partial^* \circ \partial$.

It is a crucial fact that ∂^* is P-invariant, in contrast to ∂, which is only $G_0 = \mathrm{SL}(n)$-invariant. We use ∂^* to define the spaces $Z_k = \ker \partial^* \cap C_k, B_k = \mathrm{im}\ \partial^* \cap C_k$ and $H_k = Z_k / B_k$. Using the Hodge decomposition, one can embed H_k as $\ker \Box \subset C_k$ as a $\mathrm{GL}(n)$-submodule. The corresponding P-associated spaces to Z_k, B_k and H_k are denoted by $\mathbf{Z}_k$, $\mathbf{B}_k$ and $\mathbf{H}_k$. Their spaces of sections are $\mathcal{Z}_k, \mathcal{B}_k$ and $\mathcal{H}_k$.

5.1.3.1. *The projective standard tractor bundle* **S**. The projective standard tractor bundle $\mathbf{E}^A := \mathbf{S}$ is associated to the standard representation of $P = \mathrm{GL}(n) \ltimes (\mathbb{R}^n)^* \subset \mathrm{SL}(n+1)$ on $E^A := V := \mathbb{R}^{n+1}$.

As discussed in Remark 2.2.4, we associate to V the Dynkin diagram of $\mathfrak{sl}(n+1)$ (or rather its complexification) and we write the coefficient of the fundamental weight of the complexified dual representation $(\mathbb{C}^{n+1})^*$ over each node. Thus, V corresponds to



This notation is useful for employing the algorithms of [**BE89**] for computing the homologies $H_*(\mathfrak{p}_+, V)$, which we won't discuss here.

With respect to a choice of $D \in [D]$ we have that $\mathbf{S}$ decomposes as

$$[\mathbf{S}]_D = \begin{pmatrix} \mathbf{E}[-1] \\ \mathbf{E}^a[-1] \end{pmatrix}. \tag{88}$$

The space of sections of $\mathbf{S}$ is denoted $\mathcal{S}$. Let $s \in E^A = \mathbb{R}^{n+1}$ be given by $s = \begin{pmatrix} \rho \\ \sigma^a \end{pmatrix} \in$ for $\rho \in \mathbb{R}[-1]$ and $\sigma^a \in E^a[-1]$. Then for $X^p \in E^p = \mathbb{R}^n, Y_p \in E_p = (\mathbb{R}^n)^*$

$$X^p \cdot \begin{pmatrix} \rho \\ \sigma^a \end{pmatrix} = \begin{pmatrix} 0 \\ \rho X^a \end{pmatrix},$$
$$Y_p \cdot \begin{pmatrix} \rho \\ \sigma^a \end{pmatrix} = \begin{pmatrix} -Y_p \sigma^p \\ 0 \end{pmatrix}.$$

This gives the action $\bullet$ of $\mathcal{A}M$ resp. TM and T^*M on $\mathbf{V}$, which depends on $D \in [D]$ for TM. For $s \in \mathbf{S}$ with

$$[s]_D = \begin{pmatrix} \rho \\ \sigma^a \end{pmatrix}$$

it follows from (33) that one has for $\hat{D}$ given by (78) for a $\Upsilon_a \in \mathbf{E}_a$ that

$$[s]_{\hat{D}} = \begin{pmatrix} \rho - \Upsilon_p \sigma^p \\ \sigma^a \end{pmatrix}.$$

Now, having chosen a $D \in [D]$, we have according to 2.2.7.1,

$$\partial(v)(X) = X \bullet v, \text{ for } v \in \mathbf{V}_x, X \in T_x M.$$

Let

$$\begin{pmatrix} \rho \\ \sigma^a \end{pmatrix} \in \begin{pmatrix} \mathcal{E}[-1] \\ \mathcal{E}^a[-1] \end{pmatrix}. = [\mathcal{S}]_D.$$

The tractor connection $\nabla = \partial + D + P\bullet$ on $\mathbf{E}^A$ is

$$\nabla^S_c \begin{pmatrix} \rho \\ \sigma^a \end{pmatrix} = \begin{pmatrix} D_c \rho - \mathrm{P}_{cp} \sigma^p \\ D_c \sigma^a + \rho \delta_c{}^a \end{pmatrix},$$

The curvature R^S of ∇^S is given by the algebraic action of $K \in \Omega^2(M, \mathcal{A}M)$ on $\mathbf{S}$: one has for $s \in \mathbf{S}$ that $R^S s = K \bullet s$. (Cf. (47)).

The first BGG-splitting operator $L_0^S : \mathcal{E}^a[-1] \to \mathcal{S}$ is now computed as follows. In the first step one simply has the inclusion

$$L_0^0 : \sigma^a \mapsto \begin{pmatrix} 0 \\ \sigma^a \end{pmatrix}.$$

Composing this with the tractor connection we have

$$\sigma \mapsto \nabla^S(L_0^0(\sigma)) = \begin{pmatrix} -\mathrm{P}_{cp}\sigma^p \\ D_c\sigma^p \end{pmatrix}.$$

In 2.2.7.1 we saw that for $\varphi \in \mathbf{C}_1 = \mathbf{E}_c \otimes \mathbf{V}$, $\partial^*(\varphi) = -\bullet(\varphi)$, which is the action of the $\mathbf{E}_c$-slot of $\mathbf{E}_c \otimes \mathbf{V}$ on $\mathbf{V}$. Therefore

$$\partial^*(\nabla^S(L_0^0(\sigma))) = \begin{pmatrix} -D_p\sigma^p \\ 0 \end{pmatrix}.$$

Now $\Box = \partial^* \circ \partial$ on $\mathbf{V}$ is easily computed to be

$$\begin{pmatrix} -n \\ 0 \end{pmatrix},$$

and thus, $-\Box^{-1} \circ \partial^* \circ \nabla^S \circ L_0^0 : \mathcal{E}^a[-1] \to \mathcal{V}$ is given by

$$\sigma^a \mapsto \begin{pmatrix} -\frac{1}{n}D_p\sigma^p \\ 0 \end{pmatrix},$$

which gives the complete first splitting operator

$$L_0^S : \mathcal{E}^a[-1] \to \mathcal{S},$$
$$\sigma^a \mapsto \begin{pmatrix} -\frac{1}{n}D_p\sigma^p \\ \sigma^a \end{pmatrix}.$$

Having this, we can compute $\Theta_0^S = \Pi_1 \circ \nabla^S \circ L_0$: First, we observe (either calculate it directly or use Kostant's algorithm [**Kos61**]), that $\mathbf{H_1} = \mathcal{E}_{0c}{}^a[-1]$. Then

$$\Theta_0^S : \mathcal{E}^a[-1] \to \mathcal{E}_{0c}{}^a[-1]$$
$$\sigma^a \mapsto D_c\sigma^a - \frac{1}{n}\delta_c{}^a D_p\sigma^p.$$

Thus, $\ker \Theta_0^S$ consists of vector fields which are mapped to multiples of the identity by D.

Denoting the dual of the standard tractor bundle $\mathbf{S} = \mathbf{E}^A$ by $\mathbf{S}^* = \mathbf{E}_A$ we can regard the action of the curvature $K_{ab}\bullet$ on $\mathbf{E}^A$ as an element of $\mathcal{E}_{p_1p_2}{}^C{}_D$. By (81), this action is given by

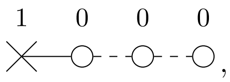
$$K_{p_1p_2}\begin{pmatrix} \rho \\ \sigma^a \end{pmatrix} = \begin{pmatrix} -A_{qp_1p_2}\sigma^q \\ C_{p_1p_2}{}^a{}_q\sigma^q \end{pmatrix}.$$

Composing this with $\partial^* : \mathcal{E}_{p_1p_2}{}^A \to \mathcal{E}_{p_1}{}^A$ gives

$$\partial^* \circ (K\bullet)(\begin{pmatrix} \rho \\ \sigma^a \end{pmatrix}) = \begin{pmatrix} C_{p_1}{}^u{}_{uq}\sigma^q \\ 0 \end{pmatrix}, \tag{89}$$

which vanishes by trace-freeness of the Weyl curvature tensor C. Thus ∇^S already satisfies condition $\partial^* \circ (K\bullet) = 0$ introduced in 3.2. Therefore, according to Theorem 4.1.1, ∇^S is the prolongation connection for the equation $0 = \Theta_0^S\sigma^a = D_c\sigma^a - \frac{1}{n}\delta_c{}^a D_p\sigma^p$ on $\sigma^a \in \mathcal{E}^a[1]$.

5.1.4. The dual standard tractor bundle. The dual standard tractor bundle, corresponding to the diagram



is $\mathbf{S}^* = \mathbf{E}_A$; with respect to a connection $D \in [D]$ it decomposes into

$$[\mathbf{S}^*]_D = \begin{pmatrix} \mathbf{E}_a[1] \\ \mathbf{E}[1] \end{pmatrix}. \tag{90}$$

Dually to the case of the standard tractor bundle, we have the following actions: Let $s = \begin{pmatrix} \varphi_a \\ \sigma \end{pmatrix} \in E_A = (\mathbb{R}^{n+1})^*$ for $\sigma \in \mathbb{R}[1]$ and $\varphi_a \in E_a[1]$. Then, with $X^p \in E^p = \mathbb{R}^n, Y_p \in E_p = (\mathbb{R}^n)^*$ one has

$$X^p \cdot \begin{pmatrix} \varphi_a \\ \sigma \end{pmatrix} = \begin{pmatrix} 0 \\ -X^p\varphi_p \end{pmatrix},$$
$$Y_p \cdot \begin{pmatrix} \varphi_a \\ \sigma \end{pmatrix} = \begin{pmatrix} \sigma Y_a \\ 0 \end{pmatrix}.$$

Thus the tractor connection is given by

$$\nabla_c^{S^*} \begin{pmatrix} \varphi_a \\ \sigma \end{pmatrix} = \begin{pmatrix} D_c\varphi_a + \mathrm{P}_{ca}\sigma \\ D_c\sigma - \varphi_c \end{pmatrix}.$$

Let $s \in \mathbf{S}^*$ with

$$[s]_D = \begin{pmatrix} \varphi_a \\ \sigma \end{pmatrix} \in \begin{pmatrix} \mathbf{E}_a[1] \\ \mathbf{E}[1] \end{pmatrix} = [\mathbf{S}^*]_D.$$

Then, for $\hat{D} \in [D]$ given by (78) for $\Upsilon_a \in \mathbf{E}_a$, one has

$$[s]_{\hat{D}} = \begin{pmatrix} \varphi_a + \Upsilon_a \\ \sigma \end{pmatrix}.$$

Since the computations are completely analogous to the previous case, we just state the results: The first splitting operator is

$$L_0^{S^*} : \mathcal{E}[1] \to \mathcal{S}^*,$$
$$\sigma \mapsto \begin{pmatrix} D_a\sigma \\ \sigma \end{pmatrix}$$

and with $\mathbf{H}_1 = \mathcal{E}_{(ab)}[1]$ we obtain

$$\Theta_0^{S^*} : \mathcal{E}[1] \to \mathcal{E}_{(ab)}[1],$$
$$\sigma \mapsto D_aD_b\sigma + \sigma\mathrm{P}_{ab}.$$

Similarly to the previous case of the projective standard tractor bundle, it is again easy to see that ∇^{S^*} is already the prolongation connection of the equation $\Theta_0^{S^*}\sigma \overset{!}{=} 0, \sigma \in \mathcal{E}[1]$. In particular, according to Corollary 4.1.2, every nontrivial solution $\sigma \in \mathcal{E}[1]$ of this equation is non-vanishing on an open dense subset.

Let therefore $0 \neq \sigma \in \ker\Theta_0^{S^*} \subset \mathcal{E}[1]$ and set $U = \{x \in M : \sigma(x) \neq 0\}$. Let $\bar{\sigma} \in \mathrm{C}^\infty(M)$ be the trivialization of $\sigma \in \mathcal{E}[1]$ and take the 1-form

$$\Upsilon_a = D_a(\log\frac{1}{|\bar{\sigma}|})$$

on U and consider the projectively equivalent connection $\hat{D}$, (78). Then

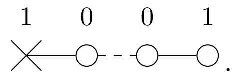
$$\begin{aligned}\Upsilon_a &= -\bar{\sigma}^{-1} D_a \bar{\sigma},\\ D_a \Upsilon_b &= \bar{\sigma}^{-2}(D_a\bar{\sigma})(D_b\bar{\sigma}) - \bar{\sigma}^{-1} D_a D_b \bar{\sigma},\\ \Upsilon_a \Upsilon_b &= \bar{\sigma}^{-2}(D_a\bar{\sigma})(D_b\bar{\sigma})\end{aligned}$$

and thus, according to (84), $\hat{\mathrm{P}} = 0$. I.e.: On U the projectively equivalent connection $\hat{D}$ is Ricci-flat.

5.1.5. The adjoint tractor bundle. Let us now treat the adjoint bundle $\mathcal{A}M$

$$\mathcal{A}M = \mathfrak{sl}(\mathbf{S}) = (\mathbf{S} \otimes \mathbf{S}^*)_0 = \mathbf{E}_0{}^A{}_B,$$

which corressponds to the diagram



Define

$$\tau_+^S := \begin{pmatrix}1\\0\end{pmatrix} \otimes 1 \in S \otimes \mathbb{R}[1],\ \tau_-^{S^*} := \begin{pmatrix}0\\1\end{pmatrix} \otimes 1 \in S^* \otimes \mathbb{R}[-1].$$

Choosing $D \in [D]$, we have decompositions (88) and (90), and can use τ_+^S and $\tau_-^{S^*}$ to define sections $\tau_+^{\mathbf{S}} \in \Gamma(\mathbf{S} \otimes \mathbf{E}[1])$, $\tau_-^{\mathbf{S}^*} \in \Gamma(\mathbf{S} \otimes \mathbf{E}[-1])$, which correspond to the respective inclusions of $\mathbf{E}[-1]$ into the top slot of (88) and $\mathbf{E}[1]$ into the bottom slot of (90).

They allow us to decompose

$$[\mathcal{A}M]_D = \begin{pmatrix}\mathrm{gr}_1(\mathcal{A}M)\\ \mathrm{gr}_0(\mathcal{A}M)\\ \mathrm{gr}_{-1}(\mathcal{A}M)\end{pmatrix} = \begin{pmatrix}\mathbf{E}_a\\ \mathbf{E}[0] \mid \mathbf{E}_0{}^a_b\\ \mathbf{E}^a\end{pmatrix}$$

via the associated bundle maps to

$$\begin{aligned}\varphi_a \in E_a &\mapsto \tau_+^S \otimes \varphi_a\\ 1 \in \mathbb{R} &\mapsto -\frac{n}{n+1}\tau_+^S \otimes \tau_-^{S^*} + \frac{1}{n+1}\delta^a_b\\ \mu^a_b \in E_0{}^a_b &\mapsto \mu^a_b\\ \sigma^a \in E^a &\mapsto \sigma \otimes \tau_-^{S^*}.\end{aligned}$$

To be precise, the identification of $\mathbf{E}^a$ with $\mathrm{gr}_{-1}(\mathfrak{g}) = \mathfrak{g}_{-1} = E^a$ used here employes the embedding $E^a \hookrightarrow S \otimes \mathbb{R}[1]$, which yields the embedding $\sigma^a \in E^a \mapsto \sigma \otimes \tau_-^{S^*} \in (S \otimes \mathbb{R}[1]) \otimes (S \otimes \mathbb{R}[-1]) = S \otimes S^*$, and completely similarly for the other components.

In this picture, $X^p \in E^p$ and $Y_p \in E_p$ act on $\mathfrak{g}$ by

$$X^p \cdot \begin{pmatrix}\varphi_a\\ \rho \mid \mu^a_b\\ \sigma^a\end{pmatrix} = \begin{pmatrix}0\\ \frac{n+1}{n}X^p\varphi_p \mid (X^a\varphi_b)_0\\ -\rho X^a - \varphi^a_p X^p\end{pmatrix}$$

$$Y_p \cdot \begin{pmatrix}\varphi_a\\ \rho \mid \mu^a_b\\ \sigma^a\end{pmatrix} = \begin{pmatrix}-\rho Y_a - \mu^p_a Y_p\\ \frac{n+1}{n}\sigma^p Y_p \mid (\sigma^a\varphi_b)_0\\ 0\end{pmatrix},$$

and this yields

$$\Box \begin{pmatrix} \varphi_a \\ \rho \mid \mu^a{}_b \\ \sigma^a \end{pmatrix} = \begin{pmatrix} -(n+1)\varphi_a \\ -(n+1)\rho \mid -\mu^a{}_b \\ 0 \end{pmatrix}.$$

It is now straightforward to compute the tractor connection and the first BGG-splitting operator:

$$\nabla^{\mathcal{A}}_c \begin{pmatrix} \varphi_a \\ \rho \mid \mu^a{}_b \\ \sigma^a \end{pmatrix} = \begin{pmatrix} D_c\varphi_a - \rho \mathrm{P}_{ca} - \mathrm{P}_{cp}\mu^p{}_a \\ D_c\rho + \frac{n+1}{n}\mathrm{P}_{cp}\sigma^p + \frac{n+1}{n}\varphi_c \mid D_c\mu^a{}_b + \mathrm{P}_{ca}\sigma^b - \frac{1}{n}\mathrm{P}_{cp}\sigma^p\delta^a{}_b + \delta^a{}_c\varphi_b - \frac{1}{n}\varphi_c\delta^a{}_b \\ D_c\sigma^a - \rho\delta^a{}_c - \mu^a{}_c \end{pmatrix}.$$

$$L_0^{\mathcal{A}} : \mathcal{E}^a \to \Gamma(\mathcal{A}M),$$

$$\sigma^a \mapsto \begin{pmatrix} -\frac{1}{n+1}(D_pD_a\sigma^p + 2\mathrm{P}_{ap}\sigma^p) \\ \frac{1}{n}D_p\sigma^p \mid (D_a\sigma^b)_0 \\ \sigma^a \end{pmatrix}.$$

Employing Kostant's algorithm [**Kos61**], one finds that $H_1 = E_{0}{}_{(ab)}^{c}$. An element $\mu_{ab}{}^c \in E_{0}{}_{(ab)}^{c}$ includes into $p_+ \otimes \mathfrak{g}_0 \subset \mathfrak{p}_+ \otimes \mathfrak{g} = C_1$ via

$$\mu_{ca}{}^b \mapsto \begin{pmatrix} 0 \\ 0 \mid \mu_{ca}{}^b \in E_c \otimes E_{0}{}_a^{b} \\ 0 \end{pmatrix}.$$

Now one computes that for a $\sigma \in \mathcal{E}^a$ the lowest slot of $\nabla^{\mathcal{A}}(L_0^{\mathcal{A}}(\sigma))$ vanishes and the $\Gamma(\mathbf{E}_c \otimes \mathbf{E}_{0}{}_a^{b})$-component is given by $(D_aD_b\sigma^c + P_{ab})_0$, with subscript 0 denoting the complete trace-free part. Now im $\partial^* \subset E_c \otimes E_{0}{}_a^{b}$ is easily seen to consist of those elements in $E_c \otimes E_{0}{}_a^{b}$ which are alternating in c and a. Therefore

$$\Theta_0^{\mathcal{A}} : \mathcal{E}^a \to \mathcal{E}_{0}{}_{(ab)}^{c},$$
$$\sigma^a \mapsto (D_{(a}D_{b)}\sigma^c + \mathrm{P}_{ab}\sigma^c)_0.$$

The infinitesimal symmetries of the projective structure are those vector fields whose flows preserve the projective class of linear connections; they are also called projective vector fields. Since the homological condition $\Pi_2(Z_2^1(\mathfrak{p}_+, \mathfrak{g})) = \{0\}$ is not satisfied for projective structures (see 4.3), $\Theta_0^{\mathcal{A}}$ is not the operator governing projective vector fields. To obtain this operator, one needs to form the first BGG-operator of the tractor connection $\nabla^{\mathcal{A}} + iK$ with K the curvature (81). One finds that this yields the operator

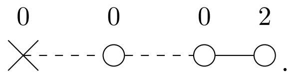
$$\sigma^a \mapsto (D_{(a}D_{b)}\sigma^c + \mathrm{P}_{ab}\sigma^c)_0 + C_{p(a}{}^c{}_{b)}\sigma^p.$$

Observe that $C_{p(a}{}^c{}_{b)}\sigma^p$ already lies in $\mathcal{E}_{0}{}_{(ab)}^{c}$ by complete trace-freeness of the Weyl curvature C.

5.2. Metrization of conformal structures

In [**EM07**] M. Eastwood and V. Matveev gave an explicit prolongation of an overdetermined equation in projective geometry which governs the metrizability of a projective class of metrics. The authors also construct a prolongation connection for this system. We will show how our algorithm of chapter 3 can be applied in this case.

We are going to construct the prolongation of the BGG-operator of the tractor bundle $\mathbf{V}$ associated to the second symmetric power $E^{\bar{A}} := V := S^2\mathbb{R}^{n+1}$ of the standard representation of P on $\mathbb{R}^{n+1}$, which corresponds to the diagram



Looking at the Hasse diagram of $(\mathfrak{g}, P)$ and employing Kostant's Version of the Bott-Borel-Weyl theorem [**Kos61**] one sees that with respect to a connection in the projective class the resulting first BGG-operator $\Theta_0 : \mathcal{H}_0 \to \mathcal{H}_1$ is

$$\begin{aligned}\Theta_0 : \mathcal{E}^{(ab)}[-2] &\to \mathcal{E}_{0c}^{(ab)}[-2],\\ \sigma^{ab} &\mapsto (D_c \sigma^{ab})_0.\end{aligned}$$

One could also proceed as in 5.1 and compute this directly.

We will write

$$E^{\bar{A}} = E^{(AB)} = \begin{pmatrix} E[-2] \\ E^a[-2] \\ E^{(ab)}[-2] \end{pmatrix}$$

via the following identification: Let $\sigma^a, \eta^a \in E^a[-1]$:

$$\begin{pmatrix} 0 \\ 0 \\ \sigma^{(a}\eta'_{b)} \end{pmatrix}^{AB} = \begin{pmatrix} 0 \\ \sigma \end{pmatrix}^A \begin{pmatrix} 0 \\ \eta \end{pmatrix}^B + \begin{pmatrix} 0 \\ \sigma \end{pmatrix}^B \begin{pmatrix} 0 \\ \eta \end{pmatrix}^A \in E^{(AB)}.$$

For $\sigma = \varepsilon\sigma'$ with $\sigma' \in E^a[-1]$ and $\varepsilon \in \mathbb{R}[-1]$,

$$\begin{pmatrix} 0 \\ \sigma^a \\ 0 \end{pmatrix}^{AB} = \begin{pmatrix} 0 \\ \sigma' \end{pmatrix}^A \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}^B + \begin{pmatrix} 0 \\ \sigma' \end{pmatrix}^B \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}^A \in E^{(AB)}.$$

For $\varepsilon, \varepsilon' \in \mathbb{R}[-1]$,

$$\begin{pmatrix} \varepsilon\varepsilon' \\ 0 \\ 0 \end{pmatrix}^{AB} = \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}^A \begin{pmatrix} \varepsilon' \\ 0 \end{pmatrix}^B \in E^{(AB)}.$$

An element $s \in E^{\bar{A}} = V$ will be written

$$s = \begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab} \end{pmatrix} \in \begin{pmatrix} E[-2] \\ E^a[-2] \\ E^{(ab)}[-2] \end{pmatrix} = \begin{pmatrix} V_2 \\ V_1 \\ V_0 \end{pmatrix}. \qquad (91)$$

One can project to the lowest and include the highest slot P-equivariantly.

As in the previous section it is completely straightforward to compute the actions of $E^a = \mathbb{R}^n$ and $E_a = \mathbb{R}_n$ on V: For $X^a \in E^a, Y_a \in E_a$ one has

$$X^a \cdot \begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab} \end{pmatrix} = \begin{pmatrix} 0 \\ \rho X^a \\ X^{(a}\mu^{b)} \end{pmatrix},$$

$$Y_a \cdot \begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab} \end{pmatrix} = \begin{pmatrix} -2Y_p\mu^p \\ -2Y_p\sigma^{pa} \\ 0 \end{pmatrix}.$$

This allows us to compute the Lie algebra differentials ∂ and ∂^*, which we will only need on the first chain spaces:

$$\begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab}. \end{pmatrix} \overset{\partial}{\mapsto} \begin{pmatrix} 0 \\ \rho\delta_c{}^a \\ \delta_c{}^{(a}\mu^{b)} \end{pmatrix}.$$

$$\begin{pmatrix} \rho_c \\ \mu_c{}^a \\ \sigma_c{}^{ab}. \end{pmatrix} \overset{\partial}{\mapsto} \begin{pmatrix} 0 \\ 2\delta_{[c_1}{}^a\rho_{c_2]} \\ 2\delta_{[c_1}{}^{(a_1}\mu_{c_2]}{}^{a_2)} \end{pmatrix}.$$

$$\begin{pmatrix} \rho_c \\ \mu_c{}^a \\ \sigma_c{}^{ab}. \end{pmatrix} \overset{\partial^*}{\mapsto} \begin{pmatrix} -2\mu_p{}^p \\ -2\sigma_p{}^{pa} \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} \rho_{c_1c_2} \\ \mu_{c_1c_2}{}^a \\ \sigma_{c_1c_2}{}^{ab}. \end{pmatrix} \overset{\partial^*}{\mapsto} \begin{pmatrix} 2\mu_{cp}{}^p \\ 2\sigma_{cp}{}^{pa} \\ 0 \end{pmatrix}.$$

As G_0-representations, V_2, V_1 and $\mathfrak{p}_+ \otimes V_2$ are irreducible and are contained in the image of ∂^*; $\mathfrak{p}_+ \otimes V_1$ decomposes into the trace-free part im $\partial^* \cap \mathfrak{p}_+ \otimes V_1$ and the trace part which lies in the image of ∂. The Kostant Laplacian $\Box$ acts by multiplication by the scalars

$$\begin{pmatrix} -2n \\ -(n+1) \\ 0 \end{pmatrix}$$

on V, by multiplication with $-2(n-1)$ on $\mathfrak{p}_1 \otimes V_2$ and by multiplication with $-n$ on the trace-free part of $\mathfrak{p}_1 \otimes V_1$. This is all the algebraic information we need to calculate the splitting operators and the prolongation.

Now take a $D \in [D]$. Then a section $s \in \mathcal{V}$ decomposes to

$$[s]_D = \begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab} \end{pmatrix} \in \begin{pmatrix} \mathcal{E}[-2] \\ \mathcal{E}^a[-2] \\ \mathcal{E}^{(ab)}[-2] \end{pmatrix}. \tag{92}$$

If we transform $D \rightsquigarrow \hat{D}$ for a $\Upsilon_a \in \mathcal{E}_a$ then according to (33),

$$[s]_{\hat{D}} = \begin{pmatrix} \rho - 2\Upsilon_p\mu^p + \Upsilon_p\Upsilon_q\sigma^{pq} \\ \mu^a - 2\Upsilon_p\sigma^{pa} \\ \sigma^{ab} \end{pmatrix}. \tag{93}$$

The tractor connection ∇ on $\mathbf{V}$ is easily calculated with the above actions of $\mathbf{E}_a$ and $\mathbf{E}^a$ on $\mathbf{V}$ together with the formula $\nabla = D + \partial + \mathrm{P}\bullet$:

$$\nabla \begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab} \end{pmatrix} = \begin{pmatrix} D_c\rho - 2\mathrm{P}_{ca}\mu^a \\ D_c\mu^a - 2\mathrm{P}_{cb}\sigma^{ab} + \rho\delta_c{}^a \\ D_c\sigma^{ab} + \delta_c{}^{(a}\mu^{b)} \end{pmatrix}.$$

It is then straightforward to calculate that the first splitting operator $L_0 : H_0 \to \mathcal{V}$ is given by

$$\sigma^{(ab)} \mapsto \begin{pmatrix} \frac{1}{n(n+1)} D_p D_q \sigma^{pq} + \frac{1}{2n}\mathrm{P}_{pq}\sigma^{pq} \\ -\frac{1}{n+1} D_p \sigma^{pa} \\ \sigma^{ab} \end{pmatrix}.$$

5.2.1. Prolongation. The prolongation procedure is also straightforward enough: The action of the curvature $K \in \Omega^2(M, \mathcal{A}M)$ is given by

$$K_{c_1c_2}\bullet \begin{pmatrix} 0 \\ 0 \\ \sigma^{ab} \end{pmatrix} = \begin{pmatrix} -2A_{pc_1c_2}\mu^p \\ -2A_{pc_1c_2}\sigma^{pa} + C_{c_1c_2}{}^a{}_p\mu^p \\ 2C_{c_1c_2}{}^{(a_1}{}_p\sigma^{a_2)p} \end{pmatrix} \tag{94}$$

Therefore we define

$$\Psi_1(\begin{pmatrix} 0 \\ 0 \\ \sigma^{ab} \end{pmatrix}) := \begin{pmatrix} 0 \\ \bar{\Psi}_1\sigma \\ 0 \end{pmatrix} := -\Box^{-1}(\partial^*(K\bullet \begin{pmatrix} 0 \\ 0 \\ \sigma^{ab} \end{pmatrix})) = \begin{pmatrix} 0 \\ \frac{2}{n}C_{cp}{}^a{}_q\sigma^{pq} \\ 0 \end{pmatrix}.$$

Now the curvature of the modified connection $\nabla + \Psi_1$ is

$$R = K\bullet + \mathrm{d}^\nabla\Psi_1$$

since $\Psi_1 \wedge \Psi_1$ vanishes. For $\xi_1, \xi_2 \in \mathfrak{X}(M)$ and $s \in \mathcal{V}$

$$\begin{aligned}(\mathrm{d}^\nabla\Psi_1)s(\xi_1,\xi_2) &= \\ &= \nabla_{\xi_1}(\Psi_1(\xi_2)s) - \Psi_1(\xi_2)(\nabla_{\xi_1}s) - \nabla_{\xi_2}(\Psi_1(\xi_1)s) + \Psi_1(\xi_1)(\nabla_{\xi_2}s) - \Psi_1([\xi_1,\xi_2])s.\end{aligned} \tag{95}$$

We may expand (95) and write $(\mathrm{d}^\nabla\Psi_1)s$ as

$$\begin{pmatrix} * \\ \begin{pmatrix} D_{\xi_1}\big(\bar{\Psi}_1(\xi_2)\sigma\big) - \bar{\Psi}_1(\xi_2)\big(D_{\xi_1}\sigma\big) - D_{\xi_2}\big(\bar{\Psi}_1(\xi_1)\sigma\big) + \bar{\Psi}_1(\xi_1)\big(D_{\xi_2}\sigma\big) \\ -\bar{\Psi}_1([\xi_1,\xi_2])\sigma \\ -\bar{\Psi}_1(\xi_2)\partial_{\xi_1}\varphi + \bar{\Psi}_1(\xi_1)\partial_{\xi_2}\varphi - \bar{\Psi}_1(\xi_2)\partial_{\xi_1}\mu + \bar{\Psi}_1(\xi_1)\partial_{\xi_2}\mu \end{pmatrix} \\ \partial_{\xi_1}\bar{\Psi}_1(\xi_2)\sigma - \partial_{\xi_2}\bar{\Psi}_1(\xi_1)\sigma \end{pmatrix}, \tag{96}$$

where we don't care about the top component since it will vanish after an application of ∂^*. The lowest component is simply $\partial(\bar{\Psi}_1\sigma) = -\partial\Box^{-1}\partial^*(K\bullet\sigma)$. Thus $\partial^*(Rs)$ lies in the top slot (i.e., in homogeneity 1). So our first adjustment had the effect of moving the expression $\partial^* \circ R$ one slot higher, and $\partial^* \circ R$ only has values in the top slot.

The new connection $\nabla + \Psi_1$ has the following curvature $R = R_{\Psi_1}$ in the middle slot: From (143) we obtain the terms $2D_{[c_1}\Psi_{1c_2]}$ and (via an application of the algebraic Bianchi identity for C), $C_{c_1c_2}{}^a{}_p\mu^p$. By (94), the contribution of $K \bullet s$ to the middle slot is $-2A_{pc_1c_2}\sigma^{pa} + C_{c_1c_2}{}^a{}_p\mu^p$. In total, we obtain the action of the curvature $R = R_{\Psi_1}$:

$$\begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab} \end{pmatrix} \mapsto \begin{pmatrix} * \\ \frac{2}{n}(D_{[c_1}C_{c_2]p}{}^a{}_q)\sigma^{pq} - 2A_{pc_1c_2}\sigma^{pa} + 2C_{c_1c_2}{}^a{}_p\mu^p \\ * \end{pmatrix}.$$

Here the lowest slot is by construction in the kernel of ∂^* and the highest slot aways lies in $\ker \partial^*$. Now define

$$\Psi_2(\begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab} \end{pmatrix}) := -\Box^{-1}\partial^*((R_{\Psi_1}(\begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab} \end{pmatrix}))$$

Using $D_pC_{c_1c_2}{}^p{}_a = (n-2)A_{ac_1c_2}$ and trace-freeness of C we calculate

$$\Psi_2(\begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab} \end{pmatrix}) = \begin{pmatrix} -\frac{4}{n}A_{pcq}\sigma^{pq} \\ 0 \\ 0 \end{pmatrix}.$$

we have that $\Psi = \Psi_1 + \Psi_2 \in \Gamma(\mathbf{E}_c \otimes \mathrm{End}(\mathbf{V}))$ is given by

$$\begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab} \end{pmatrix} \mapsto \frac{2}{n}\begin{pmatrix} -2A_{pcq}\sigma^{pq} \\ C_{cp}{}^a{}_q\sigma^{pq} \\ 0 \end{pmatrix}. \tag{97}$$

Now, with R_Ψ the curvature of $\tilde{\nabla} = \nabla + \Psi$, one has by construction $\partial^* \circ R_\Psi = 0$. Thus,by Theorem 4.1.1, $\tilde{\nabla}$ is the prolongation connection for $(D_c\sigma^{ab})_0 \stackrel{!}{=} 0$.

5.2.2. Projective invariance. The Theorem also tells us that $\tilde{\nabla}$ is natural and therefore doesn't depend on the choice of $D \in [D]$ used to construct it. In this case this is easy to see directly:

Let $\hat{D} \in [D]$ be the modified connection of (78) corresponding to $\Upsilon_a \in \mathcal{E}_a$. We use (78) to find

$$\hat{A}_{cab} = A_{cab} + \Upsilon_p C_{ab}{}^p{}_c. \tag{98}$$

Let $\widehat{\Psi} \in \Omega^1(M, \mathrm{End}(\mathbf{V}))$ be the adjustment map calculated with respect to to $\hat{D}$.

Take $s \in \mathcal{V}$. Then $\widehat{\Psi}$ only depends on $\Pi_0(s) = \sigma$, and since C is projectively invariant and A transforms according to (98) we have

$$\widehat{\Psi}[s]_{\hat{D}} = \frac{2}{n}\begin{pmatrix} -2A_{pcq}\sigma^{pq} - 2\Upsilon_u C_{cp}{}^u{}_q \\ C_{cp}{}^a{}_q\sigma^{pq} \\ 0 \end{pmatrix}.$$

On the other hand, if we first calculate $\Psi[s]_D$ with (97) and then transform according to (93), the middle slot doesn't change and the top slot changes by $-2\Upsilon_u C_{cp}{}^u{}_q$, which shows projective invariance directly.

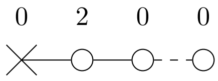
Remark 5.2.1. One can ask whether a given class of projective equivalence class of connections $[D]$ contains the Levi-Civita connection of some Riemannian metric g. It is shown in [**EM07**] that, for a $D \in [D]$ which has a parallel volume form, such Riemannian metrics correspond to positive definite $\sigma^{ab} \in \mathcal{E}^{(ab)}[-2]$ with $\Theta_0\sigma = (D_c\sigma^{ab})_0 = 0$.

5.3. $S^2\Lambda^2\mathcal{S}^*$

We now treat a more complicated example. Let V be the highest weight part of

$$S^2\Lambda^2\mathbb{R}^{n+1},$$

which corresponds to the diagram


,

and $\mathbf{V} = \mathcal{G} \times_P \mathbb{V}$. Then $\mathbf{V} \subset S^2\Lambda^2\mathbf{S}^*$, and we will treat the corresponding prolongation problem. Recall (90). We will write

$$E_{\bar{A}} = E_{[AB]} = \begin{pmatrix} E_{[ab]}[2] \\ E_a[2] \end{pmatrix}$$

via the following identification: Let $\sigma_a, \sigma'_a \in E_a[1]$:

$$\begin{pmatrix} \sigma_{[a}\sigma'_{b]} \\ 0 \end{pmatrix}_{AB} = \begin{pmatrix} \sigma \\ 0 \end{pmatrix}_A \begin{pmatrix} \sigma' \\ 0 \end{pmatrix}_B - \begin{pmatrix} \sigma \\ 0 \end{pmatrix}_B \begin{pmatrix} \sigma' \\ 0 \end{pmatrix}_A \in E_{[AB]}.$$

For $\sigma = \varepsilon\sigma'$ with $\sigma' \in E_a[1]$ and $\varepsilon \in \mathbb{R}[1]$,

$$\begin{pmatrix} 0 \\ \sigma \end{pmatrix}_{AB} = \begin{pmatrix} \sigma' \\ 0 \end{pmatrix}_A \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix}_B - \begin{pmatrix} \sigma' \\ 0 \end{pmatrix}_B \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix}_A \in E_{[AB]}.$$

We define

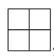
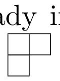
$$E_{\hat{A}} := E_{\odot(\bar{A}_1\bar{A}_2)} = \begin{pmatrix} E_{\odot[ab][cd]}[4] \\ E_{\odot[ab]c}[4] \\ E_{(ab)}[4] \end{pmatrix}.$$

Here we use the following identification: For $\sigma, \sigma' \in E_{[ab]}[2]$, $\varphi, \varphi' \in E_a[2]$,

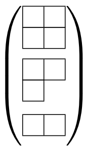
$$\begin{pmatrix} \sigma_{ab}\sigma'_{cd} + \sigma_{cd}\sigma'_{ab} \in E_{([ab][cd])} \\ 0 \\ 0 \end{pmatrix}_{\hat{A}} = \begin{pmatrix} \sigma \\ 0 \end{pmatrix}_{\bar{A}_1} \begin{pmatrix} \sigma' \\ 0 \end{pmatrix}_{\bar{A}_2} + \begin{pmatrix} \sigma \\ 0 \end{pmatrix}_{\bar{A}_2} \begin{pmatrix} \sigma' \\ 0 \end{pmatrix}_{\bar{A}_1};$$

$$\begin{pmatrix} 0 \\ \sigma_{ab}\varphi_c \\ 0 \end{pmatrix}_{\hat{A}} = \begin{pmatrix} \sigma \\ 0 \end{pmatrix}_{\bar{A}_1} \begin{pmatrix} 0 \\ \varphi \end{pmatrix}_{\bar{A}_2} + \begin{pmatrix} \sigma \\ 0 \end{pmatrix}_{\bar{A}_2} \begin{pmatrix} 0 \\ \varphi \end{pmatrix}_{\bar{A}_1};$$

$$\begin{pmatrix} 0 \\ 0 \\ \varphi_{(a}\varphi'_{b)} \end{pmatrix}_{\hat{A}} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}_{\bar{A}_1} \begin{pmatrix} 0 \\ \varphi' \end{pmatrix}_{\bar{A}_2} + \begin{pmatrix} 0 \\ \varphi \end{pmatrix}_{\bar{A}_2} \begin{pmatrix} 0 \\ \varphi' \end{pmatrix}_{\bar{A}_1}.$$

The space $E_{\odot[ab][cd]}$ has Young diagram (see e.g. [**Ful97**]) , and $\varphi_{abcd} \in E_{[ab][cd]}$ actually lies in $E_{\odot[ab][cd]}$ iff the alternation over any three indices vanishes, which already implies the symmetry $\varphi_{abcd} = \varphi_{cdab}$. The Young diagram of $E_{\odot[ab]c}$ is  and $\mu_{abc} \in E_{[ab]c}$ lies in $E_{\odot[ab]c}$ iff $\varphi_{[abc]} = 0$. We thus write, employing Young tableaux schematically,

$$V = \begin{pmatrix} V_2 \\ V_1 \\ V_0 \end{pmatrix} := \begin{pmatrix} E_{\odot[ab][cd]}[4] \\ E_{\odot[ab]c}[4] \\ E_{(ab)}[4] \end{pmatrix} =$$



Recall from 5.1 that the actions of E^a and E_a on E_A are:

$$X^p \cdot \begin{pmatrix} \sigma_a \\ \varepsilon \end{pmatrix}_A = \begin{pmatrix} 0 \\ -X^p \sigma_p \end{pmatrix}_A$$
$$Y_p \cdot \begin{pmatrix} \sigma_a \\ \varepsilon \end{pmatrix}_A = \begin{pmatrix} \varepsilon Y_p \\ 0 \end{pmatrix}_A$$

On $E_{\bar{A}} = E_{[AB]}$ one acts via:

$$X^p \cdot \begin{pmatrix} \phi_{ab} \\ \sigma_a \end{pmatrix}_{AB} = \begin{pmatrix} 0 \\ 2X^p \phi_{pa} \end{pmatrix}$$
$$Y_p \cdot \begin{pmatrix} \phi_{ab} \\ \sigma_a \end{pmatrix}_{AB} = \begin{pmatrix} -Y_{[p}\sigma_{a]} \\ 0 \end{pmatrix}$$

And finally, on $E_{\hat{A}} = E_\odot(\bar{A}\bar{B})$ we have:

$$X^p \cdot \begin{pmatrix} \phi_{abcd} \\ \mu_{abc} \\ \sigma_{ab} \end{pmatrix} = \begin{pmatrix} 0 \\ -2\phi_{abcp} X^p \in E_{\odot[ab]c}[4] \\ +2X^p \mu_{p(ab)} \in E_{(ab)} \end{pmatrix};$$
$$Y_p \cdot \begin{pmatrix} \phi_{abcd} \\ \mu_{abc} \\ \sigma_{ab} \end{pmatrix} = \begin{pmatrix} \mu_{ab[c}Y_{p]} + \mu_{cp[a}Y_{b]} \in E_{\odot[ab][cp]}[4] \\ -2Y_{[p}\sigma_{a]b} \in E_{\odot[pa]b}[4] \\ 0 \end{pmatrix}.$$

We can therefore calculate the differentials ∂ and ∂^*:

$$\partial_p \begin{pmatrix} \phi_{abcd} \\ \mu_{abc} \\ \sigma_{ab} \end{pmatrix}_{\hat{A}} = \begin{pmatrix} 0 \\ -2\phi_{abcp} \in E_{\odot[ab]c}[4] \\ 2\mu_{p(ab)} \in E_{(ab)}[4] \end{pmatrix}$$
$$\partial_{p_1} \begin{pmatrix} \phi_{p_2abcd} \\ \mu_{p_2abc} \\ \sigma_{p_2ab} \end{pmatrix}_{\hat{A}} = \begin{pmatrix} 0 \\ -4\phi_{[p1p2]cab} \in E_{\odot[ab]c}[4] \\ -4\mu_{[p1p2](ab)} \in E_{(ab)}[4] \end{pmatrix}_{\hat{A}}$$
$$\partial^* \begin{pmatrix} \phi_{pabcd} \\ \mu_{pabc} \\ \sigma_{pab} \end{pmatrix}_{\hat{A}} = \begin{pmatrix} \frac{1}{2}\left(\mu_{pabc} - \mu_{cabp} + \mu_{bcpa} - \mu_{acpb}\right) \in E_{\odot[ab][cp]}[4] \\ -\sigma_{pab} + \sigma_{apb} \in E_{\odot[pa]b}[4] \\ 0 \end{pmatrix}_{\hat{A}}$$
$$\partial^* \begin{pmatrix} \phi_{p_1p_2abcd} \\ \mu_{p_1p_2abc} \\ \sigma_{p_1p_2ab} \end{pmatrix}_{\hat{A}} = \begin{pmatrix} \frac{1}{2}\left(\mu_{p_1p_2abc} - \mu_{p_1cabp_2} + \mu_{p_1bcp_2a} - \mu_{p_1acp_2b}\right) \in E_{\odot[ab][cp_2]}[4] \\ -\sigma_{p_1p_2ab} + \sigma_{p_1ap_2b} \in E_{\odot[p_2a]b} \\ 0 \end{pmatrix}_{\hat{A}}$$

Then $\Box = \partial \circ \partial^* + \partial^* \circ \partial$ acts by the scalar multiplications

$$\Box \begin{pmatrix} \rho \\ \mu \\ \sigma \end{pmatrix} = \begin{pmatrix} -4\rho \\ -3\mu \\ 0 \end{pmatrix}$$

on V.

In schematic Young-diagram notation the space $E_c \otimes V$ splits into the following components :

$$E_c \otimes V = \begin{pmatrix} T_1 \oplus T_2 \\ M_1 \oplus M_2 \oplus M_3 \\ L_1 \oplus L_2 \end{pmatrix} = \begin{pmatrix} \yng(2,2,1) \oplus \yng(3,2) \\ \yng(2,1,1) \oplus \yng(2,2) \oplus \yng(3,1) \\ \yng(2,1) \oplus \yng(3) \end{pmatrix}. \tag{99}$$

The projections to the irreducible components (99) of an

$$s = \begin{pmatrix} \phi_{pabcd} \\ \mu_{pabc} \\ \sigma_{pab} \end{pmatrix} \in E_p^{\hat{A}}$$

are:

$$\begin{aligned}
\mathrm{proj}_{T_1}(\phi_{pabcd}) &= \frac{1}{4}(2\phi_{pabcd} + \phi_{cabdp} + \phi_{bacdp} + \phi_{abdcp} - \phi_{dabcp} - \phi_{badcp} - \phi_{abcdp}) \\
\mathrm{proj}_{T_2}(\phi_{pabcd}) &= \frac{1}{4}(2\phi_{pabcd} - \phi_{cabdp} - \phi_{bacdp} - \phi_{abdcp} + \phi_{dabcp} + \phi_{badcp} + \phi_{abcdp}) \\
\mathrm{proj}_{M_1}(\mu_{pabc}) &= \frac{1}{8}(3\mu_{pabc} + 2\mu_{abpc} + 2\mu_{bpac} + \mu_{bpca} - \mu_{apcb} - \mu_{cabp}) \\
\mathrm{proj}_{M_2}(\mu_{pabc}) &= \frac{1}{4}(\mu_{pabc} - \mu_{cabp} + \mu_{bcpa} - \mu_{acpb}) \\
\mathrm{proj}_{M_2}(\mu_{pabc}) &= \frac{1}{8}(3\mu_{pabc} + 3\mu_{cabp} - 2\mu_{abpc} - 2\mu_{bcpa} - 2\mu_{bpac} + 2\mu_{acpb} - \mu_{bpca} + \mu_{apcb}) \\
\mathrm{proj}_{L_1}(\sigma_{pab}) &= \sigma_{pab} - \sigma_{(pab)} \\
\mathrm{proj}_{L_2}(\sigma_{pab}) &= \sigma_{(pab)}.
\end{aligned}$$

One finds that $L_2 = H_2 = \ker \Box$, $L_1 \oplus M_2 = \mathrm{im}\, \partial$ and $T_1 \oplus T_2 \oplus M_1 \oplus M_3 = \mathrm{im}\, \partial^* = B_1$. We only need the Kostant Laplacian on $B_1 = \mathrm{im}\, \partial^*$: On $T_1 \oplus T_2$ the Kostant Laplacian acts by $-6\mathrm{id}_{T_1} \oplus -2\mathrm{id}_{T_2}$ and on $M_1 \oplus M_3$ it acts by $-6\mathrm{id}_{M_1} \oplus -2\mathrm{id}_{M_3}$.

Let us now fix some $D \in [D]$. An element $s \in \mathcal{E}_{\hat{A}}$ corresponding to

$$[s]_D = \begin{pmatrix} \phi_{abcd} \\ \mu_{abc} \\ \sigma_{ab} \end{pmatrix}$$

transforms to

$$[s]_{\hat{D}} = \begin{pmatrix} \hat{\phi}_{abcd} \\ \hat{\mu}_{abc} \\ \hat{\sigma}_{ab} \end{pmatrix} = \begin{pmatrix} \phi_{abcd} + \mu_{ab[c}\Upsilon_{d]} + \mu_{cd[a}\Upsilon_{b]} + \Upsilon_a \Upsilon_{[c}\sigma_{d]b} - \Upsilon_b \Upsilon_{[c}\sigma_{d]a} \\ \mu_{abc} - 2\Upsilon_{[a}\sigma_{b]c} \\ \sigma_{ab} \end{pmatrix} \tag{100}$$

under the change $D \rightsquigarrow \hat{D}$ given by (78) with $\Upsilon_a \in \mathcal{E}_a$.

The tractor connection on $\mathbf{V}$ is given by

$$[\nabla_p s]_D = \nabla_p \begin{pmatrix} \phi_{abcd} \\ \mu_{abc} \\ \sigma_{ab} \end{pmatrix} = \begin{pmatrix} D_p\phi_{abcd} + \mu_{ab[c}\mathrm{P}_{d]p} + \mu_{cd[a}\mathrm{P}_{b]p} \\ D_p\mu_{abc} - 2\phi_{abcp} - 2\mathrm{P}_{p[a}\sigma_{b]c} \\ D_p\sigma_{ab} + 2\mu_{p(ab)} \end{pmatrix}$$

Recall that the curvature form of ω is

$$K_{p_1p_2} = \left(\begin{array}{c|c} 0 & -A_{ap_1p_2} \\ \hline 0 & C_{p_1p_2}{}^{a}{}_{b} \end{array}\right).$$

Thus

$$K_{p_1p_2}\bullet s = K_{p_1p_2}\bullet \begin{pmatrix} \phi_{abcd} \\ \mu_{abc} \\ \sigma_{ab} \end{pmatrix} =$$

$$= \begin{pmatrix} -2C_{p_1p_2}{}^{q}{}_{[a}\phi_{b]qcd} + 2C_{p_1p_2}{}^{q}{}_{[c}\phi_{d]qab} + \mu_{ab[c]}A_{d]p_1p_2} + \mu_{cd[a}A_{b]p_1p_2} \\ 2C_{p_1p_2}{}^{q}{}_{[a}\mu_{b]qc} - C_{p_1p_2}{}^{q}{}_{c}\mu_{abq} - 2A_{[a|p_1p_2|}\sigma_{b]c} \\ -2C_{p_1p_2}{}^{q}{}_{(a}\sigma_{b)q} \end{pmatrix}.$$

Calculating

$$\Psi_1 := -\Box^{-1}(\partial^*(K_{c_1c_2}\bullet \begin{pmatrix} 0 \\ 0 \\ \sigma_{ab} \end{pmatrix}))$$

we obtain

$$\Psi_1(\begin{pmatrix} 0 \\ 0 \\ \sigma_{ab} \end{pmatrix}) =$$
$$-\frac{1}{12}C_{pc}{}^{q}{}_{a}\sigma_{bq} + \frac{1}{12}C_{pc}{}^{q}{}_{b}\sigma_{aq} + \frac{1}{3}C_{pa}{}^{q}{}_{c}\sigma_{bq} - \frac{1}{3}C_{pb}{}^{q}{}_{c}\sigma_{aq} - \frac{1}{4}C_{ab}{}^{q}{}_{c}\sigma_{pq}$$
$$+\frac{5}{12}C_{pa}{}^{q}{}_{b}\sigma_{cq} - \frac{5}{12}C_{pb}{}^{q}{}_{a}\sigma_{cq} \in \mathcal{E}_{pabc}[4].$$

Changing of ∇ to $\nabla + \Psi_1$ changes the curvature K to $R = R_{\Psi_1} = K + \mathrm{d}^\nabla\Psi^1 + [\Psi^1, \Psi^1]$, and the commutator expression vanishes. The terms in $\mathrm{d}^\nabla\Psi^1$ involving the Schouten tensor are either trivial or lie in the top slot of $\mathbf{E}_{c_1c_2} \otimes \mathbf{V}$, which is contained in the kernel of ∂^*. The terms in the middle slot of

$$(K\bullet + \mathrm{d}^\nabla\Psi^1 + [\Psi^1, \Psi^1]) \begin{pmatrix} \phi_{abcd} \\ \mu_{abc} \\ \sigma_{ab} \end{pmatrix}$$

have three contributers: first, the ones coming from $K_{c_1c_2}\bullet s$, which are

$$2C_{p_1p_2}{}^{q}{}_{[a}\mu_{b]qc} - C_{p_1p_2}{}^{q}{}_{c}\mu_{abq} + 2A_{[a|p_1p_2|}\sigma_{b]c}.$$

Then we have

$$2\Psi^1_{[c_1}\partial_{c_2]}\mu_{abc} = -2\Psi^1_{[c_1}\mu_{c_2]ab} - -2\Psi^1_{[c_1}\mu_{c_2]ba} =$$
$$= -\Psi^1_{c_1}\mu_{c_2ab} + \Psi^1_{c_2}\mu_{c_1ab} - \Psi^1_{c_1}\mu_{c_2ba} + \Psi^1_{c_2}\mu_{c_1ba};$$

and morever the terms $2(D_{[c_1}\Psi^1_{c_2]})\sigma$.

The computation of $\Psi_2 := -\Box^{-}1 \circ \partial^* \circ R_{\Psi_1}$ is alrady quite calculation intensive, and has been supported with computer algebra. We use the projections

$$\begin{aligned}
\mathrm{proj}_{\yng(2,2)} &: \mathbf{E}_{abcd} \to \mathbf{E}_{\odot[ab][cd]},\\
\mathrm{proj}_{\yng(2,2)}(\varphi_{abcd}) &:= \frac{1}{2}\big(\varphi_{[ab][cd]} + \varphi_{[cd][ab]}\big),\\
\mathrm{proj}_{\yng(2,1)} &: \mathbf{E}_{abc} \to \mathbf{E}_{\odot[ab]c},\\
\mathrm{proj}_{\yng(2,1)}(\mu_{abc}) &:= \frac{1}{2}\big(\mu_{a(bc)} - \mu_{b(ac)}\big).
\end{aligned}$$

One obtains $\Psi = \Psi_1 + \Psi_2 \in \Gamma(\mathbf{E}_a \otimes \mathrm{End}(\mathbf{V}))$:

$$\Psi(\begin{pmatrix} 0 \\ \mu_{abc} \\ \sigma_{ab} \end{pmatrix}) = \tag{101}$$

$$\begin{pmatrix}
\mathrm{proj}_{\yng(2,2)} \begin{bmatrix}
\frac{1}{3}C_{bc}{}^{q}{}_{d}\mu_{paq} - \frac{1}{3}C_{bd}{}^{q}{}_{c}\mu_{pqa} - \frac{1}{3}C_{pc}{}^{q}{}_{d}\mu_{abq} \\
- \frac{2}{3}C_{pc}{}^{q}{}_{b}\mu_{adq} - \frac{2}{3}C_{pd}{}^{q}{}_{b}\mu_{aqc} \\
- \frac{2}{3}(D_aC_{bd}{}^{q}{}_{c})\sigma_{pq} - \frac{2}{9}(D_dC_{pc}{}^{q}{}_{b})\sigma_{aq} \\
- \frac{5}{9}(D_cC_{pb}{}^{q}{}_{d})\sigma_{aq} - \frac{7}{9}(D_bC_{pc}{}^{q}{}_{d})\sigma_{aq} \\
- A_{bpc}\sigma_{ad} - \frac{1}{3}A_{acd}\sigma_{pb} - \frac{1}{3}A_{dbc}\sigma_{pa}
\end{bmatrix} \\
\mathrm{proj}_{\yng(2,1)}\big[- \frac{1}{4}C_{ab}{}^{q}{}_{c}\sigma_{pq} - \frac{4}{3}C_{pb}{}^{q}{}_{c}\sigma_{aq} - \frac{5}{3}C_{pb}{}^{q}{}_{a}\sigma_{cq} \big] \\
0
\end{pmatrix} \in \begin{pmatrix} \Gamma(\mathbf{E}_p \otimes \mathbf{E}_{\odot[ab][cd]}) \\ \Gamma(\mathbf{E}_p \otimes \mathbf{E}_{\odot[ab]c}) \\ 0 \end{pmatrix}.$$

By Theorem 4.1.1, $\tilde{\nabla} = \nabla + \Psi$ is the prolongation connection for $D_{(c}\sigma_{ab)} \overset{!}{=} 0$.

REMARK 5.3.1. Like in 5.2 we checked projective invariance of Ψ directly. One employes (100), (98) and

$$\hat{D}_pC_{ab}{}^{c}{}_{d} = D_pC_{ab}{}^{c}{}_{d} - 2\Upsilon_pC_{ab}{}^{c}{}_{d} + \Upsilon_qC_{ab}{}^{q}{}_{d}\delta_p{}^{c} - \Upsilon_dC_{ab}{}^{c}{}_{p} + 2\Upsilon_{[a}C_{b]p}{}^{c}{}_{d},$$

which follows from (78).

CHAPTER 6

Invariant prolongation in conformal geometry

Interesting equations in conformal geometry described by first BGG-operators are those for Einstein scales, conformal Killing forms, conformal Killing tensors and twistor spinors.

In section 6.2 we will discuss our natural prolongation procedure of chapter 4 in the conformal setting and specifically apply it to obtain an invariant prolongation of the equation for conformal Killing forms.

We then proceed to lay out the interesting case of twistor spinors in section 6.3, where similarly to the almost Einstein case no adjustment of the standard tractor connection is necessary.

6.1. Conformal structures

Two pseudo-Riemannian metrics g and $\hat{g}$ with signature (p,q) on a $n = p+q$-dimensional manifold M are said to be *conformally equivalent* if there is a function $f \in \mathrm{C}^\infty(M)$ such that $\hat{g} = e^{2f}g$. The conformal equivalence class of g is denoted by $[g]$ and $(M,[g])$ is said to be a manifold endowed with a conformal structure. If D is the Levi-Civita connection of g and $\hat{D}$ the connection of $\hat{D}$, then, according to (37),

$$\hat{D}_c\omega_a = D_c\omega_a - \Upsilon_c\omega_a - \Upsilon_a\omega_c + \Upsilon^p\omega_p\boldsymbol{g}_{ca}$$

for $\omega_a \in \mathcal{E}_a = \Omega^1(M)$.

Let $\bar{g} \in S^2(\mathbb{R}^n)^*$ be a symmetric bilinear form on $\mathbb{R}^n$ of signature (p,q). We will also regard $\bar{g}$ as a symmetric $n\times n$-matrix via the usual identification employing the standard Euclidean inner product on $\mathbb{R}^n$. The conformal class of metrics $[g]$ yields a reduction of structure group of the full frame bundle of TM to $\mathrm{CO}(\bar{g}) = \mathrm{CO}(p,q) = \mathbb{R}_+ \times \mathrm{O}(p,q)$: this $\mathrm{CO}(\bar{g})$-frame bundle will be denoted by $\mathcal{G}_0$. In fact, a reduction of structure group of TM to $\mathrm{CO}(\bar{g})$ is an equivalent description of a conformal structure of signature (p,q).

The associated bundle to $\mathcal{G}_0$ for the 1-dimensional representation $\mathbb{R}[w]$ of $\mathrm{CO}(\bar{g})$ given by

$$(c,C) \in \mathrm{CO}(\bar{g}) = \mathbb{R}_+ \times \mathrm{O}(\bar{g}) \mapsto c^w$$

for $w \in \mathbb{R}$ is called the bundle of conformal w-densities and denoted by $\mathbf{E}[w]$ with space of sections $\mathcal{E}[w]$.

Given a metric $g \in [g]$, a section $\sigma \in \mathcal{E}[w]$ trivializes to a function $[\sigma]_g \in \mathrm{C}^\infty(M)$ and one has

$$[\sigma]_{e^{2f}g} = e^{wf}[\sigma]_g.$$

The conformal class of metrics $[g]$ defines a tautological section $\boldsymbol{g}$ in $\mathcal{E}_{(ab)}[2] = \Gamma(S^2T^*M \otimes \mathbf{E}[2])$, called the *conformal metric*, such that the trivialization of $\boldsymbol{g}$ with respect to $g \in [g]$ is just g. The conformal metric $\boldsymbol{g}$ allows one to raise or lower indices with simultaneous adjustment of the conformal weight: e.g., for a vector field $\xi^p \in \mathcal{E}^p = \mathfrak{X}(M)$ one can form $\xi_p = \boldsymbol{g}_{pq}\xi^q \in \mathcal{E}_p[2] = \Gamma(T^*M \otimes \mathbf{E}[2])$, which is a 1-form of weight 2.

6.1.1. Conformal structures as parabolic geometries. Let P be the stabilizer in $\mathrm{SO}(p{+}1, q{+}1)$ of an isotropic ray in $\mathbb{R}^{p+1,q+1}$. It is a classical result of Élie Cartan [**Car23**] that an oriented conformal structure on M can be equivalently described as a parabolic geometry of type $(\mathrm{SO}(p+1, q+1), P)$, which we will now discuss. The *homogeneous model* $\mathrm{SO}(p{+}1, q{+}1)/P$ of Cartan geometries of type $(\mathrm{SO}(p + 1, q + 1), P)$ is the space of rays in $\mathbb{R}^{p+1,q+1}$, which can be identified with the *pseudo-sphere* $S_{p,q} = (S^p \times S^q)$: it is endowed with the conformal class of metrics with representative $g_p \oplus (-g_q)$, with g_p,g_q the round metrics on S^p and S^q.

It will be useful to realize the groups P and $\mathrm{SO}((p + 1, q + 1)$ explicitly via the symmetric bilinear form h of signature $(p+1, q+1)$ on $\mathbb{R}^{n+2}$ defined by

$$h = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \bar{g} & 0 \\ 1 & 0 & 0 \end{pmatrix} : \tag{102}$$

When P is realized as the stabilizer of the isotropic ray

$$\mathbb{R}_+ e_0 = (\mathbb{R}_+, 0, \ldots, 0) \subset \mathbb{R}^{n+2}$$

one computes

$$P = \{ \begin{pmatrix} c^{-1} & -c^{-1}v^t\bar{g}C & -\frac{1}{2c}\bar{g}(v, v) \\ 0 & C & v \\ 0 & 0 & c \end{pmatrix} \in \mathrm{SO}(h) \ : \ c \in \mathbb{R}_+, C \in \mathrm{SO}(\bar{g}), v \in \mathbb{R}^{p+q}\}.$$

The Lie algebra $\mathfrak{so}(p + 1, q + 1) = \mathfrak{so}(h)$ is 1-graded

$$\mathfrak{so}(p + 1, q + 1) = \mathfrak{so}(h) = \mathfrak{so}(h)_{-1} \oplus \mathfrak{so}(h)_0 \oplus \mathfrak{so}(h)_1 := \\ = \mathbb{R}^n \oplus \mathfrak{co}(\bar{g}) \oplus (\mathbb{R}^n)^*.$$

Realized in $\mathfrak{gl}(n + 2)$ it is given by matrices of the form

$$\left(\begin{array}{c|c|c} -\alpha & -Z^t\bar{g} & 0 \\ \hline X & A & Z \\ \hline 0 & -X^t\bar{g} & \alpha \end{array}\right), \alpha \in \mathbb{R}; X, Z \in \mathbb{R}^n, A \in \mathfrak{so}(\bar{g}).$$

The grading element $E \in \mathfrak{so}(h)$ which is uniquely determined by $\mathrm{ad}(E)_{|\mathfrak{g}_j} = j\,\mathrm{id}_{\mathfrak{g}_j}$ is $-\mathbb{I}_n \in \mathfrak{co}(\bar{g}) \in \mathfrak{g}_0$; explicitly,

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let P_+ denote the set of unipotent matrices in P, i.e., $P_+ = \{p \in P : (p - \mathbb{I}_{n+2})^2 = 0\}$. The exponential map induces an isomorphism between the abelian Lie groups $(\mathbb{R}^n)^*$ and P_+; and since P_+ is a normal subgroup of P one sees that P is the semidirect product $P = \mathrm{CO}(\bar{g}) \ltimes (\mathbb{R}^n)^*$.

Given a manifold M endowed with a geometry $(\mathcal{G}, \omega)$ of type $(\mathrm{SO}(p + 1, q + 1), P)$, we can see how it is endowed with a conformal structure very easily: first note that since $\mathfrak{g}_- = \mathfrak{g}_{-1}$ has just one component, the filtration of TM introduced in (28) is trivial: $TM = T^{-1}M \supset T^0M = \{0\}$. Since the filtration is trivial, TM is automatically a filtered manifold in the sense of (29). Since $T^{-2}M = T^{-1}M = TM$, $\mathrm{gr}_{-2}TM = T^{-2}M/T^{-2}M = \{0\}$, so the Levi bracket (30), $\mathcal{L} : TM \times TM \to \mathrm{gr}_{-2}(TM) = \{0\}$ is trivial. Since also $\mathfrak{g}_- = \mathfrak{g}_{-1}$ is abelian, the geometry is also automatically regular in the sense of Definition 2.2.5. Recall moreover that we showed in 2.2.6 that there is a natural isomorphism $\mathrm{gr}(TM) = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_{-1}$. But since $\mathrm{gr}(TM) = TM$ this provides a reduction of structure group of TM to $G_0 = \mathrm{CO}(\bar{g})$, which is the same as the choice of a conformal structure of signature (p, q) on M.

For inducing the conformal structure on TM from the parabolic geometry $(\mathcal{G}, \omega)$ we only needed the identification

$$TM = \mathrm{gr}(TM) = \mathcal{G} \times_P \mathrm{gr}(\mathfrak{g}/\mathfrak{p}) = \mathcal{G}_0 \times_{G_0} \times \mathfrak{g}_-. \tag{103}$$

If $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is replaced by another Cartan connection form $\omega' \in \Omega^1(\mathcal{G}, \mathfrak{g})$ such that $\omega' - \omega \in \Omega^1(\mathcal{G}, \mathfrak{g}^0)$, this won't change the identification (103), and thus there is no unique Cartan connection inducing a given conformal structure. However, using the notion of normality introduced in Definition 2.2.7 one has:

Theorem 6.1.1 ([**Car23**]). *Up to isomorphism there is a unique P-principal bundle $\mathcal{G}$ over M endowed with a normal Cartan connection form $\tilde{\omega} \in \Omega^1(\mathcal{G}, \mathfrak{so}(h)))$ such that $\mathcal{G}/P_+ = \mathcal{G}/(\mathbb{R}^n)^* = \mathcal{G}_0$ is the conformal frame bundle of $(M, [g])$.*

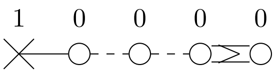
Remark 6.1.2. Let $\mathcal{G}_0$ be the conformal frame bundle of $(M, [g])$. We remark that the Cartan bundle $\mathcal{G}$ can be realized as the bundle of torsion-free principal connections on $\mathcal{G}_0$ (see for instance [**ČS09**]): In contrast to the (pseudo)-Riemannian case, there is no unique torsion-free $\mathfrak{co}(\bar{g})$-valued principal connection form on $\mathcal{G}_0$, and one can define the fiber of $\mathcal{G}_u$ of $\mathcal{G}$ over $u \in \mathcal{G}_0$ by

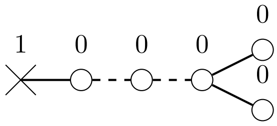
$$\begin{aligned}&\{\gamma_u \in \mathrm{L}(T_u\mathcal{G}_0, \mathfrak{co}(\bar{g}))\\ &: \ \gamma \in \Omega^1(\mathcal{G}_0, \mathfrak{co}(\bar{g})) \text{ is a torsion} - \text{free principal connection form}\}.\end{aligned}$$

Now every choice of $g \in [g]$ yields the unique Levi-Civita connection which can be extended to a $\mathrm{CO}(\bar{g})$-principal connection form $\gamma \in \Omega^1(\mathcal{G}_0, \mathfrak{co}(\bar{g}))$ on $\mathcal{G}_0$. γ can thus be regarded as a smooth section $\mathcal{G}_0 \to \mathcal{G}$. The $\mathrm{CO}(\bar{g})$-equivariancy of γ as a principal connection form means that this section is likewise $\mathrm{CO}(\bar{g})$-equivariant. We therefore have a Weyl-structure as introduced in 2.4 for every $g \in [g]$. These are in fact special Weyl structures which even yield reductions of structure group not only to $\mathrm{CO}(\bar{g})$ but to $\mathrm{O}(\bar{g})$. ◊

6.1.2. Tractor bundles for conformal structures. With $S = \mathbb{R}^{p+q+2} = \mathbb{R}^{n+2}$ the standard representation of $P \hookrightarrow \mathrm{GL}(n+2)$, the *standard tractor bundle* of conformal geometry is defined as the associated bundle $\mathbf{S} :=$

$\mathcal{G} \times_P S$. It corresponds to the diagrams (Cf. Remark 2.2.4)



resp.


for $p+q$ odd resp. $p+q$ even.

S is naturally graded $S_{-1} \oplus S_0 \oplus S_1$ via the grading element E of $\mathfrak{so}(h)$: we write

$$S = \begin{pmatrix} S_1 \\ S_0 \\ S_{-1} \end{pmatrix} = \begin{pmatrix} E[-1] \\ E_a[1] \\ E[1] \end{pmatrix}.$$

We have the canonical elements

$$\tau_+^S = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes 1 \in S \otimes \mathbb{R}[1], \ \tau_-^S = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \otimes 1 \in S \otimes \mathbb{R}[-1]. \tag{104}$$

$\tau_+^S \in S \otimes \mathbb{R}[1]$ is P-invariant and defines a canonical section, denoted by τ_+, of $\mathbf{S}[1]$. This corresponds to the canonical inclusion of $\mathbf{E}[-1]$ into $\mathbf{S}$. A choice of $g \in [g]$ provides a reduction of structure group of $\mathbf{S}$ from P to $G_0 = \mathrm{CO}(\bar{g})$, and we obtain the decomposition With respect to a metric $g \in [g]$ one has

$$[\mathbf{S}]_g = \begin{pmatrix} \mathbf{E}[-1] \\ \mathbf{E}_a[1] \\ \mathbf{E}[1] \end{pmatrix}. \tag{105}$$

Then $\tau_-^S \in S \otimes \mathbb{R}[1]$ is $\mathrm{CO}(\bar{g})$-invariant and the corresponding section of $\mathbf{E}[1]$, which depends on the choice of g, is again denoted by τ_-. This correspond to the inclusion of $\mathbf{E}[1]$ into $[\mathbf{S}]_g$.

Let $s \in \Gamma(\mathbf{S}) = \mathcal{S}$ with

$$[s]_g = \begin{pmatrix} \rho \\ \varphi_a \\ \sigma \end{pmatrix} \in \begin{pmatrix} \mathcal{E}[-1] \\ \mathcal{E}_a[1] \\ \mathcal{E}[1] \end{pmatrix}.$$

Then for $\hat{g} = \mathrm{e}^{2f} g$ one has the transformation

$$[s]_{\hat{g}} = \begin{pmatrix} \hat{\rho} \\ \hat{\varphi}_a \\ \hat{\sigma} \end{pmatrix} = \begin{pmatrix} \rho - \Upsilon_a \varphi^a - \frac{1}{2}\sigma \Upsilon^b \Upsilon_b \\ \varphi_a + \sigma \Upsilon_a \\ \sigma \end{pmatrix} \tag{106}$$

with $\Upsilon = df$.

Since $h \in S^2 T^* \mathbb{R}^{n+2}$ is P-invariant it defines a *tractor metric* $\mathbf{h}$ on $\mathbf{S} = \mathcal{G}' \times_{\mathrm{SO}(h)} \mathbb{R}^{n+2}$. With respect to $g \in [g]$ and the decomposition (105) of an element $s \in \Gamma(\mathcal{S})$

$$\mathbf{h} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{g} & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{107}$$

Via the tractor metric $\mathbf{h}$ we can identify $\Lambda^2\mathbf{S}$ with $\mathfrak{so}(\mathbf{S})$, which is the adjoint tractor bundle $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{so}(h)$ for conformal structures. The decomposition (105) via a $g \in [g]$ yields a decomposition of $\mathcal{A}M$, and employing matrix notation, we will write its elements in the fiber over $x \in M$ as

$$[\mathcal{A}M_x]_g = \tag{108}$$

$$\{\begin{pmatrix} -c & -\eta_b & 0 \\ \xi^a & C & \eta^b \\ 0 & -\xi_a & c \end{pmatrix} \text{ with } c \in \mathbb{R}, C \in \mathfrak{so}(T_xM, g_x), \xi^a \in T_xM, \eta_a \in T^*_xM\}. \tag{109}$$

One has a natural surjection (projection to ξ^a) of $\mathcal{A}M$ onto TM and an inclusion (inserting of η_b) of T^*M into $\mathcal{A}M$, while the inclusion via (108) of TM depends on the choice of g.The algebraic action $\bullet$ of $\mathcal{A}M$ on a tractor bundle $\mathbf{V}$ restricts to actions of TM and T^*M; so according to formula (51), the tractor connection ∇^S can be written as $D + \partial + \mathrm{P}\bullet$ with D be the Levi-Civita connection of a $g \in [g]$. Here

$$\mathrm{P} = \mathrm{P}(g) = \frac{1}{n-2}(\mathrm{Ric}(g) - \frac{\mathrm{Sc}(g)}{2(n-1)}g) \tag{110}$$

is the *Schouten tensor* of g, which is a trace modification of the Ricci curvature $\mathrm{Ric}(g)$ by a multiple of the scalar curvature $\mathrm{Sc}(g)$. The trace of the Schouten tensor is denoted $J = g^{pq}P_{pq}$.

Using (108) it is then easy to compute for $s \in \mathcal{S}$ with

$$[s]_g = \begin{pmatrix} \rho \\ \varphi_a \\ \sigma \end{pmatrix} \in \Gamma([\mathbf{S}]_g)$$

that

$$[\nabla^S_c s]_g = \nabla^S_c \begin{pmatrix} \rho \\ \varphi_a \\ \sigma \end{pmatrix} = \begin{pmatrix} D_c\rho - \mathrm{P}_c{}^b\varphi_b \\ D_c\varphi_a + \sigma\mathrm{P}_{ca} + \rho\boldsymbol{g}_{ca} \\ D_c\sigma - \varphi_c \end{pmatrix}. \tag{111}$$

As an $\mathfrak{so}(\mathbf{S})$-valued form $K \in \Gamma(\mathbf{E}_{[c_1c_2]} \otimes \mathcal{A}M)$, the curvature of the standard tractor connection is

$$K_{c_1c_2} = \begin{pmatrix} 0 & -A_{ec_1c_2} & 0 \\ 0 & C_{c_1c_2}{}^c{}_d & A^e{}_{c_1c_2} \\ 0 & 0 & 0 \end{pmatrix}. \tag{112}$$

Here C is the Weyl curvature and $A = A_{ec_1c_2} = 2D_{[c_1}P_{c_2]e}$ is the Cotton-York tensor. We recall that both $A_{ec_1c_2}$ and $C_{c_1c_2}{}^c{}_d$ are trace-free. Furthermore the skew-symmetrization over any 3 indices of C_{abcd} vanishes, as does the skew-symmetrization of A_{abc}. The Weyl curvature doesn't satisfy the differential Bianchi identity, however one has

$$D_{[a}C_{bc]de} = \boldsymbol{g}_{d[a}A_{|e|bc]} - \boldsymbol{g}_{e[a}A_{|d|bc]}.$$

We will later need the transformation behavior of P: According to 38, the Schouten tensor changes under $\hat{g} = e^{2f}g$, $\Upsilon_a = D_af$ as follows:

$$\hat{\mathrm{P}}_{ab} = \mathrm{P}_{ab} - D_a\Upsilon_b + \Upsilon_a\Upsilon_b - \frac{1}{2}\Upsilon^p\Upsilon_p\boldsymbol{g}_{ab}. \tag{113}$$

We note here that a reader who looks for an introduction to tractor calculus in conformal geometry and an explanation of related notational issues could also find the careful and detailed exposition in the first part of [**Šil06**] useful.

REMARK 6.1.3. The standard tractor bundle of conformal geometry provides a notion of *conformal holonomy*: we define the holonomy of the conformal class of metrics $[g]$ by

$$\mathrm{Hol}([g]) := \mathrm{Hol}(\nabla^S) \subset \mathrm{SO}(p+1, q+1).$$

(Cf. [**Arm07**], [**Bau07**]). To be precise, $\mathrm{Hol}([g])$ is only given up to $\mathrm{SO}(p+1, q+1)$-conjugacy. $\mathrm{Hol}([g])$ is well defined, since the standard tractor bundle $(\mathbf{S}, \nabla^S)$ is unique up to isomorphism. Another way to introduce $\mathrm{Hol}([g])$ is to define it as the holonomy of the principal connection form ω' on the extended bundle $\mathcal{G}' := \mathcal{G} \times_P G$ (cf. 2.3). Since $\mathbf{S} = \mathcal{G}' \times_G \mathbb{R}^{p+q+2}$ with ∇^S the linear connection induced by ω', these two definitions coincide. Conformal holonomy will play a central role in chapter 7.

6.1.3. The standard tractor bundle and almost Einstein scales. The first splitting operator for the standard tractor bundle is

$$L_0^S : \mathcal{E}[1] \to \mathcal{S}, \qquad \sigma \mapsto \begin{pmatrix} \frac{1}{n}(\triangle - J)\sigma \\ D\sigma \\ \sigma \end{pmatrix} \tag{114}$$

with the convention $\triangle = -D^p D_p$. By (111),

$$\nabla^S \circ L_0^S(\sigma) = \begin{pmatrix} \frac{1}{n} D_c(\triangle\sigma - J\sigma) - \mathrm{P}_c{}^p D_p \sigma \\ (D_a D_b \sigma + \mathrm{P}_{ab}\sigma) + \frac{1}{n}(\triangle\sigma - J\sigma)\boldsymbol{g}_{ab} \\ 0 \end{pmatrix}.$$

Since $\frac{1}{n}(\triangle\sigma - J\sigma)\boldsymbol{g}_{ab}$ is minus the trace part of $(D_a D_b \sigma + \mathrm{P}_{ab}\sigma)$ and $\mathcal{H}_1(\mathcal{S}) = \mathcal{E}_{0(ab)}$ we have that the first BGG-operator of $\mathbf{S}$ is

$$\Theta_0^S : \mathcal{E}[1] \to \mathcal{E}_{0(ab)}, \qquad \sigma \mapsto (D_a D_b \sigma + \mathrm{P}_{ab}\sigma)_0. \tag{115}$$

One has, for $s = \begin{pmatrix} \rho \\ \varphi^a \\ \sigma \end{pmatrix}$,

$$K_{c_1 c_2} \bullet s = \begin{pmatrix} -A_{p c_1 c_2} \varphi^p \\ C_{c_1 c_2}{}^a{}_p \varphi^p + \sigma A^a{}_{c_1 c_2} \\ 0 \end{pmatrix}$$

and thus

$$\partial^*(K \bullet s) = \begin{pmatrix} C_{cq}{}^q{}_p \varphi^p + \sigma A^q{}_{cq} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus ∇ satisfies $\partial^* \circ (K\bullet) = 0$, and one has $1:1$-correspondence of $\ker \Theta_0$ with ∇-parallel standard tractors.

Now take $\sigma \neq 0$ such that (116) holds and let $U = \{x \in M : \sigma(x) \neq 0\}$. By Corollary (4.1.2), U is open dense. On U we rescale g to $\hat{g} = \sigma^{-2}g$. Then

$$\Upsilon_a = -\sigma^{-1}D_a\sigma = D_a(\frac{1}{\log(|\sigma|)})$$
$$D_a\Upsilon_b = \sigma^{-2}(D_a\sigma)(D_b\sigma) - \sigma^{-1}D_aD_b\sigma = \Upsilon_a\Upsilon_b + P_{ab}$$

Thus, according to (113),

$$\hat{P}_{ab} = -\frac{1}{2}\Upsilon^p\Upsilon_p g_{ab},$$

which is purely trace, and thus $\mathrm{Ric}(\hat{g})$, which differs only by a trace from P, is a multiple of $\hat{g}$. I.e.: $\hat{g}$ is an Einstein-metric on U, or

$$(D_aD_b\sigma + \mathrm{P}_{ab}\sigma)_0 = 0 \ \Leftrightarrow\ \sigma^{-2}\boldsymbol{g} \text{ is Einstein on } U, \tag{116}$$

which is well known. U always has to be an open dense subset of M, and we call the set of solutions of (116) the space of *almost Einstein scales* ([**Gov10**]), i.e.

$$\mathbf{aEs}([g]) = \ker\Theta_0^S \subset \mathcal{E}[1]. \tag{117}$$

Thus one recovers the well known fact

Proposition 6.1.4. *The space of ∇^S-parallel sections of the standard tractor bundle is isomorphic to* $\mathbf{aEs}([g])$.

6.1.4. Conformal Killing fields. The symmetries of the associated conformal structure $[g] = [g]_{\mathbf{D}}$ are the conformal Killing fields $\mathbf{cKf}([g])$,

$$\mathbf{cKf}([g]) = \{\xi \in \mathfrak{X}(M)\ :\ \mathcal{L}_\xi g = e^{2f}g \text{ for some } g \in [g] \text{ and } f \in \mathrm{C}^\infty(M)\}. \tag{118}$$

This condition on ξ is easily seen not to depend on the choice of $g \in [g]$. Since $\mathcal{L}_\xi g$ decomposes into a multiple of g and a trace-free part one can equivalently demand that $\mathcal{L}_\xi g$ is purely trace. Now, with D the Levi-Civita connection of $g \in [g]$ one has that $\mathcal{L}_\xi g$ being purely trace is equivalent to

$$D_{(c}\xi_{a)_0} = D_{(c}g_{a)_0p}\xi^p = 0;$$

i.e., the symmetric, trace-free part of $D_c\xi_a$ vanishes.

As an equation on 1-forms of conformal weight 2 this is in fact described by the first BGG-operator of $\mathcal{A}M$: The first BGG-splitting operator and tractor connection are calculated for $\Lambda^{k+1}\mathbf{S}$ below in 6.2 for all $k \geq 0$, so we will use these results for the special case $k = 1$, since $\Lambda^2\mathbf{S} = \mathfrak{so}(\mathbf{S}, \mathbf{h}) = \mathcal{A}M$:

By (134) the splitting operator

$$L_0^{\Lambda^2 S} : \mathfrak{X}(M) = \mathcal{E}_a[2] \to \Gamma(\Lambda^2\mathbf{S})$$

is given by

$$\sigma \mapsto \begin{pmatrix} \begin{pmatrix} -\frac{1}{2n}D^pD_p\sigma_a + \frac{k}{2n}D^pD_{[a}\sigma_{p} + \frac{k}{n^2}D_aD^p\sigma_p \\ +\frac{2}{n}\mathrm{P}^p{}_a\sigma_p - \frac{1}{n}J\sigma_a \end{pmatrix} \\ D_{[a_0}\sigma_{a_1]}\ |\ -\frac{1}{n}\boldsymbol{g}^{pq}D_p\sigma_q \\ \sigma_a \end{pmatrix}. \tag{119}$$

Here indices within bars are not skewed over. Now the first BGG-operator $\Theta_0^{\Lambda^2 S}$ of $\Lambda^2 S$ defined by the composition

$$\Theta_0^{\Lambda^2 S} : \mathfrak{X}(M) = \mathcal{E}_a[2] \to \mathcal{E}_{0(ab)}[2],$$
$$\Theta_0^{\Lambda^2 S} = \Pi_1 \circ \nabla^{\Lambda^2 S} \circ L_0^{\Lambda^2 S}$$

is seen by direct calculation employing (127) to be

$$\xi^a \mapsto D_{(c}\xi_{a)_0}$$

for $\xi \in \mathfrak{X}(M)$; i.e., $\Theta_0^{\Lambda^2 S}$ is indeed conformally invariant operator governing conformal Killing fields.

We now proceed to prove a technical Lemma, which will be used in the proof of Theorem 7.4.2 below. It is a general fact that normal infinitesimal automorphisms of normal, torsion-free parabolic geometries insert trivially into curvature, as was discussed in 4.3, Lemma 4.3.2. It is however easy to see this directly for conformal Killing fields, in which case this has first been observed in [**Gov06**]. We only give a simple proof for conformal structures of dimension ≥ 4, which is all we need:

Lemma 6.1.5. *Let $s \in \Gamma(\mathcal{A}M)$ be $\nabla^{\mathcal{A}}$-parallel; i.e., $\nabla^{\mathcal{A}} s = 0$. Then $K(\Pi(s), \cdot) = 0$.*

Proof. Since $\nabla^A s = 0$ one has $R_{c_1 c_2} s = 2\nabla^{\mathcal{A}}_{[c_1} \nabla^{\mathcal{A}}_{c_2]} s = 0$, with $R \in \Omega^2(M, \mathrm{End}(\mathcal{A}M))$ the curvature of $\nabla^{\mathcal{A}}$. But since $\nabla^{\mathcal{A}}$ is the induced tractor connection of the Cartan connection form ω one thus has

$$Rs = K \bullet s = 0 \tag{120}$$

Let $s \in \Gamma(\mathcal{A}M) = \Gamma(\Lambda^2 T)$ be of the form

$$\begin{pmatrix} \rho_a \\ \varphi_{ab} \mid \mu \\ \sigma_a \end{pmatrix}.$$

Then via the projection $\Pi : \Lambda^2 \mathbf{S} \to \mathcal{E}_a[2] = \mathfrak{X}(M)$, s projects to the conformal Killing field $\sigma^a \in \mathfrak{X}(M)$. We want to show that $K(\Pi(s), \cdot) = K(\sigma^a, \cdot) = 0$. By formula (112) this is equivalent to

$$A_{apc}\sigma^p = 0 \tag{121}$$
$$C_{pc}{}^a{}_b \sigma^p = 0. \tag{122}$$

By (120), $0 = \Pi(K_{c_1 c_2} \bullet s) = C_{c_1 c_2 pa}\sigma^p$. But by the symmetries of the Weyl curvature $C_{abcd} = -C_{bacd} = -C_{abdc} = C_{cdab}$, having σ^a insert trivially into one slot is equivalent to trivial insertion into any other slot. Thus the second equation holds. Now just use the fact that $(n-3)A_{abc} = D^p C_{pabc}$, which gives the first equation in our case for $n \geq 4$. □

6.2. Invariant prolongation of conformal Killing forms.

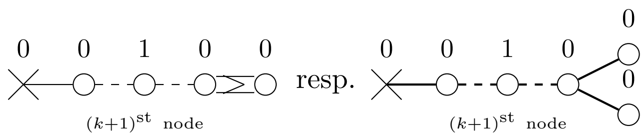
Conformal Killing forms were first prolonged by U. Semmelmann [**Sem03**], however the discussion there did not take into account conformal invariance of the equation. In [**GŠ08**] an invariant prolongation was calculated directly (see also [**Šil06**]). Here we calculate the prolongation connection of chapter 4 for this system.

We are going to proceed as follows: In 6.2.1 we describe the exterior powers of the standard tractor bundles, give explicit formulas for the Lie algebraic differentials on the first chain spaces and determine their $CO(p,q)$-decompositions. In 6.2.4 we describe explicitly how the operator governing conformal Killing k-forms comes about as the first BGG-operator for the $k+1$-st exterior power of the standard tractor bundle. In 6.2.5 we obtain a geometric prolongation by constructing a modification map Ψ with the properties called for in section Theorem 3.2.4 resp. Theorem 4.1.1. In 6.2.7 we show directly the conformal invariance of Ψ. This section is based on [**Ham08**].

6.2.1. The tractor bundle. In the following k will be ≥ 1. Let

$$V = \Lambda^{k+1}\mathbb{R}^{p+q+2},$$

which corresponds to the diagrams



for $p+q$ odd resp. $p+q$ even.

The grading element $E \in \mathfrak{so}(h)$ induces the following G_0-invariant grading on V:

$$V = \begin{pmatrix} V_1 \\ V_0 \\ V_{-1} \end{pmatrix} := \begin{pmatrix} E_{[a_1\cdots a_k]}[k-1] \\ E_{[a_0\cdots a_k]}[k+1] \mid E_{[a_2\cdots a_{k-1}]}[k] \\ E_{[a_1\cdots a_k]}[k+1] \end{pmatrix};$$

Here we use the identification

$$\begin{pmatrix} \rho_{a_1\cdots a_k} \\ \varphi_{a_0\cdots a_k} \mid \mu_{a_2\cdots a_k} \\ \sigma_{a_1\cdots a_k} \end{pmatrix} \mapsto \tau_-^S \wedge \sigma + \varphi + \tau_-^S \wedge \tau_+^S \wedge \mu + \tau_+^S \wedge \rho. \tag{123}$$

This is to be understood as follows: Since $\mathrm{gr}_0(V) = V_0 = E_a[1]$, we have that $E_{a_1\cdots a_k}[k] \hookrightarrow \mathrm{gr}_0(\Lambda^k S)$. Therefore $E_{a_1\cdots a_k}[k+1] \hookrightarrow \mathrm{gr}_0(\Lambda^k S) \otimes \mathbb{R}[1]$. With the canonical element $\tau_-^S \in \mathrm{gr}_{-1}(S) \otimes \mathbb{R}[-1]$ defined above, it is then easy to see that the map

$$E_{a_1\cdots a_k}[k+1] \to \mathrm{gr}_{-1}(V) = \mathrm{gr}_{-1}(\Lambda^{k+1} S),$$
$$\sigma \mapsto \tau_-^S \wedge \sigma$$

is an isomorphism of $\mathrm{CO}(p,q)$-modules; and completely analogously for the other irreducible components of V.

Another way to view the expression $\tau_-^S \wedge \sigma$ as an element of $V = \Lambda^{k+1} S$ would be to directly identify $E_{[a_1\cdots a_k]}[k+1]$ with $E^{[a_1\cdots a_k]}[-k+1]$ via $\bar{g}$ and then use the embedding of $\mathbf{E}^a[-1]$ into the middle slot of S.

For $\mathbf{V} = \Lambda^{k+1}\mathbf{S}$ the associated tractor bundle we have the semidirect composition series

$$\mathbf{V} = \mathbf{V}^{-1} \oplus \mathbf{V}^{0} \oplus \mathbf{V}^{1} := \tag{124}$$

$$\mathbf{E}_{[a_1\cdots a_k]}[k+1] \oplus (\mathbf{E}_{[a_1\cdots a_{k+1}]}[k+1] \oplus \mathbf{E}_{[a_1\cdots a_{k-1}]}[k-1]) \oplus \mathbf{E}_{[a_1\cdots a_k]}[k-1], \tag{125}$$

which splits into $\mathbf{V}_{-1} \oplus \mathbf{V}_0 \oplus \mathbf{V}_1$ after a choice of g in the conformal class.

With respect to $g \in [g]$, for $k \geq 0$ an element $s \in \mathbf{V} = \Lambda^{k+1}\mathbf{S}$ is written

$$[s]_g = \begin{pmatrix} \rho_{a_1\cdots a_k} \\ \varphi_{a_0\cdots a_k} \mid \mu_{a_2\cdots a_k} \\ \sigma_{a_1\cdots a_k} \end{pmatrix} \in \begin{pmatrix} \mathbf{E}_{[a_1\cdots a_k]}[k-1] \\ \mathbf{E}_{[a_0\cdots a_k]}[k+1] \mid \mathbf{E}_{[a_2\cdots a_k]}[k-1] \\ \mathbf{E}_{[a_1\cdots a_k]}[k+1] \end{pmatrix}.$$

Similarly as in (106), we have transformations

$$\begin{pmatrix} \hat{\rho}_{a_1\cdots a_k} \\ \hat{\varphi}_{a_0\cdots a_k} \mid \hat{\mu}_{a_2\cdots a_k} \\ \hat{\sigma}_{a_1\cdots a_k} \end{pmatrix} = \begin{pmatrix} \rho_{a_1\cdots a_k} - \Upsilon^b\varphi_{ba_1\cdots a_k} - k\Upsilon_{[a_1}\mu_{a_2\cdots a_k]} - \frac{1}{2}\Upsilon^b\Upsilon_b\sigma_{a_1\cdots a_k} \\ \varphi_{a_0\cdots a_k} + (k+1)\Upsilon_{[a_0}\sigma_{a_1\cdots a_k]} \mid \mu_{a_2\cdots a_k} - \Upsilon^b\sigma_{ba_2\cdots a_k} \\ \sigma_{a_1\cdots a_k} \end{pmatrix}. \tag{126}$$

The standard tractor connection (111) gives rise to the invariantly defined tractor connection ∇ on $\mathbf{V}$:

$$\nabla_c \begin{pmatrix} \rho_{a_1\cdots a_k} \\ \varphi_{a_0\cdots a_k} \mid \mu_{a_2\cdots a_k} \\ \sigma_{a_1\cdots a_k} \end{pmatrix} = \tag{127}$$

$$= \begin{pmatrix} D_c\rho_{a_1\cdots a_k} - \mathrm{P}_c{}^p\varphi_{pa_1\cdots a_k} - k\mathrm{P}_{c[a_1}\mu_{a_2\cdots a_k]} \\ \begin{pmatrix} D_c\varphi_{a_0\cdots a_k} + (k+1)\boldsymbol{g}_{c[a_0}\rho_{a_1\cdots a_k]} \\ +(k+1)\mathrm{P}_{c[a_0}\sigma_{a_1\cdots a_k]} \end{pmatrix} \mid \begin{pmatrix} D_c\mu_{a_2\cdots a_k} \\ -\mathrm{P}_c{}^p\sigma_{pa_2\cdots a_k} + \rho_{ca_2\cdots a_k} \end{pmatrix} \\ D_c\sigma_{a_1\cdots a_k} - \varphi_{ca_1\cdots a_k} + k\boldsymbol{g}_{c[a_1}\mu_{a_2\cdots a_k]} \end{pmatrix}.$$

6.2.2. Description of the first homology groups. $\partial^* : C_1 \to C_0 = V$ is given (see (86)) by $Z \otimes s \mapsto -Z \bullet s$ for $s \in V$, $Z \in \mathfrak{p}_+$. Thus $B_0 = \operatorname{im}\partial^* : C_1 \to V$ is simply $\mathfrak{p}_+ \bullet V$, which is all of V^0. Therefore $H_0 = V/V^0$. By the Hodge decomposition (22) we can embed H_0 as $V_{-1} = \ker\Box = \ker\partial \subset V$.

Also, H_i will be embedded into C_i as $\ker\Box = \ker(\partial\partial^* + \partial^*\partial)$ for $i = 1, 2$. The calculation of the $\mathrm{CO}(p,q)$-decomposition of the spaces H_i is purely algorithmic using Kostant's version of the Bott-Borel-Weyl theorem [**Kos61**]; the details of which are not important for us here. We just state the results for H_1 and H_2, which are all homologies we are going to need: We will write

$$C_k = \begin{pmatrix} E_{[c_1\cdots c_k]} \otimes V_1 \\ E_{[c_1\cdots c_k]} \otimes V_0 \\ E_{[c_1\cdots c_k]} \otimes V_{-1} \end{pmatrix},$$

and speak of the top, middle and bottom slots, which we will say have homogeneities 1, 0 and -1.

$E_c \otimes V_{-1}$ contains the highest weight part $E_{\odot c[a_1\ldots a_k]}[k+1]$, and this is all of H_1. Explicitly, $E_{\odot c[a_1\ldots a_k]}[k+1]$ sits in $E_{c[a_1\ldots a_k]}[k+1]$ as those $\sigma = \sigma_{ca_1\cdots a_k}$ which have both zero trace and vanishing alternation:

$$0 = \bar{g}^{pq}\sigma_{pqa_2\cdots a_k}, \quad 0 = \sigma_{[ca_2\cdots a_k]}.$$

If $k \geq 2$ then the analogous statement holds also for the second chain space: in this case H_2 is exactly the highest weight part of $E_{[c_1c_2]} \otimes V_{-1} = E_{[c_1c_2][a_1\ldots a_k]}[k+1]$. i.e., $H_2 = E_{\odot[c_1c_2][a_1\ldots a_k]}[k+1] \subset E_{[c_1c_2]} \otimes V_{-1}$.

In particular, for $i = 0, 1$ we have that H_i lies in the lowest grading part of C_i and if $k \geq 2$ this also holds for $i = 2$:

$$\begin{pmatrix} V_1 \\ V_0 \\ H_0 = V_{-1} \end{pmatrix} \xrightarrow{\partial} \begin{pmatrix} E_c \otimes V_1 \\ E_c \otimes V_0 \\ H_1 \oplus \text{im } \partial_{|V_0} \end{pmatrix} \xrightarrow{\partial} \begin{pmatrix} E_{[c_1c_2]} \otimes V_1 \\ E_{[c_1c_2]} \otimes V_0 \\ H_2 \oplus \text{im } \partial_{|E_c \otimes V_0} \end{pmatrix}$$

Now we describe what ∂,∂^* and $\Box$ do on the first few chain spaces $C_0 = V, C_1 = E_c \otimes V$ and $C_2 = E_{[c_1c_2]} \otimes V$:

6.2.3. Explicit formulas for ∂, ∂^* and $\Box$ on the first chain spaces.

$$\partial_c \begin{pmatrix} \rho_{a_1\cdots a_k} \\ \varphi_{a_0\cdots a_k} \mid \mu_{a_2\cdots a_k} \\ \sigma_{a_1\cdots a_k} \end{pmatrix} = \begin{pmatrix} 0 \\ (k+1)\bar{g}_{c[a_0}\rho_{a_1\cdots a_k]} \mid \rho_{ca_2\cdots a_k} \\ -\varphi_{ca_1\cdots a_k} + k\bar{g}_{c[a_1}\mu_{a_2\cdots a_k]} \end{pmatrix} \tag{128}$$

$$\partial_{c_1} \begin{pmatrix} \rho_{c_2a_1\cdots a_k} \\ \varphi_{c_2a_0\cdots a_k} \mid \mu_{c_2a_2\cdots a_k} \\ \sigma_{c_2a_1\cdots a_k} \end{pmatrix} = \begin{pmatrix} 0 \\ 2(k+1)\bar{g}_{[a_0|[c_1}\rho_{c_2]|a_1\cdots a_k]} \mid -2\rho_{[c_1c_2]a_2\cdots a_k} \\ 2\varphi_{[c_1c_2]a_2\cdots a_k} + 2k\bar{g}_{[a_1|[c_1}\mu_{c_2]|a_2\cdots a_k]} \end{pmatrix}$$

$$\partial^* \begin{pmatrix} \rho_{ca_1\cdots a_k} \\ \varphi_{ca_0\cdots a_k} \mid \mu_{ca_2\cdots a_k} \\ \sigma_{c\ a_1\cdots a_k} \end{pmatrix} = \begin{pmatrix} \bar{g}^{pq}\varphi_{pqa_1\cdots a_k} + k\mu_{[a_1\cdots a_k]} \\ -(k+1)\sigma_{[a_0\cdots a_k]} \mid \bar{g}^{pq}\sigma_{pqa_2\cdots a_k} \\ 0 \end{pmatrix}$$

$$\partial^* \begin{pmatrix} \rho_{c_1c_2a_1\cdots a_k} \\ \varphi_{c_1c_2a_0\cdots a_k} \mid \mu_{c_1c_2a_2\cdots a_k} \\ \sigma_{c_1c_2a_1\cdots a_k} \end{pmatrix} = \begin{pmatrix} -2\bar{g}^{pq}\varphi_{cpqa_1\cdots a_k} - 2k\mu_{c[a_1\cdots a_k]} \\ 2(k+1)\sigma_{c[a_0\cdots a_k]} \mid -2\bar{g}^{pq}\sigma_{cpqa_2\cdot a_k} \\ 0 \end{pmatrix}.$$

The image of ∂^* in $V = C_0$ is simply $V^0 = V_0 \oplus V_1$, and the Kostant Laplacian thus acts by positive real scalars on V_1 and the two components of V_0. It vanishes on V_{-1} by (87). Explicitly, $\Box$ is given on V by

$$\begin{pmatrix} n \\ (k+1) \mid (n-k+1) \\ 0 \end{pmatrix}. \tag{129}$$

The image of ∂^* in $\mathcal{C}_1$ contains all of $E_c \otimes V_1$ (since we have (87)). Now $E_c \otimes V_1$ decomposes into three parts: the alternating maps, $E_{[p_0\cdots p_k]}[k+1]$, the purely trace maps, $E_{p_2\cdots p_k}[k-1]$, and finally those maps which have both trivial trace and trivial alternating part, $E_{\odot c[a_1\ldots a_k]}[k+1]$. We will denote the three irreducible components of $E_c \otimes V_1$ by $(E_c \otimes V_1)_{alt}$, $(E_c \otimes V_1)_\odot$ and $(E_c \otimes V_1)_{tr}$. We will write this decomposition of $\text{gr}_1(B_1) = E_c \otimes V_1 \cap \text{im } \partial^*$ as

$$\text{gr}_1(B_1) = U_{alt} \oplus U_\odot \oplus U_{tr} \tag{130}$$

and one computes that the Kostant Laplacian $\Box$ acts by

$$2(n+k-1)\text{id}_{U_{alt}} \oplus 2(n-2)\text{id}_{U_\odot} \oplus 2(2n-k-1)\text{id}_{U_{tr}}. \tag{131}$$

Now to the middle slot: We have

$$E_c \otimes V_0 = E_{c[a_0\cdots a_k]}[k+1] \oplus E_{c[a_2\cdots a_k]}[k-1]$$

and both parts split into alternating, $\odot$ (highest weight)- and trace components. Both highest weight-components, the 'left' alternating and the 'right' trace component lie in the image of ∂^*. The only other component of im $\partial^* \cap E_c \otimes V_0$ is $E_{[a_1\cdots a_k]}[k-1]$, which embeds into $E_c \otimes V_0$ via

$$\tau_{a_1\cdots a_k} \mapsto \begin{pmatrix} 0 \\ -k(k+1)\bar{g}_{c[a_0}\tau_{a_1\cdots a_k} \mid (n-k)\tau_{ca_2\cdots a_k} \\ 0 \end{pmatrix}.$$

We will write the decomposition of $\mathrm{gr}_0(B_1) = E_c \otimes V_0 \cap \mathrm{im}\,\partial^* \subset E_c \otimes V_0$ as

$$\mathrm{gr}_0(B_1) = \begin{pmatrix} alt & \mid & * \\ \odot & \mid & \odot \\ tr & \mid & tr \end{pmatrix}, \tag{132}$$

where the component $*$ in the upper right corner is determined by the trace component in the lower left corner. One computes that the Kostant Laplacian acts by the scalars

$$\begin{pmatrix} 4(k+1) & \mid & * \\ 2k & \mid & 2(n-k) \\ 2n & \mid & 2(n-k-1) \end{pmatrix}. \tag{133}$$

6.2.4. The first BGG-operator $\Theta_0 : \mathcal{H}_0 \to \mathcal{H}_1$ and conformal Killing forms. Using (127), (128) and (129), we compute that the first BGG-splitting operator $L_0 : \mathcal{H}_0 \to \mathcal{V}$ is given by

$$\sigma \mapsto \begin{pmatrix} \begin{pmatrix} -\frac{1}{n(k+1)}D^pD_p\sigma_{a_1\cdots a_k} + \frac{k}{n(k+1)}D^pD_{[a_1}\sigma_{|p|a_2\cdots a_k]} + \frac{k}{n(n-k+1)}D_{[a_1}D^p\sigma_{|p|a_2\cdots a_k]} \\ +\frac{2k}{n}\mathrm{P}^p_{[a_1}\sigma_{|p|a_2\cdots a_k]} - \frac{1}{n}J\sigma_{a_1\cdots a_k} \end{pmatrix} \\ D_{[a_0}\sigma_{a_1\cdots a_k]} \mid -\frac{1}{n-k+1}\boldsymbol{g}^{pq}D_p\sigma_{qa_2\cdots a_k} \\ \sigma_{a_1\cdots a_k} \end{pmatrix}. \tag{134}$$

In 6.2.2 we saw that

$$\begin{aligned} \mathcal{H}_0 &= \mathcal{E}_{[a_1\cdots a_k]}[k+1], \\ \mathcal{H}_1 &= \mathcal{E}_{\odot c[a_1\cdots a_k]}[k+1], \\ \mathcal{H}_2 &= \mathcal{E}_{\odot [c_1c_2][a_1\cdots a_k]}[k+1]. \end{aligned}$$

Thus we immediately see using (127) that

$$\begin{aligned} \Theta_0 &: \mathcal{H}_0 \to \mathcal{H}_1, \\ \Theta_0 &= \Pi_1 \circ \nabla \circ L \end{aligned}$$

is given by

$$\Theta_{0c}\sigma_{a_1\cdots a_k} = D_c\sigma_{a_1\cdots a_k} - D_{[a_0}\sigma_{a_1\cdots a_k]} - \frac{k}{n-k+1}g_{c[a_1}g^{pq}D_{|p}\sigma_{q|a_2\cdots a_k]}.$$

We also denote this expression, which is the projection of $D\sigma$ to the highest weight-component in $\mathcal{E}_{c[a_1\cdots a_k]}[k+1]$, by $\left(D_c\sigma_{a_1\cdots a_k}\right)_\odot$. This is exactly the conformal Killing operator. Thus our prolongation procedure will yield an isomorphism between the space conformal Killing k-forms

$$\sigma_{a_1\cdots a_k} \in \mathcal{E}_{[a_1\cdots a_k]}[k+1],\ \left(D_c\sigma_{a_1\cdots a_k}\right)_\odot = 0 \tag{135}$$

and the space of parallel sections of a natural connection on $\mathbf{V}$.

For $k = 1$ we get exactly the operator describing conformal Killing fields, i.e., infinitesimal automorphisms of the conformal structure; see also remark 6.2.1. This case has been treated in detail in [**Gov06**]. The main result of this section, an explicit geometric prolongation, will also work for $k = 1$. We only need $k \geq 2$ for obtaining the algebraic obstruction tensor which was described in subsection 6.2.6.

6.2.5. The adjustment of the tractor connection. We are now going to construct a $\Psi \in \Gamma(\mathbf{E}_c \otimes \mathrm{End}(\mathbf{V})^1)$ with the properties called for in Theorem 4.1.1.

The calculations will be made more readable by providing beforehand the mappings which will appear: We will make use of the vector bundle maps

$$L_i : \mathbf{E}_{[a_1 \cdots a_k]}[k+1] \to \mathbf{E}_{c[a_1 \cdots a_{k+1}]}[k+1],\ i = 1, 2 \text{ and}$$
$$R_i : \mathbf{E}_{[a_1 \cdots a_k]}[-k+1] \to \mathbf{E}_{c[a_1 \cdots a_{k-1}]}[k-1],\ i = 1, 2,$$

of homogeneity 1 defined by

$$\begin{aligned} L_1(\sigma) &= C_{[a_0 a_1}{}^{p}{}_{|c}\sigma_{p|a_2 \cdots a_k]} & L_2(\sigma) &= \boldsymbol{g}_{c[a_0} C_{a_1 a_2}{}^{pq} \sigma_{|pq|a_3 \cdots a_k]} \\ R_1(\sigma) &= C_{c[a_2}{}^{pq} \sigma_{|pq|a_3 \cdots a_k]} & R_2(\sigma) &= C_{[a_2 a_3}{}^{pq} \sigma_{|cpq|a_4 \ldots a_k]}. \end{aligned} \tag{136}$$

Recall that indices within bars | are not skewed over. In homogeneity 2 we will need the maps

$$\begin{aligned} F_i, G_i &: \mathbf{E}_{[a_1 \cdots a_k]}[k+1] \to \mathbf{E}_{c[a_1 \cdots a_k]}[k-1], \\ E_i &: \mathbf{E}_{[a_1 \cdots a_{k+1}]}[k+1] \to \mathbf{E}_{c[a_1 \cdots a_k]}[k-1], \text{ and} \\ T_i &: \mathbf{E}_{[a_1 \cdots a_{k-1}]}[k-1] \to \mathbf{E}_{c[a_1 \cdots a_k]}[k-1] \end{aligned}$$

defined by

$$\begin{aligned} E_1(\varphi) &= C_{c[a_1}{}^{pq} \varphi_{|pq|a_2 \cdots a_k]} & E_2(\varphi) &= C_{[a_1 a_2}{}^{pq} \varphi_{c|pq|a_3 \cdots a_k]} \\ T_1(\mu) &= C_c{}^{p}{}_{[a_1 a_2} \mu_{|p|a_3 \cdots a_k]} & T_2(\mu) &= \boldsymbol{g}_{c[a_1} C_{a_2 a_3}{}^{pq} \mu_{|pq|a_3 \cdots a_k]} \\ F_1(\sigma) &= A_{[a_1|c|}{}^{p} \sigma_{|p|a_2 \cdots a_k]} & F_2(\sigma) &= A^{p}{}_{c[a_1} \sigma_{|p|a_2 \cdots a_k]} \\ F_3(\sigma) &= A^{p}{}_{[a_1 a_2} \sigma_{c|p|a_3 \cdots \mathfrak{a}_k]} & F_4(\sigma) &= \boldsymbol{g}_{c[a_1} A_{a_2}{}^{pq} \sigma_{|pq|a_3 \cdots \mathfrak{a}_k]} \\ G_1(\sigma) &= (D_c C_{[a_1 a_2}{}^{pq}) \sigma_{|pq|a_3 \cdots a_k]} & G_2(\sigma) &= (D^p C_c{}^{q}{}_{[a_1 a_2}) \sigma_{|pq|a_3 \cdots a_k]} \\ G_3(\sigma) &= (D_{[a_1} C_{|c|a_2}{}^{pq}) \sigma_{|pq|a_3 \cdots a_k]}. \end{aligned} \tag{137}$$

With respect to the CO(p,q)-decompositions (130) and (132) a more natural basis for the linear space formed by these maps is formed by

$$
\begin{aligned}
L_{tr} &= -\frac{k-1}{n-k}L_2 & L &= L_1 - L_{tr} \\
R_{alt} &= \frac{2}{k}R_1 + \frac{k-2}{k}R_2 & R &= \frac{k-2}{k}(R_1 - R_2) \\
E_{alt} &= \frac{2}{k+1}E_1 + \frac{k-1}{k+1}E_2 & E &= \frac{k-1}{k+1}(E_1 - E_2) \\
T_{tr} &= -\frac{k-2}{n-k+1}T_2 & T &= T_1 - T_{tr} \\
F_{tr} &= \frac{k}{n-k+1}F_4 & F_{alt} &= \frac{2}{k+1}F_2 - \frac{k-1}{k+1}F_3 \\
F_i &= F_1 - \frac{1}{k+1}F_2 + \frac{k-1}{2(k+1)}F_3 - \frac{k-1}{k}F_{tr} & F_{ii} &= \frac{k-1}{k+1}(F_2+F_3) - \frac{k-1}{2k}F_{tr} \\
G_i &= G_1 + 2F_{alt} - \frac{2}{k}(n-k-1)F_{tr} & G_{ii} &= G_2 - \frac{2(k-2)}{k}F_{tr} \\
G_{iii} &= G_3 - 2F_{alt} - \frac{n-3}{k}F_{tr}.
\end{aligned}
\tag{138}
$$

L_{tr}, T_{tr} and F_{tr} are purely trace, R_{alt},E_{alt} and F_{alt} are alternating and all other maps have both vanishing alternation and trace.

The maps of (136) and (137) can be expressed as

$$
\begin{aligned}
L_1 &= L + L_{tr} & L_2 &= -\frac{n-k}{k-1}L_{tr} & R_1 &= R + R_{alt} \\
R_2 &= -\frac{2}{k-2}R + R_{alt} & E_1 &= E + E_{alt} & E_2 &= -\frac{2}{k-1}E + E_{alt} \\
T_1 &= T + T_{tr} & T_2 &= -\frac{n-k+1}{k-2}T_{tr}
\end{aligned}
\tag{139}
$$

and

$$
\begin{aligned}
F_1 &= F_i + \frac{1}{2}F_{alt} + \frac{k-1}{k}F_{tr} & F_2 &= F_{ii} + F_{alt} + \frac{k-1}{2k}F_{tr} \\
F_3 &= \frac{2}{k-1}F_{ii} - F_{alt} + \frac{1}{k}F_{tr} & F_4 &= \frac{n-k+1}{k}F_{tr} \\
G_1 &= G_i - 2F_{alt} + \frac{2}{k}(n-k-1)F_{tr} & G_2 &= G_{ii} + \frac{2(k-2)}{k}F_{tr} \\
G_3 &= G_{iii} + 2F_{alt} + \frac{n-3}{k}F_{tr}.
\end{aligned}
$$

For $s = \begin{pmatrix} \rho_{a_1\cdots a_k} \\ \varphi_{a_0\cdots a_k} \mid \mu_{a_2\cdots a_k} \\ \sigma_{a_1\cdots a_k} \end{pmatrix}$ we have

$$
(K\bullet s) =
$$

$$
\begin{pmatrix} kC_{c_1c_2[a_1}{}^p\rho_{|p|a_2\cdots a_k]} - kA_{[a_1|c_1c_2|}\mu_{a_2\cdots a_k]} - A^p{}_{c_1c_2}\varphi_{pa_1\cdots a_k} \\ (k+1)C_{c_1c_2[a_0}{}^p\varphi_{|p|a_1\cdots a_k]} + (k+1)A_{[a_0|c_1c_2|}\sigma_{a_1\cdots a_k]} \mid (k-1)C_{c_1c_2[a_2}{}^p\mu_{|p|a_3\cdots a_k]} - A^p{}_{c_1c_2}\sigma_{pa_2\cdots a_k} \\ kC_{c_1c_2[a_1}{}^p\sigma_{|p|a_2\cdots a_k]} \end{pmatrix}.
$$

We calculate

$$\partial^*(K\bullet s) = \begin{pmatrix} 2kF_1 + 2kF_2 - kE_1 + k(k-1)T_1 \\ -k(k+1)L_1 \mid \ -(k-1)R_1 \\ 0 \end{pmatrix} \tag{140}$$

and thus have that the lowest homogeneous component of $\partial^*(K\bullet s)$, which is of homogeneity 1, is given by $(-k(k+1)L_1| \ -(k-1)R_1)$. Now we use (139),(129) and (138) to apply $-\Box^{-1}$ to this expression, which yields

$$\Psi_1 := \begin{pmatrix} 0 \\ \lambda_1 L_1 + \lambda_2 L_2 | \ \rho_1 R_1 + \rho_2 R_2 \\ 0 \end{pmatrix} \tag{141}$$

where

$$\lambda_1 = \frac{1+k}{2} \qquad \lambda_2 = \frac{(k-1)(k+1)}{2n}$$
$$\rho_1 = \frac{(k-1)(n-2)}{2(n-k)n} \qquad \rho_2 = -\frac{(k-1)(k-2)}{2(n-k)n}.$$

Now the curvature of the modified connection $\nabla + \Psi_1$ is

$$R = K\bullet + \mathrm{d}^\nabla \Psi_1 + [\Psi_1, \Psi_1],$$

but $[\Psi_1, \Psi_1]$ obviously vanishes. Let us calculate R: The only term which needs our attention is $\mathrm{d}^\nabla\Psi_1$. Take any $s = \begin{pmatrix} \rho_{a_1\cdots a_k} \\ \varphi_{a_0\cdots a_k} | \mu_{a_2\cdots a_k} \\ \sigma_{a_1\cdots a_k} \end{pmatrix} \in \Gamma(\mathcal{V})$.

Then for $\xi_1, \xi_2 \in \mathfrak{X}(M)$, we have, since Ψ_1 is a 1-form on M with values in $\mathrm{End}(\mathbf{V})$,

$$\begin{aligned}(\mathrm{d}^\nabla\Psi_1)s(\xi_1,\xi_2) = & \\ = \nabla_{\xi_1}(\Psi_1(\xi_2)s) - \Psi_1(\xi_2)(\nabla_{\xi_1}s) - \nabla_{\xi_2}(\Psi_1(\xi_1)s) + \Psi_1(\xi_1)(\nabla_{\xi_2}s) - \Psi_1([\xi_1,\xi_2])s.\end{aligned} \tag{142}$$

We expand (142) and write $(\mathrm{d}^\nabla\Psi_1)s$ as

$$\begin{pmatrix} * \\ \begin{pmatrix} D_{\xi_1}\big(\Psi_1(\xi_2)\sigma\big) - \Psi_1(\xi_2)\big(D_{\xi_1}\sigma\big) - D_{\xi_2}\big(\Psi_1(\xi_1)\sigma\big) + \Psi_1(\xi_1)\big(D_{\xi_2}\sigma\big) \\ -\Psi_1([\xi_1,\xi_2])\sigma \\ -\Psi_1(\xi_2)\partial_{\xi_1}\varphi + \Psi_1(\xi_1)\partial_{\xi_2}\varphi - \Psi_1(\xi_2)\partial_{\xi_1}\mu + \Psi_1(\xi_1)\partial_{\xi_2}\mu \end{pmatrix} \\ \partial_{\xi_1}\Psi_1(\xi_2)\sigma - \partial_{\xi_2}\Psi_1(\xi_1)\sigma \end{pmatrix}, \tag{143}$$

where we don't care about the top component since it vanishes after an application of ∂^*. The lowest component is $\partial(\Psi_1\sigma) = -\partial\Box^{-1}\partial^*(K\bullet\sigma)$. Thus $\partial^*(Rs)$ lies in the top slot (i.e., in homogeneity 1). So $\partial^* \circ R$ has values in the top slot only. This can be repeated: it is a straightforward calculation using the expression in the middle component of (143) and, in that order, (140), (139),(131) and (138) to find, with $\phi := -\Box^{-1} \circ \partial^* \circ R$,

$$\Psi = \Psi_2 = \Psi_1 - \phi = \begin{pmatrix} \begin{pmatrix} \varepsilon_1 E_1 + \varepsilon_2 E_2 + \tau_1 T_1 + \tau_2 T_2 \\ +\phi_1 F_1 + \phi_2 F_2 + \phi_3 F_3 + \phi_4 F_4 \\ +\gamma_1 G_1 + \gamma_2 G_2 + \gamma_3 G_3 \end{pmatrix} \\ \lambda_1 L_1 + \lambda_2 L_2 | \ \rho_1 R_1 + \rho_2 R_2 \\ 0 \end{pmatrix}, \tag{144}$$

with the constants

$$\varepsilon_1 = \frac{k-1}{2(n-k)} \qquad \varepsilon_2 = \frac{(k-1)k}{2(k-n)n}$$
$$\tau_1 = \frac{(k-1)(n(n-k+1)-2k)}{2(k-n)n} \qquad \tau_2 = -\frac{(k-2)(k-1)}{2n}$$
$$\phi_1 = -\frac{n+k-3}{n-2} \qquad \phi_2 = \frac{1-k}{n}$$
$$\phi_3 = \frac{(k-1)(n+k)}{2(k-n)n} \qquad \phi_4 = \frac{(k-1)(2+k-2n)}{2(k-n)(n-2)}$$
$$\gamma_1 = -\frac{k-1}{2(n-2)n} \qquad \gamma_2 = \frac{k-1}{2(n-2)}$$
$$\gamma_3 = \frac{(k-1)k}{2(k-n)n}.$$

Now the curvature R' of $\nabla + \Psi = \nabla + \Psi_1 + \phi$ is given by

$$R + \mathrm{d}^\nabla \phi + [\Psi_1, \phi] + [\phi, \Psi_1] + [\phi, \phi].$$

One sees that for every $s \in \mathcal{V}$, $([\Psi_1, \phi] + [\phi, \Psi_1] + [\phi, \phi])s$ has values only in the top component and we may therefore forget about this term when calculating $\partial^*(R's)$. As in the calculation (143), we see that $(\mathrm{d}^\nabla \phi)s$ has only values in the middle and top slots and the middle slot is given by $2\partial_{[c_1}\phi_{c_2]}s$. Therefore, by construction of ϕ, we see that $\partial^*(Rs)$ vanishes, and thus, via the considerations of chapter 4, we have solved the prolongation problem for conformal Killing forms.

We have already remarked there that this solution must already be conformally invariant by virtue of uniqueness, but we are going to check independence of the choice of metric by hand in 6.2.7.

Remark 6.2.1. For $k = 1$, we have $\mathcal{V} = \Lambda^2\mathcal{S} = \mathfrak{so}(\mathcal{S}) = \Gamma(\mathcal{A}M)$. Thus $\nabla + \Psi$ prolongs the first BGG-operator for the adjoint tractor connection in this case. Since conformal geometries are torsion-free and, as one can calculate, the first homology of the adjoint tractor bundle $\mathcal{H}_1$ is concentrated in non-positive homogeneity, the corresponding first BGG-operator describes infinitesimal automorphisms of the structure (cf. section 4.3), and the prolongation connection is $\tilde{\nabla} = \nabla + iK$. This can also be read off directly from (144). ♢

Remark 6.2.2. The invariant connection prolonging the conformal Killing equation (135) which was constructed in [**GŠ08**] differs from our result Ψ as defined in (144): it can be checked that the image of the modification map calculated there has nontrivial intersection with im ∂; but recall that our solution $\Psi \in \Omega^1(M, \mathrm{End}(\mathbf{V})^1)$ obeys in particular the condition $\Psi s \in \mathrm{im}\ \partial^*\ \forall s \in \mathcal{V}$, and thus its image has trivial intersection with im ∂ by the Hodge decomposition (87). If one wants to translate the solution (144) into the notation used in [**GŠ08**] for an explicit comparison, one has to use the automorphism

$$\begin{pmatrix} \rho_{a_1\cdots a_k} \\ \varphi_{a_0\cdots a_k}|\mu_{a_2\cdots a_k}) \\ \sigma_{a_1\cdots a_k} \end{pmatrix} \mapsto \begin{pmatrix} (k+1)\rho_{a_1\cdots a_k} \\ \varphi_{a_0\cdots a_k}| - k(k+1)\mu_{a_2\cdots a_k}) \\ (k+1)\sigma_{a_1\cdots a_k} \end{pmatrix}$$

which transforms an element of

$$\mathbf{E}_{[a_1\cdots a_k]}[-k+1] \oplus (\mathbf{E}_{[a_1\cdots a_{k+1}]}[-k-1] \oplus \mathbf{E}_{[a_1\cdots a_{k-1}]}[-k+1]) \oplus \mathbf{E}_{[a_1\cdots a_k]}[-k-1]$$

in our notation to the equivalent element in the notation of [**GŠ08**]. Then Ψ as defined in (144) has the following form with respect to the conventions of Gover-Šilhan:

$$\Psi_c(\begin{pmatrix} \rho_{a_1\cdots a_k} \\ \varphi_{a_0\cdots a_k}|\mu_{a_2\cdot s a_k}) \\ \sigma_{a_1\cdots a_k} \end{pmatrix}) = \begin{pmatrix} \begin{pmatrix} (k+1)(\varepsilon_1 E_1 + \varepsilon_2 E_2)\varphi - \frac{1}{k}(\tau_1 T_1 + \tau_2 T_2)\mu \\ +(\phi_1 F_1 + \phi_2 F_2 + \phi_3 F_3 + \phi_4 F_4 \\ +\gamma_1 G_1 + \gamma_2 G_2 + \gamma_3 G_3)\sigma \end{pmatrix} \\ \frac{1}{k+1}(\lambda_1 L_1 + \lambda_2 L_2)\sigma| - k(\rho_1 R_1 + \rho_2 R_2)\sigma \\ 0 \end{pmatrix}.$$

◊

6.2.6. Algebraic obstruction tensors obtained via the curvature of the modified connection. Since $\Psi \in \Gamma(\mathbf{E}_c \otimes \mathrm{End}(\mathbf{V})^1)$, we know that the curvature $R \in \mathcal{E}_{[c_1c_2]}(\mathrm{End}(\mathbf{V}))$ of $\tilde{\nabla} = \nabla + \Psi$ agrees with $K\bullet$ in homogeneity 0. But if $\sigma_{a_1\cdots a_k} \in \mathcal{E}_{[a_1\cdots a_k]}[k+1]$ is a conformal Killing k-form, then $L_0\sigma$ is given by (134); and thus $0 = \mathrm{d}^{\tilde{\nabla}}\tilde{\nabla}s = Rs$ agrees with $K\bullet L_0\sigma$ in $\Gamma(\mathbf{E}_{[c_1c_2]} \otimes \mathrm{gr}_{-1}(\mathbf{V}))$. But by (112) this is simply (minus) $C_{c_1c_2}{}^{p}{}_{[a_1}\sigma_{|p|a_2\cdots a_k]}$. For $k \geq 2$ we have $\mathcal{H}_2 = \mathcal{E}_{\odot[c_1c_2][a_1\cdots a_k]}[k+1]$ and projecting the previous expression to this space gives the conformally invariant tensorial map

$$\Phi : \sigma \mapsto \mathrm{proj}_{\odot}(C_{c_1c_2}{}^{p}{}_{[a_1}\sigma_{|p|a_2\cdots a_k]}).$$

This obstruction has also been constructed T. Kashiwada in [**Kas68**], U. Semmelmann in [**Sem03**] and recently by R. Gover and J. Šilhan in [**GŠ08**]. Our derivation describes this map as the composition of the first two BGG-operators for the modified connection $\tilde{\nabla}$: $\Phi = \tilde{\Theta}_1 \circ \Theta_0$. This evidently explains both conformal invariance of Φ and why it vanishes on conformal Killing forms. That Φ is tensorial has the cohomological reason that $\mathcal{H}_2$ is concentrated in lowest homogeneity. Cf. section 4.2.

Remark 6.2.3. Apart from the (trivial) cases of Einstein scales and twistor spinors where one doesn't need any adjustments and automatically has $\partial^* \circ (K\bullet) = 0$, the case of conformal Killing forms is the simplest situation in which to explicitly compute the prolongation. Another interesting case to treat will be conformal Killing tensors, for which, as far as we know, there has not yet been given any invariant prolongation, and which can be treated similarly as the form case. There the situation becomes more complicated however, since the modelling representations $S^k\mathfrak{g}$ are $2k+1$-graded. This case has interesting relations to symmetries of the Laplacian ([**Eas05**]). ◊

Remark 6.2.4. The holonomy $\mathrm{Hol}(\tilde{\nabla})$ of the thus obtained prolongation connection $\tilde{\nabla}$ describes the solution space of the operator Θ_0. For the standard tractor bundle and the spinor tractor bundle, which will be treated in the next section, one has $\nabla = \tilde{\nabla}$. In particular, the solution space is governed by the conformal holonomy of the structure, i.e., existence of Einstein scales and twistor spinors correspond to reductions of the conformal holonomy. In general, the existence of nontrivial solutions of Θ_0 doesn't imply a holonomy reduction: for instance, full conformal holonomy doesn't obstruct the existence of conformal Killing fields or conformal Killing forms. (See for instance [**Ham07**]).

Parallel sections of a tractor bundle give special solutions to Θ_0. In the case of conformal Killing Forms, those coming from parallel sections were called *normal* conformal Killing forms by F. Leitner in [**Lei05**]. This notion of *normal* solutions of first BGG-operators makes sense for every tractor bundle and they correspond to reductions in conformal holonomy, cf. Definition 4.1.5. ♢

REMARK 6.2.5. Using the tractor approach above for describing Einstein scales as parallel sections, R. Gover and P. Nurowski [**GN06**] used the curvature R of the standard tractor connection and its derivatives to obtain (under a genericity condition on the Weyl curvature) a conformally invariant system of tensors which provides a sharp obstruction against the existence of Einstein scales. For a general tractor bundle and R the curvature of the prolongation connection, one can similarly build natural systems of obstruction tensors, but it is not known whether these will be sharp; See also Remark 4.2.3. ♢

6.2.7. Conformal invariance of Ψ. For this calculation we need some transformation formulas. We will denote by $\hat{D}$ the Levi-Civita connection of the metric rescaled by e^{2f}. More generally, we will denote by a hatted symbol the corresponding quantity calculated with respect to the metric $\hat{g}$. With $\Upsilon = df$ we have

$$\begin{aligned}\hat{D}_u C_{abcd} &= D_u C_{abcd} - 2\Upsilon_u C_{abcd} - 2\Upsilon_{[a} C_{|u|b]cd} - 2\Upsilon_{[c} C_{|u|d]ab} \\ &\quad + 2(n-3) g_{u[a} A_{b]cd} + 2(n-3) g_{u[c} A_{d]ab} \\ \hat{A}_{abc} &= A_{abc} + \Upsilon^d C_{dabc}\end{aligned}$$

In the calculation the following transformation maps

$$H_i : \mathcal{E}_{[a_1 \cdots a_k]}[k+1] \to \mathcal{E}_{[a_1 \cdots a_k]}[k-1]$$

will appear:

$$\begin{aligned}
H_1(\sigma) &= \Upsilon_c C_{[a_1 a_2}{}^{pq} \sigma_{|pq|a_3 \cdots a_k]} & H_6(\sigma) &= \boldsymbol{g}_{c[a_1} \Upsilon^u C_{a_2 a_3}{}^{pq} \sigma_{|upq|a_4 \cdots a_k]} \\
H_2(\sigma) &= \Upsilon_c C_{[a_1 a_2}{}^{pq} \sigma_{|pq|\mathfrak{a}_3 \cdots a_k]} & H_7(\sigma) &= \Upsilon_d C_{[a_1 a_2}{}^{dp} \sigma_{c|p|a_3 \cdots a_k]} \\
H_3(\sigma) &= \Upsilon^p C_{[a_1 a_2 c}{}^{q} \sigma_{|pq|a_3 \cdots a_k]} & H_8(\sigma) &= \Upsilon^d C_{d[a_1 c}{}^{p} \sigma_{|p|a_2 \cdots a_k]} \\
H_4(\sigma) &= \boldsymbol{g}_{c[a_1} \Upsilon^d C_{|d|a_2}{}^{pq} \sigma_{|pq|a_3 \cdots a_k]} & H_9(\sigma) &= \Upsilon_d C_{ca_1}{}^{dp} \sigma_{|p|a_2 \cdots a_k]} \\
H_5(\sigma) &= \Upsilon_{[a_1} C_{a_2 a_3}{}^{pq} \sigma_{|cpq|a_4 \cdots a_k]}. & &
\end{aligned}$$

The maps (136) of homogeneity 1 are invariant with respect to the choice of $g \in [g]$ since the Weyl curvature is conformally invariant. It is straightforward to calculate that the individual maps (137) transform like

$$\hat{E}_1 = E_1 + 2H_9 - (k-1)H_2 \qquad \hat{E}_2 = E_2 + H_1 - 2H_7 - (k-2)H_5$$
$$\hat{G}_1 = G_1 - 2H_1 - 2H_2 - 2H_3 + 2H_4 + 2H_7 \quad \hat{G}_2 = G_2 - H_1 - H_2 - H_3 + H_7 + 2H_8$$
$$\hat{G}_3 = G_3 - H_1 - H_2 - H_3 + H_4 + 2H_9,$$

and

$$\hat{F}_1 = F_1 + H_8 \qquad \hat{F}_2 = F_2 + H_9 \qquad \hat{F}_3 = F_3 + H_7$$
$$\hat{F}_4 = F_4 + H_4 \qquad \hat{T}_1 = T_1 - H_3 \qquad \hat{T}_2 = T_2 - H_6.$$

Thus, if we switch to another metric $\hat{g}$ respectively the corresponding linear connection $\hat{D}$ and then calculate $\hat{\Psi}$ using (144), the result differs from Ψ only in the top slot of C_1, and it does so by

$$\begin{aligned}&(\varepsilon_2 - 2\gamma_1 - \gamma_2 - \gamma_3)H_1 - \tau_2 H_6 - ((k-1)\varepsilon_1 - 2\gamma_1 - \gamma_2 - \gamma_3)H_2 \\ &+ (-2\varepsilon_2 + \phi_3 + 2\gamma_1 + \gamma_2)H_7 - (\tau_1 - 2\gamma_1 - \gamma_2 - \gamma_3)H_3 + (\phi_1 + 2\gamma_2)H_8 \\ &+ (\phi_4 + 2\gamma_1 + \gamma_3)H_4 + (2\varepsilon_1 + \phi_2 + 2\gamma_3)H_9 - (k-2)\varepsilon_2 H_5.\end{aligned} \tag{145}$$

On the other hand, if we calculate Ψ with respect to g and then transform the expression via $\hat{\rho} = \rho - \Upsilon^d \varphi_{da_1 \cdots a_k} - k\Upsilon_{[a_1}\mu_{a_2 \cdots a_k]}$, the difference to Ψ also lies in homogeneity two and is

$$\begin{aligned}&-\lambda_2 \frac{1}{k+1} H_1 - \rho_1 H_2 + \frac{k-1}{k+1}\lambda_1 H_3 + \frac{2}{k+1}\lambda_2 H_4 \\ &-k\rho_2 H_5 + \frac{k-1}{k+1}\lambda_2 H_6 - \frac{2}{k+1}\lambda_1 H_8.\end{aligned} \tag{146}$$

Now it is straightforward to check that the expressions (145) and (146) coincide. Thus Ψ is seen not to depend on the choice of the metric in the conformal class used to construct it. As we already remarked this is in fact a consequence of uniqueness of Ψ with the normalization conditions of Theorem 4.1.1.

6.3. Twistor spinors

6.3.1. Algebraic description. Let $\hat{G} = \mathrm{Spin}(p+1, q+1)$ be the (connected) spin group of signature $(p+1, q+1)$, which is a double-cover of $G = \mathrm{SO}_0(p+1, q+1)$ and $\hat{G}_0^{\mathrm{ss}} := \mathrm{Spin}(p, q)$ the covering of $G_0^{\mathrm{ss}} = \mathrm{SO}_0(p, q)$ by the spin group of signature (p, q).

Conformal structures of signature (p, q) which preserve both orientations are modelled on Cartan geometries of type (G, P), with $P \subset G$ the stabilizer of the ray $\mathbb{R}_+ e_+ \in \mathbb{R}^{p+1,q+1}$.

The preimage $\hat{P}$ of P under the covering $\hat{G} \xrightarrow{\pi} G$ is the stabilizer in $\mathrm{Spin}(p+1, q+1)$ of $\mathbb{R}_+ e_+$.

We know that $P \cong (\mathbb{R}_+ \times G_0^{\mathrm{ss}}) \ltimes (\mathbb{R}^{p,q})^*$. Since $\mathrm{Spin}(p, q) = \hat{G}_0^{\mathrm{ss}}$ embeds canonically into $\mathrm{Spin}(p+1, q+1)$ and $\mathbb{R}_+$,$(\mathbb{R}^{p,q})^*$ are simply connected, we see that

$$\hat{P} = (\mathbb{R}_+ \times \mathrm{Spin}(p, q)) \ltimes (\mathbb{R}^{p,q})^*.$$

Thus $\hat{G}_0 = \mathrm{CSpin}(p,q) := \mathbb{R}_+ \times \mathrm{Spin}(p,q)$ is the conformal Spin group of signature (p,q).

Let $\Delta^{p+1,q+1}$ and $\Delta^{p,q}$ the spin representations of $\mathrm{Spin}(p+1,q+1)$ resp. $\mathrm{Spin}(p,q)$. We are going to describe the structure of $\Delta^{p+1,q+1}$ as a (pseudo)-hermitian Clifford bundle directly in terms of the representations $\Delta^{p,q}$. This is an alternative approach to the construction of [**Lei08a**].

We claim that

$$\Delta^{p+1,q+1} = \Delta^{p,q} \oplus \Delta^{p,q} = \begin{pmatrix} \Delta^{p,q} \\ \Delta^{p,q} \end{pmatrix}$$

as a $\mathrm{Spin}(p,q)$-representation. To show this we give explicitly the action of

$$\mathbb{R}^{p+1,q+1} = \mathbb{R}e_+ \oplus \mathbb{R}^{p,q} \oplus \mathbb{R}e_- :$$

$$\begin{aligned} X \cdot \begin{pmatrix} v \\ w \end{pmatrix} &:= \begin{pmatrix} -X \cdot v \\ X \cdot w \end{pmatrix} \\ e_+ \cdot \begin{pmatrix} v \\ w \end{pmatrix} &:= \begin{pmatrix} \sqrt{2}w \\ 0 \end{pmatrix} \\ e_- \cdot \begin{pmatrix} v \\ w \end{pmatrix} &:= \begin{pmatrix} 0 \\ -\sqrt{2}v \end{pmatrix}. \end{aligned}$$

Then we have, for $s = \begin{pmatrix} v \\ w \end{pmatrix}$ and $X \in \mathbb{R}^{p,q}$:

$$\begin{aligned} e_+ \cdot (e_+ \cdot s) &= 0 \\ e_- \cdot (e_- \cdot s) &= 0 \\ e_+ \cdot (e_- \cdot s) + e_- \cdot (e_+ \cdot s) &= -2s \\ e_+ \cdot (X \cdot s) + X \cdot (e_+ \cdot s) &= 0 \\ e_- \cdot (X \cdot s) + X \cdot (e_- \cdot s) &= 0 \\ X \cdot (X \cdot s) &= -\|X\|^2_{p,q} s; \end{aligned}$$

Thus indeed, for all $Z_1, Z_2 \in \mathbb{R}^{p+1,q+1}$, we have

$$Z_1 \cdot (Z_2 \cdot s) + Z_2 \cdot (Z_1 \cdot s) = -2h(Z_1, Z_2).$$

$\Delta^{p+1,q+q}$, as defined by the direct sum $\Delta^{p,q} \oplus \Delta^{p,q}$, is therefore a (complex) Clifford module for $(\mathbb{R}^{p+1,q+1}, h)$ of dimension $2\dim(\Delta^{p,q})$ and in particular a $\mathrm{Spin}(p+1,q+1)$-representation. If $p+q$ is even, there is only one complex Clifford module of this dimension and the restriction to $\mathrm{Spin}(p+1,q+1)$ must thus be $\Delta^{p+1,q+1}$. If $p+q$ is odd, there are exactly two complex Clifford modules of this dimension, one of them is thus realized via the action just defined on $\Delta^{p,q} \oplus \Delta^{p,q}$, but they are isomorphic as $\mathrm{Spin}(p+1,q+1)$-representations, and as such are both equal to $\Delta^{p+1,q+1}$.

The identity in $\hat{\mathfrak{g}}_0 = \mathfrak{g}_0 = \mathfrak{co}(p,q) = \mathbb{R}_+ \oplus \mathfrak{so}(p,q)$ is just $e_+ \wedge e_- \in \Lambda^2 \mathbb{R}^{p+q+2}$. This element acts on a spinor $s = \begin{pmatrix} v \\ w \end{pmatrix} \in \begin{pmatrix} \Delta^{p,q} \\ \Delta^{p,q} \end{pmatrix}$ by

$$\frac{1}{4}(e_+ \cdot (e_- \cdot s) - e_- \cdot (e_+ \cdot s)) = \begin{pmatrix} -\frac{1}{2}v \\ \frac{1}{2}w \end{pmatrix}.$$

We will also need

$$e_+ \wedge X \bullet \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} X \cdot w \\ 0 \end{pmatrix}, e_- \wedge X \bullet \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} X \cdot v \end{pmatrix}.$$

Now $\mathrm{CSpin}(p,q)$ is realized as $\mathbb{R}_+ \exp(e_+ \wedge e_-) \times \mathrm{Spin}(p,q) \in \hat{P} \subset \hat{G}$. Since we use the convention that the spin representation of $\mathrm{Spin}(p,q)$ on $\Delta^{p,q}$ is trivially extended to $\mathbb{R}_+ \times \mathrm{Spin}(p,q) = \mathrm{CSpin}(p,q)$, we thus obtain the decomposition

$$\Delta^{p+1,q+1} = \begin{pmatrix} \Delta^{p,q}[-\frac{1}{2}] \\ \Delta^{p,q}[\frac{1}{2}] \end{pmatrix}$$

of $\Delta^{p+1,q+1}$ as a $\mathrm{CSpin}(p,q)$-representation.

The Clifford action $\mathbb{R}^{p,q} \times \Delta^{p,q} \to \Delta^{p,q}$ is $\mathrm{Spin}(p,q)$-invariant. As a map $\mathbb{R}^{p,q} \times \Delta^{p,q} \to \Delta^{p,q} \otimes \mathbb{R}[1]$ it becomes $\mathrm{CSpin}(p,q)$-invariant.

$\Delta^{p,q}$ is endowed with a (pseudo-)Hermitian metric (cf. [**Bau81**] or [**Kat99**]) $k_{p,q}$, which satisfies that for all $X \in \mathbb{R}^{p,q}$ and $v, v' \in \Delta^{p,q}$ one has

$$k_{p,q}(X \cdot v, v') + (-1)^{p+1} k_{p,q}(v, X \cdot v') = 0.$$

This property shows that $k_{p,q}$ is $\mathrm{Spin}(p,q)$-invariant. For p odd we define

$$k_{p+1,q+1} := \begin{pmatrix} 0 & k_{p,q} \\ k_{p,q} & 0 \end{pmatrix} \tag{147}$$

and for p even we set

$$k_{p+1,q+1} := \begin{pmatrix} 0 & ik_{p,q} \\ -ik_{p,q} & 0 \end{pmatrix}. \tag{148}$$

One checks that then for all $X \in \mathbb{R}^{p+1,q+1}$ and $v, w \in \Delta^{p+1,q+1}$ one has that

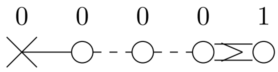
$$k_{p+1,q+1}(X \cdot v, v') + (-1)^p k_{p+1,q+1}(v, X \cdot v') = 0, \tag{149}$$

and thus $k_{p+1,q+1}$ is the $\mathrm{Spin}(p+1,q+1)$-invariant (pseudo)-Hermitian metric on $\Delta^{p+1,q+1}$ satisfying (149).

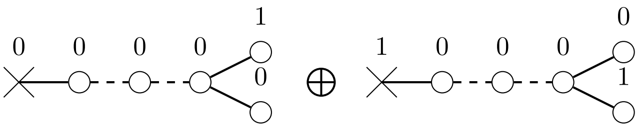
Let now $(\hat{\mathcal{G}} \to M, \omega)$ be a $(\mathrm{Spin}(p+1,q+1), \hat{P})$-structure. Then $\hat{\mathcal{G}}_0 = \mathcal{G}/(\mathbb{R}^{p,q})^*$ is a reduction of structure group of TM to $\mathrm{CSpin}(p,q)$ and therefore a conformal spin structure.

Conversely, assume that $(M, [g])$ is a conformal class of metrics together with a reduction of the $\mathrm{CO}_{p,q}$-bundle $\mathcal{G}_0$ to a a $\mathrm{CSpin}(p,q)$-bundle $\hat{\mathcal{G}}_0$. Let $(\mathcal{G} \to M, \omega)$ be the torsion-free Cartan geometry of type $(\mathrm{SO}_0(p+1,q+1), P)$ encoding $[g]$. Take some $g \in [g]$; Then the corresponding Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ yields the decomposition $\mathcal{G} = \mathcal{G}_0 \times (\mathbb{R}^{p,q})^*$. Since $P = G_0 \ltimes (\mathbb{R}^{p,q})^*$ and $\hat{P} = \hat{G}_0 \ltimes (\mathbb{R}^{p,q})^*$, the P-principal action can be translated into an action on the decomposition $\mathcal{G} = \mathcal{G}_0 \times (\mathbb{R}^{p,q})^*$; on the other hand, the extension of this definition to an action by $\hat{P} = \hat{\mathcal{G}}_0 \times (\mathbb{R}^{p,q})^*$ makes $\hat{\mathcal{G}} := \hat{\mathcal{G}}_0 \times (\mathbb{R}^{p,q})^*$ into a reduction of structure group of $\mathcal{G}$ to $\hat{\mathcal{G}}$ by the double cover $\hat{P} \to P$. We thus see that conformal spin structures on $(M, [g])$ correspond to torsion-free Cartan geometries of type $(\hat{G}, \hat{P})$.

6.3.2. The spin-tractor bundle. Let now $(M,[g])$ be a conformal class of metrics with a choice of spin structure and $(\hat{G}\to M,\omega)$ the corresponding parabolic geometry. Let $\mathbf{\Sigma}$ be the (complex) spin-tractor bundle $\hat{\mathcal{G}}\times_{\hat{P}}\Delta^{p+1,q+1}$ corresponding to



resp.



for $p+q$ odd resp. $p+q$ even. Denote by $\mathbf{\Delta}=\hat{\mathcal{G}}_0\times_{\hat{G}_0}\Delta^{p,q}$ the spin bundle of $(M,[g])$. Then it immediately follows from our discussion above that under a choice of metric $g\in[g]$ one has the decomposition

$$[\mathbf{\Sigma}]_g=\begin{pmatrix}\mathbf{\Delta}[-\frac{1}{2}]\\ \mathbf{\Delta}[\frac{1}{2}]\end{pmatrix}.$$

We encode the Clifford action $TM\to\mathrm{End}(\mathbf{\Delta})$ via the Clifford symbol $\gamma_a\in\Gamma(\mathbf{E}_a[1]\otimes\mathrm{End}(\mathbf{\Delta}))$; Spinor indices will always be suppressed. Then a vector field $\xi\in\mathfrak{X}(M)$ acts on a tractor-spinor $s\in\Gamma(\mathbf{\Sigma})$ with

$$[s]_g=\begin{pmatrix}v\\ w\end{pmatrix}\in\Gamma([\mathbf{\Sigma}]_g)=\Gamma(\begin{pmatrix}\mathbf{\Delta}[-\frac{1}{2}]\\ \mathbf{\Delta}[\frac{1}{2}])\end{pmatrix})$$

by

$$\xi\bullet s=\begin{pmatrix}0\\ \xi\cdot v\end{pmatrix}=\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\xi^p\gamma_p v\end{pmatrix}.$$

A one form $\varphi\in\Omega^1(M)$ acts by

$$\varphi\bullet\begin{pmatrix}v\\ w\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\boldsymbol{g}^{pq}\varphi_q\gamma_q w\\ 0\end{pmatrix}.$$

Accordingly,

$$[s]_{\hat{g}}=\begin{pmatrix}v+\frac{1}{\sqrt{2}}\Upsilon^p\gamma_p w\\ w\end{pmatrix}$$

The spin tractor connection ∇ is given by

$$\nabla_c\begin{pmatrix}v\\ w\end{pmatrix}=\begin{pmatrix}D_c v+\frac{1}{\sqrt{2}}P_{cp}\gamma^p w\\ D_c w+\frac{1}{\sqrt{2}}\gamma_c v\end{pmatrix}, \tag{150}$$

with D the spin connection on $\mathbf{\Delta}$. To check conformal invariance of this connection, one uses

$$\hat{D}_c\sigma=D_c\sigma-\frac{1}{2}\Upsilon^p\gamma_c\gamma_p\sigma-\frac{1}{2}\Upsilon^p\gamma_p\sigma.$$

for $\sigma\in\Gamma(\mathbf{\Delta})$.

The Lie algebra differentials are

$$\partial_c \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}\gamma_c v \end{pmatrix}$$
$$\partial^* \begin{pmatrix} v_c \\ w_c \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}\gamma^p w_p \\ 0 \end{pmatrix},$$

and thus the Kostant Laplacian acts by

$$\Box \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} -\frac{n}{2} \\ 0 \end{pmatrix}$$

and the first splitting operator $L_0 : \Gamma(\mathbf{\Delta}[\frac{1}{2}]) \to \Gamma(\mathbf{\Sigma})$ is given by

$$\sigma \mapsto \begin{pmatrix} \frac{\sqrt{2}}{n} \not{D}\sigma \\ \sigma \end{pmatrix}$$

with $\not{D} : \Gamma(\mathbf{\Delta}[\frac{1}{2}]) \to \Gamma(\mathbf{\Delta}[-\frac{1}{2}])$ the Dirac-operator, $\not{D} = \gamma^p D_p$. As an operator $\not{D} : \Gamma(\mathbf{\Delta}[-\frac{n-1}{2}]) \to \Gamma(\mathbf{\Delta}[-\frac{n+1}{2}])$ the Dirac-operator is conformally invariant. The first BGG-operator is now

$$\mathbf{Tw} := \Theta_0 : \Gamma(\mathbf{\Delta}[\frac{1}{2}]) \to \Gamma(\mathbf{E}_c \otimes \mathbf{\Sigma}[\frac{1}{2}]),$$
$$\sigma \mapsto D_c\sigma + \frac{1}{n}\gamma_c \not{D}\sigma,$$

which is called the *Twistor-operator*; $\mathbf{Tw}$ takes values in the *Twistor bundle* $\mathrm{Tw} := \mathbf{E}_a \odot \mathbf{\Delta}[\frac{1}{2}]$, which is defined by

$$\{\sigma_a \in \Gamma(\mathbf{E}_a \otimes \mathbf{\Delta}[\frac{1}{2}]) \ : \ \gamma^p \sigma_p = 0\},$$

i.e., it is the kernel of the Clifford multiplication, where one views the Clifford multiplication as a map $\mathbf{E}_a \otimes \mathbf{\Delta} \to \mathbf{\Delta}$ via the conformal metric $\boldsymbol{g}$.

The curvature acts by

$$K_{c_1c_2} \bullet \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{2}C_{c_1c_2pq}\gamma^p\gamma^q v + \frac{1}{\sqrt{2}}A_{pc_1c_2}\gamma^p w \\ \frac{1}{2}C_{c_1c_2pq}\gamma^p\gamma^q w \end{pmatrix}$$

Thus

$$\partial^*(K_{c_1c_2} \bullet \begin{pmatrix} v \\ w \end{pmatrix}) = \frac{1}{2^{\frac{3}{2}}} \begin{pmatrix} C_{pabc}\gamma^a\gamma^b\gamma^c w \\ 0 \end{pmatrix};$$

Now C_{pabc} has symmetries $\yng(2,1)_0$ in a, b, c: i.e: $C_{pabc} = -C_{pacb}$, $C_{p[abc]} = 0$ and $\boldsymbol{g}^{ab}C_{pabc} = 0$. But

$$\yng(2,1)_0 \to \mathrm{End}(\mathbf{\Delta}[\frac{1}{2}]) = \mathbf{\Delta} \otimes \mathbf{\Delta}^*,$$
$$X_{a[bc]} \mapsto X_{abc}\gamma^a\gamma^b\gamma^c$$

vanishes since $\mathbf{\Delta} \otimes \mathbf{\Delta}^* \cong \mathbf{\Delta} \otimes \mathbf{\Delta}$ does not contain $\yng(2,1)_0$. To see this directly, one writes

$$X_{abc}\gamma^a\gamma^b\gamma^c = X_{[ab]c}\gamma^a\gamma^b\gamma^c + X_{(ab)c}\gamma^a\gamma^b\gamma^c,$$

and has, since X is trace-free and $\gamma_{(a}\gamma_{b)} = -\boldsymbol{g}_{ab}$, that

$$X_{abc}\gamma^a\gamma^b\gamma^c = \frac{1}{2}(X_{abc} + X_{bca})\gamma^a\gamma^b\gamma^c;$$

Applying $X_{[abc]} = 0$ yields

$$X_{abc}\gamma^a\gamma^b\gamma^c = \frac{1}{2}X_{cba}\gamma^a\gamma^b\gamma^c.$$

Repeating this we see

$$X_{abc}\gamma^a\gamma^b\gamma^c = \frac{1}{4}X_{abc}\gamma^a\gamma^b\gamma^c$$

and thus $X_{abc}\gamma^a\gamma^b\gamma^c = 0$. This implies vanishing of $C_{pabc}\gamma^a\gamma^b\gamma^c$ since $C_{pabc} = C_{pa[bc]}$ and $C_{p[abc]} = 0$.

Thus, we see that

PROPOSITION 6.3.1. *Twistor-spinors are in 1:1-correspondence with the space of parallel sections of of the spin tractor bundle endowed with the canonical spin tractor connection* ∇ (150)*.*

This is an alternative proof of a well known fact; it has previously be shown by calculating the differential consequences of the Twistor equation directly, cf. e.g [**BFGK90**].

CHAPTER 7

Applications of BGG-techniques and prolongation connections for the Fefferman construction $G_2 \hookrightarrow SO(3,4)$.

In this chapter we are going to apply methods from tractor calculus, the BGG-machinery and prolongation connections to the (generalized) Fefferman construction which associates a conformal class of $(2,3)$-signature metrics to a generic rank two distribution in dimension 5.

There is a similar and well studied result: this is the classical Fefferman construction ([**Fef76**],[**BDS77**],[**Čap02**],[**ČG06**],[**ČG08**],[**Lei06**],[**Lei08b**],[**Bau07**]) of a (pseudo-)conformal structure on an S^1-bundle over a CR-manifold. It has been observed in [**Čap06**] that both Nurowski's and Fefferman's results admit interpretations as special cases of a very general construction relating parabolic geometries of different types. This viewpoint as a generalized Fefferman construction lays the groundwork for this chapter.

The outline of this chapter is as follows: In section 7.1 we briefly introduce generic rank 2- distributions $\mathbf{D} \subset TM$ and describe them as parabolic geometries. In section 7.2 we describe the association of conformal structures to generic rank two distributions as a generalized Fefferman construction. In section 7.3 it is shown that given a holonomy reduction of a conformal structure $[g]$ of signature $(2,3)$ to G_2, the conformal class $[g]$ is induced by a distribution $\mathbf{D} \subset TM$. This is then used to give precise criteria for a $(2,3)$-signature conformal structure $[g]$ to come from a generic distribution $\mathbf{D}$: We provide characterizations via normal conformal Killing 2-forms and twistor-spinors. Finally, in section 7.4, we show that every conformal Killing field of $[g]_{\mathbf{D}}$ decomposes into an almost Einstein scale and a symmetry of the distribution $\mathbf{D}$.

The general procedure to characterize induced Fefferman geometries and decompose their automorphisms is largely analogous to the constructions of [**ČG06**] and [**ČG08**] for the (classical) Fefferman spaces.

The results of this chapter mostly grew out of a joint project with K. Sagerschnig to apply the BGG-methods which underly this thesis and explicit calculations of 6.2 in chapter 6 resp. [**Ham08**] to the Fefferman construction studied in [**Sag08**]. A joint article is in progress, [**HS09**]. In this presentation we will mostly focus on holonomy considerations and applications of the BGG-machinery. For some algebraic calculations necessary to compare normality we will only refer to [**HS09**].

7.1. Generic rank two distributions as parabolic geometries of type (G_2, P)

7.1.1. The distributions. Let $\mathbf{D}_1$ and $\mathbf{D}_2$ be subbundles of TM. For $x \in M$ we define

$$[\mathbf{D}_1, \mathbf{D}_2]_x := \operatorname{span}(\{[\xi, \eta]_x \;:\; \xi \in \Gamma(\mathbf{D}_1), \eta \in \Gamma(\mathbf{D}_2)\}). \qquad (151)$$

In general $[\mathbf{D}_1, \mathbf{D}_2]$ does not define a subbundle of TM. One has that a subbundle $\mathbf{D} \subset TM$, $\mathbf{D}$ is integrable if and only if $[\mathbf{D}, \mathbf{D}] \subset \mathbf{D}$. We are interested in maximally non-integrable rank 2-subbundles of 5-manifolds: We say that a rank 2 subbundle $\mathbf{D} \subset TM$ is *generic* if $[\mathbf{D}, \mathbf{D}]$ is a subbundle of rank 3 and $[\mathbf{D}, [\mathbf{D}, \mathbf{D}]] = TM$. Thus, we are looking at distributions of maximal growth vector $(2, 3, 5)$ in each point. We will also say that $\mathbf{D}$ is a generic distribution. Classically, there is a well known class of second order ODEs which yield generic rank 2-distributions, but we won't discuss this aspect here (cf. e.g. [**Nur05**]).

A generic distribution $\mathbf{D}$ automatically gives rise to a filtration of the tangent bundle by subbundles

$$T^{-1}M := \mathbf{D} \qquad (152)$$

$$T^{-2}M := [\mathbf{D}, \mathbf{D}] \qquad (153)$$

$$T^{-3}M := TM. \qquad (154)$$

By construction, this filtration is compatible with the Lie bracket of vector fields in the sense that for $\xi \in \Gamma(T^iM)$ and $\eta \in \Gamma(T^jM)$ we have $[\xi, \eta] \in \Gamma(T^{i+j}M)$; i.e.: M is a filtered manifold (compare with 2.2.5). The Lie bracket of vector fields thus induces a tensorial bracket $\mathcal{L}$ on the associated graded bundle $\operatorname{gr}(TM) = \bigoplus \operatorname{gr}_i(TM)$, where $\operatorname{gr}_i(TM) = T^iM/T^{i+1}M$. This is the Levi bracket

$$\mathcal{L} : \operatorname{gr}(TM) \times \operatorname{gr}(TM) \to \operatorname{gr}(TM)$$

introduced in general in section 2.2.5. It makes the bundle $\operatorname{gr}(TM)$ into a bundle of nilpotent Lie algebras; the fiber $(\operatorname{gr}(TM)_x, \mathcal{L}_x)$ is the *symbol algebra* at the point x. Note, that we can equivalently characterize generic rank two distributions in terms of their symbol algebras: A distribution is generic if and only if the symbol algebra at each point is isomorphic to the graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3}$, where $\dim(\mathfrak{g}_{-1}) = 2$, $\dim(\mathfrak{g}_{-2}) = 1$, $\dim(\mathfrak{g}_{-3}) = 2$, and the only nontrivial components of the Lie bracket, $\mathfrak{g}_{-1} \times \mathfrak{g}_{-2} \to \mathfrak{g}_{-3}$ and $\Lambda^2\mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ define isomorphisms.

7.1.2. The embedding $G_2 \hookrightarrow \mathrm{SO}(3,4)$. Let us recall one of the possible definitions of an exceptional Lie group of type G_2: It is well known (see e.g. [**Bry87**]) that the canonical $\mathrm{GL}(7)$-representation on the space $\Lambda^3(\mathbb{R}^7)^*$ of 3-forms has two open orbits, and the stabilizer of a 3-form in either of these open orbits is a 14-dimensional Lie group. For one of these orbits it is a compact real form of the complex exceptional Lie group $G_2^{\mathbb{C}}$ and for the other orbit it is a split real form.

Consider the symmetric bilinear map

$$\mathbb{R}^7 \times \mathbb{R}^7 \to \Lambda^7\mathbb{R}^{7*},$$
$$(X, Y) \mapsto i_X\Phi \wedge i_Y\Phi \wedge \Phi$$

associated to a 3-form Φ. Using the classification of orbit-types in [**Wes81**], it turns out that this map is non-degenerate if and only if Φ is contained in an open orbit. It is positive definite if the stabilizer is the compact real form and it has signature $(3,4)$ if the stabilizer is the split real form of $G_2^{\mathbb{C}}$. By construction, the map will be GL_Φ-equivariant with GL_Φ the stabilizer of of Φ in GL(7).

Now this GL_Φ-invariant bilinear map determines an invariant element vol on $\mathbb{R}^7$ given by the 9-th root $\sqrt[9]{D} \in \Lambda^7(\mathbb{R}^{7*})$ of its determinant $D \in (\Lambda^7\mathbb{R}^{7*})^9$, see e.g. [**Hit01**]. One finds that for any Φ in one of the two open GL(7)-orbits, vol $\neq 0$, and thus $\mathrm{GL}(7)_\Phi \subset \mathrm{SL}(7)$, and we will henceforth take vol the standard-volume form on $\mathbb{R}^7$.

Thus, for any Φ in one of the two open orbits,

$$H(\Phi)(X,Y)\mathrm{vol} := i_X\Phi \wedge i_Y\Phi \wedge \Phi \tag{155}$$

defines a non-degenerate $\mathbb{R}$-valued bilinear form $H(\Phi)$ on $\mathbb{R}^7$ which is invariant under the action of the stabilizer of Φ. In particular, $\mathrm{GL}(7)_\Phi \subset \mathrm{SO}(H(\Phi))$.

Consider the standard basis $e_1, \cdots, e_7$ on $\mathbb{R}^7$ and $e_1^*, \cdots, e_7^*$ its dual basis. We fix the representative

$$\begin{aligned}\bar{\Phi} :=& \frac{1}{\sqrt{6}}e_1^* \wedge e_4^* \wedge e_7^* - \frac{1}{\sqrt{3}}e_2^* \wedge e_3^* \wedge e_7^* - \frac{1}{\sqrt{3}}e_1^* \wedge e_5^* \wedge e_6^* \\ &- \frac{1}{\sqrt{6}}e_2^* \wedge e_4^* \wedge e_5^* - \frac{1}{\sqrt{6}}e_3^* \wedge e_4^* \wedge e_6^*\end{aligned}$$

of a split real form of $G_2^{\mathbb{C}}$ and always work with $G = G_2 := \mathrm{GL}(7)_{\bar{\Phi}}$.

One computes that

$$H(\bar{\Phi})(X,Y) = \frac{1}{\sqrt{6}}(x_1y_7 + x_2y_5 + x_3y_6 - x_4y_4 + x_5y_2 + x_6y_3 + x_7y_1).$$

In matrix form, the resulting metric $H(\bar{\Phi})$ on $\mathbb{R}^7$ is (up to a factor)

$$h = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \bar{g} & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{156}$$

with

$$\bar{g} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{157}$$

Since $G_2 \subset \mathrm{SO}(h)$, we can define it equivalently as the stabilizer in GL(7) of the 3-vector corresponding to $\bar{\Phi}$ via h:

$$\Phi := -\frac{1}{\sqrt{3}}e_7 \wedge e_2 \wedge e_3 + \frac{1}{\sqrt{6}}e_5 \wedge e_4 \wedge e_2 + \frac{1}{\sqrt{6}}e_6 \wedge e_4 \wedge e_3 \tag{158}$$

$$- \frac{1}{\sqrt{6}}e_7 \wedge e_4 \wedge e_1 - \frac{1}{\sqrt{3}}e_1 \wedge_5 \wedge e_6. \tag{159}$$

The Lie algebra $\mathfrak{so}(3,4) = \mathfrak{so}(h)$ has the matrix representation

$$\left(\begin{array}{c|c|c} -\alpha & -Z^t\bar{g} & 0 \\ \hline X & A & Z \\ \hline 0 & -X^t\bar{g} & \alpha \end{array}\right), \alpha \in \mathbb{R}; X, Z \in \mathbb{R}^5, A \in \mathfrak{so}(\bar{g}).$$

The Lie algebra $\mathfrak{g}$ of $G = G_2$, i.e. the Lie algebra of elements $M \in \mathfrak{gl}(7)$ such that $\Phi(Mv, v', v'') + \Phi(v, Mv', v'') + \Phi(v, v', Mv'') = 0$, is a subalgebra in $\mathfrak{so}(h)$ consisting of matrices of the form

$$\left(\begin{array}{c|c|c|c|c} \mathrm{tr}(A) & Z & s & W & 0 \\ \hline X & A & \sqrt{2}\mathbb{J}Z^t & \frac{s}{\sqrt{2}}\mathbb{J} & -W^t \\ \hline r & -\sqrt{2}X^t\mathbb{J} & 0 & -\sqrt{2}Z\mathbb{J} & s \\ \hline Y & -\frac{r}{\sqrt{2}}\mathbb{J} & \sqrt{2}\mathbb{J}X & -A^t & -Z^t \\ \hline 0 & -Y^t & r & -X^t & -\mathrm{tr}(A) \end{array}\right) \tag{160}$$

with $A \in \mathfrak{gl}(2)$, $X, Y \in \mathbb{R}^2$, $Z, W \in \mathbb{R}^{2*}$, $r, s \in \mathbb{R}$ and $\mathbb{J} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

For later use let us note here that the complement of $\mathfrak{g}$ in $\mathfrak{so}(h)$ with respect to the Killing form is isomorphic to the seven dimensional standard representation of G_2. That means we have a G_2-module decomposition

$$\mathfrak{so}(h) = \mathfrak{g} \oplus \mathbb{R}^7. \tag{161}$$

This can be understood via the sequence

$$0 \to \mathfrak{g} \hookrightarrow \mathfrak{so}(h) \stackrel{i\Phi}{\to} \mathbb{R}^7 \to 0, \tag{162}$$

which is G_2-equivariant and exact. Here

$$i\Phi : \mathfrak{so}(h) = \Lambda^2\mathbb{R}^7 \to \mathbb{R}^7 \tag{163}$$

is the insertion of $\mathfrak{so}(h)$ into Φ. The factor of Φ as given in (158) was chosen such that the insertion

$$i\Phi : \mathbb{R}^7 \to \Lambda^2\mathbb{R}^7 = \mathfrak{so}(h) \tag{164}$$

splits sequence (162).

Let $\tilde{P}$ be the stabilizer of isotropic ray $\mathbb{R}_+e_1 \subset \mathbb{R}^7$. As we discussed in section 6.1, parabolic geometries of type $(SO(h), \tilde{P})$ correspond to (oriented) conformal structures of signature $(2,3)$.

Define $P = \tilde{P} \cap G_2$; this is now a parabolic subgroup in G_2. To describe explicitly the corresponding parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$, we introduce vector space decompositions of the Lie algebra. We consider the block decomposition

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 & \mathfrak{g}_3 & 0 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 & \mathfrak{g}_3 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & 0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \mathfrak{g}_{-3} & \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ 0 & \mathfrak{g}_{-3} & \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix},$$

of matrices (160), which defines a grading

$$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3.$$

Note that the subalgebra $\mathfrak{g}_- = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3}$ coincides with the symbol algebra of a generic rank two distribution in dimension five as discussed in 7.1.1. The grading induces a filtration $\mathfrak{g}^3 \subset \mathfrak{g}^2 \subset \mathfrak{g}^1 \subset \mathfrak{g}^0 \subset \mathfrak{g}^{-1} \subset \mathfrak{g}^{-2} \subset$

$\mathfrak{g}^{-3}$, which is preserved by the action of P on $\mathfrak{g}$. The subalgebra $\mathfrak{p} = \mathfrak{g}^0$ is the Lie algebra of the parabolic P and the subalgebra $\mathfrak{g}_0 \cong \mathfrak{gl}(2)$ is the Lie algebra of the subgroup $G_0 \subset P$ that also preserves the grading. The subgroup G_0 is isomorphic to $\mathrm{GL}_+(2) = \{M \in \mathrm{GL}(2) : \det(M) > 0\}$.

7.1.3. The homogeneous model and associated Cartan geometries. Let us look at the Lie group quotient G_2/P: The action of G_2 on the class $e\tilde{P} \in \mathrm{SO}(h)/\tilde{P}$ induces a smooth map

$$G_2/P \to \mathrm{SO}(h)/\tilde{P}.$$

Since both homogeneous spaces have the same dimension, the map is an open embedding. Since G_2/P is a quotient of a semisimple Lie group by a parabolic subgroup, it is compact, and the map is in fact a diffeomorphism. The group $\mathrm{SO}(h)$ acts transitively on the space of isotropic rays in $\mathbb{R}^7$, which can be identified with the pseudo-sphere $Q_{2,3} \cong S^2 \times S^3$. It turns out that the $(3,4)$-metric h on $\mathbb{R}^7$ defined in (156) induces the conformal class of $(g_2, -g_3)$ on $Q_{2,3}$, with g_2,g_3 being the round metrics on S^2 resp. S^3. The pullback of that conformal structure yields a G_2-invariant conformal structure on G_2/P. Thus $G_2/P = \mathrm{SO}(h)/\tilde{P} = (Q_{2,3}, [(g_2, -g_3)]$.

Explicit descriptions of the canonical rank two distribution on the pseudo-sphere $Q_{2,3} \cong S^2 \times S^3$ can be found in [**Sag06**]. In an algebraic picture, the distribution corresponds to the P-invariant subspace $\mathfrak{g}^{-1}/\mathfrak{p} \subset \mathfrak{g}/\mathfrak{p}$. Via the identification of $T(G/P)$ with $G \times_P \mathfrak{g}/\mathfrak{p}$ this invariant subspace gives rise to a rank two distribution which is generic in the sense of 7.1.1.

More generally, suppose $(\mathcal{G}, \omega)$ is any parabolic geometry of type (G_2, P). Recall from chapter 2 that the Cartan connection ω defines an isomorphism $TM \cong \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$. Hence for any such geometry the subspace $\mathfrak{g}^{-1}/\mathfrak{p}$ gives rise to a rank two distribution. In general, this distribution will not be generic. To obtain genericity, one imposes the regularity condition of definition 2.2.5 on ω. Let us recall this condition: Let

$$TM = T^{-3}M \supset T^{-2}M \supset T^{-1}M \supset \{0\}$$

be the sequence of subbundles of constant ranks 2, 3 and 5 coming from the P-invariant filtration $\mathfrak{g}/\mathfrak{p} = \mathfrak{g}^{-3}/\mathfrak{p} \supset \mathfrak{g}^{-2}/\mathfrak{p} \subset \mathfrak{g}^{-1}/\mathfrak{p}$. Consider the associated graded bundle $\mathrm{gr}(TM)$. This bundle can be naturally identified with

$$\mathcal{G} \times_P \mathrm{gr}(\mathfrak{g}/\mathfrak{p}) \cong (\mathcal{G}/P_+) \times_{G_0} \mathfrak{g}_-.$$

We recall from chapter 2 that since the Lie bracket on the nilpotent Lie algebra $\mathfrak{g}_-$ is invariant under the adjoint representation, it induces a bundle map $\{,\} : \mathrm{gr}(TM) \times \mathrm{gr}(TM) \to \mathrm{gr}(TM)$, algebraic bracket. A Cartan connection form ω was defined to be regular in section 2.2.5 if the underlying manifold M is filtered and the associated Levi bracket on $\mathrm{gr}(TM)$ coincides with the algebraic bracket. But since $\mathfrak{g}_-$ is the symbol algebra of a generic rank 2 distribution one immediately has that a regular parabolic geometry $(\mathcal{G}, \omega)$ of type (G_2, P) endows M with the structure of a generic distribution $\mathbf{D} := T^{-1}M = \mathcal{G} \times_P \mathfrak{g}_{-1}/\mathfrak{p} \subset TM$.

Moreover, we recall that regularity can be expressed as a condition on the curvature of a Cartan connection (Proposition 2.2.6): Since $\mathfrak{g}$ has a P-invariant filtration, we have a notion of maps in $\Lambda^k(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ of homogeneous degree $\geq l$, and the set of these maps is P-invariant. A Cartan connection form is regular if and only if the curvature function is homogeneous of degree ≥ 1; this means that $\kappa(u)(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1}$ for all i, j and $u \in \mathcal{G}$. Note that if the curvature function takes values in $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{p}$, i.e. if the geometry is *torsion-free*, it is automatically regular.

We have seen above that a regular parabolic geometry of type (G_2, P) determines an underlying generic rank two distribution $\mathbf{D}$, and for our choice of P the distribution turns out to be orientable. Conversely, the prolongation procedures for parabolic geometries, see e.g. [**ČS09**], show that we can always associate a regular parabolic geometry of type (G_2, P) to a generic distribution $\mathbf{D}$. However, there are many regular Cartan connection forms inducing the same underlying structure. To obtain uniqueness, one additionally imposes the normality condition of parabolic geometries, as was discussed in 2.2.7: Let $(\mathcal{G}, \omega)$ be a regular parabolic geometry of type (G_2, P). Let $K \in \Omega^2(M, \mathcal{A}M)$ be the curvature of the Cartan connection form ω. Then we said that ω is normal if $\partial^*(K) = 0$.

Now we can state Cartan's classical result in modern language. We restrict our considerations to orientable distributions. Equivalently, this means that the bundle $\Lambda^2\mathbf{D}$ be orientable. Then one has:

THEOREM 7.1.1 ([**EC10**]). *One can naturally associate a regular, normal parabolic geometry $(\mathcal{G}, \omega)$ of type (G_2, P) to an orientable generic rank two distribution in dimension five, and this establishes an equivalence of categories.*

An important point to be used in the Fefferman construction later, is that given a regular Cartan connection form ω which is already normalized up to a certain homogeneity, it only differs from the unique Cartan form by terms of higher homogeneity: This follows from the Proposition below. It will be employed in 7.3.3. Note that for two Cartan connections the difference $\omega - \tilde{\omega}$ is a P-equivariant, horizontal, $\mathfrak{g}$- valued 1-form on $\mathcal{G}$ and thus it can be described by a P-equivariant function $\Psi : \mathcal{G} \to (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$. We say that Ψ has homogeneous degree $\geq l$ if $\Psi(u)(\mathfrak{g}^i) \subset \mathfrak{g}^{i+l}$ for all i and $u \in \mathcal{G}$.

PROPOSITION 7.1.2 ([**ČS09**]). *Let $(\mathcal{G}, \omega)$ be a regular parabolic geometry with curvature function κ and suppose that $\partial^*\kappa$ is of homogeneous degree $\geq l$ for some $l \geq 1$. Then there is a normal Cartan connection $\omega_N \in \Omega^1(\mathcal{G}, \mathfrak{g})$ such that $(\omega_N - \omega)$ is of homogeneous degree $\geq l$.*

7.2. The Fefferman construction for $G_2 \hookrightarrow \mathrm{SO}(3,4)$

The relation observed in section 7.1.3 between the homogeneous models G_2/P and $\mathrm{SO}(h)/\tilde{P}$ suggests a relation between Cartan geometries of type (G_2, P) and $(\mathrm{SO}(h), \tilde{P})$, i.e. between generic rank two distributions and conformal structures. Indeed, it was Pawel Nurowski who first observed in [**Nur05**] that any generic rank two distribution on a five manifold M naturally determines a conformal class of metrics of signature $(2,3)$ on M. Starting from a system of ODEs he explicitly constructed a metric from the conformal class. A different construction of such a metric can be found in [**ČS07**].

Here we shall discuss Nurowski's result as a special case of an extension functor of Cartan geometries, called a generalized Fefferman construction; cf. [**Čap06**]. This was first done in [**Sag08**]. Let $i' : \mathfrak{g} \hookrightarrow \mathfrak{so}(h)$ denote the derivative of the inclusion $i : G_2 \hookrightarrow \mathrm{SO}(h)$. Given a Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G_2, P), we can extend the structure group of the Cartan bundle, i.e., we can form the associated bundle $\tilde{\mathcal{G}} = \mathcal{G} \times_P \tilde{P}$. Then this is a principal bundle over M with structure group $\tilde{P}$. We have a natural inclusion $j : \mathcal{G} \hookrightarrow \tilde{\mathcal{G}}$ mapping an element $u \in \mathcal{G}$ to the class $[(u, e)]$. Moreover, we can uniquely extend the Cartan connection ω on $\mathcal{G}$ to a Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \mathfrak{so}(h))$ such that $j^*\tilde{\omega} = i' \circ \omega$.

The construction defines a functor from Cartan geometries of type (G_2, P) to Cartan geometries of type $(\tilde{G}, \tilde{P})$.

We will later need the relations between the curvatures of $\tilde{\omega}$ and ω: In the next Lemma we use the inclusion of adjoint tractor bundles $\mathcal{A}M \hookrightarrow \tilde{\mathcal{A}}M$ via

$$\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g} \hookrightarrow \mathcal{G} \times_P \mathfrak{so}(h) = \tilde{\mathcal{G}} \times_{\tilde{P}} \mathfrak{so}(h) = \tilde{\mathcal{A}}M.$$

Lemma 7.2.1.

(1) *The curvature form $\tilde{\Omega} \in \Omega^2(\tilde{\mathcal{G}}, \mathfrak{so}(h))$ of $\tilde{\omega}$ pulls back to the curvature form $\Omega \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of ω:*

$$j^*(\tilde{\Omega}) = \Omega. \tag{165}$$

(2) *The factorizations $K \in \Omega^2(M, \mathcal{A}M)$ of Ω and $\tilde{K} \in \Omega^2((M, \tilde{\mathcal{A}}M)$ of $\tilde{\Omega}$ agree:*

$$\tilde{K} = K \in \Omega^2(M, \mathcal{A}M). \tag{166}$$

In particular, $K \in \Omega^2(M, \tilde{\mathcal{A}}M)$ has in fact values in $\mathcal{A}M \subset \tilde{\mathcal{A}}M$.

Proof. Since the exterior derivative d is natural, it commutes with pullbacks: $j^*d\tilde{\omega} = d(j^*\tilde{\omega}) = d\omega$. Since also $j^*([\tilde{\omega}, \tilde{\omega}]) = [j^*\tilde{\omega}, j^*\tilde{\omega}] = [\omega, \omega]$, we thus see that by Definition of curvature (13) we have $j^*\tilde{\Omega} = \Omega$.

Now the inclusion $j : \mathcal{G} \to \tilde{\mathcal{G}}$ is a reduction of structure group from $\tilde{P}$ to P. Therefore factorizing $\tilde{\Omega} \in \Omega^2_{\mathrm{hor}}(\tilde{\mathcal{G}}, \mathfrak{so}(h))^{\tilde{P}}$ to the curvature form $\tilde{K} \in \Omega^2(M, \tilde{\mathcal{A}}M)$ is the same as pulling back $\tilde{\Omega}$ via j and then factorizing. □

By Theorem 7.1.1, we can associate a canonical Cartan geometry $(\mathcal{G}, \omega)$ of type $(\tilde{G}, \tilde{P})$ to a generic rank two distribution on a five manifold M. As discussed in section 6.1 any Cartan geometry $(\tilde{\mathcal{G}}, \tilde{\omega})$ of type $(\mathrm{SO}(h), \tilde{P})$ determines a conformal structure on the underlying manifold M. Thus the above generalized Fefferman construction shows that a generic rank two distribution $\mathbf{D}$ naturally determines a conformal class $[g]$ of metrics of signature $(2, 3)$. However, a priori we do not know whether $\tilde{\omega}$ is the normal Cartan connection associated to that conformal structure; one has to prove that normality of ω implies normality of $\tilde{\omega}$:

Proposition 7.2.2. *Let $(\mathcal{G} \to M, \omega)$ be a regular Cartan geometry of type (G_2, P) and let $(\tilde{\mathcal{G}} \to M, \tilde{\omega})$ be the associated Cartan geometry of type $(\tilde{G}, \tilde{P})$. Then normality of $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ implies normality of $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \mathfrak{so}(h))$.*

For the proof we refer to [**HS09**] or [**Sag08**].

7.2.1. The parallel tractor three-form and the underlying conformal Killing 2-form. Let $\mathbf{S} = \tilde{\mathcal{G}} \times_{\tilde{P}} \mathbb{R}^7$ be the standard tractor bundle for a conformal structure $[g]$ associated to a generic rank 2-distribution $\mathbf{D}$ on a five manifold M. Then $\mathbf{S}$ is easily seen to carry an additional structure:

PROPOSITION 7.2.3.

(1) The standard tractor bundle $\mathbf{S}$ *for a conformal structure* $[g]$ *associated to a generic* 2*-distribution carries a parallel tractor* 3*-form* $\mathbf{\Phi} \in \Gamma(\Lambda^3 \mathbf{S}^*) = \Gamma(\Lambda^3 \mathbf{S})$.

(2) The tractor 3*-form* $\mathbf{\Phi}$ *determines an underlying normal conformal Killing* 2*-form* $\phi \in \mathcal{E}_{[ab]}[3]$ *which is locally decomposable, or equivalently, satisfies* $\phi \wedge \phi = 0$.

PROOF.

(1) By construction, the conformal Cartan bundle is the associated bundle $\tilde{\mathcal{G}} = \mathcal{G} \times_P \tilde{P}$, where $\mathcal{G}$ is the Cartan bundle for the distribution. Hence, the tractor bundle can be viewed as $\mathbf{S} = \mathcal{G} \times_P \mathbb{R}^7$. It follows that the P-equivariant function $f_\Phi : \mathcal{G} \to \Lambda^3 \mathbb{R}^7$ mapping constantly onto the 3-vector Φ stabilized by G_2 induces a section $\mathbf{\Phi} \in \Gamma(\Lambda^3 \mathbf{S})$. It follows from Proposition 7.2.2 that the normal tractor connection $\nabla^{\Lambda^3 \mathbf{S}}$ is induced by the normal Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. Hence, according to (46), $\nabla^{\Lambda^3 \mathbf{S}}_\xi \Phi$ corresponds to the function $u \mapsto (\xi'_u \cdot f_\Phi) + \omega_u(\xi'(u))(f_\Phi(u))$, where $u \in \mathcal{G}$ and $\xi' \in \mathfrak{X}(\mathcal{G})$ is a P-invariant lift of a vector field $\xi \in \mathfrak{X}(M)$. Since f_Φ is constant and ω takes values in the isotropy algebra $\mathfrak{g}$ of Φ, this means that $\nabla^{\Lambda^3 \mathbf{S}} \mathbf{\Phi} = 0$, i.e. $\mathbf{\Phi}$ is a parallel 3-tractor.

(2) Since $\mathbf{\Phi}$ is $\nabla^{\Lambda^3 S}$-parallel, $\phi := \Pi_0(\mathbf{\Phi})$ lies in the kernel of the first BGG-operator of $\Lambda^3 \mathbf{S}$, which, according to 6.2, is the operator governing conformal Killing 2-forms. By Definition 4.1.5, the parallel tractor Φ therefore gives rise to a *normal* conformal Killing 2-form, subject to additional equations, to be stated explicitly in 7.3, (170).

Let $V = \Lambda^3 \mathbb{R}^7$. With respect to the conformal parabolic $\tilde{P} \subset \mathrm{SO}(h)$, V is filtered $V = V^{-1} \supset V^0 \supset V^1 \supset \{\}$. We see from our explicit formula for Φ, (158), that, up to a factor, a representative in $\mathrm{gr}_{-1}(V) = V/V^0$ is given by $e_7 \wedge e_2 \wedge e_3$. Around every point $x \in M$ we can choose a local section $\sigma : U \to \mathcal{G}$; on $\sigma(U) \subset \mathcal{G} \subset \tilde{\mathcal{G}}$ we define 3 constant functions, mapping to e_7, e_2 and e_3; these correspond to sections s_7, s_2 and s_3 of the standard tractor bundle $\mathbf{S}$. s_7 is simply τ_- of (104); to be precise, we use that $\sigma : U \to \mathcal{G}$ gives in particular a trivialization of the conformal weight bundles, and we can view τ_- as an (unweighted) section of $\mathbf{S}$. The tractors s_2 and s_3 lie in $\mathcal{S}^0$, and therefore project to elements φ_2 and φ_3 in $\Gamma(\mathrm{gr}_0(\mathbf{S})) = \Gamma(\mathbf{S}^0/\mathbf{S}^1) = \mathcal{E}E_a$, where we again use the trivialization of the conformal weight bundles. Thus $\tau_- \wedge \varphi_2 \wedge \varphi_3$ is a representative of $\varphi = \Pi_0(\mathbf{\Phi}) \in \mathcal{V}/\mathcal{V}^0$, and the identification (123) of $\mathcal{V}/\mathcal{V}^0$ with $\mathcal{E}_{[ab]}$ tells us that $\varphi = \varphi_2 \wedge \varphi_3 \in \mathcal{E}_{[ab]}$.

□

REMARK 7.2.4. The parabolic subgroup P preserves a filtration of the standard representation, and thus the standard tractor bundle is naturally filtered

$$\mathbf{S} = \mathbf{S}^{-2'} \supset \mathbf{S}^{-1'} \subset \mathbf{S}^{0'} \supset \mathbf{S}^{1'} \supset \mathbf{S}^{2'} \supset \{0\}.$$

The isotropic line bundle $\mathbf{S}^{2'}$ corresponds to the subspace generated by $e_1 \in \mathbb{R}^7$. The bundle $\mathbf{S}^{-1'}$ is the orthogonal complement to $\mathbf{S}^{2'}$ with respect to the tractor metric. The explicit form of the 3-form Φ (see (158)) shows how the additional filtration components can be characterized in terms of $\mathbf{\Phi}$. Recall the canonical insertion $\tau_+ \in \Gamma(\mathbf{S} \otimes \mathbf{E}[1])$ of $\mathbf{E}[-1]$ into $\mathbf{S}^2 \subset \mathbf{S}$, which was defined in section 6.1.1. Then the subbundle $\mathbf{S}^{1'}$ can be described as the set of all tractors $s \in \Gamma(\mathbf{S})$ such that $i_s i_{\tau_+} \mathbf{\Phi} = 0$. The subbundle $\mathbf{S}^{0'}$ is the bundle orthogonal to $\mathbf{S}^{-1'}$. ♢

REMARK 7.2.5. The distribution $\mathbf{D}$ can be recovered from the conformal class $[g]$ associated to the distribution and the conformal Killing 2-form ϕ. The kernel of the 2-form is the rank three distribution $[\mathbf{D}, \mathbf{D}]$. Restricted to $[\mathbf{D}, \mathbf{D}]$, a metric $g \in [g]$ is degenerate, and in fact the rank two distribution can be recovered as the kernel of the restriction of$g$ to the rank three distribution. ♢

7.3. Holonomy reduction and characterization via conformal Killing forms and twistor-spinors

The goal of this section is to characterize conformal structures arising from generic rank 2 distributions in dimension five in terms of normal conformal Killing 2-forms satisfying certain additional equations. See Theorem 7.3.8 for a precise statement of the result. We will obtain another characterization in terms of twistor spinors in Theorem 7.3.11.

We proceed as follows. First, we prove that a conformal manifold of signature $(2, 3)$ whose conformal holonomy is contained in G_2 is obtained from a generic rank two distribution via a Fefferman construction. Then we aim for a characterization of the conformal structures in terms of underlying conformal data; we derive conditions to distinguish those normal conformal Killing 2-forms coming from parallel tractor 3-forms defining holonomy reductions to G_2. This is done analogously to [**ČG06**], where the authors arrive at a version of Sparling's characterization [**Gra87**] of Fefferman spaces in terms of a conformal Killing field.

REMARK 7.3.1. Recently there has been an interest in conformal structures with holonomy G_2, since this seems to imply interesting properties for the ambient metric construction; this has been studied so far for certain classes of examples in [**Nur08**] and [**LN09**].

7.3.1. Conformal holonomy. Let $(M, [g])$ be a conformal structure of signature $(2, 3)$ encoded in a Cartan geometry $(\tilde{\mathcal{G}}, \tilde{\omega})$ as described in section 6.1. The standard tractor bundle $\mathbf{S}$ of $[g]$ is endowed with the tractor connection ∇^S and we recall the conformal holonomy

$$\mathrm{Hol}([g]) = \mathrm{Hol}(\nabla^S).$$

introduced in 6.1.3. Now $\mathbf{S}$ comes about as associated bundle to $\tilde{\mathcal{G}}' := \tilde{\mathcal{G}} \times_{\tilde{P}} \mathrm{SO}(p+1, q+1)$, and ∇^S is just the induced connection from the principal connection form $\tilde{\omega}' \in \Omega^1(\tilde{\mathcal{G}}', \mathfrak{so}(h))$. Thus we have that $\mathrm{Hol}(\nabla^S) = \mathrm{Hol}(\tilde{\omega}')$. Recall that by construction the pullback of $\tilde{\omega}'$ to $\tilde{\mathcal{G}} \subset \tilde{\mathcal{G}}'$ is simply the Cartan connection form $\tilde{\omega}$.

In the generalized Fefferman construction of section 7.2 we started with a parabolic geometry $(\mathcal{G}, \omega)$ of type (G_2, P) encoding a generic rank 2 distribution and associated to this the parabolic geometry $(\tilde{\mathcal{G}} := \mathcal{G} \times_P \tilde{P}, \tilde{\omega})$ of type $(\mathrm{SO}(h), \tilde{P})$ by equivariantly extending ω to $\tilde{\omega}$. If we add the extended bundles $\mathcal{G}' = \mathcal{G} \times_P G_2$ and $\tilde{\mathcal{G}}' = \tilde{\mathcal{G}} \times_{\tilde{P}} \mathrm{SO}(h) = \mathcal{G} \times_P \mathrm{SO}(h)$ to the picture we obtain the commuting diagram of inclusions

$$\begin{array}{ccc} (\mathcal{G}', \omega') & \hookrightarrow & (\tilde{\mathcal{G}}', \tilde{\omega}') \\ \uparrow & & \uparrow \\ (\mathcal{G}, \omega) & \hookrightarrow & (\tilde{\mathcal{G}}, \tilde{\omega}) \end{array}$$

In particular, this yields a holonomy reduction of $(\tilde{\mathcal{G}}', \tilde{\omega}')$ to $(\mathcal{G}', \omega')$, and thus $\mathrm{Hol}(\tilde{\omega}') = \mathrm{Hol}(\omega') \subset G_2$. By Proposition 7.2.2, $\tilde{\omega}$ is normal. Hence $\mathrm{Hol}(\tilde{\omega}')$ is indeed the conformal holonomy $\mathrm{Hol}([g]_{\mathbf{D}})$, which is thus seen to be contained in G_2.

We are now going to show the converse: if, for a conformal structure $(M, [g])$ of signature $(2,3)$ one has that $\mathrm{Hol}([g]) \subset G_2$, then there is in fact a canonical generic rank 2-distribution $\mathbf{D}$ on M such that $[g] = [g]_{\mathbf{D}}$. To be precise, the group $\mathrm{Hol}(\tilde{\omega})$ is only defined up to $\mathrm{SO}(3,4)$-conjugacy: it is defined for a point $u \in \tilde{\mathcal{G}}'$, and one then writes $\mathrm{Hol}_u(\tilde{\omega})$. Since for $g \in \mathrm{SO}(3,4)$, $\mathrm{Hol}_{u \cdot g} = g \mathrm{Hol}_u g^{-1}$, one can always choose a point in the fiber of u with a given representative of $\mathrm{Hol}(\tilde{\omega})$ in the $\mathrm{SO}(3,4)$-conjugacy class.

Let $\pi : \tilde{\mathcal{G}}' \to M$ be the surjective submersion of the $\mathrm{SO}(h)$-principal bundle $\tilde{\mathcal{G}}'$. The next Theorem treats the holonomy reduction of a Cartan geometry. A similar procedure was stated in [**Alt08**].

PROPOSITION 7.3.2. *Let $(\tilde{\mathcal{G}}, \tilde{\omega})$ be such that $(\tilde{\mathcal{G}}', \tilde{\omega}')$ has holonomy in G_2 and let $\mathcal{H} \subset \tilde{\mathcal{G}}'$ be a holonomy reduction of $(\tilde{\mathcal{G}}', \tilde{\omega}')$ to G_2. Then*

(1) $\mathcal{H} \subset \tilde{\mathcal{G}}'$ and $\tilde{\mathcal{G}} \subset \tilde{\mathcal{G}}'$ intersect transversally. We denote the resulting submanifold by $\mathcal{G} := \mathcal{H} \cap \tilde{\mathcal{G}}$.
(2) For every $u \in \mathcal{G}$, $T_u\pi(T_u\mathcal{G}) = T_{\pi(u)}M$.
(3) $\mathcal{G}$ is a P-principal bundle over M.
(4) Let ω be the pullback of $\tilde{\omega} \in \Omega^1(\mathcal{G}, \mathfrak{so}(h))$ to $\mathcal{G}$. Then $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is a Cartan connection form.

PROOF. The holonomy reduction $\mathcal{H} \overset{j}{\hookrightarrow} G_2$ has by definition the following properties: $\mathcal{H}$ is a G_2-principal bundle over M and and $j : \mathcal{H} \hookrightarrow \tilde{\mathcal{G}}'$ is a G_2-equivariant embedding of $\mathcal{H}$ into $\tilde{\mathcal{G}}'$. One has that the pullback $j^*(\tilde{\omega})$ of the principal connection form $\tilde{\omega} \in \Omega^1(\mathfrak{g}, \mathfrak{so}(h)$ has values in $\mathfrak{g}_2$, i.e., $j^*(\tilde{\omega}) \in \Omega^1(\mathcal{H}, \mathfrak{g}_2)$ is a G_2-principal connection form. In the following we will regard $\mathcal{H}$ as a subbundle of $\tilde{\mathcal{G}}'$.

(1) We have that $T_u\mathcal{H} + T_u\tilde{\mathcal{G}} \supset u \cdot \mathfrak{g} + u \cdot \tilde{\mathfrak{p}} = u \cdot \mathfrak{so}(h) = \ker(T_u\pi)$. Since $T_u\pi : T_u\tilde{\mathcal{G}} \to T_{\pi(u)}M$ is surjective, we have that $\dim\ (T_u\mathcal{H} + T_u\tilde{\mathcal{G}}) = \dim\ (\mathfrak{so}(h)) + \dim\ (\mathfrak{g}/\mathfrak{p}) = \dim\ (T_u\tilde{\mathcal{G}}')$.

(2) Take $u \in \mathcal{G} = \mathcal{H} \cap \tilde{\mathcal{G}}$ and $\xi \in T_{\pi(u)}M$. Since the restriction of π to $\mathcal{H}$ and $\tilde{\mathcal{G}}$ are surjective submersions, there exist $\xi_1 \in T_u\mathcal{H}$ and $\xi_2 \in T_u\tilde{\mathcal{G}}$ such that $\xi = T_u\pi\xi_1 = T_u\pi\xi_2$. Then

$$\xi_1 - \xi_2 \in \ker T_u\pi = u \cdot \mathfrak{so}(h) = u \cdot \mathfrak{g} + u \cdot \tilde{\mathfrak{p}}.$$

Thus there exist $\eta_1 \in u \cdot \mathfrak{g}$ and $\eta_2 \in u \cdot \tilde{\mathfrak{p}}$ such that $\xi_1 - \xi_2 = \eta_1 + \eta_2$. Let

$$\xi' = \xi_1 - \eta_1 = \xi_2 + \eta_2 \in T_u\mathcal{G}.$$

Then indeed $T_u\pi\xi' = \xi$.

(3) Assume first that for an $x \in M$ there is a $u \in \mathcal{H}_x \cap \tilde{\mathcal{G}}_x = \mathcal{G}_x$. Then evidently $\mathcal{G}_x = u \cdot (G_2 \cap \tilde{P}) = u \cdot P$. It therefore remains to show that $\mathcal{H}_x \cap \tilde{\mathcal{G}}_x$ is always non-empty: let $u \in \mathcal{H}_x$. Then there is a $g \in \mathrm{SO}(h)$ such that $u \cdot g \in \tilde{\mathcal{G}}$. But since $G_2/P = \mathrm{SO}(h)/\tilde{P}$ (see 7.1.2) there is a $p \in \tilde{\mathcal{P}}$ with $gp = g' \in G_2$; then $u \cdot g' \in \mathcal{H}$ since $\mathcal{H}$ is a G_2-subbundle and $u \cdot g' = (u \cdot g) \cdot p \in \mathrm{SO}(h)$; i.e., $u \cdot g' \in \mathcal{G}$.

(4) We now consider $\mathcal{G}$ as a reduction of the $\tilde{P}$-principal bundle $\tilde{\mathcal{G}}$ to P and denote by ω the pullback of $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \mathfrak{so}(h))$. By construction, $\mathcal{H} \supset \mathcal{G}$ was obtained by holonomy reduction of $(\tilde{\mathcal{G}}', \tilde{\omega}')$ to G_2. In particular, $\tilde{\omega}'_{|T\mathcal{H}}$ has values in $\mathfrak{g}$, and thus $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. P-equivariance and reproduction of $\mathfrak{p}$-fundamental vector fields is clear since $\mathcal{G}$ is just a P-principal subbundle of $\tilde{\mathcal{G}}$ and $\tilde{\omega}$ is a Cartan connection form satisfying (C.1)-(C.2) by assumption. We thus need to check that also (C.3) holds for ω. I.e., for every $u \in \mathcal{G}$ we need that $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ is an isomorphism. We have seen that $T_u\pi(T_u\mathcal{G}) = T_{\pi(u)}M$. Since $u \cdot \tilde{\mathfrak{p}} = \ker(T_u\pi) \subset T_u\tilde{\mathcal{G}}$, we see that $T_u\mathcal{G} \subset T_u\tilde{\mathcal{G}}$ must span at least dim $(\mathfrak{g}/\mathfrak{p})$-complementary dimensions and thus already $T_u\mathcal{G} + u \cdot \mathfrak{p} = T_u\tilde{\mathcal{G}}$. But then $\omega_u(T_u\mathcal{G}/u \cdot \mathfrak{p}) = \tilde{\omega}_u(T_u\tilde{\mathfrak{g}}/u \cdot \tilde{\mathfrak{p}}) = \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} = \mathfrak{g}/\mathfrak{p}$. This, together with $\omega_u(u \cdot \mathfrak{p}) = \mathfrak{p}$ by reproduction of fundamental vector fields, gives that indeed $\omega_u(T_u\mathcal{G}) = \mathfrak{g}$.

□

Proposition 7.3.3. *Suppose $(M, [g])$ is a conformal structure of signature $(2, 3)$ such that $\mathrm{Hol}([g]) \subset G_2$. Let $(\tilde{\mathcal{G}}, \tilde{\omega})$ be the normal parabolic geometry of type $(\tilde{G}, \tilde{P})$ associated to the conformal structure and let $(\mathcal{G}, \omega)$ be the parabolic geometry of type (G_2, P) obtained via reduction as explained in Proposition 7.3.2. Then ω is regular and there is a normal Cartan connection $\omega_N \in \Omega^1(\mathcal{G}, \mathfrak{g})$ such that the difference $(\omega_N - \omega)$ is of homogeneous degree ≥ 3.*

For the proof we refer to [**Sag08**] or [**HS09**]. Having Propositions 7.3.2 and 7.3.3 we can now show:

Theorem 7.3.4. *Let $(M, [g])$ be a conformal structure of signature $(2, 3)$ with conformal holonomy $\mathrm{Hol}([g]) \subset G_2$. Then $[g]$ is canonically associated to a generic rank two distribution $\mathbf{D}$ via a generalized Fefferman construction.*

PROOF. Let $(\tilde{\mathcal{G}}, \tilde{\omega})$ be the normal parabolic geometry of type $(\mathrm{SO}(h), \tilde{P})$ associated to the conformal structure $[g]$. Let $(\mathcal{G}, \omega)$ be the Cartan geometry of type (G_2, P) constructed in Proposition 7.3.2. Then we know by Proposition 7.3.3 that ω is regular and that there is a normal Cartan connection $\omega_N \in \Omega^1(\mathcal{G}, \mathfrak{g})$ that differs from ω only in homogeneous degree ≥ 3.

Recall that the Cartan connection ω determines an isomorphism $\mathcal{G} \times_P \mathfrak{g}/\mathfrak{p} \cong TM$. Regularity of ω implies that the image of $\mathcal{G} \times_P \mathfrak{g}^{-1}/\mathfrak{p}$ under this isomorphism is a generic rank two distribution $\mathbf{D}$. Furthermore, we have a P-invariant conformal class of bilinear forms of signature $(2,3)$ on $\mathfrak{g}/\mathfrak{p}$, and the conformal structure induced via the above isomorphism on M is just $[g]$. On the other hand, the Fefferman construction associates a conformal structure $[g]_{\mathbf{D}}$ to the distribution $\mathbf{D}$. This is the conformal structure induced via the isomorphism $\mathcal{G} \times_P \mathfrak{g}/\mathfrak{p} \cong TM$ defined by the normal Cartan connection $\omega_N \in \Omega^1(\mathcal{G}, \mathfrak{g})$ associated to the distribution $\mathbf{D}$. Since $\omega_N - \omega$ is of homogeneous degree ≥ 3, the difference $(\omega - \omega_N)$ takes values in $\mathfrak{p}$. But this implies that ω and ω_N induce the same isomorphism $TM \cong \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$ and hence the same conformal structure on M; i.e., the conformal structure $[g]$ is the one induced by the distribution $\mathbf{D}$: $[g] = [g]_{\mathbf{D}}$. □

7.3.2. Characterization via the tractor 3-form. We have seen that conformal structures associated to generic rank two distributions in dimension five precisely correspond to reductions in conformal holonomy from $\mathrm{SO}(h)$ to G_2. As the next step towards the desired characterization results we show that such a holonomy reduction can be encoded in terms of a parallel tractor 3-form satisfying a certain compatibility condition with the tractor metric.

The group $G_2 \subset \mathrm{SO}(h)$ has been realized as the isotropy subgroup $\mathrm{SO}(h)_\Phi$ of $\Phi \in \Lambda^3\mathbb{R}^7$ given by (158). Let $u \in \tilde{\mathcal{G}}' = \tilde{\mathcal{G}} \times_{\tilde{P}} \mathrm{SO}(h)$ be an arbitrary point in the extended $\mathrm{SO}(h)$-principal bundle over M. Recall that we have extended the Cartan connection form equivariantly to an $\mathrm{SO}(h)$-principal connection form $\tilde{\omega}' \in \Omega^1(\tilde{\mathcal{G}}', \mathfrak{so}(h))$. We consider the conformal holonomy $\mathrm{Hol}_u = \mathrm{Hol}_u(\tilde{\omega}')$ of $[g]$. For $g \in \mathrm{SO}(h)$ one has $\mathrm{Hol}_{u \cdot g} = g \, \mathrm{Hol}_u \, g^{-1}$, and of course $\mathrm{Hol}([g])$ really is the $\mathrm{SO}(h)$-conjugacy class of Hol_u.

Let $\mathcal{H}_u \hookrightarrow \tilde{\mathcal{G}}'$, $u \in \tilde{\mathcal{G}}'$, be the reduction of the $\mathrm{SO}(h)$-bundle $\tilde{\mathcal{G}}'$ to Hol_u. If $\mathbf{\Psi} \in \Gamma(\Lambda^3\mathbf{S})$ is parallel it corresponds to a $\mathrm{SO}(h)$-equivariant function $f : \tilde{\mathcal{G}}' \to \Lambda^3\mathbb{R}^7$, which is a constant $\Psi_u \in \Lambda^3\mathbb{R}^7$ on $\mathcal{H}_u$. In particular $\mathrm{Hol}_u \cdot \Psi_u = \Psi_u$, or $\mathrm{Hol}_u \subset \mathrm{SO}(h)_{\Psi_u}$. If u' is another point in $\tilde{\mathcal{G}}'$ one has $\mathrm{Hol}_{u'} = g \, \mathrm{Hol}_u \, g^{-1}$ for some $g \in \mathrm{SO}(h)$ and $\Psi_{u'} = g \cdot \Psi_u$. Thus $f(\tilde{\mathcal{G}}') = \mathrm{SO}(h) \cdot \Psi_u$. We say that $\mathrm{SO}(h)_{\Psi_u}$ is the *orbit type* of the parallel tractor $\mathbf{\Psi}$. To be precise, the orbit type is of course only given up conjugacy under $\mathrm{SO}(h)$.

Now compatibility condition (155) singles out the unique $\mathrm{SO}(h)$-orbit of $\Lambda^3\mathbf{S}$ whose isotropy type is G_2. Hence $\mathrm{Hol}([g])$ reduces to G_2 if and only if there is a $\nabla^{\Lambda^3 S}$-parallel $\mathbf{\Phi} \in \Lambda^3\mathbf{S}$ satisfying the global version of (155), i.e.,

$$H(\mathbf{\Phi}) = \lambda \mathbf{h} \text{ for a } \lambda \in \mathbb{R}\backslash\{0\}, \tag{167}$$

where, for $s_1, s_2 \in \Gamma(\mathbf{S})$,

$$H(\mathbf{\Phi})(s_1, s_2) = i_{s_1}\mathbf{\Phi} \wedge i_{s_2}\mathbf{\Phi} \wedge \mathbf{\Phi}. \tag{168}$$

We remark that this formula at first only defines a section of in $S^2\mathbf{S}^*\otimes\Lambda^7\mathbf{S}^*$; However $\Lambda^7\mathbf{S}^*$ is trivial by orientability of $\mathbf{S}$, resp. since $\tilde{P}\subset \mathrm{SO}(h)$.

7.3.3. Characterization in terms of the underlying conformal Killing 2-form. We want to express compatibility condition (167) of the $\nabla^{\Lambda^3 S}$-parallel tractor $\mathbf{\Phi}\in\Gamma(\Lambda^3\mathbf{S})$ in terms of the underlying normal conformal Killing 2-form $\phi=\Pi_0(\mathbf{\Phi})\in\mathcal{E}_{[ab]}[3]$.

By Lemma 2.4.4 and (134), we have

$$\Phi=\begin{pmatrix}\rho_{a_1a_2}\\ \varphi_{a_0a_1a_2}\mid\ \mu_{a_2}\\ \phi_{a_1a_2}\end{pmatrix}= \tag{169}$$
$$=\begin{pmatrix}\begin{pmatrix}-\frac{1}{15}D^pD_p\phi_{a_1a_2}+\frac{2}{15}D^pD_{[a_1}\phi_{|p|a_2]}+\frac{1}{10}D_{[a_1}D^p\phi_{|p|a_2]}\\ +\frac{4}{5}\mathrm{P}^p{}_{[a_1}\phi_{|p|a_2]}-\frac{1}{5}J\phi_{a_1a_2}\end{pmatrix}\\ D_{[a_0}\phi_{a_1a_2]}\mid -\frac{1}{4}\boldsymbol{g}^{pq}D_p\phi_{qa_2}\\ \phi_{a_1a_2}\end{pmatrix}$$
$$\in\begin{pmatrix}\mathcal{E}_{[a_1a_2]}[1]\\ \mathcal{E}_{[a_0a_2]}[3]\mid\ \mathcal{E}_{a_2}[1]\\ \mathcal{E}_{[a_1\cdots a_2]}[3]\end{pmatrix}.$$

According to (127), the tractor connection $\nabla^{\Lambda^3 S}$ is given by

$$\nabla_c^{\Lambda^3 S}\begin{pmatrix}\rho_{a_1a_2}\\ \varphi_{a_0a_1a_2}\mid\mu_{a_2}\\ \phi_{a_1a_2}\end{pmatrix}=$$
$$=\begin{pmatrix}D_c\rho_{a_1a_2}-\mathrm{P}_c{}^p\varphi_{pa_1a_2}-2\mathrm{P}_{c[a_1}\mu_{a_2]}\\ D_c\varphi_{a_0a_1a_2}+3\boldsymbol{g}_{c[a_0}\rho_{a_1a_2]}+3\mathrm{P}_{c[a_0}\phi_{a_1a_2]}\mid D_c\mu_{a_2}-\mathrm{P}_c{}^p\phi_{pa_2}+\rho_{ca_2}\\ D_c\phi_{a_1a_2}-\varphi_{ca_1\cdots a_2}+2\boldsymbol{g}_{c[a_1}\mu_{a_2]}\end{pmatrix}.$$

Thus $\mathbf{\Phi}\in\Gamma(\Lambda^3\mathbf{S})$ being $\nabla^{\Lambda^3 S}$-parallel is equivalent to the following 4 equations: The equation in the lowest slot just says that $\phi_{a_1a_2}\in\mathcal{E}_{[a_1a_2]}[3]$ is a conformal Killing 2-form:

$$D_c\phi_{a_1a_2}-D_{[a_0}\phi_{a_1a_2]}-\frac{1}{2}g_{c[a_1}g^{pq}D_{|p}\phi_{q|a_2]}=0.$$

The additional 3-equations, for which we don't write out φ,μ and ρ as given by (169), are then

$$\begin{aligned} D_c\rho_{a_1a_2}-\mathrm{P}_c{}^p\varphi_{pa_1a_2}-2\mathrm{P}_{c[a_1}\mu_{a_2]}&=0\\ D_c\varphi_{a_0a_1a_2}+3\boldsymbol{g}_{c[a_0}\rho_{a_1a_2]}+3\mathrm{P}_{c[a_0}\phi_{a_1a_2]}&=0\\ D_c\mu_{a_2}-\mathrm{P}_c{}^p\phi_{pa_2}+\rho_{ca_2}&=0.\end{aligned} \tag{170}$$

We now consider the map (168). As a $\mathrm{SO}(h)$-representation $S^2\mathbb{R}^{7*}$ decomposes into the irreducible components $S^2_0\mathbb{R}^{7*}$ of trace-free symmetric 2-forms and the space $\mathbb{R}h$ of multiples of h. The corresponding decomposition on the tractor level is

$$S^2\mathbf{S}^*=S^2_0\mathbf{S}^*\oplus\mathbb{R}\mathbf{h}. \tag{171}$$

Accordingly $H(\mathbf{\Phi})$ decomposes into $H(\mathbf{\Phi})_0$ and $H(\mathbf{\Phi})_{tr}$. Compatibility condition (167) then means that $H(\mathbf{\Phi})_0=0$ and $H(\mathbf{\Phi})_{tr}\neq 0$.

LEMMA 7.3.5. *One has* $H(\mathbf{\Phi})_0 = 0$ *if and only if*

$$\phi \wedge \phi \wedge \mu = 0. \tag{172}$$

PROOF. $S^2_0\mathbf{S}^*$ is the tractor bundle associated to the irreducible representation of SO(h) on $S^2_0\mathbb{R}^{7*}$. By assumption $\mathbf{\Phi}$ is $\nabla^{\Lambda^3 S}$-parallel; the mapping $\mathbf{\Phi} \mapsto H(\Phi) \mapsto H(\Phi)_0$ is algebraic, and thus naturality of the tractor connection implies that $H(\mathbf{\Phi})_0$ is $\nabla^{S^2_0 S^*}$-parallel. By Lemma 2.4.4, the section $H(\mathbf{\Phi})_0 \in \Gamma(S^2_0\mathbf{S}^*)$ can thus be recovered via the BGG-splitting operator $L_0^{S^2_0 S^*}$ from its projection to $\mathcal{H}_0(S^2_0\mathbf{S}^*) = \mathcal{E}[2]$. This projection is achieved by inserting twice the top slot τ_+ into $H(\mathbf{\Phi})_0$, and since $\mathbf{h}(\tau_+, \tau_+) = 0$ this is the same as evaluating $H(\mathbf{\Phi})(\tau_+, \tau_+)$. Now according to (168),

$$H(\mathbf{\Phi})(\tau_+, \tau_+)\mathbf{vol} = (i_{\tau_+}(\mathbf{\Phi})) \wedge (i_{\tau_+}(\mathbf{\Phi})) \wedge \mathbf{\Phi}.$$

Here $\mathbf{vol} \in \Lambda^7\mathbf{S}$ is the canonical volume-form. We use the representation (123) with respect to a metric $g \in [g]$ and calculate

$$\begin{aligned}
&H(\mathbf{\Phi})(\tau_+, \tau_+)\mathbf{vol} = \\
&= (\phi - \tau_+ \wedge \mu) \wedge (\phi - \tau_+ \wedge \mu) \wedge (\tau_- \wedge \phi + \varphi + \tau_+ \wedge \tau_- \wedge \mu + \tau_+ \wedge \rho) = \\
&= \phi \wedge \phi \wedge \tau_+ \wedge \tau_- \wedge \mu - \tau_+ \wedge \mu \wedge \phi \wedge \tau_- \wedge \phi - \phi \wedge \tau_+ \wedge \mu \wedge \tau_- \wedge \varphi = \\
&= 3\tau_+ \wedge \tau_- \wedge \phi \wedge \phi \wedge \mu.
\end{aligned}$$

This vanishes if and only if $\phi \wedge \phi \wedge \mu = 0$. □

REMARK 7.3.6. It follows from the proof of Lemma 7.3.5 that $\phi \wedge \phi \wedge \mu \in \mathcal{E}_{[a_1 \cdots a_5]}[7]$ is conformally invariant. To see this directly observe that with a change of metric $\hat{g} = e^{2f}g$, $\hat{\mu}_{a_2} = \mu_{a_2} - \Upsilon^p \varphi_{pa_2}$ according to the transformation rules (126). But

$$0 = \phi \wedge \phi \wedge \phi = i_{\Upsilon^p}\big(\phi \wedge \phi \wedge \phi\big) = 3\phi \wedge \phi \wedge (i_{\Upsilon^p}\phi).$$

Assume now that $H(\mathbf{\Phi})_0$ vanishes, i.e. $H(\mathbf{\Phi}) = H(\mathbf{\Phi})_{tr} = \lambda\mathbf{h}$, and since $0 = \nabla^{S^2 S^*_0}(\lambda h) = (d\lambda)h$ we have that $\lambda \in \mathbb{R}$ is a constant.

LEMMA 7.3.7. *If* $H(\mathbf{\Phi})_0 = 0$*, one has* $H(\mathbf{\Phi}) = \lambda\mathbf{h}$ *for a constant* $\lambda \in \mathbb{R}$. $\lambda \neq 0$ *if and only if*

$$\phi \wedge \mu \wedge \rho \neq 0. \tag{173}$$

PROOF. We check that $\lambda \neq 0$ by inserting τ_+, τ_- since $H(\mathbf{\Phi})(\tau_+, \tau_-)\mathbf{vol} = \lambda\mathbf{h}(\tau_+, \tau_-)\mathbf{vol} = \lambda\mathbf{vol}$:

$$\begin{aligned}
&H(\mathbf{\Phi})(\tau_+, \tau_-)\mathbf{vol} = (i_{\tau_+}\mathbf{\Phi}) \wedge (i_{\tau_-}\mathbf{\Phi}) \wedge \mathbf{\Phi} = \\
&= (\phi - \tau_+ \wedge \mu) \wedge (\tau_- \wedge \mu + \rho) \wedge (\tau_- \wedge \phi + \varphi + \tau_+ \wedge \tau_- \wedge \mu + \tau_+ \wedge \rho) = \\
&= \phi \wedge \tau_- \wedge \mu \wedge \tau_+ \wedge \rho + \phi \wedge \rho \wedge \tau_+ \wedge \tau_- \wedge \mu - \tau_+ \wedge \mu \wedge \rho \wedge \tau_- \wedge \phi = \\
&= 3\tau_+ \wedge \tau_- \wedge \phi \wedge \mu \wedge \rho.
\end{aligned}$$

Thus $\lambda \neq 0$ if and only if $\phi \wedge \mu \wedge \rho \neq 0$. Note that this fixes the constant λ and $\phi \wedge \mu \wedge \rho$ either vanishes globally or nowhere. □

We are now ready to prove Theorem 7.3.8:

Theorem 7.3.8. *Let $[g]$ be a conformal class of signature $(2,3)$ metrics on M. Then $[g]$ is induced from a generic rank 2 distribution $\mathbf{D} \subset TM$ if and only if there exists a normal conformal Killing 2-form $\phi \in \mathcal{E}_{[ab]}[3]$ that is locally decomposable and satisfies the following genericity condition: for*

$$\begin{aligned}\mu_{a_2} &= \boldsymbol{g}^{pq} D_p \phi_{q a_2} \in \mathcal{E}_a[1],\\ \rho_{a_1 a_2} &= -\frac{1}{15} D^p D_p \phi_{a_1 a_2} + \frac{2}{15} D^p D_{[a_1} \phi_{|p|a_2]} + \frac{1}{10} D_{[a_1} D^p \phi_{|p|a_2]}\\ &\quad + \frac{4}{5} \mathrm{P}^p{}_{[a_1} \phi_{|p|a_2]} - \frac{1}{5} J \phi_{a_1 a_2} \in \mathcal{E}_{[ab]}[1]\end{aligned}$$

one must have

$$0 \neq \phi \wedge \mu \wedge \rho \in \mathcal{E}_{[a_1 \cdots a_5]}[5].$$

Proof. Normal conformal Killing 2-forms $\phi_{ab} \in \mathcal{E}_{[ab]}[3]$ correspond to $\nabla^{\Lambda^3 S}$-parallel sections of $\Lambda^3 \mathbf{S}$ via (169). The explicit conditions for a conformal Killing 2-form $\phi_{ab} \in \mathcal{E}_{[ab]}[3]$ to be normal are (170).

Now it was shown in 7.3.2 that a parallel $\mathbf{\Phi} \in \Gamma(\Lambda^3 \mathbf{S})$ yields a reduction to a parabolic geometry of type (G_2, P) if and only if $\mathbf{\Phi}$ satisfies compatibility condition (167). In fact, (167) is equivalent to the orbit type of the parallel tractor $\mathbf{\Phi} \in \Gamma(\Lambda^3 \mathbf{S})$ being G_2. Lemmata 7.3.7 and 7.3.5 above yield that (167) is equivalent to conditions (172) and (173) on the normal conformal Killing 2-form $\phi = \Pi_0(\mathbf{\Phi})$, resp. on its differential splitting components as given by (169). The orbit type of $\mathbf{\Phi}$ being G_2 actually implies locally decomposability of the underlying 2-form ϕ, as was shown Proposition 7.2.3, which then implies that already $\phi \wedge \phi = 0$.

We thus see that (172)+(173) is in fact equivalent to local decomposability of ϕ together with condition (173). □

Remark 7.3.9. It is a well known consequence of the classical Plücker relations (cf. [**EM00**]) that a two form ϕ is locally decomposable if and only if $\phi \wedge \phi$ vanishes globally. ◊

Remark 7.3.10. Throughout this chapter we have assumed orientability of TM. This was however only a minor point so far: If we remove this assumption and denote by $\mathcal{O}$ the 2-fold covering of M which is the orientation-bundle, we would obtain a *twisted* normal conformal Killing 2-form $\varphi \in \Lambda^2 T^*M \otimes \mathcal{E}[3] \otimes \mathcal{O}$. The real gain in orientability is that one has in fact a canonical spin structure in that case: ◊

7.3.4. Characterization via twistor-spinors. Since we work with G_2 connected, $G_2 \hookrightarrow \mathrm{SO}_0(h)$. In fact one has that this embedding lifts to $G_2 \hookrightarrow \mathrm{Spin}(h) = \mathrm{Spin}(3,4)$. It is shown in [**Kat99**] that $G_2 \subset \mathrm{Spin}(h)$ can be realized as the isotropy group of an arbitrary non-isotropic element of the spin representation $\Delta^{3,4}$.

Let now $(M,[g])$ be a conformal spin manifold of signature $(2,3)$. As discussed in 6.3, $(M,[g])$ corresponds to a Cartan geometry of type $(\mathrm{Spin}(3,4), \hat{P})$ with $\hat{P} \subset \mathrm{Spin}(3,4)$ the preimage of the conformal parabolic $\tilde{P} \subset \mathrm{SO}_0(3,4)$ under the double-covering $\mathrm{Spin}(3,4) \to \mathrm{SO}_0(3,4)$. Let $\mathbf{\Delta}$ be the spin bundle of $(M,[g])$.

Now let $\mathbf{\Sigma} = \hat{\mathcal{G}} \times_{\hat{P}} \Delta^{3,4} = \mathcal{G} \times_P \Delta^{3,4}$ be the spin tractor bundle introduced in 6.3. We showed there that with respect to any $g \in [g]$

$$[\mathbf{\Sigma}]_g = \begin{pmatrix} \mathbf{\Delta}[-\frac{1}{2}] \\ \mathbf{\Delta}[\frac{1}{2}] \end{pmatrix}.$$

Furthermore, parallel sections of $\mathbf{\Sigma}$ were shown to be in $1:1$-correspondence with twistor spinors $\chi \in \Gamma(\mathbf{\Delta}[\frac{1}{2}])$ via the projection $\Pi_0 : \mathbf{\Sigma} \to \mathbf{\Delta}[\frac{1}{2}]$ and the splitting operator

$$L : \Gamma(\mathbf{\Delta}[\frac{1}{2}]) \to \Gamma(\mathbf{\Sigma}), \tag{174}$$

$$\chi \mapsto \begin{pmatrix} \frac{\sqrt{2}}{n} \not{D} \chi \\ \chi \end{pmatrix} \tag{175}$$

with $\not{D} : \Gamma(\mathbf{\Delta}[\frac{1}{2}]) \to \Gamma(\mathbf{\Delta}[-\frac{1}{2}])$ the Dirac-operator.

THEOREM 7.3.11. *Let $(M, [g])$ be a conformal spin structure of signature $(2,3)$ and $\mathbf{\Delta}$ its (complex 4-dimensional) spin bundle, which is endowed with a pseudo-hermitian inner product $\langle \cdot, \cdot \rangle$ of signature $(2,2)$. Then $[g]$ is induced from a generic rank 2-distribution $\mathbf{D} \subset TM$ if and only if there exists a twistor spinor $\chi \in \Gamma(\mathbf{\Delta}[\frac{1}{2}])$ with nowhere vanishing imaginary part $\mathfrak{Im}(\langle \chi, \not{D}\chi \rangle)$.*

PROOF. We only give a sketch of the proof, which employs similar arguments to the characterization with normal conformal Killing 2-forms:

First, let $\mathbf{D}$ be an oriented generic rank 2-distribution and $(\mathcal{G}, \omega)$ the corresponding regular normal Cartan geometry of type (G_2, P). We regard G_2 as an embedded subgroup of $\mathrm{Spin}(3,4)$, realized as the stabilizer of an element $\mathrm{X} \in \Delta^{3,4}$. The Fefferman construction $\tilde{\mathcal{G}} := \mathcal{G} \times_P \mathrm{Spin}(3,4)$ is then seen to yield a normal Cartan geometry of type $(\mathrm{Spin}(3,4), \hat{P})$, which corresponds to a conformal spin structure of signature $(2,3)$. Now, as in the proof of Proposition 7.2.3, $\mathrm{X} \in \Delta^{3,4}$ determines a parallel spin-tractor $\mathbf{X} \in \Gamma(\mathbf{\Sigma})$, where $\mathbf{\Sigma} = \mathcal{G} \times_P \Delta^{3,4} = \hat{\mathcal{G}} \times_{\hat{P}} \Delta^{3,4}$ is the spin-tractor bundle. We have seen in 6.3 that then $\mathbf{X}$ projects to a twistor-spinor $\chi \in \Gamma(\mathbf{\Delta}[\frac{\mathbf{1}}{\mathbf{2}}]$).

Conversely, let $\chi \in \Gamma(\mathbf{\Delta}[\frac{1}{2}])$ be a twistor-spinor for a conformal spin structure of signature $(2,3)$. Let $\mathbf{X} := L(\chi) \in \Gamma(\mathbf{\Sigma})$ and recall the decomposition (148) of the pseudo-hermitian metric $\mathbf{k}^{3,4}$ on $\mathbf{\Sigma}$: with $\langle \cdot, \cdot \rangle$ the pseudo-hermitian metric on $\mathbf{\Delta}$ one has, for $v, v' \in \Gamma(\mathbf{\Delta}[\frac{1}{2}])$ and $w, w' \in \Gamma(\mathbf{\Delta}[\frac{1}{2}])$

$$\mathbf{k}_{3,4}(\begin{pmatrix} v \\ w \end{pmatrix}, \begin{pmatrix} v' \\ w' \end{pmatrix}) = i\big(\langle v, w' \rangle - \langle w, v' \rangle\big).$$

Thus $\mathbf{k}^{3,4}(\mathbf{X}, \mathbf{X})$ is a multiple of the imaginary part of $\langle \chi, \not{D}\chi \rangle$, which is nonzero by assumption on χ. Since χ is a twistor-spinor, $L(\chi)$ is ∇^{Σ}-parallel (cf. Proposition 6.3.1). Thus the holonomy of the extended conformal tractor bundle $\tilde{\mathcal{G}}' := \tilde{\mathcal{G}} \times_{\hat{P}} \mathrm{Spin}(3,4)$ is contained in the stabilizer of a non-isotropic element in $\Delta^{3,4}$ and is therefore contained in G_2. The rest is analogous to the proof of Theorem 7.3.4. □

REMARK 7.3.12. $\Delta^{3,4}$ and $\Delta^{2,3}$ admit real structures $\Delta^{3,4}_{\mathbb{R}}$ resp. $\Delta^{2,3}_{\mathbb{R}}$. $\Delta^{3,4}_{\mathbb{R}}$ is endowed with an invariant symmetric $(4,4)$-form, which can be expressed in terms of an invariant non-degenerate skew-form on $\Delta^{2,3}_{\mathbb{R}}$. Then, with $\Delta_{\mathbb{R}}$ the real spin bundle of dimension 4 over M and ω the skew-form on this bundle, one can also show that a conformal spin structure is induced by a generic distribution iff if there is a real twistor spinor $\chi \in \Gamma(\mathbf{\Delta}_{\mathbb{R}})[\frac{1}{2}]$ with $\omega(\chi, \not{D}\chi) \neq 0$. ◊

7.4. Decomposition of conformal Killing fields of $[g]_{\mathbf{D}}$

The goal of this section is to prove Theorem 7.4.3: We will show that every conformal Killing field of $[g]_{\mathbf{D}}$ decomposes into a symmetry of the distribution $\mathbf{D}$ and an almost Einstein scale. The space of almost Einstein scales $\mathbf{aEs}([g])$ was defined in (117) of section 6.1.3. Now we discuss symmetries:

Since $\mathbf{D}$ and $[g]$ are equivalently described by Cartan geometries $(\mathcal{G}, \omega)$ resp. $(\tilde{\mathcal{G}}, \tilde{\omega})$ we can determine their symmetry algebras by determining the symmetries of their corresponding Cartan geometries - in fact one can define them in this way. For this purpose, recall the general description [**Čap08**] of the Lie algebra of infinitesimal automorphisms of a parabolic geometry presented in 4.3.

Before this Cartan geometric description, let us discuss the classical notions: an infinitesimal automorphism or symmetry of the distribution $\mathbf{D} \subset TM$ is a vector field on M whose Lie derivative preserves $\mathbf{D}$, i.e.,

$$\mathbf{sym}(\mathbf{D}) = \{\xi \in \mathfrak{X}(M) \ : \ \mathcal{L}_\xi \eta = [\xi, \eta] \subset \Gamma(\mathbf{D}) \ \forall \eta \in \Gamma(\mathbf{D})\}. \tag{176}$$

Remark 7.4.1. In this text we won't show directly that the symmetries of the distribution $\mathbf{sym}(\mathbf{D})$ defined via (176) agree with the infinitesimal automorphisms $\mathbf{inf.aut}.(\omega)$ of the corresponding Cartan geometry. We just use the fact that associating a regular normal parabolic geometry of type (G_2, P) to a generic rank 2-distribution $\mathbf{D}$ is an equivalence of categories. The explicit form of the splitting from vector fields on M into the adjoint tractor bundle relating the classical and the Cartan- viewpoint is only needed in the conformal case, which was treated in 6.1.4. ◊

7.4.1. Decomposition of the conformal adjoint tractor bundle. We will use the description [**Čap08**] laid out in 4.3 of the symmetry Lie algebra of a parabolic geometry $(\mathcal{G}, \omega)$ of type (G, P). This will be applied for $(\mathcal{G}, \omega)$ the geometry of type (G_2, P) describing the generic rank 2 distribution $\mathbf{D}$ and for the conformal geometry encoded in the Cartan geometry $(\tilde{\mathcal{G}}, \tilde{\omega})$ of type $(\mathrm{SO}(h), \tilde{P})$.

Let now $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$ be the adjoint tractor bundle of the generic distribution $\mathbf{D}$ and let $\tilde{\mathcal{A}}M := \tilde{\mathcal{G}} \times_{\tilde{P}} \mathfrak{so}(h)$ be the conformal adjoint tractor bundle. The tractor connection on $\mathcal{A}M$ will be denoted by $\nabla^{\mathcal{A}}$ and the one on $\tilde{\mathcal{A}}M$ by $\nabla^{\tilde{\mathcal{A}}}$.

Recall from section 7.1.2 that as a G_2-module,

$$\mathfrak{so}(h) = \mathbb{R}^7 \oplus \mathfrak{g}, \tag{177}$$

I.e., $\mathfrak{so}(h)$ decomposes into the direct sum of the standard representation of $G_2 \subset \mathrm{SO}(h)$ on $\mathbb{R}^7$ and the adjoint representation $\mathrm{Ad} : G_2 \to \mathrm{GL}(\mathfrak{g})$. This decomposition was realized by the exact sequence (162) of section 7.1.2 and its splitting (164).

On the level of associated bundles this yields a decomposition of the conformal adjoint tractor bundle:

$$\tilde{\mathcal{A}}M = \tilde{\mathcal{G}} \times_{\tilde{P}} \mathfrak{so}(h) = \mathcal{G} \times_P \mathfrak{so}(h) = (\mathcal{G} \times_P \mathbb{R}^7) \oplus (\mathcal{G} \times_P \mathfrak{g}) = \mathbf{S} \oplus \mathcal{A}M. \tag{178}$$

I.e., the conformal adjoint tractor bundle $\tilde{\mathcal{A}}M$ is the direct sum of the standard tractor bundle $\mathbf{S}$ and the adjoint tractor bundle $\mathcal{A}M$ of the generic distribution.

Let us check that we can also decompose the tractor connection

$$\nabla^{\tilde{\mathcal{A}}} = \nabla^{\mathbf{S}} \oplus \nabla^{\mathcal{A}} : \tag{179}$$

Take a vector field $\xi \in \mathfrak{X}(M)$ and its horizontal lift ξ' to a vector field on the extended bundle $\mathcal{G}' = \mathcal{G} \times_P G_2$. Then, for $s \in \Gamma(\tilde{\mathcal{A}}M)$, the tractor derivative $\nabla_\xi s$ is defined by differentiating the G_2-equivariant function $f : \mathcal{G} \to \mathfrak{so}(h)$ corresponding to s in direction ξ'. But evidently taking this derivative commutes with the algebraic projections of f to its components $f_{\mathbf{S}} : \mathcal{G} \to \mathbb{R}^7$ and $f_{\mathcal{A}M} : \mathcal{G} \to \mathfrak{g}$; thus (179) holds.

Theorem 7.4.2. *Let $s \in \Gamma(\tilde{\mathcal{A}}M)$. s splits into $s_1 \in \Gamma(\mathbf{S})$ and $s_2 \in \Gamma(\mathcal{A}M)$ via the decomposition* (178). *Then s is parallel with respect to $\tilde{\nabla}^{\tilde{\mathcal{A}}} = \nabla^{\tilde{\mathcal{A}}} + i\tilde{K}$ if and only if s_1 is ∇^S-parallel and s_2 is $\tilde{\nabla}^{\mathcal{A}} = \nabla^{\mathcal{A}} + iK$-parallel.*

Proof. Let $\Omega \in \Omega^2_{\text{hor}}(\mathcal{G}, \mathfrak{g})^P$ be the curvature of $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, which is given by $\Omega(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$. We have the inclusion

$$j : \mathcal{G} \to \tilde{\mathcal{G}}$$

which satisfies by construction $j^*\tilde{\omega} = \omega$.

Recall that according to (166) we have $\tilde{K} = K \in \Omega^2(M, \mathcal{A}M)$. Now according to Proposition 4.3.1, we have an identification of $\mathbf{inf.aut.}(\tilde{\mathcal{G}}, \tilde{\omega})$ with parallel sections of $\tilde{\mathcal{A}}M$ with respect to $\tilde{\nabla}^{\tilde{\mathcal{A}}} = \nabla^{\tilde{\mathcal{A}}} + iK$, with $i_{\Pi^{\mathcal{A}}(s)}K = K(\Pi^{\mathcal{A}}(s), \cdot)$ and an identification of $\mathbf{inf.aut}(\mathcal{G}, \omega)$ with parallel sections of $\mathcal{A}M$ with respect to the connection $\tilde{\nabla}^{\mathcal{A}}$.

Let $s_1 \in \Gamma(\mathbf{S})$ and $s_2 \in \Gamma(\mathcal{A}M)$ be ∇^S- resp. $\tilde{\nabla}^{\mathcal{A}}$- parallel sections. Since the restriction of $\nabla^{\tilde{\mathcal{A}}}$ to $\Gamma(\mathbf{S}) \subset \Gamma(\tilde{\mathcal{A}}M)$ is just ∇^S we have that s_1 includes as a $\nabla^{\tilde{\mathcal{A}}}$-parallel section into $\Gamma(\tilde{\mathcal{A}}M)$. But by Lemma 6.1.5, we have that $K(\Pi^{\mathcal{A}}(s_1), \cdot) = 0$, and thus also $\tilde{\nabla}^{\tilde{\mathcal{A}}} s_1 = 0$. For s_2 we have $\tilde{\nabla}^{\tilde{\mathcal{A}}} s_2 = 0$ by (166), and we see that $s_1 + s_2$ corresponds to an infinitesimal automorphism of $(\tilde{\mathcal{G}}, \tilde{\omega})$.

Conversely, we take a $s \in \Gamma(\tilde{\mathcal{A}}M)$ with $\tilde{\nabla}^{\tilde{\mathcal{A}}} s = 0$ and decompose

$$s = s_1 \oplus s_2 \in \Gamma(\mathbf{S}) \oplus \Gamma(\mathcal{A}M)$$

according to (178). Since K has values in $\mathcal{A}M$ we have that $s_1 \in \Gamma(\mathbf{S})$ is parallel with respect to the standard tractor connection ∇^S by (179). We still need to show that s_2 is parallel with respect to $\tilde{\nabla}^{\mathcal{A}} = \nabla^{\mathcal{A}} + iK$, while so far we only know that

$$\nabla^{\mathcal{A}} s_2 + K(\Pi^{\mathcal{A}}(s_1), \cdot) + K(\Pi^{\mathcal{A}}(s_2), \cdot)$$

vanishes. But since s_1 is parallel as a section of $\tilde{\mathcal{A}}M$ with respect to the usual adjoint tractor connection $\nabla^{\tilde{\mathcal{A}}}$ according to (179), we can again apply Lemma 6.1.5, which tells us that s_1 inserts trivially into the curvature $\tilde{K} = K$. Thus also $\tilde{\nabla}^{\mathcal{A}} s_2 = 0$. □

We can now translate Theorem 7.4.2 into a decomposition of conformal Killing fields:

THEOREM 7.4.3. *Let* $\mathbf{D}$ *be a generic rank* 2*-distribution on a* 5*-manifold* M *and* $[g]_{\mathbf{D}}$ *the induced conformal class of signature* $(2,3)$*-pseudo-Riemannian metrics. Let* $\phi \in \mathcal{E}_{[ab]}[3]$ *be the normal conformal Killing* 2*-form of Proposition 7.2.3. Then one has:*

Every conformal Killing field decomposes into a symmetry of the distribution $\mathbf{D}$ *and an almost Einstein scale:*

$$\mathbf{cKf}([g]) = \mathbf{sym}(\mathbf{D}) \oplus \mathbf{aEs}([g]). \tag{180}$$

The mapping which associates to an almost Einstein scale $\sigma \in \mathcal{E}[1]$ *a conformal Killing field is given by*

$$\sigma \mapsto \phi_{ap} D^p \sigma - \frac{1}{4} \sigma D^p \phi_{pa} \tag{181}$$

where D *is the Levi-Civita connection of an arbitrary metric* g *in the conformal class.*

The mapping which associates to a conformal Killing field $\xi \in \mathfrak{X}(M)$ *its almost Einstein scale part with respect to the decomposition* (180) *is given by*

$$\xi_a \mapsto \phi_{pq} (D\xi)^{pq} - \frac{1}{2} \xi^p D^q \phi_{pq} \tag{182}$$

PROOF. By Proposition 4.3.1 conformal Killing fields of $[g]$ are in 1:1-correspondence with $\tilde{\nabla}^{\tilde{\mathcal{A}}}$-parallel sections of $\tilde{\mathcal{A}}M$. By Theorem 7.4.2 above, every such section decomposes into a parallel standard tractor in $\mathcal{S} = \Gamma(\mathbf{S})$ and a $\tilde{\nabla}^{\mathcal{A}}$-parallel section of $\mathcal{A}M$. By Proposition 6.1.4 and again Proposition 4.3.1, now for $\mathcal{A}M$, this yields the decomposition (180).

It is now straightforward to make this decomposition explicit in terms of the normal conformal Killing 2-form of Theorem 7.3.8 encoding the generic distribution $\mathbf{D}$:

To map an almost Einstein scale $\sigma \in \mathcal{E}[1]$ to a conformal Killing field we use the splitting operator $L_0^{\mathcal{S}} : \mathcal{E}[1] \to \mathcal{S})$ given in (114), contract this section into the characterizing $\nabla^{\Lambda^3 S}$-parallel 3-form $\mathbf{\Phi} \in \Gamma(\Lambda^3 \mathbf{S})$ given by (169) via the tractor metric $\mathbf{h}$ and project the resulting section of $\Lambda^2 \mathbf{S} = \tilde{\mathcal{A}}M$ down to $\mathfrak{X}(M)$: This yields (181).

To project a conformal Killing field $\xi \in \mathfrak{X}(M)$ to its almost Einstein scale part we proceed similarly: we split it into $\Gamma(\Lambda^2 \mathbf{S})$ via $L_0^{\Lambda^2 S}$ of (119), contract it into $\mathbf{\Phi} \in \Gamma(\Lambda^3 \mathbf{S})$ and project the resulting standard tractor to $\mathcal{E}[2]$. This gives (182). □

REMARK 7.4.4. We remark that this theorem employs the 2-form $\phi \in \mathcal{E}_{[ab]}[3]$ with the 'correct' factor, i.e, the one corresponding to (158) constructed in Proposition 7.2.3. This depends on the tractor-version of the exact sequence (162) and its splitting (164). If ϕ is a normal conformal

Killing 2-form as described by Theorem 7.3.8 may differ from the one obtained by Proposition 7.2.3, but it can be multiplied with a scalar that is unique up to sign such that one has an analogous exact sequence on the tractor-level. Then the composition of (182) with (181) is the identity.

REMARK 7.4.5. Mapping (181) actually works more generally: in the presence of an almost Einstein scale it was shown in [**GŠ08**], Corollary 5.2, that one can associate to every conformal Killing 2-form, not only to normal ones, a conformal Killing field. $\diamondsuit$

REMARK 7.4.6. In terms of the Twistor spinor $\chi \in \Gamma(\Delta[\frac{1}{2}])$ and the skew-symmetric form ω of Remark 7.3.12 this decomposition is given as follows: An almost Einstein scale $\sigma \in \mathcal{E}[1]$ corresponds to the Killing field

$$\xi_a := \omega(\frac{2}{5}\sigma \not{D}\chi + (D\sigma)^p \gamma_p \chi, \gamma_a \chi).$$

The almost Einstein scale part of a Killing field $\xi_a \in \mathcal{E}_a[2]$ is given by

$$\omega(-\frac{4}{5}\xi^p \gamma_p \not{D}\chi + (D_{[p}\xi_{q]})\gamma^{pq}\chi, \chi).$$

We remark that $\phi = \omega(\chi, \gamma_{[a}\gamma_{b]}\chi)$. $\diamondsuit$

Bibliography

[Alt08] Jesse Alt. Fefferman constructions in conformal holonomy. Thesis, Humboldt-Universität zu Berlin, 2008.

[Arm07] Stuart Armstrong. Definite signature conformal holonomy: a complete classification. *J. Geom. Phys.*, 57(10):2024–2048, 2007.

[Bau81] Helga Baum. *Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltigkeiten*, volume 41 of *Teubner-Texte zur Mathematik [Teubner Texts in Mathematics]*. BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1981. With English, French and Russian summaries.

[Bau07] Helga Baum. The conformal analog of calabi-yau manifolds. In V. Batyrev V. Cortes, editor, *Handbook of pseudo-Riemannian Geometry and Supersymmetry (to appear)*, IRMA-Series. Publishing House of the EMS, 2007.

[BČEG06] Thomas Branson, Andreas Čap, Michael Eastwood, and A. Rod Gover. Prolongations of geometric overdetermined systems. *Int. J. Math.*, 17(6):641–664, 2006.

[BDS77] D. Burns, Jr., K. Diederich, and S. Shnider. Distinguished curves in pseudoconvex boundaries. *Duke Math. J.*, 44(2):407–431, 1977.

[BE89] Robert J. Baston and Michael G. Eastwood. *The Penrose transform*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1989. Its interaction with representation theory, Oxford Science Publications.

[BEG94] T. N. Bailey, M. G. Eastwood, and A. Rod Gover. Thomas's structure bundle for conformal, projective and related structures. *Rocky Mountain J. Math.*, 24(4):1191–1217, 1994.

[BFGK90] Helga Baum, Thomas Friedrich, Ralf Grunewald, and Ines Kath. *Twistor and Killing spinors on Riemannian manifolds*, volume 108 of *Seminarberichte [Seminar Reports]*. Humboldt Universität Sektion Mathematik, Berlin, 1990.

[Bry87] Robert L. Bryant. Metrics with exceptional holonomy. *Ann. of Math. (2)*, 126(3):525–576, 1987.

[Čap02] Andreas Čap. Parabolic geometries, CR-tractors, and the Fefferman construction. *Differential Geom. Appl.*, 17(2-3):123–138, 2002. 8th International Conference on Differential Geometry and its Applications (Opava, 2001).

[Čap05] Andreas Čap. Correspondence spaces and twistor spaces for parabolic geometries. *J. Reine Angew. Math.*, 582:143–172, 2005.

[Čap06] Andreas Čap. Two constructions with parabolic geometries. *Rend. Circ. Mat. Palermo (2) Suppl.*, (79):11–37, 2006.

[Čap08] Andreas Čap. Infinitesimal automorphisms and deformations of parabolic geometries. *J. Eur. Math. Soc. (JEMS)*, 10(2):415–437, 2008.

[Car23] Élie Cartan. Les espaces á connexion conforme. *Ann. Soc. Pol. Math*, (2):172–202, 1923.

[CD01] David M.J. Calderbank and Tammo Diemer. Differential invariants and curved Bernstein-Gelfand-Gelfand sequences. *J. Reine Angew. Math.*, 537:67–103, 2001.

[ČG00] Andreas Čap and A. Rod Gover. Tractor bundles for irreducible parabolic geometries. In *Global analysis and harmonic analysis (Marseille-Luminy, 1999)*, volume 4 of *Sémin. Congr.*, pages 129–154. Soc. Math. France, Paris, 2000.

[ČG02] Andreas Čap and A. Rod Gover. Tractor calculi for parabolic geometries. *Trans. Amer. Math. Soc.*, 354(4):1511–1548 (electronic), 2002.

[ČG06] Andreas Čap and A. Rod Gover. A holonomy characterisation of Fefferman spaces. 2006. ESI Preprint 1875.

[ČG08] Andreas Čap and A. Rod Gover. CR-tractors and the Fefferman space. *Indiana Univ. Math. J.*, 57(5):2519–2570, 2008.

[ČS03] Andreas Čap and Jan Slovák. Weyl structures for parabolic geometries. *Math. Scand.*, 93(1):53–90, 2003.

[ČS05] Andreas Čap and Vladimir Souček. Subcomplexes in curved BGG-sequences. 2005. ESI Preprint 1875.

[ČS07] Andreas Čap and Katja Sagerschnig. On Nurowski's conformal structure associated to a generic rank 2 distributions in dimension 5. *Journal of Geometry and Physics (to appear)*, 2007. math.DG/0710.2208.

[ČS09] Andreas Čap and Jan Slovák. *Parabolic Geometries I: Background and General Theory v.1*. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2009.

[ČSS01] Andreas Čap, Jan Slovák, and Vladimir Souček. Bernstein-Gelfand-Gelfand sequences. *Ann. of Math.*, 154(1):97–113, 2001.

[Eas05] Michael Eastwood. Higher symmetries of the Laplacian. *Ann. of Math. (2)*, 161(3):1645–1665, 2005.

[Eas08] Michael Eastwood. Notes on projective differential geometry. In *Symmetries and overdetermined systems of partial differential equations*, volume 144 of *IMA Vol. Math. Appl.*, pages 41–60. Springer, New York, 2008.

[EC10] Élie Cartan. Les systèmes de Pfaff, à cinq variables et les équations aux dérivées partielles du second ordre. *Ann. Sci. École Norm. Sup. (3)*, 27:109–192, 1910.

[EM00] Michael G. Eastwood and Peter W. Michor. Some remarks on the Plücker relations. In *The Proceedings of the 19th Winter School "Geometry and Physics" (Srní, 1999)*, number 63, pages 85–88, 2000.

[EM07] M. Eastwood and V.S. Matveev. Metric connections in projective differential geometry. *Symmetries and Overdetermined Systems of Partial Differential Equations*, 2007.

[Fef76] Charles L. Fefferman. Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains. *Ann. of Math. (2)*, 103(2):395–416, 1976.

[Ful97] William Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.

[GN06] A. Rod Gover and Paweł Nurowski. Obstructions to conformally Einstein metrics in n dimensions. *J. Geom. Phys.*, 56(3):450–484, 2006.

[Gov06] A. Rod Gover. Laplacian operators and Q-curvature on conformally Einstein manifolds. *Math. Ann.*, 336(2):311–334, 2006.

[Gov10] A. Rod Gover. Almost Einstein and Poincaré-Einstein manifolds in Riemannian signature. *Journal of Geometry and Physics*, 60(2):182 – 204, 2010.

[Gra87] C. Robin Graham. On Sparling's characterization of Fefferman metrics. *Amer. J. Math.*, 109(5):853–874, 1987.

[GŠ08] A. Rod Gover and Josef Šilhan. The conformal Killing equation on forms—prolongations and applications. *Differential Geom. Appl.*, 26(3):244–266, 2008.

[Ham06] Matthias Hammerl. Homogeneous Cartan Geometries. Diploma thesis, Universität Wien, 2006.
`http://othes.univie.ac.at/1022/`.

[Ham07] Matthias Hammerl. Homogeneous Cartan Geometries. *Arch. Math. (Brno)*, 43(suppl.):431–442, 2007.

[Ham08] Matthias Hammerl. Invariant prolongation of BGG-operators in conformal geometry. *Arch. Math. (Brno)*, 44(5):367–384, 2008.

[Hit01] Nigel Hitchin. Stable forms and special metrics. In *Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000)*, volume 288 of *Contemp. Math.*, pages 70–89. Amer. Math. Soc., Providence, RI, 2001.

[HS09] Matthias Hammerl and Katja Sagerschnig. Conformal structures associated to generic rank 2 distributions on 5-manifolds—characterization and Killing-field decomposition. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 5:Paper 081, 29, 2009. Available at http://www.emis.de/journals/SIGMA/Cartan.html.

[Kas68] T. Kashiwada. On conformal Killing tensor. *Natur. Sci. Rep. Ochanomizu Univ.*, 19:67–74, 1968.

[Kat99] Ines Kath. Killing spinors on pseudo-Riemannian manifilds. Habilitation thesis, Humboldt-Universität zu Berlin, 1999. http://www-irm.mathematik.huberlin.de/ kath/tex/Habil.ps.

[Kos61] Bertram Kostant. Lie algebra cohomology and the generalized Borel-Weil theorem. *Ann. of Math. (2)*, 74:329–387, 1961.

[Lei05] Felipe Leitner. Conformal Killing forms with normalisation condition. *Rend. Circ. Mat. Palermo (2) Suppl.*, (75):279–292, 2005.

[Lei06] Felipe Leitner. A remark on conformal $\mathfrak{su}(p,q)$-holonomy, 2006. math.DG/0604393.

[Lei08a] Felipe Leitner. Applications of Cartan and tractor calculus to conformal and CR-geometry. Habilitation, Universität Stuttgart, 2008. http://elib.uni-stuttgart.de/opus/volltexte/2009/3922/.

[Lei08b] Felipe Leitner. A remark on unitary conformal holonomy. In *Symmetries and overdetermined systems of partial differential equations*, volume 144 of *IMA Vol. Math. Appl.*, pages 445–460. Springer, New York, 2008.

[LN09] Thomas Leistner and Paweł Nurowski. Conformal structures with G2(2)-ambient metrics. 2009. arXiv:0904.0186.

[Mic08] Peter W. Michor. *Topics in differential geometry*, volume 93 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[Nur05] Paweł Nurowski. Differential equations and conformal structures. *J. Geom. Phys.*, 55(1):19–49, 2005.

[Nur08] Paweł Nurowski. Conformal structures with explicit ambient metrics and conformal G_2 holonomy. In *Symmetries and overdetermined systems of partial differential equations*, volume 144 of *IMA Vol. Math. Appl.*, pages 515–526. Springer, New York, 2008.

[PR87] Roger Penrose and Wolfgang Rindler. *Spinors and space-time. Vol. 1*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1987. Two-spinor calculus and relativistic fields.

[Ram05] S. Ramanan. *Global calculus*, volume 65 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2005.

[Sag06] Katja Sagerschnig. Split octonions and generic rank two distributions in dimension five. *Arch. Math. (Brno)*, 42(suppl.):329–339, 2006.

[Sag08] Katja Sagerschnig. Weyl structures for generic rank two distributions in dimension five. Thesis, University of Vienna, 2008.

[Sem03] Uwe Semmelmann. Conformal Killing forms on Riemannian manifolds. *Math. Z.*, 245(3):503–527, 2003.

[Šil] Josef Šilhan. Cohomologies of real Lie algebras. `http://bart.math.muni.cz/~silhan/lie/lac/formR.php`.

[Šil04] Josef Šilhan. A real analog of Kostant's version of the Bott-Borel-Weil theorem. *J. Lie Theory*, 14(2):481–499, 2004.

[Šil06] Josef Šilhan. *Invariant operators in conformal geometry*. PhD thesis, University of Auckland, 2006. `http://www.math.muni.cz/~silhan/ps/Thesis_UoA.ps`.

[Wes81] R. Westwick. Real trivectors of rank seven. *Linear and Multilinear Algebra*, 10(3):183–204, 1981.