**DIPLOMARBEIT**

# Homogeneous Cartan Geometries

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## Contents

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# Preface

Cartan geometry provides a uniform treatment of diverse geometric structures and in the case of parabolic geometries we even have an equivalence of categories between manifolds endowed with the respective structure and the corresponding (regular, normal) Cartan geometries.
In this text we will consider several homogeneous geometric spaces and explicitly construct the corresponding (normalized) Cartan geometries.
In Chapter 1 we recall basic facts and notions of (principal) bundles.
In Chapter 2 we discuss the geometry of homogeneous spaces, introduce homogeneous principal bundles and motivate the extension of Klein geometry to Cartan geometry.
In Chapter 3 we recall general facts of Cartan geometry, discuss in particular how reductions of structure groups can be described as reductive Cartan geometries and recall induced connections.
In Chapter 4 we discuss invariant connections on homogeneous principal bundles: homogeneous principal connections and homogeneous Cartan connections are classified, explicit formulas for the curvatures derived and applications to invariant connections given.
In Chapter 5 we treat homogeneous Riemannian spaces and derive the Levi-Civita connection in this picture.
In Chapter 6 we recall basic notions of parabolic geometries resp. their underlying structures and discuss the relation of a parabolic geometry with its induced infinitesimal flag structure in the homogeneous case.
In Chapter 7 we show how a homogeneous conformal structure on a manifold is prolonged to a parabolic geometry.
In Chapter 8 we introduce contact and CR structures and prolong a family of CR structures on $\mathrm{SU}(l+2)/\mathrm{U}(l)$ to Cartan geometries.

# 1. Differential Geometric Background on Bundles

Here we recall basic facts about principal bundles and fix some notations on the way. We mostly follow [9], where one can find more details. In the following all manifolds and all maps between them are smooth.

**Definition 1.0.1.** A *fiber bundle with standard fiber $S$* is a surjective submersion $E \stackrel{\pi_M}{\to} M$ such that for every $x \in M$ there is a neighbourhood $U$ of $x$ in $M$ and a diffeomorphism $\phi_U : \pi_M^{-1}(U) \to U \times S$ such that $\pi_M = \mathrm{pr}_U \circ \phi_U$, where $\mathrm{pr}_U : U \times S$ is the projection to $U$.
$E$ is called *total space* and $M$ is the *base space.*
$(U, \phi_U)$ is called a fiber bundle chart.

Thus all bundles we consider are locally trivial.
A morphism from a fiber bundle $E_1 \stackrel{\pi_{M_1}}{\to} M_1$ to a fiber bundle $E_2 \stackrel{\pi_{M_2}}{\to} M_2$ consists of maps $f : E_1 \to E_2$ and $\check{f} : M_1 \to M_2$ which satisfy $\pi_{M_2} \circ f = \check{f} \circ \pi_{M_1}$. One says that $f$ covers $\check{f}$. Equivalently we can say that a morphism from $E_1$ to $E_2$ is a map $f : E_1 \to E_2$ such that $\pi_{M_2} \circ f$ factorizes to a map from $M_1$ to $M_2$.
We denote the sections of a fiber bundle $E \to M$ by $\Gamma(E \to M)$. The fiber over a point $x \in M$ is written $E_x$. The space of vector fields on a manifold $M$ will be denoted by $\mathfrak{X}(M)$.
Observe that the transition function from a chart $(U_1, \phi_{U_1})$ to a chart $(U_2, \phi_{U_2})$, which is a diffeomorphism of $U_1 \cap U_2 \times S$, is of the form $(x, s) \mapsto (x, \theta(x, s))$.
An atlas of the fiber bundle $E \to M$ consists of a family of fiber bundle charts $(U_\alpha, \phi_{U_\alpha})$ such that $U_\alpha$ cover $M$.
Now we introduce the notion of *vector bundle.* Consider a fiber bundle $E \to M$ with standard fiber a vector bundle $V$. A fiber bundle atlas of $E \to M$ whose transition functions are of the form $(x, v) \mapsto (x, \theta(x)v)$, where $\theta$ is a map from $U$ into $\mathrm{GL}(V)$, is called a *vector bundle atlas.* Two vector bundle atlases are equivalent when their union is again a *vector bundle atlas.*

**Definition 1.0.2.** A *vector bundle* is a fiber bundle $E \to M$ with standard fiber a vector space $V$ together with an equivalence class of vector bundle atlases.

Every fiber $E_x$ (for $x \in M$) of a vector bundle $(E \to M, V)$ is (canonically endowed with the structure of) a vector space which is isomorphic (but not naturally so) to $V$.
A morphism from a vector bundle $(E_1 \to M_1, V_1)$ to a vector bundle $(E_2 \to M_2, V_2)$ is a morphism of fiber bundles $(f : E_1 \to E_2, \check{f} : M_1 \to M_2)$ such that for every $x \in M_1$ the map $f_{|E_{1x}}$ is linear.

## 1.1. **Principal bundles.**

Consider a fiber bundle $\mathcal{G} \to M$ whose standard fiber is a Lie group $P$. A fiber bundle atlas of $\mathcal{G}$ whose transition functions are of the form

$$(x, p) \mapsto (x, \theta(x)p)$$

for a (smooth) map $\theta : U_1 \cap U_2 \to P$ is called a *principal bundle atlas* of $(\mathcal{G} \to M, P)$. Two principal bundle atlases whose union is again a principal bundle atlas are called equivalent.

**Definition 1.1.1.** A $P$-principal bundle is a fiber bundle $(\mathcal{G} \to M, P)$ together with an equivalence class of principal bundle atlases.

On a principal bundle $(\mathcal{G} \to M, P)$ one has a natural right action of $P$. In a principal bundle chart this right action is given by $(x, p) \cdot p' = (x, pp')$. Obviously the orbits of this action are exactly the fibers of the bundle and in fact for every $u \in \mathcal{G}$ the map $p \to u \cdot p$ is an embedding of $P$ into $\mathcal{G}$.
A morphism from a $P$-principal bundle $\mathcal{G}_1 \to M_1$ to a $P$-principal bundle $\mathcal{G}_2 \to M_2$ is a morphism of fiber bundles $(f : \mathcal{G}_1 \to \mathcal{G}_2, \check{f} : M_1 \to M_2)$ such that $f$ is $P$-equivariant, i.e., for every $u \in$
$G_1$: $f(u \cdot p) = f(u) \cdot p$.
More generally we can consider morphisms from a $P_1$-principal bundle $\mathcal{G}_1 \to M_1$ to a $P_2$ principal bundle $\mathcal{G}_2 \to M_2$ when we have a homomorphism of Lie groups $\Psi : P_1 \to P_2$. Then we say that a morphism of fiber bundles $f : \mathcal{G}_1 \to \mathcal{G}_1$ is a morphism between $(\mathcal{G}_1, P_1)$ and $(\mathcal{G}_2, P_2)$ over $\Psi$ if $f(u \cdot p) = f(u) \cdot \Psi(p)$ for all $u \in \mathcal{G}_1, p \in P_1$.
Let $P' < P$ be a Lie subgroup of $P$. A *reduction* of a principal bundle $\mathcal{G} \to M$ is a principal bundle $\mathcal{G}' \to M'$ together with a morphism of fiber bundles $f : \mathcal{G}' \to \mathcal{G}$ covering the identity on $M$ such that $f$ is $P'$-equivariant.
The most useful constructions one can do with a principal bundle are

1.1.1. *Associated Bundles.* Let $(\mathcal{G} \to M, P)$ be some principal bundle and $S$ some manifold which is endowed with a left action of $P$. On $\mathcal{G} \times S$ we have a free action of $P$ by

$$(u, s) \cdot p := (u \cdot p, p^{-1} \cdot s).$$

The orbits of this action are denoted

$$[[u, s]] := \{(u \cdot p, p^{-1} \cdot s), p \in P\}$$

and regarded as equivalence classes.

**Theorem 1.1.2.** *$\mathcal{G} \times_P S$ is endowed with a unique structure of a smooth manifold such that the natural surjection*

$$\begin{aligned} \mathcal{G} \times S &\xrightarrow{q} \mathcal{G} \times_P S, \\ (u, s) &\mapsto [[u, s]] \end{aligned}$$

*is a surjective submersion.*
*In fact $(\mathcal{G} \times S \xrightarrow{q} \mathcal{G} \times_P S, P)$ is a $P$-principal bundle.*
*The natural surjection*

$$\begin{aligned} \mathcal{G} \times_P S &\to M, \\ [[u, s]] &\mapsto \pi(u) \end{aligned}$$

*makes $(\mathcal{G} \times_P S \to M, S)$ to a fiber bundle with standard fiber $S$.*
*There is a unique map*

$$\tau : \mathcal{G} \times_M (\mathcal{G} \times_P S) \to S$$

*(where $\times_M$ is the fibered product) such that for $\pi_M(u) = \pi_M(u')$*

$$[[u', \tau(u', [[u, s]])]] = [[u, s]]. \tag{1}$$

*Remark* 1.1.3. Since

$$[[up, \tau(up, [[u, s]])]] = [[u, s]]$$

by (1) and

$$[[up, p^{-1} \cdot \tau(u, [[u, s]])]] = [[u, \tau(u, [[u, s]])]]$$

by definition of the equivalence relation on $\mathcal{G} \times P$ we see that $\tau$ satisfies

$$\tau(up, [[u, s]]) = p^{-1} \cdot \tau(u, [[u, s]]) \tag{2}$$

by uniqueness. ⌟

*Remark* 1.1.4. When we have a representation $\Psi : P \to \mathrm{GL}(V)$ of $P$ on a vector space $V$ the associated bundle $\mathcal{G} \times_P V \to M$ is a vector bundle with modelling vector space $V$.
When $\Psi : P \to P'$ is a homomorphism of Lie groups the associated bundle $\mathcal{G} \times_P P' \to M$ is a $P'$-principal bundle over $M$, where the $P$-principal action is given by

$$[[u, p]] \cdot p' = [[u, pp']].$$

⌟

1.1.2. *Factorizing equivariant maps and forms.* Later we will often use the following relations between functions and forms on the total- resp. base-space of a principal bundle.
Here $(\mathcal{G} \to M, P)$ is an arbitrary principal fiber bundle.

**Theorem 1.1.5.** *Let $S$ be some manifold endowed with a left action of $P$. There is a 1:1-correspondence between $P$-equivariant maps $f : \mathcal{G} \to S$ and sections of $\mathcal{G} \times_P S \to M$.*
*Denote the set of all $P$-equivariant maps from $\mathcal{G}$ to $S$ by $\mathrm{C}^\infty_P(\mathcal{G}, S)$. Then the bijection is: for a section*

$$s : M \to \mathcal{G} \times_P S \text{ with } \pi \circ s = \mathrm{id}_M;$$

*the corresponding equivariant function $f : \mathcal{G} \to S$ is given by*

$$u \mapsto \tau(u, s(\pi(u))).$$

*Proof.* First consider a section $s \in \Gamma(\mathcal{G} \times_P S)$. We need to show that

$$u \mapsto \tau(u, s(\pi(u)))$$

is $P$-equivariant. So take $u \in \mathcal{G}, p \in P$. Then

$$\tau(up, s(\pi(up))) = \tau(up, s(\pi(u))) = p^{-1} \cdot \tau(u, s(\pi(u)))$$

by (2).
Now we have to show how an equivariant $f : \mathcal{G} \to S$ conversely determines a section $s : M \to \mathcal{G} \times_P S$. But by equivariancy of $f$ the map

$$\begin{aligned} \tilde{s} : \mathcal{G} &\to \mathcal{G} \times_P S, \\ u &\mapsto [[u, f(u)]] \end{aligned}$$

is constant on the fibers of $\mathcal{G}$. Thus it factorizes to a section $M \to \mathcal{G} \times_P S$. That this section is really smooth simply follows from the fact that $\mathcal{G} \to M$ is a surjective submersion: this is equivalent to the existence of smooth local sections $\sigma : M \supset U \to \mathcal{G}$, by which one can pullback $\tilde{s} : \mathcal{G} \to \mathcal{G} \times_P S$ to $s := \tilde{s} \circ \sigma : M \to \mathcal{G} \times_P S$. Since $\tilde{s}$ is constant on the fibers of $\mathcal{G}$ we see that $s = \tilde{s} \circ \sigma$ really doesn't depend on the particular local section $\sigma : U \to \mathcal{G}$. $\square$

Now let $V$ be some finite dimensional vector space.

**Definition 1.1.6.** A $V$-valued $\ell$-form $\omega$ on $\mathcal{G}$ is called horizontal if $\omega(X_1, \ldots, X_\ell) = 0$ whenever some $X_i \in V\mathcal{G} = \{X \in T\mathcal{G} : T\pi(X) = 0\}$.

Denote by $\Omega_P(\mathcal{G}, V)^{hor}$ the set of all $P$-equivariant, horizontal, $V$-valued forms on $\mathcal{G}$ and by $\Omega(M, \mathcal{G} \times_P V)$ the set of all $\mathcal{G} \times_P V$-valued forms on $M$.

**Theorem 1.1.7.** *A $V$-valued form $\phi$ on $\mathcal{G}$ factorizes to an $\mathcal{G} \times_P V$-valued form on $M$ iff $\phi$ is horizontal and $P$-equivariant, i.e., if $\phi \in \Omega_P(\mathcal{G}, V)^{hor}$. I.e., we have an isomorphism of vector spaces between $\Omega_P(\mathcal{G}, V)^{hor}$ and $\Omega(M, \mathcal{G} \times_P V)$.*
*Explicitly, for a $\phi \in \Omega^\ell(M, \mathcal{G} \times_P V)$ we define $q^\# \phi \in \Omega_P(\mathcal{G}, V)^{hor}$ by*

$$q^\# \phi(X_1, \ldots, X_\ell) := \tau(u, \phi(T\pi_M X_1, \ldots, T\pi_M X_\ell))$$

*for $X_1, \ldots, X_\ell \in T_u\mathcal{G}$.*

## 2. Introduction to Klein and Cartan Geometries

2.1. **Homogeneous spaces.** Define $G/P := \{gP, g \in G\}$, the set of all left cosets of $P$ in $G$. $G/P$ is called a *homogeneous space* and has a unique smooth structure such that the natural surjection

$$G \overset{\pi_{G/P}}{\to} G/P$$

is a surjective submersion. (See for instance [9], Chapter II.) In fact, $G \to G/P$ is easily seen to be a $P$-principal bundle.

Additional structure on $G \to G/P$ comes from the left action: For $g \in G$ we introduce the maps

$$\lambda_g(g') := gg', \ \check{\lambda}_g(g'P) := gg'P.$$

Obviously $G$ acts thus on $G/H$ by $\check{\lambda}$ and left-multiplication is a lift of this action to an action of $G$ on itself. Also, since left and right multiplication commute, $g \in G$ acts thus by an automorphism of the $P$-principal bundle $G \to G/P$ covering $\check{\lambda}_g$. It is furthermore obvious, that this action is transitive.

Thus we found the simplest example of a homogeneous principal bundle:

**Definition 2.1.1.** Let $H$ by a Lie group and $K < H$ a closed subgroup. A *homogeneous $P$-principal bundle over $H/K$* is a $P$-principal bundle $\mathcal{G} \xrightarrow{\pi} H/K$ together with a lift of the action of $H$ on $H/K$ to an action on the principal bundle by automorphisms: we demand that for all $h \in H, u \in \mathcal{G}, p \in P$

i. $\pi(h \cdot u) = h\pi(u)$ and
ii. $h \cdot (u \cdot p) = (h \cdot u) \cdot p$.

**Definition 2.1.2.** Let $\mathcal{G}_1 \to H/K, \mathcal{G}_2 \to H/K$ be homogeneous $P$-principal bundles. A map $\Phi : \mathcal{G}_1 \to \mathcal{G}_2$ is a homomorphism of homogeneous principal bundles if for all $u \in \mathcal{G}_1$ and $p \in P$

i. $\Phi(u \cdot p) = \Phi(u) \cdot p$
ii. $\Phi(h \cdot u) = h \cdot \Phi(u)$.

2.2. **Klein Geometries or the Geometry of Homogeneous Spaces.** A pair $(G, P)$ for a closed subgroup $P < G$, is called a *Klein geometry.* In the Klein geometric picture one regards the (left-)action of $G$ on $G/P$ as automorphisms of a geometric structure, and $G$ is the full automorphism group.

**Definition 2.2.1.** A Klein geometry $(G, P)$ is called *reductive* if there is a $P$-invariant complement to $\mathfrak{p}$ in $\mathfrak{g}$,i.e., if $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{p}$ as $P$-module for some vector-space-complement of $\mathfrak{p}$ in $\mathfrak{g}$.
A Klein geometry $(G, P)$ is called *split* if there is a complement of $\mathfrak{p}$ in $\mathfrak{g}$ which is a Lie subalgebra.

The geometric study of the Klein geometry $(G, P)$ means that we find "invariants" of the $G$-action on the homogeneous space $G/P$.

We start by discussing invariant sections of appropriate (vector-) bundles. We will be able to discuss some invariant differential operators, namely connections on $T(G/P)$, later in 4.1.2 after having developed the necessary background.

2.2.1. *Homogeneous Bundles.* In this section we follow [5]. Analogously to 2.1.1 we introduce

**Definition 2.2.2.** A homogeneous fiber bundle over $G/P$ is a fiber bundle $E \overset{\pi_M}{\to} G/P$ with standard fiber $S$ together with a lift over $\pi_M$ of the action of $G$ on $G/P$ to an action on $E$.

**Definition 2.2.3.** A homogeneous vector bundle over $G/P$ is a vector bundle $E \overset{\pi_M}{\to} G/P$ with standard fiber a vector space $V$ together with a lift over $\pi_M$ of the action of $G$ on $G/P$ to an action on $E$ by vector bundle-automorphisms.

Like above for homogeneous principal bundles the morphisms of homogeneous fiber- resp. vector- bundles are morphisms of fiber- resp. vector bundles which are also $G$-equivariant.

**Theorem 2.2.4.** *Let $E \to G/P$ be a homogeneous fiber bundle with standard fiber $S$. Then there is a left action of $P$ on $S$ such that $E \to G/P$ is isomorphic to $G \times_P S \to G/P$.*

*Proof.* We give a brief sketch of the proof:

Since the restriction of the action to $P$ lets $o = P \in G/P$ invariant the fiber over $o$ is invariant as well. But $E_o$ may be identified with $S$ and one thus obtains an action of $P$ on $S$.

One has an obvious left action by $G$ on $\{[[g,s]]\}$ by $g' \cdot [[g,s]] := [[g'g,s]]$, and this is a lift of the action of $G$ on $G/P$.

Now one completes the proof by verifying that the map

$$G \times_P S \to E,$$
$$[[g,s]] \mapsto g \cdot s$$

is a $G$-equivariant diffeomorphism covering the identity. □

For the special case of homogeneous vector bundles one has

**Theorem 2.2.5.** *Let $E \to G/P$ be a homogeneous vector bundle with standard fiber $V$. Then there is a representation of $P$ on $V$ such that $(E \to G/P, V) \cong (G \times_P V \to G/P, V)$.*

Note that the general frame bundle $\mathrm{GL}^1(E)$ of $G \times_P V \to G/P$ is $G \times_P \mathrm{GL}(V) \to G/P$.
For the case of homogeneous principal bundles one has

**Theorem 2.2.6.** *Let $\mathcal{G} \to H/K$ be homogeneous $P$-principal bundle over $M = H/K$ and $u_0 \in \mathcal{G}_o$ some arbitrary point in the fiber over $o = K \in M$.*

*i. There is a unique homomorphism of Lie groups $\Psi : K \to P$ such that*

$$H \times_K P \to \mathcal{G},$$
$$[[h, p]] \mapsto h \cdot (u_0 \cdot p)$$

*is an isomorphism of homogeneous principal bundles.*

*ii. For $u_0' = u_0 \cdot p_0$ the corresponding homomorphism is*

$$\Psi' = \mathrm{conj}_{p_0{}^{-1}} \circ \Psi.$$

*iii. The isomorphism classes of homogeneous $P$-principal bundles over $H/K$ are the $N_H(K) \times P$-conjugacy-classes of $\mathrm{Hom}(K, P)$: given $\Psi_1, \Psi_2 \in \mathrm{Hom}(K, P)$, the associated homogeneous principal bundles are isomorphic iff there is an element $h_0$ in the normalisator $N_K(H)$ of $K$ in $H$ and an element $p_0$ in $P$ such that $\Psi_2 \circ \mathrm{conj}_{h_0^{-1}} = \mathrm{conj}_{p_0} \circ \Psi_1$,or equivalently, $\Psi_2 = \mathrm{conj}_{p_0} \circ \Psi_1 \circ \mathrm{conj}_{h_0}$.*

*Proof.* i. Since the action of $H$ on $H/K$ lifts to an action on $\mathcal{G}$ we see that its restriction to $K$ leaves $\mathcal{G}_o$ invariant, it commutes with the right-action of $P$. Now the map $p \mapsto u_0 \cdot p$ embeds $P$ into $\mathcal{G}$ as $\mathcal{G}_o$, in particular, every element in $\mathcal{G}_o$ my be uniquely written as $u_0 \cdot p$. Therefore the action by $K$ on $\mathcal{G}_o$ is already determined by its action on $u_o$.: We have a map $\Psi : K \to P$ such that $k \cdot u_0 = u_0 \cdot \Psi(k)$. Now, given $k \in K$ and $p \in P = \mathcal{G}_o$ we have $k \cdot (u_0 \cdot p) = (k \cdot u_0) \cdot p = (u_0 \cdot \Psi(k)) \cdot p = u_0 \cdot (\Psi(k)p)$. And it is easy to see that $\Psi$ is in fact a homomorphism of Lie groups:

$$\begin{aligned} u_0 \cdot \Psi(kk') &= (kk') \cdot u_0 = k \cdot (k' \cdot u_0) \\ &= k \cdot (u_0 \cdot Psi(k')) = (k \cdot u_0) \cdot \Psi(k') \\ &= (u_0 \cdot \Psi(k)) \cdot \Psi(k') = u_0 \cdot (\Psi(k)\Psi(k')). \end{aligned}$$

Now one describes $\mathcal{G}$ as an associated (principal) bundle of $H$: We show that $\mathcal{G} \cong H \times_\Psi P$ as homogeneous $P$-principal bundles. We already remarked in 1.1.4 that $H \times_\Psi P$ naturally carries the structure of a $P$-principal-bundle:

$$[[h, p]] \cdot p' := [[h, pp']]$$

It is also clear that it is homogeneous in the sense of 2.1.1:the lift of the action of $H$ on $H/K$ to an action on $H \times_K P$ is given by

$$h' \cdot [[h, p]] := [[h'h, p]].$$

Now the map

$$H \times_K P \to \mathcal{G},$$
$$[[h, p]] \mapsto h \cdot (u_0 \cdot p)$$

covers the identity on $H/K$ and is both $H$- and $P$-equivariant. Thus it is already an isomorphism of homogeneous principal bundles.

ii. What happens when we start with another point $u_0' = u_0 \cdot p_0 \in P$? Then

$$u_0' \cdot \Psi'(k) = k \cdot u_0' = \\ = k \cdot (u_0 \cdot p_0) = u_0 \cdot (\Psi(k)p_0) = u_0' \cdot (p_0^{-1}\Psi(k)p_0).$$

Thus $\Psi' = \mathrm{conj}_{p_0{}^{-1}} \circ \Psi$.

iii. We know that every homogeneous $P-$principal fiber bundle over $H/K$ is isomorphic to $H \times_\Psi P \to H$ for a homomorphism $\Psi : K \to P$. Given two homomorphisms $\Psi_1, \Psi_2 : K \to P$, when is there an isomorphism

$$\Phi : H \times_{\Psi_1} P \to H \times_{\Psi_2} P?$$

Take an arbitrary representative $(h_0, p_0)$ of $\Phi([[e,e]]_{\Psi_1})$. Since $\Phi$ commutes with the actions of $H$ and $P$

$$\Phi([[h,p]]_{\Psi_1}) = \Phi(h[[e,e]]_{\Psi_1}p) = h[[h_0,p_0]]_{\Psi_2}p \\ = [[hh_0, p_0p]]_{\Psi_2}.$$

Since $[[e,e]]_{\Psi_1} = [[k, \Psi_1(k^{-1})]]_{\Psi_1}$

$$[[h_0,p_0]]_{\Psi_2} = \Phi([[e,e]]_{\Psi_1}) = \\ \Phi([[k,\Psi_1(k^{-1})]]_{\Psi_1}) = [[kh_0, p_0\Psi_1(k^{-1})]]_{\Psi_2}).$$

So there is a $\tilde{k}$ such that

$$(h_0\tilde{k}, \Psi_2(\tilde{k}^{-1})p_0) = (kh_0, p_0\Psi_1(k^{-1}));$$

We see $\tilde{k} = h_0^{-1}kh_0$ and $\Psi_2(h_0^{-1}k^{-1}h_0) = p_0\Psi_1(k^{-1})p_0^{-1}$.

□

So we described an arbitrary homogeneous $P$-principal bundle as a quotient of the trivial bundle $H \times P$. We have a $K$-principal bundle whose base is a $P$-principal bundle:

$$\begin{tikzcd} H \times P \arrow[d] & K \arrow[l] \\ H \times_K P \arrow[d] & P \arrow[l] \\ H/K & \end{tikzcd}$$

2.2.2. *Invariant Sections of Homogeneous Vector Bundles.* Take some vector space $V$ and a representation of $P$ on $V$, i.e., the data defining a homogeneous vector bundle of $G/P$. Then we ask whether there are sections $s : G/P \to G \times_P V$ which are invariant under the action of $G$, i.e.:

$$s(g'gP) = g' \cdot s(gP).$$

Given such an invariant section $s$ it is obviously already completely determined by its value at $o = P \in G/P$ since then at $gP$ by invariance $s(gP) = g \cdot s(o)$. But by invariance under $P$ it is necessary that for $p \in P$

$$p \cdot s(o) = s(o). \tag{3}$$

Let $s(o)$ be $[[e, v_0]]$. Then (3) reads as

$$[[e, v_0]] = s(o) = p \cdot s(o) = [[p, v_0]] = [[e, p \cdot v_0]]$$

which is equivalent to $p \cdot v_0 = v_0$; i.e.: $v_0$ is invariant under $P$.
Since one can conversely construct a (unique) invariant section $s$ which is given by $[[e, v_0]]$ at $o$ we have shown

**Theorem 2.2.7.** *$G$-invariant sections of the vector bundle $(G \times_P V \to G/P, V)$ are in 1:1-correspondence with $P$-invariant elements of $V$.*

(One can also employ Theorem 1.1.5 to show this fact: sections of $G \times_P V \to G/P$ correspond to $P$-equivariant functions from $G$ to $V$; now invariance of a section is easily seen to be equivalent to the corresponding function to be constant, and thus the criteria that it is $P$-invariant simply means that its value is $P$-invariant.)

*Example* 2.2.8. Consider some representation $\Psi : P \to \mathrm{GL}(V)$ Then for $i, j \in \mathbb{N}_0$

$$\begin{aligned} T^i_j(G \times_P V) := & \left(\otimes^i (G \times_P V)^*\right) \otimes \left(\otimes^j (G \times_P V)\right) = \\ & = G \times_P \left(\otimes^i V^*\right) \otimes \left(\otimes^j V\right) =: G \times_P T^i_j V \end{aligned}$$

and thus $G$-invariant $(i, j)$-tensors on the vector bundle $G \times_P V \to G/P$ are in 1:1-correspondence with $P$-invariant elements of $\left(\otimes^i V^*\right) \otimes \left(\otimes^j V\right)$. ⌟

*Example* 2.2.9. Since

$$T(G/K) = G \times_P \mathfrak{g}/\mathfrak{p},$$

it follows from the previous example that invariant (pseudo-) Riemannian metrics on $G/P$ are $P$-invariant (pseudo-) inner products $\beta$ on $\mathfrak{g}/\mathfrak{p}$.
Every such $\beta$ endows $\mathfrak{g}/\mathfrak{p}$ with the structure of a euclidean space and since $\beta$ is invariant under $P$ we have in fact that $\Psi : P \to V$ has values in $O(V, \beta)$. We can thus reduce the general frame bundle of this vector bundle to the orthogonal frame bundle

$$G \times_P O(V, \beta).$$

⌟

2.3. **From Klein to Cartan.** When studying the geometry of homogeneous spaces one regards $G$ as the automorphism group of some geometric structure on $G/P$. In 2.1 we already noted that $(G \to G/P, P)$ is a $P$-principal bundle.
So far the choice of automorphism-group is a bit arbitrary and rather extrinsic since not every principal-bundle automorphism of $G \to G/P$ is a left-multiplication. We want to get intrinsic geometrical data of the $P$-principal bundle $(G \to G/P, P)$:

2.3.1. *Encoding the geometry as an explicit structure on the bundle.* What we want to do is to encode the (Klein-) geometric structure into a form on $G$: i.e.: a form $\omega$ on $G$ such that those principal-bundle automorphisms of $G \to G/P$ which preserve this form are really exactly left multiplications by elements in $G$. Once we have done this, we have thus described the (Klein-) geometric structure on $G/P$ by this form, and this description of the structure as $(G \to G/P, \omega)$ we will then generalize from homogeneous spaces $G/P$ to arbitrary spaces.

2.3.2. *The Maurer-Cartan form.* The answer to our problem is the Maurer-Cartan form $\omega^{MC}$, which is a way to write the left-trivialization:

$$\omega^{MC}(T_e\lambda_h X) := X \text{ or}$$
$$\omega^{MC}(\xi_h) = (T_e\lambda_h)^{-1}\xi_h = T_h\lambda_{h^{-1}}\xi_h.$$

So $\omega^{MC}$ is an $\mathfrak{g}$-valued 1-form on $G$, or $\omega^{MC} \in \Omega^1(G, \mathfrak{g})$. When we view it as a diffeomorphism of $TG$ with $G \times \mathfrak{h}$ it is simply left-trivialization.

**Theorem 2.3.1.** *Consider a Klein geometry $(G, P)$ with connected $G/P$. An automorphism $\Psi$ of the $P$-principal bundle $G \to G/P$ preserves $\omega^{MC}$, i.e.,*

$$\Psi^*\omega^{MC} = \omega^{MC} \tag{4}$$

*iff $\Psi$ is left-multiplication by some $g \in G$.*

*Proof.* In the following $\omega = \omega^{MC}$. First take some $g' \in G$. We show that $\lambda^*_{g'}\omega = \omega$: This is equivalent to

$$\omega_{g'g}(T_g\lambda_{g'}\xi_g) = \omega_g(\xi_g)$$

for $\xi_g \in TG_g$. By definition of $\omega$

$$\omega_{g'g}(T_g\lambda_{g'}\xi_g) = (T\lambda_{gg'})^{-1}T_g\lambda_{g'}\xi_g.$$

Since $\lambda_{g'g} = \lambda_{g'} \circ \lambda_g$ we have $\lambda^{-1}_{g'g} = \lambda^{-1}_g \circ \lambda^{-1}_{g'}$ and thus

$$(T\lambda_{g'g})^{-1}T_g\lambda_{g'} = T_g\lambda^{-1}_{\mathfrak{g}}(T_{g'g}\lambda_{g'})^{-1}T_g\lambda_{g'} = T_g\lambda^{-1}_g.$$

Thus indeed

$$\omega_{g'g}(T_g\lambda_{g'}\xi_g) = T_g\lambda^{-1}_g\xi_g = \omega_g.$$

Now conversely consider an automorphism $\psi$ of the $P$-principal bundle $G \to P$ which satisfies (4). For $X \in \mathfrak{g}$ and $g \in G$ we define $L_X(g) := T_e\lambda_g X$, i.e., $L_X$ is the unique left-invariant vector field with $L_X(e) = X$. Now (4)

$$\omega_{\Psi(g)}(T_g\Psi(T_e\lambda_g X)) = \omega_g(T_e\lambda_g X) = X,$$

reads

$$T_g\Psi L_X(g) = L_X(\Psi(g)),$$

which just says that $L_X$ is related to itself by $\Psi$. Thus it follows for the flow of $L_X$ that

$$\Psi(\mathrm{Fl}^{L_X}_t(g)) = \mathrm{Fl}^{L_X}_t(\Psi(g)). \tag{5}$$

But $\mathrm{Fl}_t^{L_X}(g) = g\exp(tX)$, and thus (5) is equivalent to

$$\Psi(g\exp(tX)) = \Psi(g)\exp(tX).$$

Every element of the identity component $G_o$ of $G$ may be written $\exp(X_1)\cdots\exp(X_k)$, and thus for $g_1 \in G, g' \in G_o$ $\Psi(g_1 g) = \Psi(g_1)g'$. Since $\Psi$ is assumed to be an automorphism of the $P$-principal bundle $G \to G/P$ it is $P$-equivariant. But that $G/P$ is connected is equivalent to $P$ intersecting every connected component of $G$, and thus every element $g$ of $G$ may be written $g = g'p'$ with some $g' \in G_o, p' \in P$. Thus $\Psi(g_1 g) = \Psi(g_1)g$, and thus one has

$$\Psi(g) = \Psi(e)g;$$

i.e., $\Psi$ is simply left-multiplication by $\Psi(e) \in G$. □

Thus the Maurer-Cartan-form solves our problem of describing the (Klein geometric) automorphism group $G$ intrinsically: The automorphisms of a Klein geometry $((G,P),\omega^{MC})$ are the principal-bundle automorphisms of $G \to G/P$ which preserve $\omega^{MC}$.
Our next aim is to generalize the Klein geometric notion of geometric structure to general, not necessarily homogeneous, manifolds. For this we want to find properties of $\omega^{MC} \in \Omega(G,\mathfrak{g})$ as strong as possible which still make sense in the general setting. Writing $M = G/P$, $\mathcal{G} = G$, $\omega = \omega^{MC}$, these properties are

i. $\omega$ is $P$-equivariant
ii. $\omega(\frac{d}{dt}_{|t=0} u\exp(tX)) = X$ for all $u \in \mathcal{G}$, $X \in \mathfrak{p}$
iii. $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ is an isomorphism for all $u \in \mathcal{G}$.

Now we use these properties to generalize Klein geometries to

### 2.4. **Cartan Geometries.**

**Definition 2.4.1.** A $P$-principal bundle $\mathcal{G} \to M$ together with a form $\omega \in \Omega^1(\mathcal{G},\mathfrak{g})$ is called a Cartan geometry of type $(G,P)$ if $\omega$ satisfies (i),(ii) and (iii). $\omega$ is called a Cartan connection.

**Definition 2.4.2.** Let $(\mathcal{G}_1 \to M_1, \omega_1)$ and $(\mathcal{G}_2 \to M_2, \omega_2)$ be Cartan geometries of type $(G,P)$. A morphism of Cartan geometries of type $(G,P)$ from $(\mathcal{G}_1 \to M_1, \omega_1)$ to $(\mathcal{G}_2 \to M_2, \omega_2)$ is a morphism of principal bundles which pulls back $\omega_2$ to $\omega_1$.

We will regard a Cartan geometry of type $(G,P)$ to be modeled on the Klein geometry $(G,P)$, and we call $(G,P)$ equipped with $\omega^{MC}$ the homogeneous model of Cartan geometries of type $(G,P)$.
If $\omega = \omega^{MC}$ is the Maurer-Cartan form on $G$ it is well known that it satisfies the Maurer-Cartan-equation

$$d\omega(\xi,\eta) + [\omega(\xi),\omega(\eta)] = 0.$$

**Definition 2.4.3.** The curvature $K \in \Omega^2(\mathcal{G},\mathfrak{g})$ of a Cartan geometry $(\mathcal{G} \to M, \omega)$ of type $(G,P)$ is the failure of $\omega$ to satisfy the Maurer-Cartan-equation:

$$K(\xi,\eta) := d\omega(\xi,\eta) + [\omega(\xi),\omega(\eta)].$$

The picture that a Cartan geometry of type $(G, P)$ is a 'curved analogon' of the Klein geometry $(G, P)$ is based on the following

**Theorem 2.4.4.** *A Cartan geometry* $(\mathcal{G} \to M, \omega)$ *is locally isomorphic (as Cartan geometry) to the homogeneous model* $(G \to G/P, \omega^{MC})$ *iff its curvature vanishes.*

(For a proof see for instance [5] or [12]).

2.5. **The general setting of this text.** There are two general geometric problems related to Cartan geometries: First: Interpret the geometric structure described by a Cartan connection. and Second: Given some geometric structure on a manifold $M$, can one prolong it (uniquely) to a Cartan geometry?
We will mostly be concerned with the problem of prolonging a given geometric structure to a Cartan geometry. This we will do for cases of *homogeneous* Cartan geometries:

**Definition 2.5.1.** Let $M = H/K$ be a homogeneous space. We define the notion of a homogeneous Cartan geometry on $H/K$: Let $(\mathcal{G} \to H/K, \omega)$ be a Cartan geometry (of some type $(G, P)$) on the homogeneous space $H/K$. It is called *homogeneous* if $H$ acts on $\mathcal{G}$ by automorphisms $\lambda_h$ of the Cartan geometry $(\mathcal{G} \to H/K, \omega)$ which cover $\check{\lambda}_h$.

In this simpler setting we will be able to explicitly describe several prolongations.

# 3. Some Background on Principal and Cartan Connections

3.1. **Principal connections.** Let $\pi_M : \mathcal{P} \to M$ be a $P$-principal bundle and denote the principal right action of an element $p \in P$ on $\mathcal{P}$ by $r^p$; i.e.: $r^p(u) = u \cdot p$ for $u \in \mathcal{P}$. The fundamental vector fields on $\mathcal{P}$ are

$$\zeta_Y(u) := \frac{d}{dt}_{|t=0} u \cdot \exp(tY)$$

for $Y \in \mathfrak{p}$.

**Definition 3.1.1.** A $\mathfrak{p}$-valued 1 form $\gamma$ on $\mathcal{P}$ is called a *principal connection* on $\mathcal{P}$ if the following two conditions hold:

i. $(r^p)^*\gamma = \mathrm{Ad}(p^{-1}) \circ \gamma$;
ii. $\gamma(\frac{d}{dt}_{|t=0} u \cdot \exp(tX)) = X$ for all $u \in \mathcal{P}$ and $X \in \mathfrak{p}$.

I.e.: $\gamma$ is $P$-equivariant and reproduces the generators of fundamental vector fields.
The kernel of a principal connection $\gamma \in \Omega^1(\mathcal{P}, \mathfrak{p})$ is a smooth subbundle of $T\mathcal{P}$, called the *horizontal bundle* $H\mathcal{P}$. $H\mathcal{P}$ is complementary to the *vertical bundle* $V\mathcal{P} = \ker T\pi_M$ and both subbundles are $P$-invariant.
By definition, $\ker T_u\pi_M = V\mathcal{P}_u$, thus $T_u\pi_M$ is an isomorphism of $H\mathcal{P}_u$ with $T_{\pi_M(u)}M$. This allows us to lift vector fields $\xi$ on $M$ uniquely to horizontal fields $\xi^{hor}$ on $\mathcal{P}$.
For a principal connection we have a natural notion of curvature, namely the failure of the horizontal bundle to be integrable; this we encode in the principal curvature form

$$\rho(\xi, \eta) := -\gamma([\xi_{hor}, \eta_{hor}]) \text{ for } \xi, \eta \in \mathfrak{X}(\mathcal{P})$$

where subscripts denote projections to the respective subbundles.
$\rho$ is in fact a two-form; take $u \in \mathcal{P}$: that $\rho_u(\xi, \eta)$ really depends only on $\xi(u), \eta(u)$ is equivalent for the map

$$\begin{aligned} \mathfrak{X}(\mathcal{P}) \times \mathfrak{X}(\mathcal{P}) &\to \mathfrak{p}, \\ \xi, \eta &\mapsto \gamma(\xi, \eta) \end{aligned}$$

$\xi, \eta \mapsto \gamma(\xi, \eta)$ to be linear not only over $\mathbb{R}$, but also over $C^\infty(\mathcal{P})$. For this, take a $f \in C^\infty(\mathcal{P})$; note that $(f\eta)_{hor} = f(\eta)_{hor}$ since horizontal projection is algebraic and thus

$$[\xi_{hor}, f\eta_{hor}] = f[\xi_{hor}, \eta_{hor}] + (\xi_{hor} \cdot f)\eta_{hor}.$$

But the latter term is horizontal, and thus lies in the kernel of $\gamma$.

By definition,

$$\rho(\xi, \eta) = d\gamma(\xi - \zeta_{\gamma(\xi)}, \eta - \zeta_{\gamma(\eta)}).$$

Lets calculate $d\gamma(\zeta_Y, \xi_u)$ for $Y$ in $\mathfrak{p}$. For this, note that $\mathrm{Fl}_t^{\zeta_Y}(u) = u\exp(tY)$. So

$$\begin{aligned}(\mathcal{L}_{\zeta_Y}\gamma)(\xi_u) &= \frac{d}{dt}_{|t=0}\gamma(T\mathrm{Fl}_t^{\zeta_Y}\xi_u) = \\ &= \frac{d}{dt}_{|t=0}\gamma(Tr^{\exp(tY)}\xi_u) = \frac{d}{dt}_{|t=0}\mathrm{Ad}(\exp(-tY))\gamma(\xi_u) = \\ &= -\mathrm{ad}(Y)\gamma(\xi_u).\end{aligned}$$

But since also $\mathcal{L}_{\zeta^Y}\gamma = i_{\zeta^Y}d\gamma + d(\gamma(\zeta(Y))) = d\gamma(\zeta_Y, \cdot) + 0$, we have $d\gamma(\zeta_Y, \xi_u) = -[\gamma(Y), \gamma(\xi_u)]$. Therefore

$$\begin{aligned}&d\gamma(\xi - \zeta_{\gamma(\xi)}, \eta - \zeta_{\gamma(\eta)}) = \\ &= d\gamma(\xi,\eta) - d\gamma(\zeta_{\gamma(\xi)}, \eta) + d\gamma(\zeta_{\gamma(\eta)}, \xi) + d\gamma(\zeta_{\gamma(\xi)}, \zeta_{\gamma(\eta)}) = \\ &= d\gamma(\xi,\eta) + [\gamma(\xi)\gamma(\eta)] - [\gamma(\eta),\gamma(\xi)] - [\gamma(\xi),\gamma(\eta)] = d\gamma(\xi,\eta) + [\gamma(\xi),\gamma(\eta)].\end{aligned}$$

So

$$\rho(\xi,\eta) = d\gamma(\xi,\eta) + [\gamma(\xi),\gamma(\eta)].$$

3.2. **Induced linear connections.** Let $\Phi : P \to \mathrm{GL}(V)$ be a representation of $P$ on $V$. Then we can construct the associated vector bundle $\mathcal{P}\times_P V$. We induce a linear connection (a covariant derivative) on $\mathcal{P}\times_P V$: given a vector fields $\xi \in \mathfrak{X}(M)$ and a section $s \in \Gamma(\mathcal{P}\times_P V)$, let $g$ be the $P$-equivariant function from $\mathcal{P} \to V$ corresponding to $s$; then we define the section $\nabla_\xi \eta$ of $\mathcal{P}\times_P V$ as the section corresponding to the $P$-equivariant function $\xi^{hor}\cdot g$. It's easy to check that this defines indeed a linear connection.

In fact one automatically has a linear connection on every tensor power of $\mathcal{P}\times_P V$ in the same way: for some section $s \in \Gamma(\mathcal{P}\times_P (\otimes^i V^*)\otimes(\otimes^j V))$ take the corresponding $P$-invariant function $f : \mathcal{P} \to (\otimes^i V^*)\otimes(\otimes^j V)$. Then $\nabla_\xi s$ is the section of $\mathcal{P}\times_P (\otimes^i V^*)\otimes(\otimes^j V)$ corresponding to $\xi^{hor}\cdot f$.

Lets calculate the curvature of the induced connection: Take $\xi,\eta \in \mathfrak{X}(M)$ and $\zeta \in \Gamma(\mathcal{P}\times_P V)$. Then the curvature of $\nabla$ is defined as

$$R(\xi,\eta)\zeta = \nabla_\xi\nabla_\eta\zeta - \nabla_\eta\nabla_\xi\zeta - \nabla_{[\xi,\eta]}\zeta.$$

$R$ is skew-symmetric, bilinear and has values in $\mathrm{End}(\mathcal{P}\times_P V)$; i.e., $R \in \Omega^2(\mathcal{P}\times_P V, \mathrm{End}(\mathcal{P}\times_P V))$. It is the failure of the map $\xi \mapsto \nabla_\xi$ from $\mathfrak{X}(M) \to \mathrm{End}(\Gamma(\mathcal{P}\times_P V))$ to be a homomorphism of Lie-Algebras.

Lets calculate it: let $g$ be the function $\mathcal{P} \to V$ corresponding to $\zeta$; then $R(\xi,\eta)\zeta$ corresponds to

$$\begin{aligned}&\xi^{hor}\cdot(\eta^{hor}\cdot g) - \eta^{hor}\cdot(\xi^{hor}\cdot g) - [\xi,\eta]^{hor}\cdot g = [\xi^{hor},\eta^{hor}]_{vert}\cdot g = \\ &= -\zeta_{\rho(\xi^{hor},\eta^{hor})}\cdot g = \Phi'(\rho(\xi^{hor},\eta^{hor}))\circ g.\end{aligned}$$

Thus

$$R(\xi,\eta)\zeta = \Phi'(\rho(\xi^{hor},\eta^{hor}))\zeta;$$

But $\rho$ is horizontal, thus one may take arbitrary lifts; and since it is also $P$-equivariant we have

$$\Phi' \circ \rho \text{ induces } R.$$

Especially, for injective $\Phi'$, flatness of the induced linear connection is equivalent to flatness of the principal connection.

3.3. **Reductive Cartan geometries.** Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type $(G, P)$, with $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{p}$ as $P$-module. To $u \in \mathcal{G}$ we associate the isomorphism $\Theta(u) := T_u\pi \circ (\omega_u)^{-1}_{|\mathfrak{n}} : \mathfrak{n} \to T_{\pi_M(u)}M$. We have $\omega_{up}(T_u r^p(\xi_u)) = \mathrm{Ad}(p^{-1})\omega_u(\xi_u)$ by equivariancy of $\omega$. Thus, for $X \in \mathfrak{n}$, $\omega_{up}^{-1}(X) = T_u r^p \omega_u^{-1}(Ad(p)X)$. Since $\frac{d}{dt}_{|t=0}\pi(c(t)p) = \frac{d}{dt}_{|t=0}\pi(c(t))$, $T_{up}\pi \circ Tr^p = T_u\pi$. Thus $\Theta(up) = \Theta(u) \circ \mathrm{Ad}(p)$. Therefore the map

$$\begin{gathered}(u, X) \mapsto \Theta(u)X \\ \mathcal{G} \times \mathfrak{n} \to TM\end{gathered}$$

factorizes to an isomorphism $\mathcal{G} \times_P \mathfrak{n} \cong TM$. This shows that a reductive Cartan geometry of type $(G, P)$ over $M$ is a Cartan connection on a reduction of structure group of $TM$ to $P$. By composing $\omega$ with the projection to $\mathfrak{p}$ we get a principal connection $\gamma = \omega_{\mathfrak{p}}$ on the reduction of structure group $\mathcal{G} \to M$. The projection of $\omega$ to $\mathfrak{n}$ is the soldering form $\theta = \omega_{\mathfrak{n}}$.
Now $\gamma$ induces a linear connection on $TM$. From above we know that the curvature of the induced linear connection is $\check{\rho} \in \Omega^2(M, \mathrm{End}(TM))$. From above we know that $\rho(\xi, \eta) = d\gamma(\xi, \eta) + [\gamma(\xi), \gamma(\eta)]$; Now $K(\xi, \eta)_{\mathfrak{p}} = d\gamma(\xi, \eta) + [\gamma(\xi), \gamma(\eta)] + [\theta(\xi), \theta(\eta)]_{\mathfrak{p}}$; Thus

$$\rho(\xi, \eta) = K(\xi, \eta)_{\mathfrak{p}} - [\theta(\xi), \theta(\eta)]_{\mathfrak{p}}.$$

Since this is a linear connection on $TM$ itself, we also have the notion of torsion: It is defined by

$$\begin{gathered}T \in \Lambda^2(TM^*) \otimes TM, \\ T(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta],\end{gathered}$$

where $\xi, \eta \in \Gamma(TM)$. To calculate it we first note that the function $f : \mathcal{G} \to \mathfrak{n}$ which corresponds to a vector field $\xi$ on $M$ is given by $\theta \circ \xi^{hor}$. For equivariance of $f$ note that $\theta(\xi^{hor}(up)) = \theta(T_u r^p \xi^{hor}) = Ad(p^{-1})\theta(\xi^{hor}(u))$,where we first used invariance of the horizontal subbundle and then equivariance of $\theta$. That $f$ induces $\xi$ follows directly from the definition of the isomorphism $\mathcal{G} \times_P \mathfrak{n} \cong TM$: it is induced by $(u, X) \mapsto T_u p\ \omega^{-1}(X)$, so $(u, \theta(\xi^{hor}(u))) \mapsto T_u p\ \omega^{-1}(\theta(\xi^{hor}(u))) = T_u\pi_M\ \xi^{hor}(u) = \xi(\pi_M(u))$. Now let $f, g : \mathcal{G} \to \mathfrak{n}$ be the functions corresponding to $\xi$ respectively $\eta$. Then the function corresponding to $T(\xi, \eta)$ is

$$\xi^{hor} \cdot g - \eta^{hor} \cdot f - \theta([\xi, \eta]^{hor})$$

Note that $\theta([\xi, \eta]^{hor}) = \theta([\xi^{hor}, \eta^{hor}])$, since both arguments of $\theta$ project to $[\xi, \eta]$ and thus only differ by a vertical field. But so by definition of the exterior derivative

$$\begin{aligned}
&\xi^{hor}\cdot\theta(\eta^{hor})-\eta\cdot\theta(\xi^{hor})-\theta([\xi^{hor},\eta^{hor}])= \\
&=d\theta(\xi^{hor},\eta^{hor})=d\theta(\xi-\zeta_{\gamma(\xi)},\eta-\zeta_{\gamma_\eta})= \\
&=d\theta(\xi,\eta)-d\theta(\zeta_{\gamma(\xi)},\eta)+d\theta(\zeta_{\gamma(\eta)},\xi);
\end{aligned}$$

Now

$$d\theta(\zeta_{\gamma(\xi)},\eta)=\zeta_{\gamma(\xi)}\cdot\eta-0-\theta([\zeta_{\gamma(\xi)},\eta]).$$

By equivariancy $\mathcal{L}_{\zeta_{\gamma(\xi)}}\theta=-ad_{\gamma(\xi)}\circ\theta$. But since also $\mathcal{L}_{\zeta_{\gamma(\xi)}}\theta=i_{\zeta_{\gamma(\xi)}}d\theta+0$ we see $d\theta(\zeta_{\gamma(\xi)},\eta)=-[\gamma(\xi),\eta]$. Thus

$$d\theta(\xi^{hor},\eta^{hor})=d\theta(\xi,\eta)+[\gamma(\xi),\theta(\eta)]-[\gamma(\eta),\theta(\xi)]=K(\xi,\eta)_{\mathfrak{n}}-[\theta(\xi),\theta(\eta)]_{\mathfrak{n}}.$$

So the failure of the induced linear connection on $TM$ to be torsion free is

$$\hat{T}(\xi,\eta)=K(\xi,\eta)_{\mathfrak{n}}-[\theta(\xi),\theta(\eta)]_{\mathfrak{n}}.$$

**Theorem 3.3.1.** *Let $(\mathcal{G}\to M,\omega)$ be a reductive Cartan geometry of type $(G,P)$, where $\mathfrak{g}=\mathfrak{n}\oplus\mathfrak{p}$ as a $P$-module and $\omega=\theta\oplus\gamma=\omega_{\mathfrak{n}}\oplus\omega_{\mathfrak{p}}$. Then*

*i. $\mathcal{G}\to M$ is a reduction of structure group of $TM$ to $P$ for which $\theta$ is the soldering form.*
*ii. $\gamma$ is a principal connection with principal curvature form*

$$\rho(\xi,\eta)=K(\xi,\eta)_{\mathfrak{p}}-[\theta(\xi),\theta(\eta)]_{\mathfrak{p}}.$$

*iii. The curvature of the induced linear connection on $TM$ is obtained by factorizing*

$$\hat{R}=ad\circ\rho$$

*and*
*iv. The torsion of the induced connection on $TM$ is obtained by factorizing*

$$\hat{T}(\xi,\eta)=K(\xi,\eta)_{\mathfrak{n}}-[\theta(\xi),\theta(\eta)]_{\mathfrak{n}}.$$

Before we discuss an exemplary situation we briefly discuss

3.3.1. *Affine extensions of linear automorphisms.* Given a vector space $V$ and a subgroup $P$ of $GL(V)$ we have the standard representation $\Phi$ of $P$ on the abelian Lie Group $V$. Thus we can extend $P$ affinely to the semidirect product $V\rtimes_\Phi P$ (or $V\rtimes_{aff}P$), where the composition is given by $(v,p)(v',p')=(v+\Phi(p)v',pp')$. Of course this is the composition of $(v,p)$ and $(v',p')$ regarded as affine maps from $V\mapsto V$, where $(v,p)$ corresponds to the map $x\mapsto px+v$. The inverse of $(v,p)$ is $(-p^{-1}v,p^{-1})$ and we see $\mathrm{conj}_{(v,p)}(v')=(v,p)(v',e)(v,p)^{-1}=(v+pv',p)(-p^{-1}v,p^{-1})=(v+pv'-pp^{-1}v,e)=(pv',e)\in V\lhd(V\rtimes P)$.

$$\begin{aligned}
&\mathrm{conj}_{(v,p)}((\exp(tX),\exp(tY)))= \\
&=(v,p)(tX,\exp(tY))(v,p)^{-1}=(v+tpX,p\exp(tY))(-p^{-1}v,p^{-1})= \\
&=(v+tpX-p\exp(tY)p^{-1}v,p\exp(tY)p^{-1})
\end{aligned}$$

so

$$\begin{aligned}
&\mathrm{Ad}((v,p))(X,Y) = \\
&= \frac{d}{dt}_{|t=0} \mathrm{conj}_{(v,p)}((\exp(tX), \exp(tY)) = (pX - (\mathrm{Ad}(p)Y)v, \mathrm{Ad}(p)Y),
\end{aligned}$$

and

$$\begin{aligned}
&([(X',Y'),(X,Y)] = \mathrm{ad}((X',Y'))(X,Y) = \\
&= \ \frac{d}{dt}_{|t=0} \mathrm{Ad}((tX', \exp(tY')))(X,Y) = \\
&= \frac{d}{dt}_{|t=0} (\exp(tY')X - (\mathrm{Ad}(\exp(tY'))Y)tX', \mathrm{Ad}(\exp(tY'))Y) = \\
&= (Y'X - YX', [Y', Y]),
\end{aligned}$$

i.e.

$$[(X,Y),(X',Y')] = (YX' - Y'X, [Y,Y']).$$

All of the above works identically for coverings $P$ of a virtual subgroup of $\mathrm{GL}(V)$.

*Example* 3.3.2. If $\mathcal{G} \to M$ is a $P$-principal bundle describing a reduction of structure group of $TM$ to $P$ by a soldering form $\theta : \mathcal{G} \to \mathfrak{n}$ (,where $\mathfrak{n}$ is some modeling vector space), we can affinely extend $\mathfrak{p}$ to $\mathfrak{n} \rtimes \mathfrak{p}$ (see 3.3.1 above), and any principal connection $\gamma$ on $\mathcal{G}$ puts us into the situation of theorem 3.3.1 with $\omega = \theta \oplus \gamma$. Since $\mathfrak{n}$ is abelian, the curvature and torsion of the induced linear connection correspond to the $\mathfrak{p}$, respectively $\mathfrak{n}$,-parts of the curvature of $\omega$. ⌟

*Example* 3.3.3. Take a reductive Klein geometry: that is, Lie Groups $G$ and $P < G$, such that $\mathfrak{p}$ has a $P$-invariant complement $\mathfrak{n}$, i.e. $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{p}$ as $P$-module. Then the Maurer-Cartan-form $\omega^{MC}$ is a Cartan connection on $G$, and by the Maurer-Cartan equation this Cartan geometry is flat. Thus by theorem 3.3.1 we have: at $o = P \in G/P$ the curvature of the induced linear connection on $TH/K$ is

$$\begin{gathered}
\mathfrak{n} \times \mathfrak{n} \to \mathfrak{gl}(\mathfrak{p}), \\
(X,Y) \mapsto -\mathrm{ad}([X,Y]_{\mathfrak{p}})_{|\mathfrak{n}}
\end{gathered}$$

and its torsion is

$$\begin{gathered}
\mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}, \\
(X,Y) \mapsto -[X,Y]_{\mathfrak{n}}.
\end{gathered}$$

⌟

In the special case of homogeneous spaces $H/K$ we will explicitly calculate $R$ and $T$ also in the situation of the first example. (See Corollary 4.2.4.)

# 4. Connections on Homogeneous Principal Bundles

We describe invariant principal and Cartan- connections on homogeneous principal bundles. As an application we will get a complete description of invariant connections on homogeneous vector bundles in section 4.1.2.

## 4.1. Invariant principal connections.

Now we ask what invariant principal connections on $H \times_K P \to H/K$ look like: let $\gamma$ be a principal connection form on $H \times_K P \to H/K$ which is invariant under the action of $H$. We lift $\gamma$ to a one form $\hat{\gamma}$ on $H \times P$ by $\frac{d}{dt}(h(t), p(t)) \mapsto \gamma(\frac{d}{dt}[[h(t), p(t)]])$. (I.e., when one denotes the natural surjection from $H \times P$ to $H \times_K P$ by $q$, $\hat{\gamma} = q^*\gamma$, the pullback of $\gamma$ under $q$). We have $\hat{\gamma}(\frac{d}{dt}(h,p)\exp(tY) = \gamma(\frac{d}{dt}[[h,p]]\exp(tY)) = Y$, so $\hat{\gamma}$ reproduces fundamental vector fields. Since

$$\hat{\gamma}(\frac{d}{dt}(h(t), p(t)p')) = \gamma(\frac{d}{dt}[[h(t), p(t)]]p') =$$
$$= \mathrm{Ad}(p'^{-1})\gamma(\frac{d}{dt}[[h(t), p(t)]]) = \mathrm{Ad}(p'^{-1})\hat{\gamma}(\frac{d}{dt}(h(t), p(t)))$$

it is also $p$-equivariant, which shows that it is a principal connection on $H \times P \to H$. $H$-invariance is shown analogously. So $\hat{\gamma}$ is an invariant principal connection on $H \times P$. We left trivialize

$$T(H \times P) = H \times P \times \mathfrak{h} \times \mathfrak{p}.$$

Since $\hat{\gamma}$ reproduces fundamental vector fields $(h, p, 0, Y) \mapsto Y$. By $H$-invariance $\hat{\gamma}(h, p, X, 0) = \hat{\gamma}(e, p, X, 0)$. We have

$$(e, p, X, 0) \mapsto \gamma(\frac{d}{dt}[[exp(tX), p]]) = \mathrm{Ad}(p^{-1})\gamma(\frac{d}{dt}[[exp(tX), e]]) =$$
$$= \mathrm{Ad}(p^{-1})\hat{\gamma}(e, e, X, 0).$$

So, $\hat{\gamma}$ is given by

$$(h, p, X, Y) \to \mathrm{Ad}(p^{-1})\alpha(X) + Y$$

where $\alpha(X) = \hat{\gamma}(e, e, X, 0)$. In fact, for any linear $\alpha : \mathfrak{h} \to \mathfrak{p}$ this formula defines an invariant principal connection on the trivial bundle $H \times P \to H$, and any such connection is of this form. However, for arbitrary $\alpha$ the resulting connection need not factorize to $H \times_K P$.
But recall from 1.1.7 that the form $\hat{\gamma}$ factorizes over the $K$-principal bundle $H \times P \to H \times_K K$ iff $\hat{\gamma}$ is horizontal and $K$-invariant, which means that for $Z \in \mathfrak{k}$ and $X \in \mathfrak{h}$ we must have

$$\hat{\gamma}(e, e, Z, -\Psi'(Z)) = \alpha(Z) - \Psi'(Z) = 0$$

and

$$\alpha(X) = \hat{\gamma}(e, e, X, 0) \stackrel{!}{=} \hat{\gamma}(e, e, X, 0) \cdot k =$$
$$= \hat{\gamma}(k, \Psi(k^{-1}), \mathrm{Ad}(k^{-1})X, 0) = \mathrm{Ad}(\Psi(k))\alpha(\Psi(k^{-1})X).$$

So $\alpha : \mathfrak{h} \to \mathfrak{p}$ must be a $K$-equivariant extension of $\Psi' : \mathfrak{k} \to \mathfrak{p}$. Lets calculate the curvature form of the principal connection $\gamma$. It's defined by

$$\rho_u(X, Y) := -\gamma([\xi_{hor}, \eta_{hor}]),$$

where $\xi_{hor}$ and $\eta_{hor}$ are the horizontal projections of arbitrary vector fields $\xi, \eta \in \mathfrak{X}(H \times_K P)$ which extend $X_u$ and $Y_u$.
The left action of $H$ on $H/K$ gives vector fields $\check{R}_X(hK) = \frac{d}{dt}\exp(tX)hK$ for $X \in \mathfrak{h}$. They are related to the fields $R_X(h,p) = (h, p, \mathrm{Ad}(h^{-1})X, 0)$ on $T(H \times P)$. The horizontal projection $\check{H}_X$ of $\check{R}_X$ is related to

$$H_X = (h,p) \mapsto (h, p, \mathrm{Ad}(h^{-1})X, -\mathrm{Ad}(p^{-1})\alpha(\mathrm{Ad}(h^{-1})X)).$$

By definition of the principal curvature form

$$\rho(\check{R}_X, \check{R}_{X'}) = -\gamma([\check{H}_X, \check{H}_Y]).$$

Since $[H_X, H_Y]$ is related to $[\check{H}_X, \check{H}_Y]$ we have $\gamma([\check{H}_X, \check{H}_Y]) = \hat{\gamma}([H_X, H_Y])$. But to we can take arbitrary horizontal fields which coincide with $H_X$ and $H_Y$ at $(e,e)$ to calculate $\hat{\rho}(H_X(e,e), H_Y(e,e))$, we better take fields of the form

$$\tilde{H}_X(h,p) := (h, p, X, -\mathrm{Ad}(p^{-1})\alpha(X)).$$

We see $[\tilde{H}_X, \tilde{H}_Y](e,e) = (e, e, [X,Y], -[\alpha(X), \alpha(Y)])$, so

$$\begin{aligned}\rho_{[[e,e]]}(\check{H}_X, \check{H}_Y) = \hat{\rho}_{(e,e)}(H_X, H_Y) = \\ = -\hat{\gamma}_{(e,e)}([\tilde{H}_X, \tilde{H}_Y]) = [\alpha(X), \alpha(Y)] - \alpha([X,Y]).\end{aligned}$$

Since $\rho$ is horizontal and $P$-equivariant it factorizes to a (invariant) $\mathcal{G} \times_P \mathfrak{p}$-valued 2-form $\check{\rho}$ on $H/K$. Over $o = K \in H/K$ we have a distinguished point in $\mathcal{G}_o$, namely $[[e,e]]$, and thus we may regard $\check{\rho}_o$ as an element of $\Lambda^2(\mathfrak{h}/\mathfrak{k}^*, \mathfrak{p})$. It is given by

$$\check{\rho}_o(X + \mathfrak{k}, Y + \mathfrak{k}) = [\alpha(X), \alpha(Y)] - \alpha([X,Y]).$$

We proved

**Theorem 4.1.1.** *(1) Invariant principal connections on $H \times_K P \to H/K$ are in 1:1-correspondence with invariant principal connections on $H \times P \to H$ of the form*

$$\hat{\gamma}(h, p, X, Y) = \mathrm{Ad}(p^{-1})\alpha(X) + Y$$

*where $\alpha : \mathfrak{h} \to \mathfrak{p}$ is a $K$-equivariant extension of $\Psi' : \mathfrak{k} \to \mathfrak{p}$, i.e.,*

*i. $\alpha(Z) = \Psi'(Z)$ for $Z \in \mathfrak{k}$,*

*ii. $\alpha \circ \mathrm{Ad}(k) = \mathrm{Ad}(\Psi(k)) \circ \alpha$.*

*If there is such a connection, the resulting space of connections is affine and modeled on* $\mathrm{Hom}_K(\mathfrak{h}/\mathfrak{k}, \mathfrak{p})$.

*(2) The curvature of such a connection is the failure of $\alpha$ to be a homomorphism of Lie algebras. The curvature form $\rho$ is invariant and $P$-equivariant.*
*It factorizes to an invariant, $\mathcal{G} \times_P \mathfrak{p}$-valued 2-form $\check{\rho}$ on $H/K$. At $o = K \in H/K$ it is given by*

$$\check{\rho}_o(X_1 + \mathfrak{k}, X_2 + \mathfrak{k}) = [\alpha(X_1), \alpha(X_2)] - \alpha([X_1, X_2]).$$

This reproduces a result of H.C. Wang from 1958, cf. [14].

4.1.1. *Transformation of connections under isomorphisms.* Given a connection on $H \times_K P \to H/K$, how does it transform under an isomorphism? We know from above that the isomorphism is given by a map $(h,p) \mapsto (hh_0, p_0p)$, for some $p_0 \in P$ and $h_0 \in N_H(K)$. On the tangent bundle $(h,p,X,Y) \mapsto (hh_0, p_0p, \mathrm{Ad}(h_0^{-1})X, Y)$, and thus, if the connection is induced by $\alpha : \mathfrak{h} \to \mathfrak{p}$, the pullback of this connection over the isomorphism at the identity is $(e,e,X,Y) \mapsto (h_0, p_0, \mathrm{Ad}(h_0^{-1})X, Y) \mapsto \mathrm{Ad}(p_0^{-1})\alpha(\mathrm{Ad}(h_0^{-1})X) + Y$; i.e, the pullback of $\alpha$ is $\mathrm{Ad}(p_0^{-1}) \circ \alpha \circ \mathrm{Ad}(h_0^{-1})$.

4.1.2. *Invariant connections on homogeneous vector bundles.* As we saw in Theorem 2.2.5 every homogeneous vector bundle $E$ with modeling vector space $V$ over $H/K$ is of the form $H \times_K V \to H/K$ for a representation $\Psi : K \to \mathrm{GL}(V)$.
Then the frame bundle of $H \times_K V \to H/K$ is $H \times_K \mathrm{GL}(V) \to H/K$.
Recall from 3.2 that every principal connection on $H \times_K \mathrm{GL}(V)$ induces a linear connection on $H \times_K V$ in the following way: for $\xi \in \mathfrak{X}(M)$, $s \in \Gamma(H \times_K V)$, take the horizontal lift $\hat{\xi}$ and the $K$-equivariant function $f : H \to V$ corresponding to $s$. (For this correspondence see Theorem 1.1.5.) Then $\nabla_\xi s$ corresponds to the function $\hat{\xi} \cdot f$.
Now it's easy to see

**Lemma 4.1.2.** *For an invariant principal connection $\gamma$ on $H \times_K P$ and a representation of $K$ on $V$ the induced linear connection on $H \times_K V$ is invariant:*

$$(\check{\lambda}_{h_0})^* \nabla = \nabla. \tag{6}$$

*Proof.* We check that then $(\check{\lambda}_{h_0})^* \nabla = \nabla$ for $h_0 \in H$. Since $s(hK) = [[h, f(h)]]$

$$(\check{\lambda}_{h_0})^* s(hK) = \check{\lambda}_{h_0} s(h_0^{-1}hK) =$$
$$\check{\lambda}_{h_0}[[h_0^{-1}h, f(h_0^{-1}h)]] = [[h, f \circ \check{\lambda}_{h_0^{-1}}]].$$

Thus the function corresponding to $(\check{\lambda}_{h_0})^* s$ is $f \circ \check{\lambda}_{h_0^{-1}}$. Now take $c(t)$ with $\frac{d}{dt}_{|t=0} c(t) = \hat{\xi}(h_0^{-1}h)$, i.e. $\frac{d}{dt}_{|t=0} c(t)K = \xi(c(0)K)$ and $\gamma(\frac{d}{dt}_{|t=0} c(t)) = 0$, which is horizontality. By invariance of the horizontal bundle (or invariance of $\gamma$), also $\frac{d}{dt}_{|t=0} h_0 c(t)$ is horizontal. Therefore, since $\frac{d}{dt}_{|t=0} h_0 c(t)K = (\check{\lambda}_{h_0})^* \xi(h)$, the horizontal lift of $(\check{\lambda}_{h_0})^* \xi$ is $\lambda_{h_0}^* \hat{\xi}$. Now

$$\begin{aligned}
&(\lambda_{h_0}^* \hat{\xi}(h)) \cdot (f \circ \lambda_{h_0^{-1}})(h) = \\
&= \frac{d}{dt}_{|t=0} f(h_0^{-1} h_0 c(t)) = \frac{d}{dt}_{|t=0} f(c(t)) = \\
&= (\hat{\xi} \cdot f)(h_0^{-1}h) = (\hat{\xi} \cdot f) \circ \lambda_{h_0^{-1}}(h);
\end{aligned}$$

this can be rewritten

$$\nabla_{(\check{\lambda}_{h_0})^* \xi} (\check{\lambda}_{h_0})^* s = (\check{\lambda}_{h_0})^* (\nabla_\xi s),$$

which is (6). $\square$

Now, if the representation is infinitesimally injective, all linear connections are induced by principal connections on the frame bundle. So Theorem 4.1.1 tells us that for injective $\Psi'$ invariant linear connections on $H \times_K V \to H/K$ are in 1:1-correspondence with $K$-equivariant linear maps $\alpha$ from $\mathfrak{h} \to \mathfrak{gl}(V)$ which extend $\Psi'$. And in this case (as we saw in section 3.1) the resulting linear connection is flat iff the corresponding principal connection $\gamma$ is flat; and from theorem 4.1.1 we know that this is the case iff $\alpha : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{h})$ is a homomorphism of Lie algebras.
If the vector bundle is endowed with some additional ($H$-invariant) structure we can ask for special (invariant) connections compatible with this structure: Namely, take some invariant $(i,j)$-tensor $\Theta$ on $H \times_K V \to H/K$; then, as discussed in example 2.2.8 the tensor $\Theta$ is induced by a (unique) $K$-invariant $\theta \in (\otimes^i V^*) \otimes (\otimes^j V)$.
From our discussion above we know that every invariant linear connection on $H \times_K V \to H/K$ is induced by a $K$-equivariant extension of $\Psi' : \mathfrak{k} \to \mathfrak{gl}(V)$ to a map $\mathfrak{h} \to \mathfrak{gl}(V)$.
Now we ask for which $\alpha$ the resulting linear connection satisfies

$$\nabla\Theta = 0.$$

Now $\Theta$ corresponds to the function

$$\begin{aligned} H \times_K \mathrm{GL}(V) &\to T^i_j V, \\ [[h,g]] &\mapsto g^{-1}\theta. \end{aligned}$$

This function lifts to

$$f : H \times \mathrm{GL}(V) \to T^i_j V, h, g \mapsto g^{-1}\theta.$$

Now take the horizontal vector $(e, e, X, -\alpha(X)) \in T(H \times \mathrm{GL}(V))$. Then $(e, e, X, -\alpha(X)) \cdot f = -\alpha(X)\theta$. Thus we see that $\nabla\Theta = 0$ iff $\alpha : \mathfrak{h} \to \mathfrak{gl}(V)$ has in fact values in the Lie algebra $\mathfrak{p}$ of $P$, where $P = \mathrm{GL}(V)_\theta$ is the isotropy group of $\theta$ for the action of $\mathrm{GL}(V)$ on $T^i_j V$.
We summarize our findings in

**Theorem 4.1.3.** *Consider a representation $\Psi : K \to \mathrm{GL}(V)$ with $\Psi'$ injective.*

*i. Every invariant connection on $H \times_K V$ is induced by a $K$-equivariant extension $\alpha : \mathfrak{h} \to \mathfrak{gl}(V)$ of $\Psi'$.*

*ii. Let $\theta \in T^i_j V$ be $K$-invariant. Denote the invariant tensor on $H \times_K V$ corresponding to $\theta$ by $\Theta$.*
*There is a canonical action of $\mathrm{GL}(V)$ on the tensor power $T^i_j V$ of $V$. Define*

$$P := \mathrm{GL}(V)_\theta = \{g \in \mathrm{GL}(V) : g \cdot \theta = \theta\}$$

*the isotropy subgroup of $\theta$ under this action.*
*Then an invariant connection $\nabla$ which is induced by a map $\alpha : \mathfrak{h} \to \mathfrak{gl}(V)$ respects $\Theta$ in the sense that*

$$\nabla\Theta = 0$$

*iff $\alpha$ has in fact values in $\mathfrak{p}$. I.e., iff $\alpha$ defines in fact a principal connection on the reduction $H \times_K P \to H/K$ of $\mathrm{GL}^1(E)$.*

*Example* 4.1.4. Let $\Theta$ be an invariant Riemannian metric on $H\times_K V \to H/K$ which is induced by a $K$-invariant inner product $\theta$ on $\mathfrak{h}/\mathfrak{k}$. Then $\theta$ endows $V$ with the structure of a Euclidean space and that $\theta$ is invariant under $K$ simply means that $\Psi : K \to \mathrm{GL}(V)$ has in fact values in $O(V,\theta)$.
We say that a connection is *Euclidean* if it satisfies

$$\begin{aligned} \nabla\Theta &= 0 \\ &\Leftrightarrow \\ \zeta\cdot\Theta(\xi_1,\xi_2) &= \Theta(\nabla_\zeta\xi_1,\xi_2)+\Theta(\xi_1,\nabla_\zeta\xi_2) \\ &\forall\zeta\in\mathfrak{X}(H/K), \xi_1,\xi_2\in\Gamma(E). \end{aligned}$$

Now Theorem 4.1.3 tells us that invariant, linear connections correspond to $K$-equivariant extensions of $\psi : \mathfrak{k}\to\mathfrak{so}(V,\theta)$ to maps $\alpha:\mathfrak{h}\to\mathfrak{so}(V,\theta)$. ⌟

4.2. **Invariant Cartan connections.** Let $\omega$ be an invariant Cartan connection of type $(G,P)$ on $H\times_K P\to H/K$; i.e.: $G,P$ are Lie groups with $P<G$ and $\omega$ is an $H$-invariant $\mathfrak{g}$-valued one form which satisfies

i. $\omega$ is $P$-equivariant
ii. $\omega(\frac{d}{dt}_{|t=0}u\exp(tX)) = X$ for all $u\in H\times_K P$, $X\in\mathfrak{p}$
iii. $\omega_u : T_u(H\times_K P)\to\mathfrak{g}$ is an isomorphism for all $u\in H\times_K P$.

Obviously its lift $\hat\omega$ is of the form

$$(h,p,X,Y)\to \mathrm{Ad}(p^{-1})\alpha(X)+Y$$

for a linear map $\alpha:\mathfrak{h}\to\mathfrak{g}$. Exactly as in the case of invariant principal connection one sees that for a linear map $\alpha:\mathfrak{h}\to\mathfrak{g}$ the resulting two-form $\hat\omega$ factorizes to a form on $H\times_K P$ iff $\alpha=\Psi'$ on $\mathfrak{k}$ and $\alpha$ is $K$-equivariant.
But we also need that $\omega$ is an absolute parallelism, i.e., that $\omega_u$ is an isomorphism of $T_u(H\times_K P)$ with $\mathfrak{h}$ for any $u$. But since $\omega$ is invariant, it suffices to check this at $u=[[e,e]]$.
We know that $\omega_{[[e,e]]}$ is an isomorphism of $V_{[e,e]}\mathcal{G}=\{X\in T_{[[e,e]]}\mathcal{G} : T\pi_{[[e,e]]}X=0\}$ with $\mathfrak{p}$. Thus it only remains to check that $\pi_{\mathfrak{g}/\mathfrak{p}}\circ\omega_{[[e,e]]}$ is surjective and has kernel $V_{[e,e]}\mathcal{G}$; here $\pi_{\mathfrak{g}/\mathfrak{p}}:\mathfrak{g}\to\mathfrak{g}/\mathfrak{p}$ is the natural surjection. Recall the natural surjection $Tq : T(H\times P) = H\times P\times\mathfrak{h}\times\mathfrak{p}\to T\mathcal{G}$. $\beta := \pi_{\mathfrak{g}/\mathfrak{p}}\circ\omega\circ T_{(e,e)}q$ vanishes on $\mathfrak{k}\oplus\mathfrak{p}$. The restriction of $\beta$ to $\mathfrak{h}$ is $\pi_{\mathfrak{g}/\mathfrak{p}}\circ\alpha$. That $\beta$ is surjective means that $\pi_{\mathfrak{g}/\mathfrak{p}}\circ\alpha$ is surjective, and that the kernel of $\pi_{\mathfrak{g}/\mathfrak{p}}\circ\omega_{[[e,e]]}$ is no more than $V_{[e,e]}\mathcal{G}$ means that for $X\in\mathfrak{h}\backslash\mathfrak{k}$ we have $\beta(X)\neq 0$; i.e., the condition on $\alpha$ is that it factorizes to an isomorphism of $\mathfrak{h}/\mathfrak{k}$ with $\mathfrak{g}/\mathfrak{p}$.
The curvature $K$ of a Cartan connection $\omega$ is its failure to satisfy the Maurer-Cartan-equation; i.e., for $X,Y\in T_u(H\times_K P)$:

$$K_u(X,Y) := d\omega(X,Y)+[\omega(X),\omega(Y)].$$

Since the exterior differential is compatible with pullbacks the pullback of $K$ is given by

$$\hat K(\hat X,\hat Y) = d\hat\omega(\hat X,\hat Y)+[\hat\omega(\hat X),\hat\omega(\hat Y)].$$

We want to calculate $K$ for the tangent vectors

$$\frac{d}{dt}_{|t=0}[[\exp(tX)K,e]], \frac{d}{dt}_{|t=0}[[\exp(tY)K,e]]$$

at the point $[[e,e]]$. For these we may take arbitrary lifts, and we choose fields of the form

$$L_X(h,p) := (h,p,X,0).$$

Now $\hat\omega(L_Y(h,p)) = \hat\omega((h,p,Y,0)) = \mathrm{Ad}(p^{-1})\alpha(Y)$, So $L_X \cdot \hat\omega(L_Y)(h,p) = 0$. Thus $d\hat\omega(L_X,L_Y)(e,e) = -\hat\omega([[L_X,L_Y](e,e)]) = -\alpha([X,Y])$. Therefore $\hat K(X,Y) = [\alpha(X),\alpha(Y)] - \alpha([X,Y])$. Thus, factorizing $K$ to a (invariant) $\mathcal{G}\times_P \mathfrak{p}$-valued 2-form $\check K$ on $H/K$, we have

$$\check K_o(X+\mathfrak{k}, Y+\mathfrak{k}) = [\alpha(X),\alpha(Y)] - \alpha([X,Y]).$$

We summarize

**Theorem 4.2.1.** *Let $H\times_K P \to H/K$ be the homogeneous $P$-principal bundle induced by a homomorphism $\Psi: K\to P$.*
*Let $G$ be a Lie group which contains $P$ as a subgroup.*
*Then invariant Cartan connections of type $(G,P)$ on $H\times_K P\to H/K$ are induced by maps $\alpha:\mathfrak{h}\to\mathfrak{g}$ which satisfy*

*i.* $\alpha_{|\mathfrak{k}} = \Psi'$,
*ii.* $\alpha(\mathrm{Ad}(k)X) = \mathrm{Ad}(\Psi(k))\alpha(X)$
*iii.* *$\alpha$ factorizes to an isomorphism of $\mathfrak{h}/\mathfrak{k}$ with $\mathfrak{g}/\mathfrak{p}$.*

*For such an $\alpha$, the corresponding (lift of) the Cartan connection is*

$$\hat\omega\big((h,p,X,Y)\big) = \mathrm{Ad}(p^{-1})\alpha(X) + Y.$$

*Its curvature form $K \in \Omega^(H\times_K P,\mathfrak{p})$ is $H$-invariant and $P$-equivariant. So it factorizes to an invariant $\mathcal{G}\times_K \mathfrak{g}$-valued 2-form $\check K$ on $H/K$, which is given by*

$$\check K_o(X_1+\mathfrak{k}, X_2+\mathfrak{k}) = [\alpha(X_1),\alpha(X_2)] - \alpha([X_1,X_2])$$

*at $o = K\in H/K$.*
*Given such an $\alpha$, the space of all connections inducing the same isomorphism between $\mathfrak{h}/\mathfrak{k}$ and $\mathfrak{g}/\mathfrak{p}$ is affine and is modeled on $\mathrm{Hom}_K(\mathfrak{h}/\mathfrak{k},\mathfrak{p})$.*

*Remark* 4.2.2. Thus a homogeneous Cartan geometry of type $(G,P)$ over $(H/K)$ is equivalent to a pair $(\Psi: K\to P, \alpha:\mathfrak{h}\to\mathfrak{g})$ satisfying i-iii of theorem 4.2.1.
Now let $(H\times_K P,\omega)$ be the Cartan geometry corresponding to a pair $(\Psi,\alpha)$: Then we have a morphism of principal bundles over $\Psi$ from $H\to H/K$ to $H\times_K P\to H/K$, namely

$$\begin{aligned} j: H &\to H\times_K P,\\ h &\mapsto [[(h,e)]]. \end{aligned}$$

This allows us to pull pack the Cartan connection $\omega$ on $H\times_K P$ to a 1-form on $H$: one calculates

$$\begin{aligned} &j^*\omega(\frac{d}{dt}_{|t=0} h\exp(tX)) = \\ &= \hat\omega(h,e,X,0) = \alpha(X) = \alpha\circ\omega^{MC}(X) \end{aligned}$$

with $\omega^{MC}$ the Maurer-Cartan form on $H$. Thus

$$j^*\omega = \alpha \circ \omega^{MC}.$$

But this equation already determines $\omega$ by equivariancy of $\omega$ under $P$.
In fact we can generalize this picture:

**Theorem 4.2.3.** *Let $(\Psi : K \to P, \alpha : \mathfrak{h} \to \mathfrak{g})$ satisfy i-iii of theorem 4.2.1 and let $(\mathcal{G}, \omega)$ be a Cartan connection of type $(H, K)$. Now consider the $P$-principal bundle $\mathcal{G}' := G \times_\Psi P$: Then the map*

$$j : \mathcal{G} \to \mathcal{G} \times_K P, \tag{7}$$
$$u \mapsto [[u, e]] \tag{8}$$

*is a homomorphism of principal bundles over $\Psi$ covering the identity and there is a unique Cartan connection $\omega' \in \Omega^1(\mathcal{G}', \mathfrak{g})$ satisfying*

$$j^*\omega' = \alpha \circ \omega. \tag{9}$$

*$(\mathcal{G}, \omega) \mapsto F(\mathcal{G}, \omega) := (\mathcal{G}', \omega')$ is a functor from the category of Cartan geometries of type $(H, K)$ to the category of Cartan geometries of type $(G, P)$.*

*Proof.* Its clear that $\mathcal{G}' := \mathcal{G} \times_K P$ is a $P$-principal fiber bundle over $M$. The Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{h})$ and the Maurer-Cartan form $\omega^{MC} \in \Omega^1(P, \mathfrak{p})$ allow us to trivialize

$$\mathcal{G} \times P \times \mathfrak{h} \times \mathfrak{p} \cong T(\mathcal{G} \times P),$$
$$(u, p, X, Y) \mapsto (\omega_u^{-1}(X), \omega_p^{MC^{-1}}(Y)).$$

On $\mathcal{G} \times P$ we define the $\mathfrak{g}$-valued form

$$\hat{\omega}' \in \Omega^1(H \times P, \mathfrak{g}),$$
$$\hat{\omega}'(u, p, X, Y) := \mathrm{Ad}(p^{-1})\alpha(X) + Y.$$

Then one sees exactly as above in 4.2 that horizontality of $\hat{\omega}'$ is equivalent to $\alpha_{|\mathfrak{k}} = \Psi'$ (i), $K$-invariance of $\hat{\omega}'$ is equivalent to $K$-equivariance of $\alpha$ (ii). Thus $\hat{\omega}'$ factorizes to a $\mathfrak{g}$-valued 1-form $\omega'$ on $\mathcal{G}'$ and iii implies that $\omega'$ is a Cartan connection on $\mathcal{G}' \to M$. (9) holds.
A morphism $f : (\mathcal{G}_1, \omega_1) \to (\mathcal{G}_2, \omega_2)$ of Cartan geometries of type $(H, K)$,

$$f : \mathcal{G}_1 \to \mathcal{G}_2,$$
$$f(u \cdot k) = f(u) \cdot k \quad \forall u \in \mathcal{G}_1, k \in K,$$
$$f^*\omega_2 = \omega_1$$

is of course mapped to

$$F(f) : \mathcal{G}_1' \to \mathcal{G}_2',$$
$$[u, p] \mapsto [f(u), p]$$

by the functor $F$. $F(f)$ is $P$-equivariant. With $j_i : \mathcal{G}_i \to \mathcal{G}'_i$ we defined by by (7) for $i = 1, 2$ we have the commutative diagram

$$\begin{tikzcd} \mathcal{G}'_1 \arrow[r, "F(f)"] & \mathcal{G}'_2 \\ \mathcal{G}_1 \arrow[u, "j_1"] \arrow[r, "f"] & \mathcal{G}_2 \arrow[u, "j_2"'] \end{tikzcd}$$

and thus

$$\begin{aligned} &j_1^*(F(f)^*(\omega'_2)) = f^*(j_2^*(\omega'_2)) = \\ &= f^*(\alpha \circ \omega_2) = \alpha \circ f^*(\omega_2) = \alpha \circ \omega_1 = \alpha \circ j_1^*(\omega'_1), \end{aligned}$$

which shows $F(f)^*(\omega'_2) = \omega'_1$ by uniqueness of $\omega'_1$ with $j_1^*\omega'_1 = \alpha \circ \omega_1$. Thus $F(f)$ is indeed a morphism of Cartan geometries of type $(G, P)$. Functoriality of $F$ is clear. □

We can apply Theorem 4.2.1 to connections on reductions of structure groups of homogeneous spaces: (Recall also 3.3.2).

**Corollary 4.2.4.** *Let $\Psi : K \to P$ be a homomorphism to a covering of a virtual subgroup $P$ of $\mathrm{GL}(\mathfrak{h}/\mathfrak{k})$, i.e., a reduction of structure group of $H/K$ to $P$.*
*Let $\alpha_\gamma : \mathfrak{h} \to \mathfrak{p}$ describe a principal connection on $H \times_K P$. Then the natural surjection $\pi_{\mathfrak{h}/\mathfrak{k}}$ extends $\alpha_\gamma$ to $\alpha_\omega = \pi_{\mathfrak{h}/\mathfrak{k}} \oplus \alpha_\gamma$, which induces a Cartan geometry of type $(\mathfrak{h}/\mathfrak{k} \rtimes P, P)$ on $H \times_K P \to H/K$.*
*The curvature of the principal connection $\gamma$ is*

$$\rho_o(X + \mathfrak{k}, Y + \mathfrak{k}) = [\alpha_\gamma(X), \alpha_\gamma(Y)] - \alpha_\gamma([X, Y]).$$

*The curvature function $\kappa$ of this Cartan connection factorizes to an invariant $\mathcal{G} \times_P \Lambda^2(\mathfrak{h}/\mathfrak{k}^*, \mathfrak{g})$-valued function $\check{\kappa}$ on $H/K$ and*

$$\check{\kappa}(o) = T_o \oplus \rho_o.$$

*For the linear connection induced by $\gamma$ on $TH/K$ the curvature and torsion are*

$$R_o(X + \mathfrak{k}, Y + \mathfrak{k}) = \mathrm{ad}\big([\alpha_\gamma(X), \alpha_\gamma(Y)] - \alpha_\gamma([X, Y])\big)$$

*and*

$$\begin{aligned} T_o(X + \mathfrak{k}, Y + \mathfrak{k}) &= \\ \mathrm{ad}(\alpha_\gamma(X))(Y + \mathfrak{k}) - \mathrm{ad}(\alpha_\gamma(Y))(X + \mathfrak{k}) &- ([X, Y] + \mathfrak{k}). \end{aligned}$$

## 5. Riemannian Geometry on Homogeneous Spaces

In this chapter we discuss the frame bundle- (or Cartan-) picture of Riemannian geometries. This is in fact a prolongation of a given geometric data on $M = H/K$, namely a (invariant) Riemannian metric to a connection on a principal bundle over $M$. In the case of Riemannian geometries discussed in section 5.2 we simply get the Levi-Civita connection on the orthonormal frame bundle $O^1(M)$. Later in the case of a conformal geometry discussed in section 7 we will get a Cartan connection on a certain principal bundle over $M$.

5.1. **Prologue to homogeneous (pseudo-)Riemannian spaces.** Let $(M, g_M)$ be a (pseudo-)Riemannian space: i.e., we have a section $g_M \in \Gamma(TM^* \otimes TM^*)$ which is bilinear, symmetric and non-degenerate.

**Definition 5.1.1.** Let $(M_1, g_1)$,$(M_2, g_2)$ be (pseudo-)Riemannian manifolds. An isometry between $M_1$ and $M_2$ is diffeomorphism of $M_1$ with $M_2$ which pulls back $g_2$ to $g_1$.

By the theorem of Myers-Steenrod ([8]) the isometry group $\mathrm{Isom}(M, g_M)$ of $(M, g_M)$ is a Lie group We say that $(M, g_M)$ is homogeneous if its isometry-group acts transitively. In this case $(M, g_M) \cong (H/K, g_{H/K})$ with $H = \mathrm{Isom}(M, g_M)$ and $K$ the isotropy-subgroup of some point $x \in M$. But we know from theorem 2.2.7 that such an $H$-invariant (pseudo-) Riemannian metric on $H/K$ is induced by a unique $K$-invariant (pseudo-)inner product $g$ on $\mathfrak{h}/\mathfrak{k}$. Thus we write $(H/K, g_{H/K}) = (H/K, g)$.
When $(M, g_M)$ is Riemannian, i.e., when $g_M$ is positive definite at every point of $M$, $g_M$ it induces a metric $d$ on $M$, namely

$$d(x_1, x_2) := \inf_{c \in \mathrm{C}^\infty(\mathbb{R}, M): c(0)=x_1, c(1)=x_2} \int_0^1 \sqrt{g_M(\dot{c}(t), \dot{c}(t))} dt.$$

In the Riemannian case it follows from the theorem of Arzelà-Ascoli that the isotropy subgroup $K$ of $\mathrm{Isom}(M, g_M)$ is compact (for any point $x \in M$). Since compact representations are completely reducible there is thus a $K$-invariant complement $\mathfrak{n}$ to $\mathfrak{k}$ in $\mathfrak{h}$; and by $\mathfrak{n} \cong \mathfrak{h}/\mathfrak{k}$ as $K$-modules (or directly by $T(H/K) = H \times_K \mathfrak{n}$) we have a $K$-invariant inner product $g$ on $\mathfrak{n}$ which induces the $H$-invariant Riemannian metric $g_M$. However in the general, pseudo-Riemannian case, we cannot expect $\mathfrak{h}$ to be reductive.

5.2. **Prolongations of (pseudo-)Riemannian geometries.** We just saw that every homogeneous pseudo-Riemannian space is isometric to $(H/K, g)$ with $g$ being a $K$-invariant inner product of signature $(p, q)$ on $\mathfrak{h}/\mathfrak{k}$.
Recall from Riemannian geometry:

**Theorem 5.2.1.** *On a (pseudo-)Riemannian manifold there is a unique linear connection which is compatible with the Riemannian metric and is torsion free. This connection is called the* Levi-Civita connection.

We construct the Levi-Civita connection on $T(H/K)$ in the frame bundle picture.
Denote by $O^1(H/K)$ the orthonormal frame bundle of $H/K$; We have $O^1(H/K) = H \times_{\mathrm{Ad}} O(\mathfrak{h}/\mathfrak{k})$. In 4.1.2 we saw that an invariant linear connection on $H/K$ is compatible with the Riemannian metric on $T(H/K)$ iff it is induced by a $K$-equivariant extension of $\mathrm{ad} : \mathfrak{k} \to \mathfrak{so}(\mathfrak{h}/\mathfrak{k}, g)$ to a map $\alpha : \mathfrak{h} \to \mathfrak{so}(\mathfrak{h}/\mathfrak{k}, g)$.
Now, given such a $K$-equivariant extension $\alpha$, we saw in corollary 4.2.4 that the torsion of the induced linear connection on $T(H/K)$ vanishes iff $T_o = 0$, where

$$T_o \in L(\Lambda^2(\mathfrak{h}), \mathfrak{h}/\mathfrak{k}),$$
$$T_o(X, Y) = \alpha(X)(Y + \mathfrak{k}) - \alpha(Y)(X + \mathfrak{k}) - ([X, Y] + \mathfrak{k}).$$

We know from Theorem 5.2.1 that there is a unique $\alpha$ such that $T_o$ vanishes. (We use the theorem only as a motivation, both existence and uniqueness of such an $\alpha$ will be shown directly).
Lets solve the equation (in $\alpha$)

$$T_o(X, Y) = \alpha(X)(Y + \mathfrak{k}) - \alpha(Y)(X + \mathfrak{k}) - ([X, Y] + \mathfrak{k}) = 0 \ \forall\ X, Y \in \mathfrak{h}; \tag{10}$$

Consider $g(T_o(X, Y), (Z + \mathfrak{k}))$ for $X, Y, Z \in \mathfrak{h}$ and notice that if (10) holds

$$\begin{aligned}
0 &= g(T_o(X, Y), Z + \mathfrak{k}) - g(T_o(Y, Z), X + \mathfrak{k}) + g(T_o(Z, X), Y + \mathfrak{k}) = \\
&= g(\alpha(X)(Y + \mathfrak{k}), Z + \mathfrak{k}) - g(\alpha(Y)(X + \mathfrak{k}), Z + \mathfrak{k}) - g([X, Y] + \mathfrak{k}, Z + \mathfrak{k}) \\
&\quad - g(\alpha(Y)(Z + \mathfrak{k}), X + \mathfrak{k}) + g(\alpha(Z)(Y + \mathfrak{k}), X + \mathfrak{k}) + g([Y, Z] + \mathfrak{k}, X + \mathfrak{k}) \\
&\quad + g(\alpha(Z)(X + \mathfrak{k}), Y + \mathfrak{k}) - g(\alpha(X)(Z + \mathfrak{k}), Y + \mathfrak{k}) - g([Z, X] + \mathfrak{k}, Y + \mathfrak{k}) = \\
&= 2g(\alpha(X)(Y + \mathfrak{k}), Z + \mathfrak{k}) \\
&\quad - g([X, Y] + \mathfrak{k}, Z + \mathfrak{k}) + g([Y, Z] + \mathfrak{k}, X + \mathfrak{k}) - g([Z, X] + \mathfrak{k}, Y + \mathfrak{k})
\end{aligned}$$

since $\alpha$ has values in $\mathfrak{so}(\mathfrak{h}/\mathfrak{k}, g)$. This is equivalent to

$$\begin{aligned}
&g(\alpha(X)(Y + \mathfrak{k}), Z + \mathfrak{k}) \\
&= \frac{1}{2}\big(g([X, Y] + \mathfrak{k}, Z + \mathfrak{k}) - g([X, Z] + \mathfrak{k}, Y + \mathfrak{k}) - g([Y, Z] + \mathfrak{k}, (X + \mathfrak{k})\big).
\end{aligned} \tag{11}$$

Thus, if we have an $\alpha$ satisfying (10) it is already uniquely determined by (11) since $g$ is non-degenerate.
In lemma 5.2.2 we show that this equation conversely determines a $K$-equivariant $\alpha : \mathfrak{h} \to \mathfrak{so}(\mathfrak{h}/\mathfrak{k}, g)$ extending ad and satisfying (10). For (10) to uniquely define a linear map $\alpha : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{h}/\mathfrak{k})$ we need to check that

$$g([X, Y] + \mathfrak{k}, Z + \mathfrak{k}) - g([X, Z] + \mathfrak{k}, Y + \mathfrak{k}) - g([Y, Z] + \mathfrak{k}, X + \mathfrak{k})$$

doesn't depend on the representatives of $Y+\mathfrak{k}$ and $Z+\mathfrak{k}$. But this is easily seen: Let $W\in\mathfrak{k}$, then

$$\begin{aligned}
&g([X,Y+W]+\mathfrak{k},Z+\mathfrak{k})-g([X,Z]+\mathfrak{k},Y+W+\mathfrak{k})\\
&-g([Y+W,Z]+\mathfrak{k},X+\mathfrak{k})=\\
&=g([X,Y]+\mathfrak{k},Z+\mathfrak{k})-g([X,Z]+\mathfrak{k},Y+W+\mathfrak{k})-g([Y,Z]+\mathfrak{k},X+\mathfrak{k})\\
&-\big(g(\mathrm{ad}_W(X+\mathfrak{k}),Z+\mathfrak{k})+g(\mathrm{ad}_W(Z+\mathfrak{k}),X+\mathfrak{k})\big);
\end{aligned}$$

But $g(\mathrm{ad}_W(X+\mathfrak{k}),Z+\mathfrak{k})+g(\mathrm{ad}_W(Z+\mathfrak{k}),X+\mathfrak{k})=0$ since $\mathrm{ad}_W\in\mathfrak{so}(\mathfrak{h}/\mathfrak{k},g)$. The case of $Z+W$ instead of $Z$ is done analogously.

**Lemma 5.2.2.** *For the map defined by* (11) *we have*

*i.* $\alpha(X)=\mathrm{ad}_X$ for all $X\in\mathfrak{k}$
*ii.* $\alpha(\mathrm{Ad}(k)X)=\mathrm{Ad}(k)\circ\alpha(X)\circ\mathrm{Ad}(k^{-1})$
*iii.* $\alpha(X)$ is skew symmetric for all $X\in\mathfrak{h}$
*iv.* $\alpha(X)(Y+\mathfrak{k})-\alpha(Y)(X+\mathfrak{k})-([X,Y]+\mathfrak{k})=0.$

*Proof.* For (i), let $X\in\mathfrak{k}$ and $Y,Z\in\mathfrak{h}$; then

$$\begin{aligned}
&g(\alpha(X)(Y+\mathfrak{k}),Z+\mathfrak{k})=\\
&=\frac{1}{2}\big(g([X,Y]+\mathfrak{k},Z+\mathfrak{k})-g([X,Z]+\mathfrak{k},Y)-g([Y,Z],0+\mathfrak{k})\big)=\\
&=\frac{1}{2}\big(g(\mathrm{ad}_X(Y+\mathfrak{k}),Z+\mathfrak{k})-g(\mathrm{ad}_X(Z+\mathfrak{k}),Y+\mathfrak{k})\big)=\\
&=\frac{1}{2}\big(g(\mathrm{ad}_X(Y+\mathfrak{k}),Z+\mathfrak{k})+g(Z,\mathrm{ad}_X(Y+\mathfrak{k}))\big)=g(\mathrm{ad}_X(Y+\mathfrak{k}),Z+\mathfrak{k}),
\end{aligned}$$

since $\mathrm{ad}_X\in\mathfrak{so}_{\mathfrak{g}}(\mathfrak{h}/\mathfrak{k})$.
For (ii) take $k\in K$ and $X,Y,Z\in\mathfrak{h}$. Note first that projection to $\mathfrak{h}/\mathfrak{k}$ commutes with $\mathrm{Ad}(k)$. Using $\mathrm{Ad}(k)^*=\mathrm{Ad}(k)^{-1}$ we see

$$\begin{aligned}
&g(\alpha(\mathrm{Ad}(k)X)(Y+\mathfrak{k}),Z+\mathfrak{k})=\\
&\frac{1}{2}\big(g([\mathrm{Ad}(k)X,Y]+\mathfrak{k},Z+\mathfrak{k})-g([\mathrm{Ad}(k)X,Z]+\mathfrak{k},Y+\mathfrak{k})\\
&-g([Y,Z]+\mathfrak{k},\mathrm{Ad}(k)X+\mathfrak{k})\big)=\\
&=\frac{1}{2}\big(g(\mathrm{Ad}(k)[X,\mathrm{Ad}(k^{-1})Y]+\mathfrak{k},Z+\mathfrak{k})\\
&-g(\mathrm{Ad}(k)[X,\mathrm{Ad}(k^{-1})Z]+\mathfrak{k},Y+\mathfrak{k})-g(\mathrm{Ad}(k^{-1})[Y,Z]+\mathfrak{k},X+\mathfrak{k})\big)=\\
&=\frac{1}{2}\big(g([X,\mathrm{Ad}(k^{-1})Y]+\mathfrak{k},\mathrm{Ad}(k^{-1})Z+\mathfrak{k})\\
&-g([X,\mathrm{Ad}(k^{-1})Z]+\mathfrak{k},\mathrm{Ad}(k^{-1})Y+\mathfrak{k})\\
&-g([\mathrm{Ad}(k^{-1})Y,\mathrm{Ad}(k^{-1})Z]+\mathfrak{k},X+\mathfrak{k})\big)=\\
&=g(\alpha(X)\mathrm{Ad}(k^{-1})Y+\mathfrak{k},\mathrm{Ad}(k^{-1})Z+\mathfrak{k})=\\
&=g(\mathrm{Ad}(k^{-1})\alpha(X)(\mathrm{Ad}(k^{-1})Y+\mathfrak{k}),Z+\mathfrak{k}).
\end{aligned}$$

For (iii) we need to check $g(\alpha(X)(Y+\mathfrak{k}), Y+\mathfrak{k}) = 0$:

$$g(\alpha(X)(Y+\mathfrak{k}), Y+\mathfrak{k}) = \\ = \frac{1}{2}\big(g([X,Y]+\mathfrak{k}, Y+\mathfrak{k}) - g([X,Y]+\mathfrak{k}, Y+\mathfrak{k}) - g([Y,Y], X+\mathfrak{k})\big) = 0.$$

Finally we need (iv), which is torsion-freeness. For this, let $X, Y \in \mathfrak{h}$, then

$$g(\alpha(X)(Y+\mathfrak{k}) - \alpha(Y)(X+\mathfrak{k}), Z) = \\ = \frac{1}{2}\big(g([X,Y]+\mathfrak{k}, Z) - g([X,Z]+\mathfrak{k}, Y+\mathfrak{k}) - g([Y,Z]+\mathfrak{k}, X+\mathfrak{k}) \\ - g([Y,X]+\mathfrak{k}, Z+\mathfrak{k}) + g([Y,Z]+\mathfrak{k}, X+\mathfrak{k}) + g([X,Z]+\mathfrak{k}, Y+\mathfrak{k})\big) = \\ = g([X,Y]+\mathfrak{k}, Z+\mathfrak{k}).$$

□

By Theorem 3.3.1 $R_o = \check{\rho}_o$. So by Theorem 4.1.1 or Corollary 4.2.4

$$R_o(X+\mathfrak{k}, Y+\mathfrak{k}) = \check{\rho}_o(X,Y) = \mathrm{ad}\big([\alpha(X), \alpha(Y)] - \alpha([X,Y]\big)$$

for $X+\mathfrak{k}, Y+\mathfrak{k} \in \mathfrak{h}+\mathfrak{k} = T_o(H/K)$.
For $X, Y, Z \in \mathfrak{h}/\mathfrak{k}$ we write

$$(R_o(X,Y)Z)^i = R_{sr}{}^i{}_j X^s Y^r Z^j.$$

The Ricci-curvature is defined as

$$R_{rj} := R_{ir}{}^i{}_j.$$

For an orthonormalbasis $v_1 \dots v_m \in \mathfrak{h}/\mathfrak{k}$

$$R_{ir}{}^i{}_j Y^r Z^j = \sum_i g(R(v_i, Y)Z, v_i) = \\ \sum_i g(-R(Y,Z)v_i - R(Z,v_i)Y, v_i) = \sum_i g(R(v_i, Z)Y, v_i) = \\ = R_{ij}{}^i{}_r Z^r Y^j$$

by the Bianchi-identity, which we proof below in Lemma 5.2.3; so the Ricci-curvature $R_{rj}$ is symmetric. The scalar curvature is defined as

$$R := g^{rj} R_{rj}.$$

**Lemma 5.2.3.** *With $\alpha$ defined by* (11) *and* $X_1, X_2, X_3 \in \mathfrak{h}$,

$$\sum_{cyclic} R(X_1, X_2)(X_3 + \mathfrak{k}) = 0.$$

*Proof.*

$$\sum_{cyclic} R(X_1, X_2)(X_3 + \mathfrak{k}) =$$

$$= \sum_{cyclic} \big(\alpha(X_1)(\alpha(X_2)(X_3 + \mathfrak{k}) - \alpha(X_2)(\alpha(X_1)(X_3 + \mathfrak{k})) - \alpha([X_1, X_2])(X_3 + \mathfrak{k})\big) =$$

$$= \sum_{cyclic} \big(\alpha(X_1)(\alpha(X_2)(X_3 + \mathfrak{k}) - \alpha(X_1)(\alpha(X_3)(X_2 + \mathfrak{k})) - \alpha([X_1, X_2])(X_3 + \mathfrak{k})\big)$$

by cyclic permutation. And by using torsion-freeness resp. (10) and more cyclic permutations we thus see

$$\sum_{cyclic} R(X_1, X_2)(X_3 + \mathfrak{k}) =$$

$$= \sum_{cyclic} \big(\alpha(X_1)([X_2, X_3] + \mathfrak{k}) - \alpha([X_1, X_2])(X_3 + \mathfrak{k})\big) =$$

$$= \sum_{cyclic} \big(\alpha([X_2, X_3])(X_1 + \mathfrak{k}) + [X_1, [X_2, X_3]] + \mathfrak{k} - \alpha([X_1, X_2])(X_3 + \mathfrak{k})\big) = 0,$$

where we used the Jacobi-identity in the last step. $\square$

*Remark* 5.2.4. In tensor-notation

$$\alpha_s{}^i{}_j = \frac{1}{2}\big(\tau^i_{sj} - g^{il} g_{is} \tau^i_{lj} - g^{il} g_{ij} \tau^i_{ls}\big),$$

where for $X, Y, Z \in \mathfrak{h}$

$$([X, Y] + \mathfrak{k})^i = \tau^i_{sr} X^s Y^r \text{ and}$$
$$(\alpha(X)(Z + \mathfrak{k}))^i = \alpha_s{}^i{}_j X^s (Z + \mathfrak{k})^j.$$

⌟

We summarize our discussion in

**Theorem 5.2.5.** *Every homogeneous (pseudo-)Riemannian space is isometric to $(H/K, g)$, where $g$ is some $K$-invariant (pseudo-)inner product on $\mathfrak{h}/\mathfrak{k}$.*
*There is a unique invariant Cartan connection of type*

$$(\mathfrak{h}/\mathfrak{k} \rtimes_{aff} O_g(\mathfrak{h}/\mathfrak{k}), O_g(\mathfrak{h}/\mathfrak{k}))$$

*on $O^1(H/K)$ such that the induced linear connection on $T(H/K)$ is torsion free and compatible with the metric; i.e, the Levi-Civita connection.*
*The unique $K$-equivariant extension of* $\mathrm{ad} : \mathfrak{k} \to \mathfrak{so}_g(\mathfrak{h}/\mathfrak{k})$ *which induces this principal connection is defined by* (11).
*The curvature of the induced linear connection $R$ is invariant, and at $o$*

$$R_o(X + \mathfrak{k}, Y + \mathfrak{k}) = \mathrm{ad}\big([\alpha(X), \alpha(Y)] - \alpha([X, Y])\big)$$

*for $X + \mathfrak{k}, Y + \mathfrak{k} \in \mathfrak{h}/\mathfrak{k} = T_oH/K$.*

*Remark* 5.2.6. As we saw in 3.3.2 we can regard principal connections on reductions of the frame bundle equivalently as reductive Cartan connections by affinely extending the structure group by the modeling vector space of the underlying manifold. In our Riemannian situation we get a Cartan geometry of type $(\mathbb{R}^n \rtimes O(n) = Euc(n), O(n))$. The curvature function $\kappa$ of the Cartan connection corresponding to $\alpha_\omega = \pi_{\mathfrak{n}} \oplus \alpha_\gamma$ is just $\kappa_o = 0 \oplus R_o$.

But instead of this affine extension of $O(n)$ we could also extend to $O(n+1)$. Its Lie algebra is of the form $\mathbb{R}^n \oplus \mathfrak{so}(n)$ as $K$-module: it consists of matrices of the form

$$\begin{pmatrix} 0 & -X^t \\ X & A \end{pmatrix} \text{ with } X \in \mathbb{R}^n \text{ and } A \in \mathfrak{so}(n).$$

But $\mathfrak{so}(n+1)$ is not a semidirect product, the $\mathbb{R}^n$-component brackets into $\mathfrak{so}(n)$: for $X_1, X_2$ in $\mathbb{R}^n \subset \mathfrak{so}(n+1)$ we have

$$\left[\begin{pmatrix} 0 & -X_1^t \\ X_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -X_2^t \\ X_2 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ 0 & X_2X_1^t - X_1X_2^t \end{pmatrix}.$$

We can take the same $\alpha_\gamma$ as calculated above, since by Theorem 3.3.1 the resulting torsion is the same (namely 0), (and of course the induced Riemannian curvature doesn't change), but the curvature of the Cartan connection is different: it is

$$\kappa_o(X,Y) = [\alpha(X), \alpha(Y)] - \alpha([X,Y]) + [X,Y]_{\mathfrak{k}} = [\alpha(X), \alpha(Y)] - \alpha([X,Y]_{\mathfrak{n}}).$$

⌟

*Example* 5.2.7. Consider the Riemannian Sphere $O(n+1)/O(n)$. In our terminology $H = O(n+1), K = O(n)$.
Per definition the standard metric on $\mathfrak{n} = \mathbb{R}^n$ is $K = O(n)$-invariant. Of course the induced invariant Riemannian metric on the sphere is just the standard metric: in fact every $O(n)$-invariant symmetric bilinear forms on $\mathbb{R}^n$ is a positive scalar multiple of the standard inner product: let such an invariant form be given by a matrix $B$, then $B$ has to be a multiple of the identity since it commutes with all orthogonal matrices $O$, and it has to be a positive multiple since $B$ is positive definite.
First note that as noted in Remark 5.2.6, we can model this Riemannian space as a Cartan geometry of type $(O(n+1), O(n))$ - but the homogeneous model of this type is itself the sphere $O(n+1)/O(n)$. Thus our construction above must yield a vanishing Cartan curvature. Since the adjoint action of $O(n)$ on $\mathbb{R}^n \subset \mathfrak{so}(n+1)$ is just the standard representation we can regard $\mathrm{Ad}_{O(n)}$ simply as the identity on $O(n)$; therefore $H \times_{\mathrm{Ad}} O(n) = O(n+1) \times_{O(n)} O(n) = O(n+1)$. And by formula (11) $\alpha_\gamma$ vanishes on $\mathfrak{n}$. Thus the resulting $\alpha_\omega = \mathrm{id}_{\mathbb{R}^n} \oplus \mathrm{id}_{\mathfrak{so}(n)}$ is just the identity, and therefore the induced Cartan connection has zero curvature; of course it is just the Maurer-Cartan-form, since both forms are invariant and coincide at the identity.
Now we model the Euclidean sphere on the Euclidean plane, i.e, we describe the sphere as a Cartan geometry of type $(Euc(n), O(n))$.

As before formula (11) tells us to extend $\mathrm{ad}_{\mathfrak{k}} = \mathrm{id}_{\mathfrak{so}(n)}$ trivially, and thus the map $\mathfrak{so}(n+1) = \mathbb{R}^n \oplus \mathfrak{so}(n) \to \mathbb{R}^n \rtimes_{aff} \mathfrak{so}(n)$ is simply the identity. But these two spaces have a different Lie algebra structure, which is measured by the curvature $\kappa_o = -[X, Y]$. The Riemannian curvature at $o$ is

$$R_o(X, Y) = -\mathrm{ad}(X, Y) = XY^t - YX^t.$$

In tensor-notation

$$R_{ij}{}^{r}{}_{s} = \delta_i^r g_{js} - \delta_j^r g_{is}.$$

The Ricci-curvature at $o$ is

$$R_{ij}{}^{i}{}_{s} = \delta_i^i g_{js} - \delta_j^i g_{is} = (m-1) g_{js}$$

and the scalar curvature is $m(m-1)$. ⌟

# 6. Infinitesimal Flag Structures and Parabolic Geometries

In this chapter we consider the relation between homogeneous parabolic geometries and their underlying geometric structures.
We give a short exposition of filtered manifolds and associated notions since this is the type of geometric structure naturally obtained from a parabolic geometry. For a more in-depth treatment confer to [10, 13, 4].
Since the basic notions get no simpler in the homogeneous case we introduce them for general manifolds in section 6.1. Also the notion of a parabolic geometry is introduced in general in sections 6.2 and 6.3.

6.1. **Filtrations, the associated graded and the Levi-bracket.** Let $V$ be a finite-dimensional vector space. A *filtration* of $V$ is given by subspaces $V^i \subset V, i \in \mathbb{Z}$, such that $V^i \supset V^{i+1}$ such that there are $l < r \in \mathbb{Z}$ with $V^i = V$ for $i \leq l$ and $V^i = \{0\}$ for $i > r$.
Given such a *filtered vector space* $V$ we can construct its *associated graded* $\mathrm{gr}(V)$: denote $\mathrm{gr}_i(V) := V^{i+1}/V^i$ and

$$\begin{aligned}\mathrm{gr}(V) := V^l/V^{l+1} \oplus \ldots \oplus V^r/V^{r+1} = \\ \mathrm{gr}_l(V) \oplus \ldots \oplus \mathrm{gr}_r(V).\end{aligned}$$

It's clear how these notions extend to *filtrations* of vector bundles by smooth subbundles and their associated gradeds.
Now, given a manifold $M$ together with a filtration of its tangent bundle, we can demand that this filtration is compatible with the Lie-bracket on $\mathfrak{X}(M)$; Denote by $\mathfrak{X}(M, T^iM)$ the space of $T^iM$-valued vector-fields on $M$.

**Definition 6.1.1.** A *filtered manifold* is a manifold $M$ together with a filtration $TM = T^lM \supset \ldots T^rM = M$ such that for sections $\xi_1 \in \mathfrak{X}(M, T^iM), \xi_2 \in \mathfrak{X}(M, T^jM)$

$$[\xi_1, \xi_2] \in \mathfrak{X}(M, T^{i+j}M).$$

Now, for a filtered manifold $M$, $x \in M$ and $i, j \in \mathbb{Z}$ consider the map

$$\begin{aligned}\mathfrak{X}(M, T^iM) \times \mathfrak{X}(M, T^jM) \to \mathrm{gr}_{i+j}(TM)_x, \\ \xi_1, \xi_2 \mapsto [\xi_1, \xi_2]_x + T^{i+j+1}M_x.\end{aligned}$$

Since for a $f \in \mathrm{C}^\infty(M)$

$$[\xi_1, f\xi_2] = f[\xi_1, \xi_2] + (\xi_1 \cdot f)\xi_2$$

the map $\xi_1, \xi_2 \mapsto [\xi_1, \xi_2]_x + T^{i+j}M_x$ is in fact bilinear over $\mathrm{C}^\infty(M)$ and thus only depends on the values of $\xi_1, \xi_2$ in $x$.
And again by our condition on the Lie-bracket of a filtered manifold this map factorizes in fact to a skew-symmetric bilinear map

$$\mathrm{gr}_i(TM)_x \times \mathrm{gr}_j(TM)_x \to \mathrm{gr}_{i+j}(TM)_x.$$

These maps, for all relevant $(i, j) \in \mathbb{Z}^2$, define a map

$$\mathcal{L} \in \Lambda^2(\mathrm{gr}(TM)^*) \otimes \mathrm{gr}(TM)$$

the *Levi-bracket.* One can check that $\mathcal{L}$ satisfies the Jacobi-identity and thus $\mathrm{gr}(TM)_x$ is endowed with the structure of a nilpotent graded Lie algebra for every $x \in M$. I.e., we have a (not necessarily locally trivial) bundle of nilpotent graded Lie algebras $\mathrm{gr}(TM)$.

6.2. **Basic facts about $|k|$-graded Lie algebras.**

**Definition 6.2.1.** Let $\mathfrak{g}$ be a semisimple Lie algebra. A grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_0 \oplus \ldots \oplus \mathfrak{g}_k$ which is compatible with the Lie-bracket in the sense that for $X \in \mathfrak{g}_i, X' \in \mathfrak{g}_j$ $[X, X'] \in \mathfrak{g}_{i+j}$ is called a $|k|$-grading on $\mathfrak{g}$.

For an element $X \in \mathfrak{g}$ we denote the projection of $X$ to $\mathfrak{g}_i$ by $X_{\mathfrak{g}_i}$. The projection to $\mathfrak{g}_-$ will simply be denoted by $X_-$.
The filtration which comes from this grading is $\mathfrak{g}^i = \mathfrak{g}_i \oplus \ldots \oplus \mathfrak{g}_k$.
For such a $\mathfrak{g}$, we have nilpotent subalgebras $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1}$ and $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$. $\mathfrak{p}_+$ is an ideal in the Lie subalgebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \ldots \oplus \mathfrak{g}_k$. Note that $\mathfrak{p}$ respects the filtration under its adjoint action.

6.2.1. *Grading element.* We have grading element $E \in \mathfrak{g}_0$, in the sense that the eigenspace of $\mathrm{ad}_E$ to the eigenvalue $i$ is $\mathfrak{g}_i$: For this consider the derivation $D$ on $\mathfrak{g}$ which is defined by $D(X) := iX$ for $X \in \mathfrak{g}_i$; it is a derivation, since for $X \in \mathfrak{g}_i, X' \in \mathfrak{g}_j$ $D([X, X']) = (i+j)[X, X'] = [iX, X'] + [X, jX'] = [DX, X'] + [X, DX']$. By semisimplicity of $\mathfrak{g}$ there is an element $E \in \mathfrak{g}$ for which $\mathrm{ad}_E = D$; To see that in fact $E \in \mathfrak{g}_0$ write $E = E_{-i} \oplus \ldots \oplus E_i$ with $E_i \in \mathfrak{g}_i$. Then $0 = [E, E] = \sum_{i=-k}^{k}[E, E_i] = \sum_{i=-k}^{k} iE_i$, which shows that $E_i = 0$ for $i \neq 0$. (Note that since for $X_0 \in \mathfrak{g}_0$ $[E, X_0] = 0X_0 = 0$ the grading element $E$ lies in the center of $\mathfrak{g}_0$.)

6.2.2. *Duality.* Fix any non-degenerate invariant bilinear form $B$ on $\mathfrak{g}$. Then for $X \in \mathfrak{g}_i, X' \in \mathfrak{g}_j$ $B([E, X], X') = -B(X, [E, X'])$ by invariance, and thus by definition of $E$ $B(iX, X') = -B(X, jX')$, or $(i+j)B(X, X') = 0$. But $B$ is non-degenerate. Thus for $X \in \mathfrak{g}_i$ there is a $X' \in \mathfrak{g}$ with $B(X, X') \neq 0$; but from $(i+j)B(X, X') = 0$ it follows that the pairing of $\mathfrak{g}_i$ with any $\mathfrak{g}_j$ with $j \neq -i$ is trivial. Thus the pairing of $\mathfrak{g}_{-i}$ with $\mathfrak{g}_i$ under $B$ must be non-degenerate; i.e., once we fix such a form $B$ the subspaces $\mathfrak{g}_{-i}$ and $\mathfrak{g}_i$ are dual. By invariance of $B$ this is a duality of $\mathfrak{g}_0$-modules. (Also note that it follows that $B$ is non-degenerate on $\mathfrak{g}_0$.)
Since $\mathfrak{g}^{i+1}$ has the canonical complement $\mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-i}$ we can identify $\mathfrak{g}/\mathfrak{g}^{i+1}$ with $\mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-i}$. (This is an identification of $\mathfrak{g}_0$-modules). Also note that since the dual space of $\mathfrak{g}/\mathfrak{p}$ is the annihilator of $\mathfrak{p}$, which is $\mathfrak{p}_+$ under $B$, we have a duality of $\mathfrak{p}$-modules between $\mathfrak{g}/\mathfrak{p}$ and $\mathfrak{p}_+$.

6.2.3. *Group-level.* For some Lie group $G$ which has Lie algebra $\mathfrak{g}$ the Lie subgroup $P$ defined as the group of all elements in $G$ which respect the filtration $(\mathfrak{g}^i)$ has Lie algebra $\mathfrak{p}$. The Lie subgroup of all elements of $G$ which respect the grading is denoted $G_0$ and has Lie algebra $\mathfrak{g}_0$.

The exponential map on $\mathfrak{g}$ restricts to a diffeomorphism of $\mathfrak{p}_+$ with its corresponding Lie subgroup $P_+$ which is formed by all elements $p \in G$ which satisfy $\mathrm{Ad}(p)\mathfrak{g}^i \subset \mathfrak{g}^{i+1}$ for all $i \in \mathbb{Z}$. We even have: the map $(g_0, Z) \mapsto g_0 \exp(Z)$ is a diffeomorphism between $G_0 \times \mathfrak{p}_+$ and $P$.
$P_+$ is a normal subgroup of $P$ and $P/P_+ = G_0$.

6.2.4. *The associated graded.* If we regard $\mathfrak{g}$ as filtered, then the associated graded $\mathrm{gr}(\mathfrak{g})$ is of course canonically isomorphic to $\mathfrak{g}$ as vector space. Since $P$ respects the filtration-components it descends to an action on $\mathrm{gr}(P)$. As $G_0$ modules $\mathfrak{g} \cong \mathrm{gr}(\mathfrak{g})$. However the $\mathcal{P}_+$ actions differ: it acts trivially on $\mathrm{gr}(\mathfrak{g})$.

6.3. **Parabolic Cartan Geometries.** Let $G$ be a semisimple Lie group whose Lie algebra $\mathfrak{g}$ is $|k| - \textit{graded}$ and let $P$ be the subgroup of $G$ formed by all elements which preserve the filtration of $\mathfrak{g}$ under their adjoint action. Then a Cartan geometry $(\mathcal{G} \to M, \omega)$ of type $(G, P)$ on a manifold $M$ is called a *parabolic geometry.*
We have $TM = \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$, and so the $P$-invariant filtration of $\mathfrak{g}/\mathfrak{p}$ induces a filtration of $TM$. Furthermore $\mathrm{gr}(TM) = \mathcal{G} \times_P \mathrm{gr}(\mathfrak{g}/\mathfrak{p})$. But $P_+$ acts trivially on $\mathrm{gr}(\mathfrak{g}/\mathfrak{p})$ and is a normal Lie-Subgroup of $P$ with $P/P_+ = G_0$; we have $P = G_0 \rtimes P_+$. Thus we can factor out the action of $P_+$: $\mathrm{gr}(TM) = (\mathcal{G}/P_+) \times_{G_0} \mathrm{gr}(\mathfrak{g}/\mathfrak{p})$. But as a $G_0$-module $\mathrm{gr}(\mathfrak{g}/\mathfrak{p}) = \mathfrak{g}_-$, and with $\mathcal{G}_0 := \mathcal{G}/P_+$ we have $\mathrm{gr}(TM) = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_-$. This makes $\mathrm{gr}(TM)$ into a bundle of nilpotent graded Lie algebras, and $\mathcal{G}_0 \to M$ is a reduction of structure group of this bundle to $G_0$.
Now, if the induced filtration of $TM$ makes $M$ to a filtered manifold, we have two brackets on $\mathrm{gr}(TM)$: the Levi-bracket defined in section 6.1 and the bracket induced by the Cartan-structure. In this case, if these brackets coincide, the parabolic geometry is called *regular.*
The general theory of parabolic geometries ([5]) provides us with a simple condition on the curvature of a Cartan connection for the induced geometry to be regular: $\kappa(\mathfrak{g}_i, \mathfrak{g}_j) \subset \mathfrak{g}^{i+j+1}$. Under this condition $TM$ is filtered and the two brackets coincide.
Let $M$ be a filtered manifold $M$ such that some regular parabolic geometry of type $(G, P)$ on $M$ induces the same filtration as the given one. Then we say that we have an *infinitesimal flag structure of type $G/P$* on $M$.
There are different (non-isomorphic) regular parabolic geometries inducing the same infinitesimal flag structure on $M$. Thus for being able to (naturally) prolong infinitesimal flag structures to parabolic geometries we need a normalization condition on the geometry. This is provided by the Kostant-codifferential $\partial^*$,

$$\partial^* : \Lambda^i(\mathfrak{g}/\mathfrak{p}^*) \otimes \mathfrak{g} \to \Lambda^{i-1}(\mathfrak{g}/\mathfrak{p}^*) \otimes \mathfrak{g}.$$

Since the curvature function $\mathfrak{k}$ of a Cartan connection has values in $\Lambda^2(\mathfrak{g}/\mathfrak{p}^*) \otimes \mathfrak{g}$ it makes sense to consider $\partial^*\kappa$ and we say that a parabolic geometry is *normal* if

$$\partial^*\kappa = 0. \tag{12}$$

An equivalent condition to $\partial^*\kappa = 0$ is shown in [4]:
Take a basis $X_1, \dots, X_m$ of $\mathfrak{g}_-$ and its dual basis $Z_1, \dots, Z_m$ of $\mathfrak{p}_+$. Then we demand that for every $X \in \mathfrak{g}_-$.

$$2\sum_{i=1}^{m}[Z_i, \kappa(X, X_i)] - \sum_{i=1}^{m}\kappa([Z_i, X]_{\mathfrak{g}-}, X_i) = 0. \tag{13}$$

(Here $[Z_i, X]_{\mathfrak{g}-}$ is the projection of $[Z_i, X]$ to $\mathfrak{g}_-$). We will use this condition in our calculations below in chapters 7 and 8. Now we restrict ourselves to the homogeneous case.

6.4. **Homogeneous Infinitesimal Flag Structures.** Denote $M = H/K$ and let $TM = T^{-k}M \supset T^{-k+1}M \supset \dots \supset T^{-1}M \supset T^0M = M$ be an $H$-invariant filtration of the tangent bundle; i.e.: $T^{-i}_{h'hK}M = T_{hK}\check{\lambda}_{h'}T^{-1}_{hK}M$. Then the filtration is of course determined by its filtration of the tangent space at $o = K \in H/K$, which we write as $\mathfrak{h}/\mathfrak{k} = F^{-k} \supset \dots \supset F^0 = \{0\}$. This filtration of $\mathfrak{h}/\mathfrak{k}$ must be $K$-invariant; and it is clear that any $K$-invariant filtration of $\mathfrak{h}/\mathfrak{k}$ extends to an $H$-invariant filtration of $T(H/K)$. We further demand compatibility of the Lie-bracket with this filtration; i.e., we want $H/K$ to be a filtered manifold, as defined in 6.1.1.
What is the condition on the filtration of $\mathfrak{h}/\mathfrak{k}$, such that the resulting filtration of the tangent bundle is compatible with the bracket? We help ourselves by considering the (invariant) filtration of $TH$ which one gets by extending $\hat{F}^i := \pi^*_{\mathfrak{h}/\mathfrak{k}}F^i$, or $T^i_hH := (T_h\pi_{H/K})^{-1}T^i_{hK}M$. (Since $\hat{F}^0 = \mathfrak{k}$, there is one more filtration-component, and $\hat{F}^1 = \{0\}$). Then $\frac{d}{dt}_{|t=0}h\exp(tX) \in T^iH \Leftrightarrow \frac{d}{dt}_{|t=0}h\exp(tX)K \in T^iH/K$. We claim that $H/K$ is filtered iff $H$ is filtered.
First note that compatibleness with the brackets is a local claim, and by invariance it may be checked as well around $o = eK$. Now fix some complement $\mathfrak{l}$ of $\mathfrak{k}$ in $\mathfrak{h}$. Then for some neighborhood $W = W_\mathfrak{l} \times W_\mathfrak{k}$ of $\{0, 0\} \in \mathfrak{l} \times \mathfrak{k}$ the map

$$\theta : W \to H$$
$$(X, Y) \mapsto \exp(X)\exp(Y)$$

is a chart of some neighborhood $U$ of $e$; The map $\pi_{H/K} \circ \theta$ is a chart of $Uo = UK \subset H/K$.
$s = \exp \circ (\pi_{H/K} \circ \theta_{|\mathfrak{l}})^{-1}$ is a local section of $H \to H/K$. Less formally, around some neighborhood of $o$ every $hK$ can be uniquely written as $lK$ with $l \in \exp(W_\mathfrak{l})$.
Now take some field $\xi$ which is defined on the neighborhood $Uo$ of $o$, i.e., $\xi \in \Gamma(T(Uo))$. With

$$\tilde{\xi} : W_\mathfrak{l} \to \mathfrak{l},$$
$$\tilde{\xi}(X) = T\check{\lambda}_{\theta((X,0))}\xi(\pi_{H/K}(\theta((X, 0))))$$

we have

$$\xi(\exp(X)K) = \frac{d}{dt}_{|t=0} l\exp(t\tilde{\xi}(X))K.$$

Define $\hat{\xi} \in \Gamma(TU)$,

$$\hat{\xi}(\exp(X)\exp(Y)) := \frac{d}{dt}_{|t=0} \exp(X)\exp(t\tilde{\xi}(X))\exp(Y) =$$
$$= T_e\lambda_{\exp(X)\exp(Y)}\mathrm{Ad}(\exp(-Y))\tilde{\xi}(\exp(X)).$$

This is a lift of $\xi$ and most importantly $\xi \in \Gamma(Uo, T^i(Uo))$ iff $\hat{\xi} \in \Gamma(U, T^iU)$. (In the same way we can extend any $X \in \mathfrak{h}$ to a projectable field around $e$, and if $X \in \hat{F}^i$, this field will lie in $\Gamma(T^iH)$.) Now take some fields $\xi \in \Gamma(T^iH/K), \eta \in \Gamma(T^jH/K)$; these we can lift to fields $\hat{\xi}$, $\hat{\eta}$ on $U \subset H$, which lie in the $i$-th and $j$-th filtration-components of $H$ by definition. Now $[\hat{\xi}, \hat{\eta}]$ is related to $[\xi, \eta]$, and thus $[\hat{\xi}, \hat{\eta}]$ lies in the $i+j$-th filtration component iff $[\xi, \eta]$ does. This proves our claim, since from above we also know that there are local frames of projectable fields of a filtration component. But the condition for the filtration $\hat{F}^i$ to make $H$ into a filtered manifold is easy: we need that $\mathfrak{h}$ with the filtration $\hat{F}^i$ is a filtered Lie algebra:

**Definition 6.4.1.** A Lie algebra $\mathfrak{h}$ together with a filtration $\hat{F}^i$ is called filtered if for for all $i, j \in \mathbb{Z}$

$$[\hat{F}^i, \hat{F}^j] \subset \hat{F}^{i+j}.$$

This is easily translated into a condition on the filtration $F^i$ of $\mathfrak{h}/\mathfrak{k}$: for $X + \mathfrak{k} \in F^i$, $X' + \mathfrak{k} \in F^j$ it is necessary that $[X, X'] + \mathfrak{k} \in F^{i+j}$. Now if $H/K$ is filtered we have the Levi-bracket on the associated graded. Lets calculate it. Let $X + \mathfrak{k} \in F^i, X' + \mathfrak{k} \in F^j$. Then $\mathcal{L}_o(X + F^{i+1}, X' + F^{j+1}) = [\xi, \eta](o) + F^{i+j+1}$, where $\xi$, $\eta$ are arbitrary extensions of $X, X'$ into the same filtration components. Let $\hat{\xi}, \hat{\eta}$ be local lifts as above. Then by relatedness $[\xi, \eta](o) = [\hat{\xi}, \hat{\eta}](e) + \mathfrak{k}$, and thus $[\xi, \eta](o) + F^{i+j+1} = [\hat{\xi}, \hat{\eta}](e) + \hat{F}^{i+j+1}$. Since $\mathrm{gr}_{-k}(TH) \oplus \cdots \mathrm{gr}_{-1}(TH)$ is just the pullback $\pi^*(\mathrm{gr}(TM))$ of $\mathrm{gr}(T/H)$ under $\pi : H \to H/K$, this means that also the Lie algebra-structure on this part of the grading (i.e., its Levi-Bracket) is pulled back. But $[\hat{\xi}, \hat{\eta}](e) + \hat{F}^{i+j+1}$ only depends on the values of $\hat{\xi}, \hat{\eta}$ at $e$, and we may take arbitrary extensions to fields which stay in the respective filtration components. Thus

$$\mathcal{L}_o(X + F^i, X' + F^j) = [\hat{\xi}, \hat{\eta}](e) + \hat{F}^{i+j+1} =$$
$$[L_X, L_{X'}](e) + \hat{F}^{i+j+1} = ([X, X'] + \mathfrak{k}) + F^{i+j+1}.$$

Summarizing, we see

**Theorem 6.4.2.** *i. Every invariant infinitesimal flag structure on $H/K$ is described by a $K$-invariant filtration $F^i$ of $\mathfrak{h}/\mathfrak{k}$.*

*ii. This makes $H/K$ into a filtered manifold iff for $X + \mathfrak{k} \in F^i$, $X' + \mathfrak{k} \in F^j$ $[X, X'] + \mathfrak{k} \in F^{i+j}$. This is equivalent to $\mathfrak{h}$ together with the filtration $\hat{F}^i := \pi^*_{\mathfrak{h}/\mathfrak{k}}F^i$ being a filtered Lie algebra.*

*iii. In this case the Levi-bracket equips the associated graded $\mathrm{gr}(T(H/K))$ with the structure of a nilpotent graded Lie algebra. Since $T(H/K) = H \times_K \mathfrak{h}/\mathfrak{k}$ the associated graded is $\mathrm{gr}(TM) = H \times_K \mathrm{gr}(\mathfrak{h}/\mathfrak{k})$, and the Lie algebraic structure on $\mathrm{gr}(\mathfrak{h}/\mathfrak{k})$, resp. the Levi-bracket at $o = K \in H/K$,* *is given by:*

$$\mathcal{L}_o(X+\mathfrak{k}, X'+\mathfrak{k}) = ([X,X']+\mathfrak{k}) + F^{i+j} \tag{14}$$

*for* $X+\mathfrak{k} \in F^i$, $X'+\mathfrak{k} \in F^j$.

When is this a regular infinitesimal flag structure of type $(G,P)$? Consider a homomorphism $\Psi : K \to P$ and a map $\alpha : \mathfrak{h} \to \mathfrak{g}$ describing a Cartan connection. Especially, $\tilde{\alpha} : \mathfrak{h}/\mathfrak{k} \to \mathfrak{g}/\mathfrak{p}$ is an isomorphism of vector spaces. For the filtration of $\mathfrak{h}/\mathfrak{k}$ to be induced by $(\mathcal{G} \to M, \omega)$ we need to have $F^i = \tilde{\alpha}^{-1}(\mathfrak{g}^i + \mathfrak{p}) = \alpha^{-1}(\mathfrak{g}^i)$. Now $\tilde{\alpha}$ induces an isomorphism of $K$-modules between $\mathrm{gr}(\mathfrak{h}/\mathfrak{k})$ and $\mathfrak{g}_-$ which is given simply by $(X+\mathfrak{k}) + F^{i+1} \to \alpha(X)_{\mathfrak{g}_i}$ for $X+\mathfrak{k} \in F^i$. Take an $X'+\mathfrak{k} \in F^j$; The Levi-bracket of $(X+\mathfrak{k})+F^i$ and $(X'+\mathfrak{k})+F^j$ is $\alpha([X,X'])_{i+j}$ under this isomorphism. it has to coincide with $[\alpha(X)_i, \alpha(X')_j]$. Since $\alpha : \mathfrak{h} \to \mathfrak{g}$ is filtration-preserving $\alpha(X) \in \mathfrak{g}^i$, $\alpha(X') \in \mathfrak{g}^j$ and since $[X,X']+\mathfrak{k} \in F^{i+j}$ also $\alpha([X,X']) \in \mathfrak{g}^{i+j}$. Thus the condition that the Levi-bracket coincides with the bracket induced by the Cartan geometry on $M$ is that $\kappa(X,X') \in \mathfrak{g}^{i+j+1}$ for $X \in \mathfrak{g}^i$ and $X' \in \mathfrak{g}^j$. Of course this follows immediately from the condition mentioned earlier without proof for the general (non-homogeneous) case.

Assume now that we have a $K$-invariant complement $\mathfrak{n}$ of $\mathfrak{h}$ and that $\Psi$ has values in $G_0$. Then an $\alpha : \mathfrak{h} \to \mathfrak{g}$ describing a Cartan connection induces a $K$-invariant isomorphism $\mathfrak{n} = \mathfrak{h}/\mathfrak{k} \cong \mathfrak{g}/\mathfrak{p} = \mathfrak{g}_-$, where we use that $\mathfrak{g}/\mathfrak{p} = \mathfrak{g}_-$ as a $G_0$-module. Now $\mathfrak{n}$ becomes a graded $G_0$-module by the isomorphism $\mathfrak{n} \overset{\alpha_0}{\cong} \mathfrak{g}_-$ and conversely $\mathfrak{g}_-$ becomes a $K < G_0$-module. By equivariance $\mathrm{Ad} : K \to \mathrm{GL}(\mathfrak{n})$ has in fact values in $G_0$ and we can now regard $\Psi$ simply as $\mathrm{Ad} : K \to G_0$. $\alpha$ may be written $\alpha_0 + \phi \circ \alpha_0$, where $\phi : \mathfrak{g}_- \to \mathfrak{g}$ is of positive homogeneity.

In fact, in the examples below, we will start with an identification $\mathfrak{n} \overset{\alpha_0}{\cong} \mathfrak{g}_-$ and under this identification $\mathrm{Ad}_{|K}$ will have values in $G_0$. Regularity means that $[\alpha_0(X), \alpha_0(X')] - \alpha_0([X,X']_{i+j}) = 0$ for $X \in F^i$ and $X' \in F^j$. All of this data will come from a geometric structure on $H/K$.

Our problem will be to find a change of $\alpha_0$ to a map $\alpha : \mathfrak{h} \to \mathfrak{g}$ which induces the same regular parabolic geometry as $\alpha_0$ and satisfies (12). From the general theory we know that such an $\alpha$ exists and is unique up to equivalency. Cf. [4, 5].

# 7. Conformal Structures

We start with the same situation as in chapter 5: we have a $K$-invariant (pseudo-)inner product $g$ on $\mathfrak{n} := \mathfrak{h}/\mathfrak{k}$. But now we consider the induced conformal class of (pseudo-)Riemannian metrics on $H/K$. We will prolong this structure to a Cartan geometry of type

$$\left(\mathrm{PO}\left(\mathbb{R}\oplus\mathfrak{n}\oplus\mathbb{R}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & g & 0 \\ 1 & 0 & 0 \end{pmatrix}\right), P\right).$$

Here $P$ is the stabilizer of the isotropic line through $e_1 := (1,0,0) \in \mathbb{R}\oplus\mathfrak{n}\oplus\mathbb{R}$. We first describe the Lie algebra of $G = \mathrm{PO}(\mathbb{R}\oplus\mathfrak{n}\oplus\mathbb{R})$ and see below (in 7.1.2) that the underlying structure of a parabolic geometry of type $(G,P)$ is in fact simply a conformal class of Riemannian metrics.

## 7.1. $\mathrm{PO}(\mathbb{R}\oplus\mathfrak{n}\oplus\mathbb{R})$.

Let $\mathfrak{n}$ be a real vector space equipped with a pseudo-inner product $g$. We extend $g$ to a (pseudo-)inner product $\tilde{g} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & g & 0 \\ 1 & 0 & 0 \end{pmatrix}$ on $\mathbb{R}\oplus\mathfrak{n}\oplus\mathbb{R}$. If $g$ has signature $(p,q)$ then $\tilde{g}$ has signature $(p+1,q+1)$.

**Theorem 7.1.1.** *i.* $\mathfrak{g} = \mathfrak{so}(\mathbb{R}\oplus\mathfrak{n}\oplus\mathbb{R},\tilde{g})$ *is a* $|1|$*-graded Lie algebra:* $\mathfrak{g} = \mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$ *with* $\mathfrak{g}_{-1} = \mathfrak{n}, \mathfrak{g}_1 = \mathfrak{n}^*$ *and* $\mathfrak{g}_0 = co(\mathfrak{n}, g)$*. (Here* $\mathfrak{g}_{-1}$ *and* $\mathfrak{g}_1$ *are abelian.) We will write*

$$\mathfrak{g} = \{X \oplus (\alpha, A) \oplus Z^t | X, Z \in \mathfrak{n},\ \alpha \in \mathbb{R},\ A \in \mathfrak{so}_g(\mathfrak{n})\}.$$

*Here*

$$X \oplus (\alpha, A) \oplus Z^t$$

*corresponds to the matrix*

$$\begin{pmatrix} -\alpha & Z^t & 0 \\ X & A & -Z \\ 0 & -X^t & \alpha \end{pmatrix}.$$

*For an element* $X$ *of* $\mathfrak{g}$ *we will denote the projection of* $X$ *to* $\mathfrak{g}_i$ *by* $X_{\mathfrak{g}_i}$*. The nontrivial brackets are*

$$\begin{aligned}
[(\alpha, A), (\alpha', A')] &= (0, [A, A']) \\
[(\alpha, A), X] &= (A + \alpha)X \\
[(\alpha, A), Z^t] &= -Z^t(A + \alpha) \\
[X, Z^t] &= (g(X, Z), XZ^t - ZX^t);
\end{aligned}$$

*especially, the central part of the pairing* $\mathfrak{g}_{-1}\times\mathfrak{g}_1 \to \mathfrak{g}_0$ *establishes a duality between* $\mathfrak{g}_{-1}$ *and* $\mathfrak{g}_1$*.*

*ii. The subalgebra* $\mathfrak{g}_0\oplus\mathfrak{g}_1 < \mathfrak{g}$ *we denote by* $\mathfrak{p}$*. It is the stabilizer of the isotropic line* $\{(t,0,0)|t\in\mathbb{R}\}$ *and the Lie algebra of* $P = N_G(\mathfrak{p}) < G$*.*

*iii.* $\mathfrak{g}_0$ *is the Lie algebra of* $G_0 = \mathrm{CO}(\mathfrak{n}) < G$*.*

*Remark* 7.1.2. Since $\mathfrak{g}$ is 1-graded the filtration induced on $TM = T(H/K)$ by a parabolic geometry $(\mathcal{G} \to M, \omega)$ of type $(G, P)$ is trivial and $\mathrm{gr}(TM) = TM$. In 6.3 we saw that $(\mathcal{G} \to M, \omega)$ induces a reduction of structure group of $\mathrm{gr}(TM) = TM$ to $G_0 = CO(\mathfrak{h}/\mathfrak{k})$. But this is the same as a conformal class of a metric on $T(H/K)$. ⌟

7.2. **Conformal normalization.** Assume now that the signature of the metric $g$ on $\mathfrak{n}$ is $(p, q)$ with $p + q \geq 3$. Take some orthonormal basis $X_i$ of $\mathfrak{n}$. Then its dual basis is $Z_i := \varepsilon_i X_i^t$, where $\varepsilon_i = g(X_i, X_i)$ is 1 or $-1$. Then $\partial^*\kappa = 0$ is equivalent to (recall 13)

$$2\sum_{i=1}^{m}[Z_i, \kappa(X, X_i)] - \sum_{i=1}^{m}\kappa([Z_i, X]_{\mathfrak{g}-}, X_i) = 0. \tag{15}$$

(Here $[Z_i, X]_{\mathfrak{g}-}$ is the projection of $[Z_i, X]$ to $\mathfrak{g}_-$).

Recall that $\mathfrak{so}(\mathbb{R} \oplus \mathfrak{n} \oplus \mathbb{R})$ is $\mathfrak{n} \oplus \mathfrak{co}(\mathfrak{n}) \oplus \mathfrak{n}^*$, and in this case $\mathfrak{g}_- = \mathfrak{n}$ and $\mathfrak{p}_+ = \mathfrak{n}^*$. Since $[\mathfrak{g}_-, \mathfrak{p}_+] \subset \mathfrak{g}_0$ the second sum always vanishes.
Thus the normalization condition is

$$\sum_{i=1}^{m}\varepsilon_i[X_i^t, \kappa(X, X_i)_-] = 0 \text{ and } \sum_{i=1}^{m}\varepsilon_i[X_i^t, \kappa(X, X_i)_{\mathfrak{g}_0}] = 0.$$

Since the adjoint action of $\mathfrak{g}_0 = \mathfrak{co}(\mathfrak{n})$ on $\mathfrak{g}_- = \mathfrak{n}$ is just the dual of the standard action, the second equation is equivalent to

$$\sum_{i=1}^{m}\varepsilon_i\kappa(X, X_i)_{\mathfrak{g}_0}X_i = 0.$$

Pairing this term with $X'$ and using that $\mathfrak{g}_0 = \mathfrak{co}_g(\mathfrak{n})$ we see that this is equivalent to

$$\sum_{i=1}^{m} g(\varepsilon_i X_i, \kappa(X, X_i)_{\mathfrak{g}_0}X') = 0$$

for all $X' \in \mathfrak{n}$. But this just says that for all $X, X' \in \mathfrak{n}$ the map

$$\mathfrak{n} \to \mathfrak{n},$$
$$X \mapsto \kappa(X, X_1)X_2$$

is trace-free. I.e., the Ricci type contraction of the $\mathfrak{g}_0$-part of the curvature function vanishes. In tensor-notation, with $R := \kappa_{\mathfrak{g}_0}$,

$$R_{ij}{}^{i}{}_{s} = 0.$$

The first equation is more subtle. Recall that for $X, Y \in \mathfrak{n}$ the bracket of $X$ with $Y^t$ lies in $\mathfrak{g}_0 = \mathfrak{co}(\mathfrak{n}) = \mathbb{R} \oplus \mathfrak{so}(n)$ and is given by $[X, Y^t] = (g(X, Y), XY^t - YX^t)$. Thus we have (especially), with $X = X_j$, $\sum_{i=1}^{m}\varepsilon_i\kappa(X_j, X_i)_- X_i^t - \varepsilon_i X_i\kappa(X_j, X_i)_-^t = 0$. Now pair this expression with $X_l$: $\sum_{i=1}^{m}\varepsilon_i\kappa(X_j, X_i)_- g(X_i, X_l) - \varepsilon_i X_i g(\kappa(X_j, X_i)_-, X_l) = 0$. Since $X_i$ are orthonormal, the first part of this sum is simply $\kappa(X_j,X_l)_- = \sum_{i=1}^{m} \varepsilon_i X_i g(\kappa(X_j,X_l)_-,X_i)$. Thus, since the $X_i$ are linearly independent, we see $g(\kappa(X_j,X_l)_-,X_i) = g(\kappa(X_j,X_i)_-,X_l)$. But since $\kappa$ is skew-symmetric, this is equivalent to vanishing of $\kappa_-$:

$$\begin{aligned}
&g(\kappa(X_j,X_l)_-,X_i) = -g(\kappa(X_l,X_j)_-,X_i) = \\
&= -g(\kappa(X_l,X_i)_-,X_j) = g(\kappa(X_i,X_l)_-,X_j) = \\
&= g(\kappa(X_i,X_j)_-,X_l) = -g(\kappa(X_j,X_i)_-,X_l) = -g(\kappa(X_j,X_l)_-,X_i).
\end{aligned}$$

Thus we see $\partial^*\kappa = 0 \Leftrightarrow$

$$\kappa \text{ has values in } \mathfrak{p} \text{ and} \tag{16}$$

$$\text{the Ricci type contraction of } \kappa_{\mathfrak{g}_0} \text{ vanishes.} \tag{17}$$

7.3. **The prolongation.** Since (the restriction of) $\mathrm{Ad}_{\mathfrak{k}}$ has values in $O(\mathfrak{n})$, we can construct the $P$-principal bundle $H \times_{\mathrm{Ad}} P$. (Note that $\big(H \times_{\mathrm{Ad}} P\big)/P_+ = H \times_{\mathrm{Ad}} CO_g(\mathfrak{n})$, so the conformal geometry we started with appears as a reduction of structure group to $\mathrm{CO}(\mathfrak{n})$). Now every Cartan connection $\omega$ on $H \times_{\mathrm{Ad}} P$ is obtained from a $K$-equivariant extension $\alpha : \mathfrak{h} \to \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathfrak{n} \oplus \mathfrak{g}_0 \oplus \mathfrak{n}^*$ of $\mathrm{ad}_{|\mathfrak{k}}$.
By (16) the curvature of the normalized Cartan connection has values in $\mathfrak{p}$. Let $A : \mathfrak{h} \to \mathfrak{h}/\mathfrak{k}$ be some $K$-equivariant map which vanishes on $\mathfrak{k}$. Let $\alpha_\gamma$ defined by equation (11) of theorem 5.2.5. We define

$$\begin{aligned}
\alpha_\omega &= \pi_{\mathfrak{n}} \oplus \alpha_\gamma \oplus A \\
\mathfrak{h} &\to \mathfrak{g} = \mathfrak{n} \oplus \mathfrak{co}_g(\mathfrak{n}) \oplus \mathfrak{n}^*;
\end{aligned}$$

Lets calculate the curvature of the Cartan connection induced by an $\alpha$ of this form:

$$\begin{aligned}
&\kappa(X,X') = [\alpha_\omega(X),\alpha_\omega(X')] - \alpha_\omega([X,X']) = \\
&= [(X+\mathfrak{k}) + \alpha_\gamma(X) + A(X), (X'+\mathfrak{k}) + \alpha_\gamma(X') + A(X')] \\
&- (([X,X']+\mathfrak{k}) + \alpha_\gamma([X,X']) + A([X,X'])) = \\
&= 0 \oplus \big([\alpha_\gamma(X),\alpha_\gamma(X')] - \alpha_\gamma([X,X']) + [X+\mathfrak{k}, A(X')] - [X'+\mathfrak{k}, A(X)]\big) \\
&\oplus \big(-A([X,X']) + [\alpha_\gamma(X), A(X')] - [\alpha_\gamma(X'), A(X)]\big).
\end{aligned}$$

So

$$\begin{aligned}
&\kappa(X,X')_{\mathfrak{g}_0} = \\
&= [\alpha_\gamma(X),\alpha_\gamma(X')] - \alpha_\gamma([X,X']) + [X+\mathfrak{k}, A(X')^t] - [X'+\mathfrak{k}, A(X)^t] = \\
&= \big(\big(A(X')(X+\mathfrak{k}) - A(X)(X'+\mathfrak{k}), \\
&\alpha_\gamma(X)\alpha_\gamma(X') - \alpha_\gamma(X')\alpha_\gamma(X) - \alpha_\gamma([X,X']) \\
&+ \big((X+\mathfrak{k})A(X') - A(X')^t(X+\mathfrak{k})^t - (X'+\mathfrak{k})A(X) + A(X)^t(X'+\mathfrak{k})^t\big)\big).
\end{aligned}$$

We need to find a $K$-equivariant map $A : \mathfrak{h}/\mathfrak{k} \to \mathfrak{h}/\mathfrak{k}$ such that for all $X_1, X_2 \in \mathfrak{h}$

$$X \mapsto \kappa_{\mathfrak{g}_0}(X_1, X)X_2$$

is trace-free. If $A = 0$,

$$\kappa_{\mathfrak{g}_0}(X, X') = \alpha_\gamma(X)\alpha_\gamma(X') - \alpha_\gamma(X')\alpha_\gamma(X) - \alpha_\gamma([X, X']).$$

The change due to $A$ is

$$\partial A = \big(A(X')X_{\mathfrak{n}} - A(X)X'_{\mathfrak{n}},\\ (X+\mathfrak{k})A(X') - A(X')^t(X+\mathfrak{k})^t - (X'+\mathfrak{k})A(X) + A(X)^t(X'+\mathfrak{k})^t\big).$$

In tensor-notation

$$\partial A_{ij}{}^r{}_s = \delta^r_s(A_{ij} - A_{ji}) + \big(\delta^r_i A_{sj} - \delta^r_j A_{si}\big) + \big(g^{lr}A_{li}g_{sj} - g^{lr}A_{lj}g_{is}\big).$$

So the change in trace is

$$\partial A_{ij}{}^i{}_s = \big(A_{sj} - A_{js}\big) + \big(mA_{sj} - A_{sj}\big) + \big(g^{li}A_{li}g_{sj} - A_{sj}\big) =\\ -A_{js} + (m-1)A_{sj} + g^{li}A_{li}g_{sj}.$$

So if $A_{ij}$ is symmetric and trace-free the change in trace is $(m-2)A_{ij}$ If $A_{ij} = cg_{ij}$, the change in trace is $2(m-1)cg_{ij}$. If $A_{ij}$ is skew-symmetric the change in trace is $-mA_{ij}$. Thus the unique $A$ for which the trace of the $\mathfrak{g}_0$-component of the curvature of the Cartan connection induced by $\alpha_\omega = \pi_{\mathfrak{n}} \oplus \alpha_\gamma \oplus A$ vanishes is

$$\begin{aligned} A_{ij} &= -\frac{1}{m-2}\big(R_{ij} - \frac{R}{m}g_{ij}\big) - \frac{1}{2(m-1)}Rg_{ij} = \\ &= -\frac{1}{m-2}\big(R_{ij} + R(\frac{(m-2)}{2(m-1)} - \frac{1}{m})g_{ij}\big) = \\ &= -\frac{1}{m-2}\big(R_{ij} + R\frac{(m-2)-2m+2}{2m(m-1)}g_{ij}\big) = \\ &= -\frac{1}{m-2}\big(R_{ij} - \frac{R}{2(m-1)}g_{ij}\big) \end{aligned}$$

where $R_{ij} = R_{ai}{}^a{}_j$ is the Ricci-curvature and $R = g^{ij}R_{ij}$ is the scalar-curvature. Since $\pi_{\mathfrak{n}}$ and $\alpha_\gamma$ are already known to be $K$-equivariant, only $K$-equivariancy of $A$ remains to be seen. But this is clear, since both the (Riemannian) curvature $R$ and the (pseudo-)inner product are $K$-equivariant, and thus also the Ricci-type contraction of $R$ and the scalar-curvature are $K$-equivariant.
The change of the $\mathfrak{g}_0$-component of the curvature is

$$\partial A_{ij}{}^r{}_s = \big(\delta^r_i A_{sj} - \delta^r_j A_{si}\big) + \big(g^{lr}A_{li}g_{sj} - g^{lr}A_{lj}g_{is}\big).$$

So, recalling 5.2.4, the curvature of $\omega$ is

$$(0, R_{ij}{}^r{}_s + \big(\delta^r_i A_{sj} - \delta^r_j A_{si}\big) + \big(g^{lr}A_{li}g_{sj} - g^{lr}A_{lj}g_{is}\big),\\ - R_{la}\tau_{ij}{}^a + R_{ai}\alpha_j{}^a{}_l - R_{aj}\alpha_i{}^a{}_l).$$

*Example* 7.3.1. For the Riemannian sphere already discussed above (so $H = O(n+1)$ and $K = O(n)$) we have $\alpha_{\gamma|_{\mathfrak{n}}} = 0$; i.e., $\alpha_\gamma = \mathrm{ad} \circ \pi_{\mathfrak{k}}$, where $\pi_{\mathfrak{k}} : \mathfrak{so}(n+1) = \mathfrak{h} \to \mathfrak{k} = \mathfrak{so}(n)$ is the projection to $\mathfrak{so}(n)$. Notice that since both $\tau$ and $\alpha$ vanish, the Cartan curvature has values in $\mathfrak{g}_0$. We calculated the curvature tensor ${R_{ij}}^r{}_s = \delta_i^r g_{js} - \delta_j^r g_{is}$, the Ricci-curvature $R_{ij} = (m-1)g_{ij}$ and the scalar curvature $R = m(m-1)$. Thus

$$\begin{aligned} A_{ij} &= -\frac{1}{m-2}\big(R_{ij} - \frac{R}{2(m-1)}g_{ij}\big) = \\ &= -\frac{1}{m-2}\big((m-1) - \frac{m(m-1)}{2(m-1)}\big)g_{ij} = \\ &= -\frac{1}{m-2}\big(\frac{m}{2} - 1\big)g_{ij} = -\frac{1}{2}g_{ij}. \end{aligned}$$

Therefore

$$g^{lr}A_{li}g_{sj} = g^{lr}g_{li}\big(-\frac{1}{2}g_{sj}\big) = \delta_i^r A_{sj}.$$

Thus the $\mathfrak{g}_0$-component of the Cartan curvature is ${R_{ij}}^r{}_s - \big(\delta_i^r g_{sj} - \delta_j^r g_{si}\big) = 0$. i.e: The Cartan curvature is zero. This reflects the fact that the Euclidean sphere is locally conformally flat. ⌟

# 8. Contact and CR Structures

In [2] D. Alekseevsky and A. Spiro classified all compact simply connected homogeneous CR manifolds of hypersurface type with non-degenerate Levi-bracket.
One result of this chapter will be an explicit prolongation one such family of CR manifolds to Cartan geometries; We calculate the Cartan curvature and find out which of these CR manifolds are spherical.
We begin by introducing contact structures in 8.1, CR structures in 8.2 and discuss the relation of CR structures with the corresponding parabolic geometries in 8.3.1 and 8.4.1.

8.1. **Contact structures.** Consider a manifold $M$ endowed with a co-dimension 1-distribution $\mathcal{D}$ of $TM$. Then $T^{-2}M = TM, T^{-1}M = \mathcal{D}, T^0M = M$ makes $M$ into a filtered manifold (cf. 6.4). Thus we have the Levi-bracket $\mathcal{L}$ on the associated graded of $TM$. The nontrivial part of $\mathcal{L}$ is an element of

$$\Lambda^2(\mathcal{D}^*) \otimes TM/\mathcal{D}.$$

At every point $x \in M$ the Levi-bracket is a skew-symmetric bilinear form, and when this form is non-degenerate we say that $\mathcal{D}$ is a *contact distribution* on $TM$ or that we have a *contact structure* on $M$.
(Since non-degenerate skew-symmetric bilinear forms only exist on even-dimension vector spaces it follows that $M$ is odd-dimensional.)
For $U \subset M$ open consider a one-form $\Theta \in \Omega^1(U)$ which vanishes on $\mathcal{D}$ and is not zero at any point of $U$. $\Theta$ defines a local trivialization of $\mathrm{gr}_{-2}(TM)$.
One can check that non-degeneracy of the (trivialized) Levi-bracket

$$\mathcal{L}_x : \ker(\Theta)_x \times \ker(\Theta)_x \to \mathbb{R},$$
$$(X_1, X_2) \mapsto \Theta([\xi_1, \xi_2]) \text{ for } \xi_i \in \mathfrak{X}(U, \mathcal{D}), \xi_i(x) = X_i$$

is equivalent to

$$\Theta \wedge (d\Theta)^j \neq 0, \qquad (18)$$

where $j$ is half the dimension of $\mathcal{D}$. A nowhere vanishing form $\Theta$ satisfying (18) is called a (local) *contact form.* Note that a co-dimension one distribution $\mathcal{D}$ of $TM$ is a contact distribution iff there are local contact forms with kernel $\mathcal{D}$. Of course every global contact form describes a contact distribution.
Two contact manifolds $(M_1, \mathcal{D}_1)$ and $(M_2, \mathcal{D}_2)$ are equivalent when there is a diffeomorphism $f : M_1 \to M_2$ which satisfies $Tf(\mathcal{D}_1) = \mathcal{D}_2$.

8.1.1. *Invariant Contact Structures.* We now consider invariant contact structures on a homogeneous space $H/K$. Invariant co-dimension one subbundles $\mathcal{D}$ of $T(H/K)$ are exactly the extensions of $K$-invariant co-dimension one subspaces $D$ of $\mathfrak{n}$.

We know from theorem 6.4.2 that the Levi-bracket on the induced co-dimension 1-distribution of $T(H/K)$ is non-degenerate iff the bracket

$$[\cdot,\cdot] : D \times D \to \mathfrak{n}/D \text{ is non} - \text{degenerate}. \qquad (19)$$

If this holds we say that $D$ is a *contact subspace* of $\mathfrak{n}$ and in this case we have an invariant contact structure $\mathcal{D}$ on $H/K$.
Assume that we have a non-degenerate $K$-invariant bilinear form $\theta$ on $\mathfrak{n}$. Then a $K$-invariant element $Z \in \mathfrak{n}$ whose orthogonal complement (in $\mathfrak{n}$) is a contact subspace is called *contact element.* The $K$-invariant 1-form $\theta(Z, \cdot)$ on $\mathfrak{n}$ induces an invariant 1 form $\Theta$ on $H/K$. In fact, since $\ker(\theta) = \mathcal{D}$ this is a (global) invariant contact form.
Also note that having chosen such a contact element $Z$ we may regard the associated graded $\mathrm{gr}(T(H/K)$ as $H \times_K (\mathbb{R}Z \oplus D)$; here $\mathbb{R}Z \oplus D$ is a nilpotent graded Lie algebra with $(\mathbb{R}Z \oplus D)_{-2} = \mathbb{R}Z, (\mathbb{R}Z \oplus D)_{-1} = D$; The only nontrivial bracket being the Levi-bracket $D \times D \to \mathbb{R}Z$, which is given by

$$\mathcal{L}_o : \mathcal{D} \times \mathcal{D} \to \mathbb{R}Z,$$
$$X_1, X_2 \mapsto [X_1, X_2]_{\mathbb{R}Z},$$

where $[X_1, X_2]_{\mathbb{R}Z}$ is the projection of $[X_1, X_2] + \mathfrak{k}$ to $\mathbb{R}Z$.

8.2. **CR structures.** Let $\mathcal{D}$ be an even-dimensional, co-dimension 1 distribution of the tangent bundle of a manifold $M$, which shall be endowed with an almost complex structure $J \in \mathcal{D}^* \otimes \mathcal{D}$; i.e., $J^2 = -\mathrm{id}_{\mathcal{D}}$. (The existence of such an anti-involution on $\mathcal{D}$ implies that $M$ is odd-dimensional.)
Then we say that $(\mathcal{D}, J)$ is an *almost CR structure* of hypersurface type on $M$. When $\mathcal{D}$ is also a contact distribution, i.e., when the induced Levi-bracket $\mathcal{L} \in \Lambda^2(\mathcal{D}) \otimes TM/\mathcal{D}$ is non-degenerate, we say that this almost CR structure is non-degenerate.

**Definition 8.2.1.** i. An almost CR structure $(\mathcal{D}, J)$ on a manifold $M$ is *partially integrable* if for $\xi_1, \xi_2 \in \mathfrak{X}(M, \mathcal{D})$

$$[J\xi_1, \xi_2] + [\xi_1, J\xi_2] \in \mathfrak{X}(M, \mathcal{D}).$$

This is equivalent to the Levi-bracket $\mathcal{D} \times \mathcal{D} \to TM/\mathcal{D}$ being the imaginary part of an hermitian form on $\mathcal{D}$ for every trivialization of $TM/\mathcal{D}$. Non-degeneracy of the CR structure is equivalent to non-degeneracy of the hermitian form. If $TM/\mathcal{D}$ is oriented we can define the *signature* of a partially integrable almost CR structure as the signature of the induced hermitian form(s). If the hermitian form is positive definite (for an orientation of $TM/\mathcal{D}$) we say that the partially integrable almost CR structure is *strictly pseudoconvex.*

ii. When $(M, \mathcal{D}, J)$ is a partially integrable almost CR structure one has the Nijenhuis-tensor $N_J \in \Lambda^2(\mathcal{D}^*) \otimes \mathcal{D}$. It is defined

$$N(\xi_1, \xi_2) := [\xi_1, \xi_2] - [J\xi_1, J\xi_2] + J([J\xi_1, \xi_2] + [\xi_1, J\xi_2]) \tag{20}$$

for $\xi_1, \xi_2 \in \mathfrak{X}(M, \mathcal{D})$.

iii. An almost CR structure $(\mathcal{D}, J)$ on a manifold $M$ is *integrable* if it is partially integrable and its Nijenhuis-tensor $N$ vanishes. We then say that $(\mathcal{D}, J)$ is a CR structure on $M$.

*Remark* 8.2.2. That (20) really defines a tensor follows from $N$ being bilinear over $\mathrm{C}^\infty(M)$.

Note that

$$\begin{aligned}N(J\xi_1,\xi_2) &= [J\xi_1,\xi_2]+[\xi_1,J\xi_2]-J([\xi_1,\xi_2]-[J\xi_1,J\xi_2]) = \\ &= -J\big([\xi_1,\xi_2]-[J\xi_1,J\xi_2]+J([J\xi_1,\xi_2]+[\xi_1,J\xi_2])\big) = \\ &= -JN(\xi_1,\xi_2)\end{aligned}$$

and $N(\xi_1,J\xi_2) = -N(J\xi_2,\xi_1) = J(\xi_2,\xi_1) = -J(\xi_1,\xi_2)$. Thus $N$ is anti-complex linear in both arguments. ⌟

**Definition 8.2.3.** Two almost CR manifolds $(M_1,\mathcal{D}_1,J_1)$ and $(M_2,\mathcal{D}_2,J_2)$ are equivalent if there is a diffeomorphism $f: M_1 \to M_2$ which satisfies $Tf\mathcal{D}_1 = \mathcal{D}_2$ and $Tf\ J_1(X) = J_2\ TfX$ for all $X \in \mathcal{D}_1$.

8.2.1. *Invariant almost CR structures.* As above for contact structures consider a homogeneous space $H/K$ and denote its Lie algebra by $\mathfrak{n} = \mathfrak{h}/\mathfrak{k}$. The data on $\mathfrak{n}$ defining an almost CR structure are:

i. a contact subspace $D$ of $\mathfrak{n}$,
ii. a $K$-invariant complex structure $J$ on $D$.

The Levi-bracket is given by

$$\begin{aligned}\mathcal{L}_o : \mathcal{D}\times\mathcal{D} &\to \mathfrak{n}/D, \\ X_1, X_2 &\mapsto ([X_1,X_2]+\mathfrak{k})+D;\end{aligned}$$

it is skew-symmetric and non-degenerate.
Partial integrability of an invariant almost CR structure means that

$$0 = \mathcal{L}_o(JX_1,X_2)+\mathcal{L}_o(X_1,JX_2) = ([JX_1,X_2]+[X_1,JX_2]+\mathfrak{k})+D. \quad (21)$$

In this case the Levi-bracket is the imaginary part of a unique hermitian inner product on $D$.
Integrability of a invariant partially integrable almost CR structure reads:

$$[X_1,X_2]-[JX_1,JX_2]+J([JX_1,X_2]+[X_1,JX_2]) \in \mathfrak{k} \quad (22)$$

for all $X_1, X_2 \in D$.
Lets discuss the Lie-group and -algebra of the parabolic geometry modeling non-degenerate partially integrable almost CR structures of hypersurface type:

8.3. $\mathrm{PSU}(\mathbb{C}\oplus\mathfrak{m}\oplus\mathbb{C})$**.** Let $\mathfrak{m} = \mathbb{C}^p\oplus\mathbb{C}^q$ and denote

$$\mathbb{I}_{p,q} := \begin{pmatrix}\mathbb{I}_p & 0\\ 0 & -\mathbb{I}_q\end{pmatrix}. \quad (23)$$

Then we regard $\mathfrak{m}$ with the standard hermitian form $g = \mathbb{I}_{p,q}$ of signature $(p,q)$. For a vector $v \in \mathfrak{m}$ its dual vector $g(v,\cdot) = v^{(*,g)} \in \mathfrak{m}^*$ may be also be written as the row matrix

$$g(v,\cdot) = v^{(*,g)} = v^*\mathbb{I}_{p,q} \quad (24)$$

and the dual matrix of an $A \in \mathfrak{u}(\mathfrak{m})$ with respect to $g$ is

$$A^{(*,g)} = A^*\mathbb{I}_{p,q}. \quad (25)$$

We endow $\mathbb{C} \oplus \mathfrak{m} \oplus \mathbb{C}$ with the form $\tilde{g}$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & g & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{26}$$

If $g$ has signature $(p, q)$ then $\tilde{g}$ has signature $(p+1, q+1)$.

**Theorem 8.3.1.** *i. Elements of*

$$\mathfrak{g} = \mathfrak{su}(\mathbb{C} \oplus \mathfrak{m} \oplus \mathbb{C}, \tilde{g})$$

*are of the form*

$$\begin{pmatrix} -\alpha & Z^* \mathbb{I}_{p,q} & iz \\ X & A & -Z \\ ix & -X^* \mathbb{I}_{p,q} & \bar{\alpha} \end{pmatrix} \tag{27}$$

*with $A \in \mathfrak{u}(\mathfrak{m})$, $X, Z \in \mathfrak{m}$, $\alpha \in \mathbb{C}$ and $x, z \in \mathbb{R}$ such that* $\mathrm{tr}(A) - \alpha + \bar{\alpha} = 0$.
$\mathfrak{g}$ *is a $|2|$-graded Lie algebra whose components are:* $\mathfrak{g}_{-2} = \mathbb{R}X_{-2}$, $\mathfrak{g}_2 = \mathbb{R}X_2$, *where*

$$X_{-2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\mathfrak{g}_{-1} = \{ \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & -X^* \mathbb{I}_{p,q} & 0 \end{pmatrix} : X \in \mathbb{C}^l \}$$

$$\mathfrak{g}_0 = \{(\alpha, A) = \begin{pmatrix} -\alpha & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \bar{\alpha} \end{pmatrix} : \alpha \in \mathbb{C}, A \in \mathfrak{u}(\mathbb{C}^l \oplus \mathbb{C}^l \oplus \mathbb{C}) \text{ with } \mathrm{tr}(A) = 2i\mathrm{Im}\alpha\},$$

$$\mathfrak{g}_1 = \{ \begin{pmatrix} 0 & Z^* \mathbb{I}_{p,q} & 0 \\ 0 & 0 & -Z \\ 0 & 0 & 0 \end{pmatrix} : Z \in \mathbb{C}^l \}.$$

*Thus* $\mathfrak{g}_{-2} \cong \mathbb{R}i \subset \mathbb{C}$, $\mathfrak{g}_{-1} \cong \mathfrak{m}$, $\mathfrak{g}_0 \cong s(\mathbb{C} \oplus su_g(\mathfrak{m}))$, $\mathfrak{g}_1 \cong \mathfrak{m}^*$ *and* $\mathfrak{g}_2 \cong \mathbb{R}i \subset \mathbb{C}$, *and we will write*

$$\mathfrak{g} = \{xi \oplus X \oplus (\alpha, A) \oplus Z^{(*,g)} \oplus zi | X, Z \in \mathfrak{m},\ \alpha \in \mathbb{C}, x, z \in \mathbb{R},\ A \in \mathfrak{u}_g(\mathfrak{m}) \text{ with } \mathrm{tr}(A) = 2\mathrm{Im}(\alpha)i.\}.$$

*ii. In this notation the nontrivial brackets are*

$[(\alpha, A), xi \oplus X \oplus (\alpha', A') \oplus Z^{(*,g)} \oplus iz] =$

$= 2\mathrm{Re}(\alpha)xi \oplus (A+\alpha)X \oplus (0, [A, A']) \oplus ((A+\alpha)Z)^{(*,g)} \oplus -2\mathrm{Re}(\alpha)zi;$

$[xX_{-2}, zX_2] = (-xz, 0);$

$[zX_2, 0 \oplus X \oplus 0 \oplus 0 \oplus 0] = 0 \oplus 0 \oplus 0 \oplus (ziX)^{(*,g)} \oplus 0;$

$[xX_{-2}, 0 \oplus 0 \oplus 0 \oplus Z^{(*,g)} \oplus 0] = 0 \oplus xiZ \oplus 0 \oplus 0 \oplus 0;$

$[0 \oplus X \oplus 0 \oplus 0 \oplus 0, 0 \oplus 0 \oplus 0 \oplus Z^* \oplus 0] =$

$= 0 \oplus 0 \oplus (g(Z,X), XZ^{(*,g)} - ZX^{(*,g)}) \oplus 0 \oplus 0,$

$[0 \oplus X_1 \oplus 0 \oplus 0 \oplus 0, 0 \oplus X_2 \oplus 0 \oplus 0 \oplus 0] = -2\mathrm{Im}g(X_1, X_2)X_{-2};$

$[0 \oplus 0 \oplus 0 \oplus Z_1^{(*,g)} \oplus 0, 0 \oplus 0 \oplus 0 \oplus Z_2^{(*,g)} \oplus 0] = -2\mathrm{Im}g(Z_1, Z_2)X_2.$

*iii. We can naturally regard $\mathfrak{u}(\mathfrak{m}, g)$ as a Lie subalgebra of $\mathfrak{g}$ by using the embedding*

$$\mathrm{emb}_{\mathfrak{u}} : \mathfrak{u}(\mathfrak{m}, \langle,\rangle_{(l,l+1)}) \to \mathfrak{g}_0,$$
$$A \mapsto \begin{pmatrix} -\mathrm{tr}(A)/2 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -\mathrm{tr}(A)/2 \end{pmatrix}.$$

*We will thus simply write $A$ for $\mathrm{emb}_{\mathfrak{u}}(A)$. Using this embedding we have $\mathfrak{g}_0 = \mathbb{R} \oplus \mathfrak{u}(\mathfrak{m}, g)$.*

*iv. $\mathfrak{g}_0$ is reductive; a decomposition of $\mathfrak{g}_0$ into a semisimple part $\mathfrak{g}_0^{ss}$ and its center $\mathfrak{g}_0^c$ is*

$$\mathfrak{g}_0 = \mathfrak{su}(\mathfrak{m}) \oplus \mathbb{C}$$

*where $\mathfrak{su}(\mathfrak{m})$ embeds into $\mathfrak{g}_0$ by $\mathrm{emb}_{\mathfrak{u}|\mathfrak{su}(\mathfrak{m})}$ and $\mathbb{C}$ embeds by*

$$x + iy \mapsto \begin{pmatrix} x - \frac{n}{n+2}yi & 0 & 0 \\ 0 & \frac{2}{n+2}yi & 0 \\ 0 & 0 & -x - \frac{n}{n+2}yi \end{pmatrix}.$$

*This is an isomorphism of representations of the standard representation of $\mathfrak{su}(\mathfrak{m}) \oplus \mathbb{C}$ on $\mathfrak{m}$ and the adjoint representation of $\mathfrak{g}_0$ on $\mathfrak{m} = \mathfrak{g}_{-1}$.*

*v.*

$$\mathfrak{su}(\mathbb{C} \oplus \mathfrak{m} \oplus \mathbb{C}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & g & 0 \\ 1 & 0 & 0 \end{pmatrix}) \cong \mathfrak{su}(\mathbb{C} \oplus \mathfrak{m} \oplus \mathbb{C}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & -g & 0 \\ 1 & 0 & 0 \end{pmatrix})$$

*by*

$$\begin{pmatrix} -\alpha & Z^*\mathbb{I}_{p,q} & iz \\ X & A & -Z \\ ix & -X^*\mathbb{I}_{p,q} & \bar{\alpha} \end{pmatrix} \mapsto \begin{pmatrix} -\alpha & Z^*\mathbb{I}_{p,q} & -iz \\ X & A & Z \\ -ix & X^*\mathbb{I}_{p,q} & \bar{\alpha} \end{pmatrix}.$$

Denote $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. The subalgebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 < \mathfrak{g}$ we denote by $\mathfrak{p}$. We have the standard representation of both $\mathfrak{g}$ and $G$ on $\mathbb{C} \oplus \mathfrak{m} \oplus \mathbb{C}$. We denote by $P$ the stabilizer of the isotropic line $\{(c, 0, 0) | c \in \mathbb{C}\} \subset \mathbb{C} \oplus \mathfrak{m} \oplus \mathbb{C}$. $P$ is a Lie subgroup of $G$ and its Lie algebra is $\mathfrak{p}$.
Another characterization of $P$ is $P = N_G(\mathfrak{p}) = \{p \in G : \mathrm{Ad}(p)\mathfrak{p} \subset \mathfrak{p}\}$:

$N_G(\mathfrak{p})$ is a closed subgroup of $G$, and thus a Lie subgroup. Its Lie algebra is $N_{\mathfrak{g}}(\mathfrak{p}) = \{X \in \mathfrak{g} : \mathrm{ad}(X)\mathfrak{p} \subset \mathfrak{p}\}$, and obviously $N_{\mathfrak{g}}(\mathfrak{p})$ contains $\mathfrak{p}$. But for $X \in \mathfrak{g}_-$ with $X \neq 0$ there is always some element $Y$ in $\mathfrak{g}_0 \subset \mathfrak{p}$ with $[Y, X] \notin \mathfrak{p}$. Thus indeed $N_{\mathfrak{g}}(\mathfrak{p}) = \mathfrak{p}$.

**Theorem 8.3.2.** *Every automorphism $\phi$ of the nilpotent graded Lie subalgebra $\mathfrak{g}_-$ which satisfies $\Phi(iX) = i\Phi(X)$ on $\mathfrak{g}_{-1} = \mathfrak{m}$, i.e., which is also complex-linear on $\mathfrak{m}$, is in fact the restriction of the adjoint action of some element $g_0 \in G_0$ to $\mathfrak{g}_-$.*

*Proof.* Consider an automorphism $\phi$ of the graded Lie algebra $\mathfrak{g}_-$. On the real, one-dimensional vector space $\mathfrak{g}_{-2} = \mathbb{R}i$ the map $\phi$ acts by multiplication with some real non-zero scalar; But every such action on $\mathfrak{g}_{-2}$ can be realized as the adjoint action of some element in $G_0$. Thus, by composing $\phi$ with an appropriate element we may assume that it is the identity on $\mathfrak{g}_{-2}$.
Then, for elements $X_1, X_2 \in \mathfrak{m} = \mathfrak{g}_{-1}$, we have

$$\phi([X_1, X_2]) = \phi(-2\mathrm{Im} g(X_1, X_2)) = -2\mathrm{Im} g(X_1, X_2)$$

but also

$$\phi([X_1, X_2]) = [\phi(X_1), \phi(X_2)] = -2\mathrm{Im} g(\phi(X_1), \phi(X_2)).$$

Thus, since we assumed that $\phi$ is complex-linear, it follows that it is unitary on $\mathfrak{m}$. But since $G_0$ contains (a two-fold covering of) $U(\mathfrak{m})$ the automorphism $\phi$ may indeed be realized by the adjoint action of some element in $G_0$. $\square$

### 8.4. **The homogeneous model of non-degenerate partially integrable almost CR structures of hypersurface type.**

Regard $\mathfrak{m} = \mathbb{C}^p \oplus \mathbb{C}^q$ endowed with the standard hermitian form $g = \mathbb{I}_{p,q}$ of signature $p, q$. Then we have the hermitian form $\tilde{g}$ of signature $(p+1, q+1)$ on $\mathbb{C} \oplus \mathfrak{m} \oplus \mathbb{C}$, which is given by

$$\tilde{g} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & g & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $G = \mathrm{PSU}(\mathbb{C} \oplus \mathfrak{m} \oplus \mathbb{C})$ and $P < G$ the stabilizer of the isotropic complex line $\mathbb{C}(1, 0, 0) \subset \mathbb{C} \oplus \mathfrak{m} \oplus \mathbb{C}$. Recall that the Lie algebra of $P$ is $\mathfrak{p} = \mathfrak{g}^0 = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. We will show how $G/P$ becomes a homogeneous CR manifold.

As we saw in theorem 8.3.1 $\mathfrak{g}$ is 2-graded and $\mathfrak{g}_{-1} = \mathfrak{m}$. Now by 8.1.1 an invariant contact structure on $G/P$ is obtained by a $P$-invariant codimension one subspace $D \subset \mathfrak{n} := \mathfrak{g}/\mathfrak{p}$ such that the Levi-bracket

$$\mathcal{L}_o : D \times D \to \mathfrak{n}/D$$

is non-degenerate. We check that the subspace $D := \mathfrak{g}_{-1} + \mathfrak{p} = \mathfrak{m} + \mathfrak{p} \subset \mathfrak{g}/\mathfrak{p}$ satisfies this non-degeneracy condition. First recall that the restriction of the Lie bracket to $\mathfrak{m} \times \mathfrak{m}$ is

$$[\cdot, \cdot] : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{g}_{-2} = \mathbb{R}Z, \tag{28}$$

$$[X, Y] = -2\mathrm{Im}(g(X, Y)). \tag{29}$$

Therefore, according to 8.1.1, the Levi-bracket is given by

$$\mathcal{L}_o : D \times D \to \mathfrak{n}/D, \tag{30}$$

$$(X, Y) \mapsto -2\mathrm{Im}(g(X, Y))X_{-2} + \mathfrak{g}^{-1}. \tag{31}$$

Thus, since $g$ is a non-degenerate hermitian form, we see that $D = \mathfrak{m} + \mathfrak{p}$ is a contact subspace of $\mathfrak{g}/\mathfrak{p}$.

Now by 8.2.1 this invariant contact structure on $G/P$ may be extended to an invariant almost CR structure by specifying a $P$-invariant complex structure $J$ on $D$. But since $D = \mathfrak{g}_{-1} = \mathfrak{m} = \mathbb{C}^p \oplus \mathbb{C}^q$ we have a canonical complex structure. For invariance of this complex structure under $P$ just note that $\mathfrak{p}_+ = \mathfrak{g}^1$ acts trivially on $\mathfrak{g}/\mathfrak{p}$ and that $\mathfrak{g}_0$ acts on $\mathfrak{g}_{-1} = \mathfrak{m}$ by maps of the form $\alpha \mathrm{id}_{\mathfrak{m}} + A$ with $\alpha \in \mathbb{R}$ and $A \in \mathfrak{u}(\mathfrak{m})$.

Thus we have an invariant non-degenerate almost CR structure on $G/P$. From (30) we immediately see that condition (21) of partial integrability of this almost CR structure is satisfied. Also (22), the integrability condition, follows at once from (28). Thus we have indeed an invariant CR structure of signature $(p, q)$ on $G/P$.

A more vivid realization of this homogeneous CR manifold may be obtained as follows: Consider the light cone $C \subset \mathbb{C} \oplus \mathfrak{m} \oplus \mathbb{C}$ formed by all isotropic non-zero vectors $v \in \mathbb{C}^{p+q+2}, \tilde{g}(v, v) = 0$. This is a co-dimension one submanifold of $\mathbb{C}^{p+q+2}$.

On $C$ we have a natural right action of $\mathbb{C}^*$ by multiplication. Denote by $p : C \to C/\mathbb{C}^* =: M$ the natural surjection to the orbit space. The action by $\mathbb{C}^*$ on $C$ is smooth and free and $C \to M$ is thus a $\mathbb{C}^*$ principal bundle. Since $G = \mathrm{PSU}(\mathbb{C}^{p+q+2}) = \mathrm{SU}(\mathbb{C}^{p+q+2})/\Delta$, with $\Delta$ a finite subgroup of diagonal matrices, $G$ acts on $M = C/\mathbb{C}^*$. Obviously $\mathrm{SU}(\mathbb{C}^{p+q+2})$ acts transitively on $C$, and thus also $G$ acts transitively on $M$. Let $e_0 := (1, 0, 0) \in \mathbb{C} \oplus \mathfrak{m} \oplus \mathbb{C} \in C$. The isotropy group of $\mathbb{C}e_0 \in M$ is (by definition) $P < G$, and thus $M = C/\mathbb{C}^* = G/P$.

$M$ has a simpler description as $M = C/\mathbb{C}^* = (S^{2p+1} \times S^{2q+1})/\mathrm{U}(1)$: take an orthonormal basis $v_1, \ldots v_{p+1}, w_1, \ldots w_{q+1}$ of $\mathbb{C}^{p+q+2}$, where $\tilde{g}(v_i, v_i) = 1, \tilde{g}(w_i, w_i) = -1$. Denote by $V$, resp. $W$, the subspaces spanned by the vectors $v_i$, resp. the vectors $w_i$. Then $\tilde{g}$ is the standard, positive definite, hermitian form on $V \cong \mathbb{C}^{p+1}$ and the negative of the standard hermitian form on $W \cong \mathbb{C}^{q+1}$. Denote the standard norms on $V \cong \mathbb{C}^{p+1}$ and $W \cong \mathbb{C}^{q+1}$ simply by $|| \cdot ||$. Writing $\mathbb{C}^{p+q+2} = V \oplus W$ we have

$$C = \{z \oplus w \in V \oplus W : ||z|| = ||w||\},$$

and the map

$$S^{2p+1} \times S^{2q+1} \to V \oplus W = \mathbb{C}^{p+q+2},$$

$$(z, w) \mapsto z \oplus w$$

obviously has values in $C$ and hits every $\mathbb{C}^*$-orbit; It factorizes to an injective map

$$(S^{2p+1} \times S^{2q+1})/\mathrm{U}(1) \to M = C/\mathbb{C}^*,$$

and thus indeed $M \cong (S^{2p+1} \times S^{2q+1})/\mathrm{U}(1)$.

Now the $G$-equivariant diffeomorphism

$$\begin{aligned}\Theta : G/P &\cong M,\\ gP &\mapsto \mathbb{C}ge_0\end{aligned}$$

induces an invariant CR structure on $M$. To find an explicit description it is enough to calculate the tangent map of $\Theta$ at $o = P \in G/P$. First note that the tangent space at a point $v \in C$ is

$$T_vC = \{w \in \mathbb{C}^{p+q+2} : \mathrm{Re}(\tilde{g}(v,w)) = 0\}.$$

Now

$$\begin{aligned}T_o\Theta : \mathfrak{g}/\mathfrak{p} &\to T_{\mathbb{C}e_0}M,\\ Y + \mathfrak{p} &\mapsto T_{e_0}p(Ye_0).\end{aligned}$$

Thus the contact subspace $\mathfrak{m} + \mathfrak{p} = \mathfrak{g}_{-1} + \mathfrak{p}$ of $\mathfrak{g}/\mathfrak{p}$ is mapped to

$$T_{e_0}p(\{((0,X,0) \in \mathbb{C} \oplus \mathfrak{m} \oplus \mathbb{C}\}) \tag{32}$$

under $\Theta$. Denoting the contact subbundle of $TM$ induced by $\Theta$ by $\mathcal{D}$, we thus find by using (32) and $G$-invariance that

$$\mathcal{D}_{p(v)} = T_vp(v^{\perp}) \subset T_vC.$$

And (32) also shows that the complex structure of $\mathcal{D}_{p(v)}$ is simply

$$JT_{p(v)}X = T_{p(v)}iX.$$

Thus the homogeneous model of partially integrable non-degenerate almost CR structures of signature $(p,q)$ is an (invariant) CR structure on $(S^{2p+1} \times S^{2q+1})/\mathrm{U}(1)$. An (almost) CR manifold of signature $(p,q)$ which is locally isomorphic to the CR manifold $(S^{2p+1} \times S^{2q+1})/\mathrm{U}(1)$ is called *spherical*. Note that with $q = 0$ we thus see that the homogeneous models for strictly pseudoconvex almost CR structures are the CR-spheres $S^{2p+1} \subset \mathbb{C}^{p+1}$.

*Remark* 8.4.1. Let $(\mathcal{G} \to H/K, \omega)$ be a homogeneous regular parabolic geometry of type $(G,P)$ as in 8.3.1 above, which is induced by some $\alpha : \mathfrak{h} \to \mathfrak{g}$. Then we saw in chapter 6 that $\alpha$ induces a filtration on $\mathfrak{n} = \mathfrak{h}/\mathfrak{k}$ and further endows $\mathrm{gr}(\mathfrak{n})$ with the structure of a nilpotent graded Lie algebra by an isomorphism of $\mathrm{gr}(\mathfrak{n}) \cong \mathfrak{g}_-$. Since $\mathfrak{g}$ is a 2-graded Lie algebra and $\mathfrak{g}_{-2}$ is one-dimensional the induced filtration is just some $K$-invariant co-dimension 1 subspace $D$ of $\mathfrak{n}$. The isomorphism $\alpha$ restricts to an isomorphism of $D$ with $\mathfrak{m}$, and we can thus pull back the complex structure on $\mathfrak{m}$ to a complex structure $J$ on $D$. From 8.3.2 it follows that $J$ is $K$-invariant. Furthermore, by regularity of $\alpha$, one sees that the Levi-bracket $D \times D \to \mathfrak{n}/D$ is non-degenerate.
Thus $(\mathcal{G} \to H/K, \omega)$ endows $H/K$ with an invariant non-degenerate almost CR structure of hypersurface type, and by using regularity of $\alpha$ one sees that this almost CR structure is partially integrable. ⌟

We now come to an example of invariant CR structures. In 8.5 we introduce the underlying homogeneous space, in 8.6 we discuss an invariant contact structure on this space and in 8.7 we endow this contact distribution with a (family of) complex structures and show that we thus get a (family of) integrable, invariant CR structures.
In 8.8 this family of CR structures will be prolonged to Cartan geometries.

8.5. *$SU(l+2)/U(l)$***.** Consider $H = \mathrm{SU}(l+2)$ resp. $\mathfrak{h} = \mathfrak{su}(l+2)$. We will write elements in $\mathfrak{h}$ as

$$\begin{pmatrix} b & -v^* & \gamma \\ v & A & w \\ -\bar{\gamma} & -w^* & c \end{pmatrix},$$

where $b, c$ are purely imaginary, $A \in \mathfrak{u}(l)$ and $b + \mathrm{tr}(A) + c = 0$. In $\mathfrak{h}$ we have the subalgebra $\mathfrak{k}$ consisting of elements of the form

$$\begin{pmatrix} -a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -a \end{pmatrix} =: (a, A) \tag{33}$$

with $a = \frac{\mathrm{tr}(A)}{2}$. Of course, $\mathfrak{k} = \mathfrak{u}(l)$. The corresponding virtual Lie subgroup $K$ is in fact closed and thus a Lie subgroup, and one sees immediately that $K$ is a two-fold covering of $\mathrm{U}(l)$. Elements of $K$ are of the form

$$\begin{pmatrix} c^{-1} & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & c^{-1} \end{pmatrix}$$

with $c \in U(1)$ and $C \in U(l)$ such that $c^{-2}\det(C) = 1$. Note that since $K$ is connected and $H$ is simply connected also $H/K = SU(l+2)/U(l)$ is simply connected.

8.6. **The contact structure on** *$SU(l+2)/U(l)$***.** The standard hermitian inner product on $\mathfrak{h} \subset \mathbb{C}^{(l+2)^2}$ is $K$-invariant, and by restricting it to $\mathfrak{n} := \mathfrak{k}^{\perp}$, the orthogonal complement of $\mathfrak{k}$, we can use the notion of a contact element as discussed in 8.1.1.

**Lemma 8.6.1.**

$$Z = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}$$

*is a contact element and thus defines an invariant contact structure on $SU(l+2)/U(l)$.*

*Proof.* Since $K$ is connected it suffices to check $\mathfrak{k}$-invariance of $Z$:

$$\left[ \begin{pmatrix} -a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -a \end{pmatrix}, \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix} \right] = 0.$$

For non-degeneracy of the Levi-bracket we calculate

$$\left[\begin{pmatrix} 0 & -v_1^* & \gamma_1 \\ v_1 & 0 & w_1 \\ -\bar\gamma_1 & -w_1^* & 0 \end{pmatrix}, \begin{pmatrix} 0 & -v_2^* & \gamma_2 \\ v_2 & 0 & w_2 \\ -\bar\gamma_2 & -w_2^* & 0 \end{pmatrix}\right] =$$

$$= \begin{pmatrix} -v_1^*v_2-\bar\gamma_2\gamma_1 & -\gamma_1 w_2^* & -v_1^* w_2 \\ -\bar\gamma_2 w_1 & -v_1v_2^*-w_1w_2^* & \gamma_2 v_1 \\ -w_1^*v_2 & \bar\gamma_1 v_2^* & -\bar\gamma_1\gamma_2-w_1^*w_2 \end{pmatrix} - \begin{pmatrix} -v_2^*v_1-\bar\gamma_1\gamma_2 & -\gamma_2 w_1^* & -v_2^* w_1 \\ -\bar\gamma_1 w_2 & -v_2v_1^*-w_2w_1^* & \gamma_1 v_2 \\ -w_2^*v_1 & \bar\gamma_2 v_1^* & -\bar\gamma_2\gamma_1-w_2^*w_1 \end{pmatrix} =$$

$$= \begin{pmatrix} v_2^*v_1-v_1^*v_2+\bar\gamma_1\gamma_2-\bar\gamma_2\gamma_1 & \gamma_2 w_1^*-\gamma_1 w_2^* & v_2^*w_1-v_1^*w_2 \\ \bar\gamma_1 w_2-\bar\gamma_2 w_1 & v_2v_1^*-v_1v_2^*+w_2w_1^*-w_1w_2^* & \gamma_2v_1-\gamma_1v_2 \\ w_2^*v_1-w_1^*v_2 & \bar\gamma_1v_2^*-\bar\gamma_2v_1^* & \bar\gamma_2\gamma_1-\bar\gamma_1\gamma_2+w_2^*w_1-w_1^*w_2 \end{pmatrix}.$$

Thus the $\mathbb{R}Z$-part of $[(v_1, w_1, \gamma_1), (v_2, w_2, \gamma_2)]$ is

$$\big(\mathrm{Im}(\langle w_1, w_2\rangle) - \mathrm{Im}(\langle v_1, v_2\rangle) + 2\mathrm{Im}(\bar\gamma_1\gamma_2)\big)Z. \tag{34}$$

Especially, the Levi-bracket is non-degenerate. □

For later use we note that

$$[(v_1, w_1, \gamma_1), (v_2, w_2, \gamma_2)]_D = \begin{pmatrix} 0 & \gamma_2 w_1^*-\gamma_1 w_2^* & v_2^*w_1-v_1^*w_2 \\ \bar\gamma_1 w_2-\bar\gamma_2 w_1 & 0 & \gamma_2v_1-\gamma_1v_2 \\ w_2^*v_1-w_1^*v_2 & \bar\gamma_1v_2^*-\bar\gamma_2v_1^* & 0 \end{pmatrix}, \tag{35}$$

$$[(v_1, w_1, \gamma_1), (v_2, w_2, \gamma_2)]_{\mathfrak{k}} = \tag{36}$$

$$= \begin{pmatrix} -\mathrm{Im}(\langle v_1\oplus w_1, v_2\oplus w_2\rangle)i & 0 & 0 \\ 0 & v_2v_1^*-v_1v_2^*+w_2w_1^*-w_1w_2^* & 0 \\ 0 & 0 & -\mathrm{Im}(\langle v_1\oplus w_1, v_2\oplus w_2\rangle)i \end{pmatrix}.$$

The orthogonal complement of $\mathbb{R}Z$ in $\mathfrak{n}$ is the real subspace $D$ which is formed by elements

$$\begin{pmatrix} 0 & -v^* & \gamma \\ v & 0 & w \\ -\bar\gamma & -w^* & 0 \end{pmatrix}$$

We will often write $(v, w, \gamma)$ or $v \oplus w \oplus \gamma$ for elements of $D$.
We have $\mathfrak{n} = \mathfrak{k}^\perp = \mathbb{R}Z \oplus D$.

## 8.7. **A family of CR Structures on $SU(l+2)/U(l)$.** 

Our family of CR structures on $H$ will be parametrized by $t \in \{z \in \mathbb{C} : |z| < 1\}$.

8.7.1. *The complex structure on $D$.* Denote $D_V := \{v \oplus 0 \oplus 0\}$, $D_W := \{0 \oplus w \oplus 0\}$, $D_\Gamma := \{0 \oplus 0 \oplus \gamma\}$,

$$E := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \text{ and } F := \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}.$$

The complex structure $J_t$ leaves $D_V, D_W, D_\Gamma$ invariant and doesn't depend on the parameter $t$ on $D_V$ and $D_W$: here it is given simply by

$$J(v \oplus w \oplus 0) := iv \oplus iw \oplus 0.$$

The complex structure on $D_\Gamma$ depends on the parameter $t$ and is given by

$$JE = \beta_E F - \alpha E,$$
$$JF = -\beta_F E + \alpha F$$

where

$$\alpha := \frac{2\mathrm{Im}(t)}{1-|t|^2};$$
$$\beta_E := \frac{\mathrm{Im}(t)^2 + (\mathrm{Re}(t)-1)^2}{1-|t|^2};$$
$$\beta_F := \frac{\mathrm{Im}(t)^2 + (\mathrm{Re}(t)+1)^2}{1-|t|^2}.$$

**Lemma 8.7.1.** *For every $t \in D$*

*i. $J_t$ is an anti-involution on $D$*
*ii. $J_t$ is $K$-invariant*
*iii. The Levi-bracket $\mathcal{L} \in \Lambda^2(D) \otimes \mathbb{R}Z$ is $-2\mathrm{Im}g$, where*

$$g(v_1 \oplus w_1 \oplus r_1 E \oplus s_1 F, v_2 \oplus w_2 \oplus r_2 E \oplus s_2 F) = \tag{37}$$
$$= \tfrac{1}{2}\langle v_1, v_2\rangle - \tfrac{1}{2}\langle w_1, w_2\rangle - r_1 r_2 \beta_E - s_1 s_2 \beta_F - r_1 s_2(\alpha + i) - r_2 s_1(\alpha - i).$$

*Proof.* i. Lets first check that $J$ is indeed an anti-involution on $D_\Gamma = \mathbb{R}E \oplus \mathbb{R}F$:

$$J(JE) = \beta_E JF - \alpha JE = \beta_E(-\beta_F E + \alpha F) - \alpha(\beta_E F - \alpha E) =$$
$$= -\beta_E \beta_F E + \beta_E \alpha F - \alpha \beta_E F + \alpha^2 E = (\alpha^2 - \beta_E \beta_F)E.$$

$$J(JF) = -\beta_F JE + \alpha JF = -\beta_F(\beta_E F - \alpha E) + \alpha(-\beta_F E + \alpha F) =$$
$$= -\beta_F \beta_E F + \alpha \beta_F E - \alpha \beta_F E + \alpha^2 F = (\alpha^2 - \beta_E \beta_F)F.$$

Thus we need to check that $\alpha^2 - \beta_E \beta_F = -1$.

$$(\mathrm{Im}(t)^2 + (\mathrm{Re}(t)-1))^2 * (\mathrm{Im}(t)^2 + (\mathrm{Re}(t)+1)^2) =$$
$$= \mathrm{Im}(t)^4 + \mathrm{Im}(t)^2(\mathrm{Re}(t)+1)^2 + \mathrm{Im}(t)^2(\mathrm{Re}(t)-1)^2 + (\mathrm{Re}(t)^2-1)^2 =$$
$$= \mathrm{Im}(t)^4 + \mathrm{Im}(t)^2(\mathrm{Re}(t)^2 + 2\mathrm{Re}(t) + 1)$$
$$+ \mathrm{Im}(t)^2(\mathrm{Re}(t)^2 - 2\mathrm{Re}(t) + 1) + \mathrm{Re}(t)^4 - 2\mathrm{Re}(t)^2 + 1 =$$
$$= \mathrm{Re}(t)^4 + \mathrm{Im}(t)^4 + 2\mathrm{Re}(t)^2\mathrm{Im}(t)^2 + 2\mathrm{Im}(t)^2 - 2\mathrm{Re}(t)^2 + 1 =$$
$$= (\mathrm{Re}(t)^2 + \mathrm{Im}(t)^2)^2 + 2\mathrm{Im}(t)^2 - 2\mathrm{Re}(t)^2 + 1;$$

Thus

$$\alpha^2 - \beta_E\beta_F = \frac{2\mathrm{Im}(t)^2 + 2\mathrm{Re}(t)^2 - 1 - (\mathrm{Re}(t)^2 + \mathrm{Im}(t)^2)^2}{1-|t|^2} =$$
$$= \frac{2|t|^2 - 1 - |t|^4}{(1-|t|^2)^2} = -1.$$

ii.

$$\left[\begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix}, \begin{pmatrix} 0 & -v^* & \gamma \\ v & 0 & w \\ -\bar{\gamma} & -w^* & 0 \end{pmatrix}\right] =$$
$$= \begin{pmatrix} 0 & -av^* & a\gamma \\ Av & 0 & Aw \\ -a\bar{\gamma} & -aw^* & 0 \end{pmatrix} - \begin{pmatrix} 0 & -v^*A & a\gamma \\ av & 0 & aw \\ -a\bar{\gamma} & -w^*A & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} 0 & -v^*(A + a\mathrm{id}) & 0 \\ (A + a\mathrm{id})v & 0 & (A + a\mathrm{id})w \\ 0 & -w^*(A + a\mathrm{id}) & 0 \end{pmatrix}.$$

So an element $(a, A)$ of $\mathfrak{k}$ acts by

$$[(a, A), (v \oplus w \oplus \gamma)] = (A + a)v \oplus (A + a)w \oplus 0. \tag{38}$$

But now $K$-equivariance is clear: both $J$ and $(\mathrm{ad}_{(a,A)})_{|D}$ respect the decomposition of $D$ into $\mathbb{C}^n \oplus \mathbb{C}^n \oplus (\mathbb{R}E \oplus \mathbb{R}F)$. But $(\mathrm{ad}_{(a,A)})_{|D}$ acts by complex-linear maps on the $\mathbb{C}^n$-parts, and it acts trivially on $\mathbb{R}E \oplus \mathbb{R}F$; thus it indeed commutes with $J$.

iii. The $Z$-part of the Lie-bracket, $[\cdot,\cdot]_{\mathbb{R}Z} : D \times D \to \mathbb{R}Z$, is skew-symmetric and non-degenerate. Thus, for it being the imaginary part of an hermitian form on the complex vector space $(D, J)$ it remains to check that $[JX_1, X_2]_{\mathbb{R}Z} + [X_1, JX_2]_{\mathbb{R}Z} = 0$ for all $X_1, X_2$ in $D$. This is exactly the partial integrability of the almost CR structure induced by $(D, J)$.

Since

$$\begin{aligned} &[v_1 \oplus w_1 \oplus \gamma_1, v_2 \oplus w_2 \oplus \gamma_2]_{\mathbb{R}Z} = \\ &= \big(\mathrm{Im}(\langle w_1, w_2\rangle) - \mathrm{Im}(\langle v_1, v_2\rangle) + 2\mathrm{Im}(\bar{\gamma}_1\gamma_2)\big)Z \end{aligned} \tag{39}$$

the only nontrivial equation for partial integrability is

$$[J\gamma_1, \gamma_2]_{\mathbb{R}Z} + [\gamma_1, J\gamma_2]_{\mathbb{R}Z} = 0;$$

But for $\gamma_1 = \gamma_2$ this expression vanishes by skew-symmetry of the Lie-bracket and for $\gamma_1 = E, \gamma_2 = F$ we have

$$[JE, F] + [E, JF] = [-\alpha E, F] + [E, \alpha F] = 0.$$

Now we check (37). Since $J$ leaves $D_V, D_W$ and $D_\Gamma$ invariant (39) implies that $D_V, D_W$ and $D_\Gamma$ are orthogonal with respect to the unique hermitian form $g$ on $D$ with $\mathrm{Im} g = -2[\cdot,\cdot]_{\mathbb{R}Z}$.

Since $J$ is the standard complex structure on $D_V \cong \mathbb{C}^l, D_W \cong \mathbb{C}^l$ wee see that

$$g(v_1 \oplus 0 \oplus 0, v_2 \oplus 0 \oplus 0) = \frac{1}{2}\langle v_1, v_2\rangle,$$
$$g(0 \oplus w_1 \oplus 0, 0 \oplus w_2 \oplus 0) = -\frac{1}{2}\langle w_1, w_2\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the standard hermitian inner product on $\mathbb{C}^l$.

Thus we only need to calculate the real part of $g$ for elements $0 \oplus 0 \oplus \gamma_1, 0 \oplus 0 \oplus \gamma_2 \in \Gamma$: From

$$-2\mathrm{Im}(g(0 \oplus 0 \oplus \gamma_1, 0 \oplus 0 \oplus \gamma_2)) = 2\mathrm{Im}(\bar{0} \oplus 0 \oplus \gamma_1 0 \oplus 0 \oplus \gamma_2);$$

it follows that

$$\begin{aligned}
&-2\mathrm{Re}(g(0\oplus 0\oplus\gamma_1, 0\oplus 0\oplus\gamma_2)) =\\
&= \mathrm{Im}(Jg(0\oplus 0\oplus\gamma_1, 0\oplus 0\oplus\gamma_2)) =\\
&= \mathrm{Im}(g(0\oplus 0\oplus\gamma_1, J0\oplus 0\oplus\gamma_2)) =\\
&= 2\mathrm{Im}(0\oplus 0\oplus\overline{\gamma_1}, 0\oplus 0\oplus J\gamma_2).
\end{aligned}$$

So on $D_\Gamma$

$$\begin{aligned}
g(E,E) &= -\beta_E;\\
g(F,F) &= -\beta_F;\\
g(E,F) &= -\alpha - i.
\end{aligned}$$

□

Thus $(D, J_t)$ endows $SU(l+2)/U(l)$ with an invariant, partially integrable almost CR structure.

*Remark* 8.7.2. We have $\beta_E, \beta_F > 0$; Also $\beta_E, \beta_F < 0$ and appropriate $\alpha$ would define a complex structure on $D_\Gamma$, but

$$\begin{aligned}
v\oplus w\oplus r+is &\mapsto w\oplus v\oplus -r+is,\\
xZ &\mapsto -xZ,\\
\mathfrak{k} &\xrightarrow{\mathrm{id}} \mathfrak{k}
\end{aligned}$$

is an automorphism (in fact, an involution) of $\mathfrak{su}(l+2)$ and an isomorphism of the (almost) CR structures induced by $\beta_E, \beta_F, \alpha$ resp. $-\beta_E, -\beta_F, \alpha$.
If $\mathrm{Im}(t) = 0$ $\alpha = 0$, $\beta_E = \frac{1-t}{1+t}$ and $\beta_F = \frac{1+t}{1-t}$. ⌟

**Lemma 8.7.3.** *This is a CR structure: the partially integrable invariant almost CR structure $(D, J_t)$ on $SU(l+2)/U(l)$ is in fact integrable.*

*Proof.* We already showed partial integrability in iii of lemma 8.7.1. This meant that

$$[JX_1, X_2]_{\mathbb{R}Z} + [X_1, JX_2]_{\mathbb{R}Z} = 0$$

for $X_1, X_2 \in D$. But in fact one can directly see from (36) that $([JX_1, X_2] + [X_1, JX_2])_{\mathfrak{k}} = 0$, and thus $[JX_1, X_2] + [X_1, JX_2] \in D$. Replacing $X_1$ by $JX_1$ we see that the Nijenhuis-tensor $N$ has in fact values in $D$:

$$N \in \Lambda^2(D^*) \otimes D,$$

$$N(X_1, X_2) = [X_1, X_2] - [JX_1, JX_2] + J([JX_1, X_2] + [X_1, JX_2].$$

Since $N$ is anti-complex linear in both arguments and skew-symmetric we immediately see that $N$ vanishes on $D_\Gamma \times D_\Gamma$. Furthermore

$$\begin{aligned}
&[v_1\oplus w_1\oplus 0, v_2\oplus w_2\oplus 0] - [Jv_1\oplus Jw_1\oplus 0, Jv_2\oplus Jw_2\oplus 0] =\\
&= \begin{pmatrix} v_2^*v_1 - v_1^*v_2 & 0 & v_2^*w_1 - v_1^*w_2\\ 0 & v_2v_1^* - v_1v_2^* + w_2w_1^* - w_1w_2^* & 0\\ w_2^*v_1 - w_1^*v_2 & 0 & w_2^*w_1 - w_1^*w_2 \end{pmatrix}\\
&- \begin{pmatrix} v_2^*v_1 - v_1^*v_2 & 0 & v_2^*w_1 - v_1^*w_2\\ 0 & v_2v_1^* - v_1v_2^* + w_2w_1^* - w_1w_2^* & 0\\ w_2^*v_1 - w_1^*v_2 & 0 & w_2^*w_1 - w_1^*w_2 \end{pmatrix} = 0.
\end{aligned}$$

Replacing $v_1 \oplus w_1 \oplus 0$ by $Jv_1 \oplus Jw_1 \oplus 0$ this implies that also

$$[Jv_1 \oplus Jw_1, v_2 \oplus w_2] + [v_1 \oplus w_1, Jv_2 \oplus Jw_2] = 0.$$

Now

$$\begin{aligned}
&[v_1 \oplus w_1 \oplus 0, E] - [Jv_1 \oplus Jw_1 \oplus 0, JE] \\
&= \begin{pmatrix} 0 & w_1^* & 0 \\ -w_1 & 0 & v_1 \\ 0 & -v_1^* & 0 \end{pmatrix} - \begin{pmatrix} 0 & (\beta_E+\alpha i)w_1^* & 0 \\ -(\beta_E-\alpha i)w_1 & 0 & -(\beta_E+\alpha i)v_1 \\ 0 & (\beta_E-\alpha i)v_1^* & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & (1-\beta_E-\alpha i)w_1^* & 0 \\ (\beta_E-1-\alpha i)w_1 & 0 & (\beta_e+1+\alpha i)v_1 \\ 0 & -(\beta_E+1-\alpha i)v_1^* & 0 \end{pmatrix},
\end{aligned}$$

and thus, replacing $v_1 \oplus w_1 \oplus 0$ by $Jv_1 \oplus Jw_1 \oplus 0$,

$$\begin{aligned}
&[Jv_1 \oplus Jw_1 \oplus 0, E] + [v_1 \oplus w_1 \oplus 0, JE] = \\
&= \begin{pmatrix} 0 & (-\alpha+(\beta_E-1)i)w_1^* & 0 \\ (\alpha+(\beta_E-1)i)w_1 & 0 & (-\alpha+(1+\beta_E)i)v_1 \\ 0 & (\alpha+(\beta_E+1)i)v_1^* & 0 \end{pmatrix}.
\end{aligned}$$

Thus

$$[v_1 \oplus w_1, E] - [Jv_1 \oplus Jw_1, JE] + J([Jv_1 \oplus Jw_1, E] + [v_1 \oplus w_1, JE]) = 0.$$

Analogously one shows this for $F$ instead of $E$. So our CR structure is indeed integrable. $\square$

8.8. **The prolongation of the above family of CR structures to Cartan geometries.** The hermitian form on $D$ has signature $(l, l+1)$; Denote by $\mathfrak{m}$ the complex vector space $\mathbb{C}^l \oplus \mathbb{C}^l \oplus \mathbb{C}$ endowed with the standard hermitian form the standard hermitian form $\mathbb{I}_{(l,l+1)}$ of signature $(l, l+1)$,

$$\mathbb{I}_{(l,l+1)} = \begin{pmatrix} \mathbb{I}_l & 0 & 0 \\ 0 & -\mathbb{I}_l & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

One calculates

**Theorem 8.8.1.** $\mathfrak{u}(\mathbb{C}^l \oplus \mathbb{C}^l \oplus \mathbb{C}, (l, l+1))$ *consists of matrices of the form*

$$\begin{pmatrix} A_V & B & b_V \\ B^* & A_W & b_W \\ b_V^* & -b_W^* & yi \end{pmatrix}$$

*with $A_V, A_W$ being unitary, $y \in \mathbb{R}$ and arbitrary $B \in \mathbb{C}^{l^2}$, $b_V, b_W \in \mathbb{C}^l$.*

The Cartan geometry corresponding to our CR structures on $SU(l+2)/U(l)$ is of type $(G, P)$, where $G = \mathrm{PSU}(\mathbb{C} \oplus (\mathfrak{m}, \mathbb{I}_{(l,l+1)}), \oplus\mathbb{C})$ and $P$ the stabilizer of the isotropic line $\mathbb{C}(1, 0, 0) \subset \mathbb{C} \oplus (\mathbb{C}^l \oplus \mathbb{C}^l \oplus \mathbb{C}) \oplus \mathbb{C}$.
By 8.3 and 8.8.1 elements of $\mathfrak{g}$ are of the form

$$\begin{pmatrix} -\alpha & \begin{pmatrix} \tilde{v}^* & -\tilde{w}^* & -\bar{\tilde{\gamma}} \end{pmatrix} & zi \\ \begin{pmatrix} v \\ w \\ \gamma \end{pmatrix} & \begin{pmatrix} A_V & B & b_V \\ B^* & A_W & b_W \\ b_V^* & -b_W^* & yi \end{pmatrix} & -\begin{pmatrix} \tilde{v} \\ \tilde{w} \\ \tilde{\gamma} \end{pmatrix} \\ xi & \begin{pmatrix} -v^* & w^* & \bar{\gamma} \end{pmatrix} & \bar{\alpha} \end{pmatrix},$$

where $A_V, A_W \in \mathfrak{u}(l); B \in \mathbb{C}^{l^2}; v, w, \tilde{v}, \tilde{w}, b_V, b_W \in \mathbb{C}^l; x, y, z \in \mathbb{R}; \gamma, \tilde{\gamma} \in \mathbb{C}$. Since the matrices above contain much redundancy we will simply write them

$$\begin{pmatrix} -\alpha & \begin{pmatrix} * & -* & -* \end{pmatrix} & zi \\ \begin{pmatrix} v \\ w \\ \gamma \end{pmatrix} & \begin{pmatrix} A_V & B & b_V \\ * & A_W & b_W \\ * & -* & yi \end{pmatrix} & -\begin{pmatrix} \tilde{v} \\ \tilde{w} \\ \tilde{\gamma} \end{pmatrix} \\ xi & \begin{pmatrix} -* & * & * \end{pmatrix} & * \end{pmatrix},$$

**Theorem 8.8.2.** *i. The map*

$$\alpha_0(\begin{pmatrix} xi - ai & -v^* & r + si \\ v & A & w \\ -r + si & -w^* & -xi - ai \end{pmatrix}) =$$

$$= \begin{pmatrix} -\frac{2(l+2)}{2l+3}a & 0 & 0 \\ \begin{pmatrix} \frac{1}{\sqrt{2}}v \\ \frac{1}{\sqrt{2}}w \\ r\sqrt{\beta_E} + \frac{s\alpha}{\sqrt{\beta_E}} + \frac{s}{\sqrt{\beta_E}}i \end{pmatrix} & \begin{pmatrix} A - \frac{1}{2l+3}a & 0 & 0 \\ 0 & A - \frac{1}{2l+3}a & 0 \\ 0 & 0 & -\frac{2(l+2)}{2l+3}a \end{pmatrix} & 0 \\ xi & \begin{pmatrix} -* & * & * \end{pmatrix} & -\frac{2(l+2)}{2l+3}a \end{pmatrix} \tag{40}$$

*induces an isomorphism between the graded (nilpotent) Lie algebras* $\mathrm{gr}(\mathfrak{g}_-)$ *and* $\mathrm{gr}(\mathfrak{n}) = \mathbb{R}Z \oplus D$ *endowed with the Levi-bracket; The restriction of* $\alpha_0$ *to* $D$ *is complex linear; here the complex structure on* $D$ *is the anti-involution* $J_t$ *corresponding to parameters* $\alpha, \beta_E, \beta_F$ *with* $\beta_E\beta_F - \alpha^2 = 1$ *as discussed in 8.7.1.*

*ii. The map*

$$\Psi : K \to G_0, \tag{41}$$
$$\Psi(k) := \alpha_0 \circ \mathrm{Ad}(k) \circ \alpha_0^{-1}$$

*is a homomorphism of Lie groups and under the induced action of* $K$ *on* $\mathfrak{g}$ *the map* $\alpha_0 : \mathfrak{h} \to \mathfrak{g}$ *is* $K$*-equivariant.*

*iii. Every other map* $\alpha_0'$ *which satisfies properties i and ii is of the form* $\mathrm{Ad}(g_0) \circ \alpha_0$ *for some* $g_0 \in G_0$*.*

*iv. The map* $\alpha = \alpha_0 + \phi \circ \alpha_0$*, prolongs the CR structure on* $SU(l+2)/U(l)$ *induced by* $(D, J_t)$ *to a regular, normal Cartan geometry. i.e.: the curvature* $\kappa$ *of the Cartan connection induced by* $\alpha$ *satisfies* $\partial^*\kappa = 0$*. Here*

$$\phi(xi \oplus (v \oplus w \oplus \gamma)) =$$

$$= \begin{pmatrix} -\frac{1}{2}x(a+l(c_V+c_W))i & \begin{pmatrix} * & -* & -* \end{pmatrix} & xci \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} xc_V i & \big(\mathrm{Re}(\gamma)z_E + \mathrm{Im}(\gamma)z_F\big)\mathbb{I}_l & \beta_{VW}w \\ * & xc_W i & \beta_{WV}v \\ * & -* & xai \end{pmatrix} & -\begin{pmatrix} P_V v \\ P_W w \\ P_E\mathrm{Re}(\gamma) + P_F\mathrm{Im}(\gamma) \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} & -\frac{1}{2}x(a+l(c_V+c_W))i \end{pmatrix} \tag{42}$$

*where* $\beta_{VW}, \beta_{WV}, z_E, z_F, p_E, p_F$ *are complex constants and* $P_V, P_W, a, c, c_V, c_W$ *are real constants;*

$$\beta_{VW} = \sqrt{\beta_E} - \frac{1}{\sqrt{\beta_E}}(1+\alpha i), \qquad \beta_{WV} = \sqrt{\beta_E} + \frac{1}{\sqrt{\beta_E}}(1+\alpha i), \tag{43}$$
$$z_E = \frac{1}{\sqrt{\beta_E}} + \beta_{VW}, \qquad z_F = -\frac{\alpha}{\sqrt{\beta_E}} + (\beta_{VW} - \sqrt{\beta_E})i;$$

$$\begin{aligned} c_V &= \frac{2\beta_E(1+l) - \beta_E^2(3+2l) - (3+2l)(1+\alpha^2)}{2\beta_E(3+5l+2l^2)}, \\ c_W &= \frac{-2\beta_E(1+l) - \beta_E^2(3+2l) - (3+2l)(1+\alpha^2)}{2\beta_E(3+5l+2l^2)}, \\ a &= \frac{(1+2l)(1+\beta_E^2+\alpha^2)}{2\beta_E(1+l)}, \end{aligned} \tag{44}$$

$$\begin{aligned} P_V &= -\frac{1}{2}(2+a+(l+2)c_V+lc_W), \\ P_W &= -\frac{1}{2}(-2+a+lc_V+(l+2)c_W), \\ P_E &= \frac{2}{\beta_E} - \frac{1}{2}(3a+l(c_V+c_W)) - \frac{2\alpha}{\beta_E}i, \\ P_F &= -\frac{1}{2}(3a+l(c_V+c_W))i + 2(\beta_E + \frac{\alpha^2}{\beta_E})i - \frac{2\alpha}{\beta_E}, \\ c &= \big(16\beta_E^2(1+l)^2(1+2l)(3+2l)^2\big)^{-1}\cdot \\ &\big(-\beta_E^4(3+2l)^2(15+2l(15+8l)) \\ &-(3+2l)^2(15+2l(15+8l))(1+\alpha^2)^2\big)\cdot \\ &\big(-2\beta_E^2(-153-2l(383+2l(347+2l(145+8l(7+l))))) \\ &+(3+2l)^2(15+2l(15+8l))\alpha^2)\big). \end{aligned} \tag{45}$$

v. *For the resulting family of Cartan geometries on* $SU(l+2)/U(l)$ *the following holds:*
*When* $l=0$ *this is a family of CR structures on* $SU(2)=S^3$ *and for* $t=0$ *it's the standard CR structure on* $S^3$.
*For* $t \neq 0$ *or* $l > 0$ *this structure is not locally isomorphic to the homogeneous model of partially integrable almost CR structures of hypersurface type of signature* $(l, l+1)$*; i.e., it is not spherical.*

vi. *The curvature function* $\kappa_o \in L(\Lambda^2(\mathfrak{g}_-), \mathfrak{g})$ *of the Cartan connection has values in* $\mathfrak{g}^0 = \mathfrak{p}$. *It is given by*

$$\kappa(X_{-2}, v \oplus w \oplus \gamma)_{\mathfrak{g}_0} = \tag{46}$$
$$\begin{pmatrix} 0 & \begin{pmatrix} \frac{(i(c_V-c_W)\beta_E - 2\alpha)z_E - 2z_F}{\beta_E}\mathrm{Re}(\gamma) \\ \frac{i(c_V-c_W)\beta_E - 2(\alpha^2+\beta_E^2)}{\beta_E}\mathrm{Im}(\gamma) \end{pmatrix} & (c_V-a-1)\beta_{VW}iw \\ * & 0 & (c_W-a+1)\beta_{WV}iv \\ * & -* & 0 \end{pmatrix},$$

$$
\kappa(X_{-2}, v \oplus w \oplus \gamma)_{\mathfrak{g}_1} =
$$

$$
\begin{pmatrix}
0 & \begin{pmatrix} * & -* & -* \end{pmatrix} & 0 \\
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & 0 & -\begin{pmatrix} ((m_V+1)P_V+c)i\ v \\ ((m_W-1)P_W+c)i\ w \\ m_\Gamma\ i(p_E \mathrm{Re}(\gamma) + p_F \mathrm{Im}(\gamma)) + c\ i\gamma \\ -2p_E\big(\frac{\alpha}{\beta_E}\mathrm{Re}(\gamma) - (\beta_E + \frac{\alpha^2}{\beta_E})\mathrm{Im}(\gamma)\big) \\ -\frac{2p_F}{\beta_E}\big(\mathrm{Re}(\gamma) - \alpha \mathrm{Im}(\gamma)\big) \end{pmatrix} \\
0 & \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} & 0
\end{pmatrix} \tag{47}
$$

*with*

$$
\begin{aligned}
m_V &= \frac{1}{2}(a + (l+2)c_V + lc_W)) \\
m_W &= \frac{1}{2}(a + lc_V + (l+2)c_W)) \\
m_\Gamma &= \frac{1}{2}(3a + lc_V + lc_W)).
\end{aligned} \tag{48}
$$

$$
\kappa(X_{-2}, v \oplus w \oplus \gamma)_{\mathfrak{g}_2} = 0; \tag{49}
$$

$$
\kappa(v_1 \oplus w_1 \oplus \gamma_1, v_2 \oplus w_2 \oplus \gamma_2)_{\mathfrak{g}_0} =
$$
$$
\tilde{\kappa}(v_1 \oplus w_1 \oplus \gamma_1, v_2 \oplus w_2 \oplus \gamma_2)_{\mathfrak{g}_0} - \tilde{\kappa}(v_2 \oplus w_2 \oplus \gamma_2, v_1 \oplus w_1 \oplus \gamma_1)_{\mathfrak{g}_0},
$$

*where*

$$
\tilde{\kappa}(v_1 \oplus w_1 \oplus \gamma_1, v_2 \oplus w_2 \oplus \gamma_2)_{\mathfrak{g}_0} =
$$

$$
\begin{pmatrix}
\begin{pmatrix} 2\Big((\mathrm{Re}(P_V)+1)v_1 v_2^* + w_1 w_2^*\Big) \\ +\Big(\frac{1}{2l+3}\mathrm{Im}(\langle v_1 \oplus w_1, v_2 \oplus w_2\rangle) \\ +c_V \mathrm{Im}\langle v_1 \oplus w_1 \oplus \gamma_1, \\ v_2 \oplus w_2 \oplus \gamma_2\rangle\Big)\mathbb{I}_l i \end{pmatrix} & \begin{pmatrix} +2\sqrt{\beta_E} z_E \mathrm{Re}(\langle v_1, w_2\rangle) \\ +2\frac{\alpha z_E + z_F}{\sqrt{\beta_E}}\mathrm{Im}(\langle v_1, w_2\rangle) \\ -(P_V + \overline{P_W})v_1 w_2^* \end{pmatrix} & \begin{pmatrix} (P_V \overline{\gamma_1} + \overline{P_E}\mathrm{Re}(\gamma_1) + \overline{P_F}\mathrm{Im}(\gamma_1))v_2 \\ +\beta_{VW}\alpha_0^{-1}(\gamma_1)v_2 \end{pmatrix} \\
(*) & \begin{pmatrix} 2\Big((\mathrm{Re}(P_W)-1)w_1 w_2^* + v_1 v_2^*\Big) \\ +\Big(\frac{1}{2l+3}\mathrm{Im}(\langle v_1 \oplus w_1, v_2 \oplus w_2\rangle) \\ +c_W \mathrm{Im}\langle v_1 \oplus w_1 \oplus \gamma_1, \\ v_2 \oplus w_2 \oplus \gamma_2\rangle\Big)\mathbb{I}_l i \end{pmatrix} & \begin{pmatrix} (P_W \overline{\gamma_1} + \overline{P_E}\mathrm{Re}(\gamma_1) + \overline{P_F}\mathrm{Im}(\gamma_1))w_2 \\ -\beta_{WV}\overline{\alpha_0^{-1}(\gamma_1)}w_2 \end{pmatrix} \\
(*) & (-*) & \begin{pmatrix} 2\mathrm{Im}(\overline{\gamma_1}(\mathrm{Re}(\gamma_2)P_E + \mathrm{Im}(\gamma_2)P_F)) \\ +\frac{2(l+2)}{2l+3}\mathrm{Im}(\langle v_1 \oplus w_1, v_2 \oplus w_2\rangle) \\ +a\mathrm{Im}\langle v_1 \oplus w_1 \oplus \gamma_1, v_2 \oplus w_2 \oplus \gamma_2\rangle \end{pmatrix} i
\end{pmatrix}. \tag{50}
$$

$$
\kappa(v_1 \oplus w_1 \oplus \gamma_1, v_2 \oplus w_2 \oplus \gamma_2)_{\mathfrak{g}_1} =
$$
$$
= \tilde{\kappa}(v_1 \oplus w_1 \oplus \gamma_1, v_2 \oplus w_2 \oplus \gamma_2)_{\mathfrak{g}_1} - \tilde{\kappa}(v_2 \oplus w_2 \oplus \gamma_2, v_1 \oplus w_1 \oplus \gamma_1)_{\mathfrak{g}_1},
$$

*where*

$$\tilde{\kappa}(v_1 \oplus w_1 \oplus \gamma_1, v_2 \oplus w_2 \oplus \gamma_2)_{\mathfrak{g}_1} =$$
$$\begin{pmatrix} 0 & \begin{pmatrix} * & -* & -* \end{pmatrix} & 0 \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & 0 & -\begin{pmatrix} \big(\beta_{VW}(P_E \mathrm{Re}(\gamma_2) + P_F \mathrm{Im}(\gamma_2)) + P_V \overline{\alpha_0^{-1}(\gamma_2)}\big) w_1 \\ +(\mathrm{Re}(\gamma_1) z_E + \mathrm{Im}(\gamma_1) z_F) P_W w_2 \\ \big(\beta_{WV}(P_E \mathrm{Re}(\gamma_2) + P_F \mathrm{Im}(\gamma_2)) - P_W \alpha_0^{-1}(\gamma_2)\big) v_1 \\ +(\mathrm{Re}(\gamma_1) \overline{z_E} + \mathrm{Im}(\gamma_1) \overline{z_F}) P_V v_2 \\ \big(2p_E \sqrt{\beta_E} \mathrm{Re}(\langle v_1, w_2 \rangle) \\ +\big(\frac{\alpha p_E}{\sqrt{\beta_E}} + \frac{2p_F}{\sqrt{\beta_E}}\big) \mathrm{Im}(\langle v_1, w_2 \rangle) \\ +P_V \overline{\beta_{VW}} \langle w_1, v_2 \rangle - P_W \overline{\beta_{WV}} \langle v_1, w_2 \rangle\big) \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} & 0 \end{pmatrix}; \tag{51}$$

$$\kappa(v_1 \oplus w_1 \oplus \gamma_1, v_2 \oplus w_2 \oplus \gamma_2)_{\mathfrak{g}_2} =$$
$$2 \begin{pmatrix} (c - P_V^2) \mathrm{Im}(\langle v_1, v_2 \rangle) - (c - P_W^2) \mathrm{Im}(\langle w_1, w_2 \rangle) \\ -c \mathrm{Im}(\overline{\gamma_1} \gamma_2) \\ +\mathrm{Im}((\mathrm{Re}(\gamma_1) P_E + \mathrm{Re}(\gamma_1) P_F)(\mathrm{Re}(\gamma_2) P_E + \mathrm{Re}(\gamma_2) P_F)) \end{pmatrix} X_2. \tag{52}$$

**Proof:**

i.

$$\alpha_0^{-1}(v \oplus w \oplus \gamma) = \tag{53}$$
$$= \sqrt{2} v \oplus \sqrt{2} w \oplus \frac{1}{\sqrt{\beta_E}} \mathrm{Re}(\gamma) - \frac{\alpha}{\sqrt{\beta_E}} \mathrm{Im}(\gamma) + \sqrt{\beta_E} \mathrm{Im}(\gamma) i, \alpha_0^{-1}(X_{-2}) = Z. \tag{54}$$

Note that writing $r + si = \gamma$,

$$\alpha_0(0 \oplus 0 \oplus \gamma) = 0 \oplus 0 \oplus \frac{1}{2} \sqrt{\beta_E} \big((1 + \frac{1}{\beta_E}(1 - \alpha i))\gamma + (1 - \frac{1}{\beta_E}(1 - \alpha i))\overline{\gamma}\big).$$

Lets check that the isomorphism $\alpha_0 : \mathfrak{n} = \mathbb{R}Z \oplus D \cong \mathfrak{g}_-$ induces an isomorphism of the associated graded Lie algebras: We have

$$\begin{aligned} &[(v_1, w_1, \gamma_1), (v_2, w_2, \gamma_2)]_{-2} = \\ &= -2 \mathrm{Im} g((v_1, w_1, \gamma_1), (v_2, w_2, \gamma_2)) = \\ &= -\mathrm{Im}\langle v_1, v_2 \rangle + \mathrm{Im}\langle w_1, w_2 \rangle + 2 \mathrm{Im} \overline{\gamma_1} \gamma_2 \\ &\overset{\alpha_0}{\mapsto} \big(-\mathrm{Im}\langle v_1, v_2 \rangle + \mathrm{Im}\langle w_1, w_2 \rangle + 2 \mathrm{Im} \overline{\gamma_1} \gamma_2\big) X_2 \end{aligned}$$

and

$$
\begin{aligned}
&[\alpha_0(v_1, w_1, \gamma_1), \alpha_0(v_2, w_2, \gamma_2)] = \\
&= [(\frac{1}{\sqrt{2}} v_1, \frac{1}{\sqrt{2}} w_1, \sqrt{\beta_E}\mathrm{Re}\gamma_1 + \frac{\alpha}{\sqrt{\beta_E}}\mathrm{Im}\gamma_1 + \frac{1}{\sqrt{\beta_E}}\mathrm{Im}\gamma_1 i), \\
&(\frac{1}{\sqrt{2}} v_2, \frac{1}{\sqrt{2}} w_2, \sqrt{\beta_E}\mathrm{Re}\gamma_2 + \frac{\alpha}{\sqrt{\beta_E}}\mathrm{Im}\gamma_2 + \frac{1}{\sqrt{\beta_E}}\mathrm{Im}\gamma_2 i)] = \\
&= -2\mathrm{Im}\langle(\frac{1}{\sqrt{2}} v_1, \frac{1}{\sqrt{2}} w_1, \sqrt{\beta_E}\mathrm{Re}\gamma_1 + \frac{\alpha}{\sqrt{\beta_E}}\mathrm{Im}\gamma_1 + \frac{1}{\sqrt{\beta_E}}\mathrm{Im}\gamma_1 i), \\
&(\frac{1}{\sqrt{2}} v_2, \frac{1}{\sqrt{2}} w_2, \sqrt{\beta_E}\mathrm{Re}\gamma_2 + \frac{\alpha}{\sqrt{\beta_E}}\mathrm{Im}\gamma_2 + \frac{1}{\sqrt{\beta_E}}\mathrm{Im}\gamma_2 i)\rangle_{(l,l+1)} = \\
&= -\mathrm{Im}\langle v_1, v_2\rangle + \mathrm{Im}\langle w_1, w_2\rangle - 2(\mathrm{Re}\gamma_2\mathrm{Im}\gamma_1 - \mathrm{Re}\gamma_1\mathrm{Im}\gamma_2) = \\
&= -\mathrm{Im}\langle v_1, v_2\rangle + \mathrm{Im}\langle w_1, w_2\rangle + 2\mathrm{Im}\overline{\gamma_1}\gamma_2.
\end{aligned}
$$

Thus $\alpha_0$ is indeed regular.

ii. $\mathrm{Ad}_{|G_0}$ induces an embedding of $G_0$ as a closed subgroup into $\mathrm{GL}(\mathfrak{g}_-)$. We need to show that the homomorphism of Lie groups

$$
\Psi : K \to \mathrm{GL}(\mathfrak{g}_-), \qquad \Psi(k) := \alpha_0 \circ \mathrm{Ad}(k) \circ \alpha_0^{-1}
$$

has values in $G_0$. Since the exponential map $\exp : \mathfrak{k} \to K$ is surjective this is equivalent to $\Psi' : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{g}_-)$ having values in $\mathfrak{g}_0$. Once we have shown this it is tautological that $\alpha_{0|\mathfrak{n}}$ is $K$-equivariant; and on $\mathfrak{k}$ we simply defined $\alpha_{0|\mathfrak{k}} := \psi'$.
We have: $Z$ is invariant under $\mathfrak{k}$ and recall from (38) that

$$
[(a, A), (v \oplus w \oplus \gamma)] = (A + a)v \oplus (A + a)w \oplus 0
$$

(here we use notation (33)).
Now one sees that for $X \in \mathfrak{g}_-$ and $A \in \mathfrak{k}$

$$
\alpha_0(\mathrm{ad}((a, A))(\alpha_0^{-1}(X))) =
$$

$$
= \mathrm{ad}\left(\left(\begin{matrix} -\frac{2(l+2)}{2l+3}a & 0 & 0 \\ 0 & \begin{pmatrix} A-\frac{1}{2l+3}a & 0 & 0 \\ 0 & A-\frac{1}{2l+3}a & 0 \\ 0 & 0 & -\frac{2(l+2)}{2l+3}a \end{pmatrix} & 0 \\ 0 & 0 & -\frac{2(l+2)}{2l+3}a \end{matrix}\right)\right)(X);
$$

Thus $\alpha_0 \circ \mathrm{ad}((a, A)) \circ \alpha_0^{-1}$ does indeed have values in $\mathfrak{g}_0$.

iii. The assertion that every other solution $\alpha_0'$ is of the form $\mathrm{Ad}(g_0) \circ \alpha_0$ for some $g_0 \in G_0$ follows from 8.3.2.

iv. Now we need to show that $\partial^*\kappa = 0$. According to theorem 4.2.1 the curvature function $\kappa \in \mathrm{C}^\infty(\mathrm{SU}(l+2) \times_K P, L(\Lambda^2(\mathfrak{g}_-), \mathfrak{g}))$ of the Cartan connection induced by the map $\alpha$ is $\mathrm{SU}(l+2)$-invariant and $P$-equivariant and thus factorizes to an invariant section of $\mathrm{SU}(l+2) \times_K L(\Lambda^2(\mathfrak{g}_-), \mathfrak{g})$. At $o = K$ it is

$$
\kappa_o(X_1, X_2) = [X_1 + \phi(X_1), X_2 + \phi(X_2)] - \alpha([\alpha_0^{-1}(X_1), \alpha_0^{-1}(X_2)]).
$$

It is straightforward to calculate (46)-(52).
The explicit equations in $\phi$ for $\partial^*\kappa = 0$ are obtained as follows: Let $e_j$ denote the $j$-th standard-basis-vector of $\mathbb{C}^l$.

$$e_{Vj} = e_j \oplus 0 \oplus 0 \in \mathfrak{g}_{-1},$$
$$e_{Wj} = 0 \oplus e_j \oplus 0 \in \mathfrak{g}_{-1};$$

As a real basis of $\mathfrak{g}_{-1}$ we take

$$\mathfrak{B} := \big(e_{V1}, \dots, e_{Vl}, ie_{V1}, \dots ie_{Vl}, e_{W1}, \dots, e_{Wl}, ie_{W1}, \dots ie_{Wl},$$
$$0 \oplus 0 \oplus 1, 0 \oplus 0 \oplus i\big).$$

$X_{-2}$ completes it to a basis of $\mathfrak{g}_-$ whose dual basis is $(-X_2, \mathfrak{C})$ with

$$\mathfrak{C} = \big(\tilde{e}^*_{V1}, \dots, \tilde{e}^*_{Vl}, i\tilde{e}_{V1}{}^*, \dots i\tilde{e}_{Vl}{}^*, -\tilde{e}^*_{W1}, \dots, -\tilde{e}^*_{Wl}, -(i\tilde{e}_{W1})^*, \dots (i\tilde{e}_{Wl})^*,$$
$$-(0 \oplus 0 \oplus 1)^*, -(0 \oplus 0 \oplus i)^*\big).$$

Now, as we discussed in 13, $\partial^*\kappa = 0$ is equivalent to:

$$\sum_{i=1}^{4l+2} [\mathfrak{C}_i, \kappa(X, \mathfrak{B}_\mathfrak{i})] - [X_2, \kappa(X, X_{-2})] = 0 \tag{55}$$

and

$$\sum_{i=1}^{4l+2} \kappa([\mathfrak{C}_i, X]_-, \mathfrak{B}_\mathfrak{i}) - \kappa([\alpha_0^{-1}(X_2), \alpha_0^{-1}(X)]_-, X_{-2}) = 0 \tag{56}$$

for all $X \in \mathfrak{g}_-$. (Recall that for an element $B \in \mathfrak{g}$ we denote the projection of $B$ to $\mathfrak{g}_-$ by $B_{\mathfrak{g}_-}$ or simply by $B_-$.)
Note that $[X_2, X]_- = 0$ on the whole of $\mathfrak{g}_-$ and $[\mathfrak{C}_i, X]_- = 0$ for $X \in \mathfrak{g}_{-1}$. Thus equation (56) reduces to

$$\sum_{i=1}^{4l+2} \kappa([\mathfrak{C}_i, X_{-2}], \mathfrak{B}_\mathfrak{i}) = 0,$$

but on can show that this is already implied by (55).
Next one calculates that (55) does hold for

$$X = X_{-2}, X = v \oplus 0 \oplus 0, X = 0 \oplus w \oplus 0, X = E, X = F.$$

This task consist only of taking commutators and summing up.

□

*Remark on how we found the solution.* From the general theory of parabolic geometries we know that there is a unique (up to equivalency) regular normal Cartan connection on $H \times_\Psi P$ inducing the same CR structure on $M = H/K$. Now we saw in i that our $\alpha_0$ is regular and in iii we saw that any other regular map $\alpha_0'$ differs by an isomorphism of $\mathfrak{g}$ from $\alpha_0$. Thus, in our search for an $\alpha$ with $\partial^*\kappa = 0$, we may restrict ourselves to maps $\alpha_0 + \phi \circ \alpha$ for $K$-equivariant maps $\phi : \mathfrak{g}_- \to \mathfrak{g}$ of homogeneity greater zero.
We found the solution for $\phi$ by making an ansatz for a $K$-equivariant map

of homogeneity greater one. We describe the decomposition of $\mathfrak{g}$ as a $K$-module: under $\Psi$ an element $A \in \mathfrak{su}(l) < \mathfrak{k} < \mathfrak{h}$ acts on an element of $\mathfrak{g}$ by

$$\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -\alpha & \begin{pmatrix} \tilde{v}^* & -\tilde{w}^* & -\bar{\tilde{z}} \end{pmatrix} & \beta i \\ \begin{pmatrix} v \\ w \\ z \end{pmatrix} & \begin{pmatrix} A_V & B & b_V \\ B^* & A_W & b_W \\ b_V^* & -b_W^* & z \end{pmatrix} & -\begin{pmatrix} \tilde{v} \\ \tilde{w} \\ \tilde{z} \end{pmatrix} \\ \gamma i & \begin{pmatrix} -v^* & w^* & \bar{z} \end{pmatrix} & \bar{\alpha} \end{pmatrix}\right] =$$

$$= \begin{pmatrix} 0 & \begin{pmatrix} (A\tilde{v})^* & -(A\tilde{w})^* & 0 \end{pmatrix} & 0 \\ \begin{pmatrix} Av \\ Aw \\ 0 \end{pmatrix} & \begin{pmatrix} [A, A_V] & [A, B] & Ab_V \\ [A, B]^* & [A, A_W] & Ab_W \\ (Ab_V)^* & -(Ab_W)^* & 0 \end{pmatrix} & -\begin{pmatrix} (A\tilde{v})^* \\ (A\tilde{w})^* \\ 0 \end{pmatrix} \\ 0 & \begin{pmatrix} -(Av^*) & (Aw)^* & 0 \end{pmatrix} & 0 \end{pmatrix}.$$

Thus we get the following decomposition of $\mathfrak{g}$ into irreducible $K$-modules: the grading-components $\mathfrak{g}_i$ of $\mathfrak{g}$ are $K$-invariant, thus we describe their decomposition: $\mathfrak{g}_{-2}$ and $\mathfrak{g}_2$ are already 1-dimensional, real spaces.
$\mathfrak{g}_{-1} = \mathbb{C}^l \oplus \mathbb{C}^l \oplus \mathbb{C}$ decomposes into $\mathbb{C}^l \oplus \mathbb{C}^l \oplus (\mathbb{R} \oplus \mathbb{R})$ as $K$-module. The representation of $K$ on $\mathfrak{g}_1$ is the dual representation of $K$ on $\mathfrak{g}_{-1}$ and thus similarly $\mathfrak{g}_1 = (\mathbb{C}^l \oplus \mathbb{C}^l \oplus (\mathbb{R} \oplus \mathbb{R}))^*$.
The decomposition of $\mathfrak{g}_0$ is given as follows: $\mathfrak{g}_0 = \mathbb{R} \oplus \mathfrak{u}(l, l+1)$ as $K$-module (recall theorem 8.3.1, iii) and $\mathfrak{u}(l, l+1)$ decomposes into 9 irreducible subrepresentations; these are

$$\mathbb{C}^l \oplus \mathbb{C}^l \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \cong \begin{pmatrix} \mathbb{R}i & 0 & \mathbb{C}^l \\ 0 & \mathbb{R}i & \mathbb{C}^l \\ * & (-*) & \mathbb{R}i \end{pmatrix},$$

$$\mathfrak{su}(l) \oplus \mathfrak{su}(l) \cong \begin{pmatrix} \mathfrak{su}(\mathfrak{l}) & 0 & 0 \\ 0 & \mathfrak{su}(l) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and}$$

$$\mathfrak{su}(l) \oplus \mathfrak{su}(l) \cong \begin{pmatrix} 0 & \mathfrak{su}(l) \oplus i\mathfrak{su}(l) & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We then used this decomposition to make an equivariant ansatz for $\phi$. The map $\phi$ decomposes into $\phi = \phi_1 \oplus \phi_2 \oplus \phi_3 \oplus \phi_4$, where $\phi_i$ is a map of homogeneity $i$.
Then, since higher homogeneities of $\phi$ don't contribute to lower homogeneities of $\partial^*\kappa$ one can solve $\phi$ one homogeneity after the other.
One furthermore knows from the general theory (cf. [5]) that the curvature of the Cartan connection corresponding to an integrable CR structure has in fact values in $\mathfrak{p}$.

Using a $K$-equivariant ansatz for $\phi$ we first found and verified the solution for finitely many dimensions $l \in \mathbb{N}_0$ by using Mathematica.
Then we saw what the general solution for arbitrary $l \in \mathbb{N}_0$ is and checked (55) by hand; here computer algebra was still very helpful for simplifying expressions. ⌟

*Remark* 8.8.3. Let us briefly consider the special case $l = 0$. Here, with $\alpha = 0, \beta_E = \lambda \in \mathbb{R}, \lambda > 0$

$$\alpha(\begin{pmatrix} ix & \gamma \\ -\bar{\gamma} & -ix \end{pmatrix}) =$$

$$\begin{pmatrix} -\frac{1+\lambda^2}{4\lambda}xi & \frac{\mathrm{Re}(\gamma)\lambda(3\lambda^2-5)+\mathrm{Im}(\gamma)(5\lambda^2-3)i}{4\lambda^{\frac{3}{2}}} & \frac{-15\lambda^4+34\lambda^2-15}{16\lambda^2} \\ \frac{\lambda\mathrm{Re}(\gamma)+\mathrm{Im}(\gamma)i}{\sqrt{\mathfrak{l}}} & \frac{1+\lambda^2}{2\lambda}xi & \frac{\mathrm{Re}(\gamma)\lambda(3\lambda^2-5)-\mathrm{Im}(\gamma)(5\lambda^2-3)i}{4\lambda^{\frac{3}{2}}} \\ ix & \frac{\mathrm{Re}(\gamma)\lambda-\mathrm{Im}(\gamma)i}{\sqrt{\lambda}} & -\frac{1+\lambda^2}{4\lambda}xi \end{pmatrix}.$$

The map

$$E \mapsto -E, F \mapsto F, Z \mapsto -Z$$

is an automorphism of $\mathfrak{su}(2)$ and an isomorphism of the CR structure of signature $(0, 1)$ induced by

$$\mathbb{C} \cong \{\begin{pmatrix} 0 & \gamma \\ -\bar{\gamma} & 0 \end{pmatrix}\}$$

and the CR structure of signature $(1, 0)$ induced by

$$\mathbb{C} \cong \{\begin{pmatrix} 0 & -\bar{\gamma} \\ \gamma & 0 \end{pmatrix}\}.$$

By taking the composition of this isomorphism, $\alpha$, and the isomorphism given in 8.3,v one sees that our result here is the same as the one in [3]. ⌟

*Remark* 8.8.4. In [1] D. Alekseevsky and A. Spiro classified all compact homogeneous non-degenerate CR manifolds of hypersurface type and found for the above example in particular that for $t_1, t_2$ in the unit disc of $\mathbb{C}$ the CR structures on $SU(l + 2)/U(L)$ induced by $(D, J_{t_1})$ and $(D, J_{t_2})$ are isomorphic iff $|t_1| = |t_2|$. ⌟

## References


[1] Dmitry V. Alekseevsky and Andrea F. Spiro. Compact Homogeneous CR Manifolds. *The Journal of Geometric Analysis*, 12(2), 2002.

[2] Dmitry V. Alekseevsky and Andrea F. Spiro. Invariant CR structures on compact homogeneous manifolds. *Hokkaido Math. J.*, 32(2):209–276, 2003.

[3] Andreas Čap. On left invariant CR structures on SU(2). `ftp://ftp.esi.ac.at/pub/Preprints/esi1794.pdf`, 2006. ESI Preprint 1794.

[4] Andreas Čap and Hermann Schichl. Parabolic Geometries and Canonical Cartan Connections. *Hokkaido Math. J.*, 29(3):453–505, 2000.

[5] Andreas Čap and Jan Slovák. *Parabolic Geometries*. Book in preparation.

[6] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine. *Riemannian Geometry*. Springer-Verlag, 2004.

[7] R.V. Gamkrelidze (Ed.). *Geometry I*. Springer-Verlag, 1991.

[8] Sigurdur Helgason. *Differential Geometry, Lie Groups, and Symmetric Spaces*. AMS, 2001.

[9] Peter Michor. Topics in Differential Geometry. `http://www.mat.univie.ac.at/~michor/dgbook.ps`, Book in preparation.

[10] T. Morimoto. Geometric Structures on Filtered Manifolds. *Hokkaido Math. J.*, pages 263–347, 1993.

[11] Katsumi Nomizu. Studies on Riemannian Homogeneous Spaces. *Nagoya Math. J.*, 9, 1955.

[12] R.W. Sharpe. *Differential Geometry - Cartan's Generalisation of Klein's Erlangen Program*. Springer-Verlag, 1997.

[13] N. Tanaka. On the Equivalence Problem Associated with Simple Graded Lie algebras. *Hokkaido Math. J.*, pages 23–84, 1979.

[14] Hsien-Chung Wang. On invariant connections over a principal fibre bundle. *Nagoya Math. J.*, 13:1–19, 1958.

[15] K. Yamaguchi. Differential Systems Associated with Simple Graded Lie algebras. *Advanced Studies in Pure Math.*, pages 413–494, 1993.

# Bisheriger Lebenslauf

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