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CATEGORIES OF BISTOCHASTIC MEASURES, AND REPRESENTATIONS OF SOME INFINITE-DIMENSIONAL GROUPS

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ABSTRACT. The following groups are considered: the automorphism group of a Lebesgue measure space (with finite or σ-finite measure), groups of measurable functions with values in a Lie group, and diffeomorphism groups of manifolds. It turns out that the theory of representations of all these groups is closely related to the theory of representations of some category, which will be called "the category of G-polymorphisms". Objects of this category are measure spaces, and a morphism from M to N is a probability measure on $M \times N \times G$, where G is a fixed Lie group. For some of the above-mentioned infinite-dimensional groups $\mathfrak{G}$ it is shown that any representation of $\mathfrak{G}$ extends canonically to a representation of some category of G-polymorphisms. For automorphism groups of measure spaces this makes it possible to obtain a classification of all unitary representations. Also "new" examples of representations of groups of area-preserving diffeomorphisms of two-dimensional manifolds are constructed.

Let G be an infinite-dimensional group having some supply of unitary representations. It turns out that, associated with G, there is usually (always?) a semigroup $\Gamma \supset G$ (the "mantle" of G), which cannot be seen with the naked eye, such that all unitary representations of G extend in a rigid way to Γ (there are good reasons to think that G is dense in Γ; this does not mean that Γ is a completion of G in the formal sense of the word "completion").

The principle was formulated by G. I. Ol′shanskiĭ about 1980. As the structure of such semigroups was being clarified, it became clear that it is not semigroups but categories that are concerned (see [12]–[16] and [24]). Namely, to an infinite-dimensional group G there is usually (always?) associated a category $\mathscr{K}$ (the train of G), and the theory of representations of G is in a sense the theory of representations of $\mathscr{K}$. The group G itself plays the role of the automorphism group of one of the objects of $\mathscr{K}$, and the semigroup Γ plays the role of the endomorphism semigroup of this same object.

One knows three methods of searching for the mantle and the train of a group G: "multiplicativity theorems" (going back to [27] and [6]; see §3), weak closure (see §§4.1 and 4.2), and holomorphic continuation (introduced by Ol′shanskiĭ). None of these methods gives a direct algorithm for constructing the mantle and the train. The explicit construction contains, as a rule, some volitional steps which can be justified only afterwards.

At present the most beautiful constructions of trains (the symplectic category, the orthogonal category, the category Shtan; see [12]–[16], [23], and [24] have been obtained by means of holomorphic continuation, which method is not used in this paper.

The motive for writing this paper was the fact that, in some strange way, one of the universal categories connected with infinite-dimensional groups, the category of

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G-polymorphisms, has not even been mentioned in the literature, whereas it forms a basis for unifying several beautiful theories which seem to be completely different. An object of this category is a measure space (M, μ), and a morphism $(M, \mu) \to (N, \nu)$ is a measure κ on $M \times N \times G$ (where G is a fixed group) such that the projection of κ on M is μ and its projection on N is ν (see §2). In §3 we show that the theory of representations of the group of measurable functions on a Lebesgue space with a group-valued measure is in some sense equivalent to the theory of representations of the category of G-polymorphisms. On the other hand, one can be fairly optimistic about describing all representations of this category; unfortunately, in studying this problem, we did not manage to go as far as we wanted to (i.e. to obtain the complete classification of representations, which has been done only in the case of a compact group G). In §4 we show that studying the theory of representations of diffeomorphism groups leads in a strange way to similar categories. As an application we construct a new series of representations of the group of volume-preserving diffeomorphisms of a compact two-dimensional manifold.

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0. Notation and preliminaries

0.1. **Measure spaces.** By a measure space M we shall always mean a Lebesgue space, i.e. a space isomorphic to the union of an interval (finite or infinite) and a set (empty, finite, or countable) of points of nonzero measure. We assume that the measure of the Lebesgue space is defined on the σ-algebra of Borel sets. We shall say that the measure is continuous if each point has measure 0.

By a finite partition $\mathfrak{h}$ of M we shall mean a partition of M into a finite number of measurable sets $M = \bigcup M_i$. We shall write $\mathfrak{h}_1 \le \mathfrak{h}_2$ if the partition $\mathfrak{h}_2$ is finer than $\mathfrak{h}_1$. We shall call the sequence of partitions $\mathfrak{h} = \mathfrak{h}_1 \le \mathfrak{h}_2 \le \cdots$ *basic* if, for any sequence $M_{i_1}^{(1)} \supset M_{i_2}^{(2)} \supset \cdots$ such that $M_{j_k}^{(k)}$ is an element of the kth partition, the intersection $\bigcap_k M_{j_k}^{(k)}$ consists of a single point.

Let (M, μ) and (N, ν) be measure spaces and π a Borel measure on $M \times N$ whose projection on M coincides with μ. Then by $\pi_m(n)$ we shall denote the conditional measures on N relative to the partition $\{m \times N\}$ of $M \times N$.

0.2. **The group** $\mathrm{Aut}(M)$. We denote by $\mathrm{Aut}(M)$ the automorphism group of the space M with a continuous probability measure. There is apparently only one reasonable topology on this group. It can be described in the following three ways.

1. The group $\mathrm{Aut}(M)$ acts on $L^2(M)$ by transformations of the form $f(m) \mapsto f(gm)$, and consequently the weak operator topology on $L^2(M)$ induces a topology on $\mathrm{Aut}(M)$.

2. Let g_i, $g \in \mathrm{Aut}(M)$. Then $g_i \to g$ if, for any measurable sets A, $B \subset M$, we have $\mu(g_i A \cap B) \to \mu(gA \cap B)$.

3. Let $\mathfrak{h}: M = \bigcup M_i$ and $\mathfrak{h}': M = \bigcup M_j'$ be finite partitions and $g \in \mathrm{Aut}(M)$. We define, following [2], the "intersection matrix" $P(g)$ with entries $\mu(gM_i \cap M_j')$. Our topology is the weakest topology in which all the functions $g \mapsto P(g)$ are continuous.

0.3. **The group** $\mathfrak{B}(G)$. Let G be a Lie group and M a space with a continuous probability measure. Let the "current" group $\mathscr{F}(M, G)$ be the group of measurable functions $f: M \to G$ such that the image $f(M)$ is contained in a compact subset of G (this is a technical requirement whose purpose is to make the group as small as possible and hence to enlarge the number of its representations). The group $\mathfrak{B}(G)$ is the semidirect product of $\mathrm{Aut}(M)$ and $\mathscr{F}(M, G)$. Representations of $\mathfrak{B}(G)$ may be regarded as representations of the current group $\mathscr{F}(M, G)$ invariant under the automorphisms of M (see, for instance, [11] for similar groups connected with affine algebras). We have still to introduce some kind of convergence $f_i \to f$ in $\mathscr{F}(M, G)$. To be specific we require that

(a) the sequence f_i converges to f pointwise, and that

(b) $\bigcup f_i(M)$ be contained in a compact subset of G.

A representation ρ of $\mathfrak{B}(G)$ will be called continuous if its restriction to $\mathrm{Aut}(M)$ is continuous and its restriction to $\mathscr{F}(M, G)$ is continuous in the sense of Heine. We shall not deal with a thorough analysis of the dependence of the supply of representations on the topology; we only remark that, as one can see from the considerations of §3.4, the given convergence guarantees the largest supply of unitary representations (but there are many other topologies which guarantee the same supply of representations).

0.4. **Categories and representations of categories.** Let $\mathscr{K}$ be a category and $\mathrm{Ob}(\mathscr{K})$ the set of objects of $\mathscr{K}$. Let V and W be objects of $\mathscr{K}$. Then $\mathrm{Mor}_{\mathscr{K}}(V, W)$ is the set of morphisms $V \to W$, $\mathrm{End}_{\mathscr{K}}(V) = \mathrm{Mor}_{\mathscr{K}}(V, V)$, and $\mathrm{Aut}_{\mathscr{K}}(V)$ is the group of automorphisms of V.

Let $\mathscr{K}$ be a category. By a representation $T = (T, \tau)$ of $\mathscr{K}$ we mean a functor from $\mathscr{K}$ to the category of vector spaces and linear operators; in other words, to each object V of $\mathscr{K}$ we associate a Hilbert space $T(V)$ and to each morphism $\varphi\colon V \to W$ an operator $\tau(\varphi)\colon T(W) \to T(V)$ such that

$$\tau(\psi\varphi) = \tau(\varphi)\tau(\psi) \tag{0.1}$$

for all $\varphi \in \mathrm{Mor}(V, W)$ and $\psi \in \mathrm{Mor}(W, Y)$. If (0.1) is replaced by

$$\tau(\psi\varphi) = \lambda(\varphi, \psi)\tau(\varphi)\tau(\psi),$$

where $\lambda(\varphi, \psi)$ is a complex number, we shall say that τ is a projective representation. In all cases to be considered below the group $\mathrm{Aut}(V)$ is nonempty. We shall always require that the identity element of $\mathrm{Aut}(V)$ be mapped to the identity operator E. Of course, we also require that the representation be continuous.

Suppose, further, that the set of morphisms of $\mathscr{K}$ is equipped with an involution $\varphi \mapsto \varphi^*$ from $\mathrm{Mor}(V, W)$ to $\mathrm{Mor}(W, V)$ so that $(\varphi\psi)^* = \psi^*\varphi^*$. A representation (T, τ) will be called a **-representation* if $\tau(\varphi^*) = \tau(\varphi)^*$.

We omit the more or less obvious definitions of irreducible representation, equivalent representations, direct sum, tensor product and so on; see [16].

0.5. **Schur-Weyl functors** SW_λ. Let (T, τ) be the identity representation of the category Op of Hilbert spaces and bounded operators. Let λ be an irreducible representation of the symmetric group S_n. By $(\mathrm{SW}_\lambda, \sigma w_\lambda)$ we denote the representation of Op on type λ tensors. That is, to each $V \in \mathrm{Ob}(\mathrm{Op})$ we associate the space $\mathrm{Hom}_{S_n}(\lambda, V^{\otimes n})$ of type λ tensors (the group S_n acts on $V^{\otimes n}$ by permuting the factors). The operators $\sigma w_\lambda(A)$ are the natural transformations of type λ tensors, (for details, see [16]).

1. Araki's scheme

1.1. **The affine category.** An object of the affine category Aff is a Hilbert space. By a morphism $H_1 \to H_2$ we mean a pair $(\varphi, \psi) = [A, b, c]$ of inhomogeneous linear mappings $\varphi\colon H_1 \to H_2$ and $\psi\colon H_2 \to H_1$ of the form

$$\varphi(z) = Az + b, \qquad \psi(z) = A^*u + c,$$

where $A\colon H_1 \to H_2$ is a linear operator with $\|A\| \le 1$. If $(\varphi, \psi) \in \mathrm{Mor}(H_1, H_2)$ and $(\varphi', \psi') \in \mathrm{Mor}(H_2, H_3)$, then their composition equals $(\varphi'\varphi, \psi\psi')$.

Remark. The affine category is a part of the affine symplectic category, and the Fock representation of Aff constructed below is the restriction of the "Weil representation" of the affine symplectic category (see [14]).

1.2. **The Fock representation of** Aff. We assign to each $H \in \mathrm{Ob}(\mathrm{Aff})$ the boson Fock space $F(H)$, i.e. the space of holomorphic functions on H with the scalar product

$$(f, g) = \iint f(z)\overline{g(z)}\exp(-\|z\|^2)\prod \frac{dz_i d\overline{z}_i}{\pi}$$

(for details of this definition in the case where H is infinite-dimensional, see [20]).

To a morphism $[A, b, c]: H_1 \to H_2$ we assign the operator

$$\varphi([A, b, c])f(z) = f(Az + b)\exp\langle z, c\rangle \tag{1.1}$$

from $F(H_2)$ to $F(H_1)$. It is easy to see that

$$\varphi([A_1, b_1, c_1] \circ [A_2, b_2, c_2]) = \exp\langle b_2, c_2\rangle \varphi([A_1, b_1, c_1])\varphi([A_2, b_2, c_2]).$$

Thus, (F, φ) is a projective representation of the category Aff.

Remark. If $\|A\| < 1$, the operator (1.1) is bounded (see [14]). Otherwise it may not be bounded, but it is still correctly defined. To prove that, take the dense subspace $F_0(H)$ of $F(H)$ spanned by functions of the form

$$\exp\left(\tfrac{1}{2}\sum p_{ij}z_i z_j + \sum h_i z_i\right),$$

where $\sum |h_i|^2 < \infty$, $\sum |p_{ij}|^2 < \infty$, $p_{ij} = p_{ji}$, and the norm of the matrix P with entries p_{ij} is less than 1. It is easy to see that $\varphi([A, b, c])$ maps $F_0(H_2)$ to $F_0(H_1)$; hence both the operators (1.1) and their products are correctly defined.

1.3. **Araki's scheme for groups.** Let G be a group and ρ its unitary representation on a Hilbert space H. Suppose there exists a continuous function $\gamma: G \to H$ (we shall call it cocycle) such that

$$\gamma(g_1 g_2) = \rho(g_1)\gamma(g_2) + \gamma(g_1).$$

Then the formula

$$A(g)v = \rho(g)v + \gamma(g) \tag{1.2}$$

defines an action of G on the Hilbert space H by affine isometries. The group of all isometries of H will be denoted by $\mathrm{Isom}(H)$.

Further, the group $\mathrm{Isom}(H)$ can be embedded into $\mathrm{Aut}_{\mathrm{Aff}}(H)$ in a natural way; namely, to each mapping $z \mapsto Uz + b$ one associates the morphism $[U, b, U^{-1}b]$ of the category Aff. Moreover, the mapping

$$\mathrm{Exp}: (U, b) \mapsto \exp\left(-\tfrac{1}{2}\langle b, b\rangle\right) \varphi([U, b, U^{-1}b])$$

is a unitary projective representation of the group $\mathrm{Isom}(H)$. We shall call it the Fock representation.

Restricting the Fock representation of $\mathrm{Isom}(H)$ to the subgroup G, embedded into $\mathrm{Isom}(H)$ by means of (1.2), we obtain a unitary projective representation of G.

Suppose that the action (1.2) is equivalent to a linear action, i.e. it can be made linear by shifting the origin. This means that

$$\gamma(g) = \rho(g)w - w \tag{1.3}$$

for some $w \in H$. In such a case we get nothing interesting—the representation of G we have constructed will be equivalent to the direct sum of all symmetric powers of ρ.

One very effective method of constructing representations of infinite-dimensional groups, the so-called Araki's scheme, consists in the following: Construct an affine action of the group G, not equivalent to a linear one, and apply the above construction.

Remark. Usually the cocycle $\gamma(g)$ still has the form (1.3) but with w not lying in H. For instance, let ρ be the natural representation of the group S_∞ of all finite permutations of positive integers on the space l_2. Take as w any vector from $\mathbf{C}^\infty \setminus l_2$. Then (1.3) defines a nontrivial cocycle on S_∞.

Remark. The restriction of the Fock representation Exp to the isometry group of a real Hilbert space is a linear (not projective) representation.

1.4. **The Fock representation of the group $\mathfrak{B}(G)$ (see §0.3).** Let ρ be a unitary representation of G on H. Then the group $\mathfrak{B}(G)$ acts naturally on the space $L^2(M, H)$: the group $\operatorname{Aut}(M)$ acts by coordinate changes

$$f(m) \mapsto f(\rho(m)), \tag{1.4}$$

and the group $\mathcal{F}(M, G)$ acts by transformations of the form

$$T_\rho(g(m))f(m) = \rho(g(m))f(m),$$

where $g \in \mathcal{F}(M, G)$. Thus, we obtain a unitary representation T_ρ of $\mathfrak{B}(G)$. Such representations of $\mathfrak{B}(G)$ will be said to be the simplest.

Now let G act on the Hilbert space H by transformations of the form (1.2). Then an action of $\mathfrak{B}(G)$ on $L^2(M, H)$ by affine transformations is also defined. $\operatorname{Aut}(M)$ acts as before by (1.4), and $\mathcal{F}(M, G)$ by

$$A_\rho(g(m))f(m) = \rho(g(m))f(m) + \gamma(g(m)).$$

Thus, we can define a unitary projective representation $\varphi_{\rho,\gamma} = \operatorname{Exp} \circ A_\rho$ of $\mathfrak{B}(G)$ on $F(L^2(M, H))$.

A vast literature has been devoted to this construction (we indicate only [3], [5], [21], and [26]). Let us note that the existence of nontrivial affine cocycles is a fairly rare phenomenon; it is connected with whether or not the trivial representation is Hausdorff separated from the other ones in the topology of the dual object $\widehat{G}$ of G. In the case of simple Lie groups a one-dimensional representation is, as a rule, an isolated point in the dual object, the only exceptions being the groups $\mathrm{SU}(n, 1)$ and $\mathrm{SO}(n, 1)$, and these are the only groups for which affine cocycles exist ([5]).

We shall need the explicit form of the cocycle in the case $G = \mathrm{SL}_2(\mathbf{R})$. Let $\mathrm{SL}_2(\mathbf{R})$ be realized as the group of matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix},$$

where $|\alpha|^2 - |\beta|^2 = 1$. Let H be the space of real-valued functions on the circle $z = e^{i\varphi}$ with zero mean value and with the scalar product

$$\langle f_1, f_2 \rangle = -\int_0^{2\pi}\int_0^{2\pi} \ln\left|\sin\frac{\varphi - \psi}{2}\right| f_1(\varphi)f_2(\psi)\,d\varphi\,d\psi.$$

Let ρ be the unitary representation of $\mathrm{SL}_2(\mathbf{R})$ on H given by

$$\rho\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} p(z) = p\left(\frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}}\right) |\beta z < +\alpha|^{-2}.$$

Then the affine cocycle equals

$$\gamma\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = |\beta z < +\alpha|^{-2} - 1 .$$

Note that it has the form $\rho(g)1 - 1$, but $1 \notin H$.

The representation $\varphi_{\rho\gamma}$ of the group $\mathfrak{B}(\mathrm{SL}_2(\mathbf{R}))$ is irreducible (see [3]).

2. G-POLYMORPHISMS

2.1. **Polymorphisms (see [2]).** The objects of the category $\mathscr{P}$ are probability spaces. Let (M, μ) and (N, ν) be objects of $\mathscr{P}$. A morphism of $\mathscr{P}$ (a polymorphism) $M \to N$ is a measure π on $M \times N$ such that the projection of π on M is μ and the projection of π on N is ν.

For instance, if the spaces M and N are finite, i.e. consist of a finite number of points with measures $m_1, \dots, m_\alpha$ and $n_1, \dots, n_\beta$ respectively, then a polymorphism is simply an $\alpha \times \beta$ matrix such that $\sum_i \pi_{ij} = m_j$ and $\sum_j \pi_{ij} = n_i$. Let $\pi: M \to N$ and $\psi: N \to K$ be polymorphisms of finite measure spaces. Then their composition $\psi \circ \pi$ can be expressed in terms of ordinary matrix product by the formula

$$\psi \circ \pi = \psi \Lambda^{-1}(N) \pi ,$$

where $\Lambda(N)$ is the diagonal matrix with eigenvalues $n_1, \dots, n_\beta$.

We now define the composition of $\pi: M \to N$ and $\psi: N \to K$ in the general case. Let π be a polymorphism $M \to N$; associated to it is a family of conditional measures $\pi_m(n)$ on N (m running over M). We put

$$(\psi \circ \pi)_m(k) = \int_N \psi_n(k)\, d\pi_m(n) .$$

It is natural to regard the polymorphism $\pi: M \to N$ as a multivalued mapping which "smears" each point $m \in M$ into a probability measure $\pi_m(n)$ on N.

A sequence of polymorphisms $\pi_i: M \to N$ converges to $\pi: M \to N$ if, for arbitrary measurable sets $A \subset M$ and $B \subset N$, we have $\pi_i(A \times B) \to \pi(A \times B)$.

We define an involution in the category $\mathscr{P}$. One and the same measure π on $M \times N$ may be regarded both as a polymorphism $M \to N$ and a polymorphism $N \to M$. These two polymorphisms will be considered to be conjugate to each other.

Next, to each $p \in \mathrm{Aut}(M)$ we can canonically associate a polymorphism $\pi_p: M \to M$, the measure π_p being concentrated on the graph of p. Thus, $\mathrm{Aut}(M) \hookrightarrow \mathrm{End}_{\mathscr{P}}(M)$. It is not hard to check (see [2]) that the group $\mathrm{Aut}(M)$ is dense in the semigroup $\mathrm{End}_{\mathscr{P}}(M)$.

We now define the simplest *-representation (T, τ) of the category $\mathscr{P}$. To this end we put $T(M) = L^2(M)$ and, for each polymorphism $\varphi: M \to N$, we define an operator

$$r(\varphi) f(m) = \int_N f(n)\, d\varphi_m(n) .$$

This operator is especially transparent in the case where M and N have finite measures. In such a case $\tau(\varphi)$ has matrix $\Lambda^{-1}(M)\varphi$.

The representation (T, τ) is reducible, since $\tau(\varphi) \cdot 1 = 1$ for any φ. Take in each space $T(M)$ the orthogonal complement $T_0(M)$ to the subspace of constants. This will give us an irreducible representation (as we shall see later, irreducibility is obvious), of the category $\mathscr{P}$, which we shall denote by (T_0, τ_0).

2.2. **The classification of representations of the category $\mathscr{P}$.** Denote by $\mathscr{P}_0$ the category of the finite measure spaces of polymorphisms.

Lemma 2.1 (Approximation lemma). (a) *Any *-representation of $\mathscr{P}$ is uniquely determined by its restriction to $\mathscr{P}_0$.*

(b) *Any *-representation (T, τ) of $\mathscr{P}_0$ such that $\|\tau(\varphi)\| \le 1$ for all φ can be extended to $\mathscr{P}$.*

Remark. The condition $\|\tau(\varphi)\| \le 1$ is not burdensome. It holds for any *-representation of $\mathscr{P}$, since any polymorphism φ can be approximated by elements $g \in \mathrm{Aut}(M)$ and the operators $\tau(g)$ are unitary. On the other hand, it follows from the proof of Proposition 2.1 that any representation (T, τ) of $\mathscr{P}_0$ satisfies the condition $\|\tau(\varphi)\| \le 1$.

Proof. Let (M, μ) be a measure space and $\mathfrak{h}$ a finite partition of it, $M = \bigcup M_i$. Let $M(\mathfrak{h}) = M/\mathfrak{h}$ be the quotient space, i.e. a finite space consisting of points m_i whose measures are $\mu(M_i)$. We define a canonical polymorphism $\kappa(\mathfrak{h})\colon M \to M(\mathfrak{h})$ by setting $\kappa(m_j \times M_j) = \mu(M_j)$ and $\kappa(m_i \times M_j) = 0$ for $i \ne j$. We also define a canonical polymorphism $\theta(\mathfrak{h})\colon M \to M$ by setting $\theta(M_i \times M_j) = 0$ for $i \ne j$ and on $M_j \times M_j$ taking $\mathfrak{h}$ to be $(1/\mu(M_j))\mu \times \mu$. It is easy to see that

$$(2.1)\qquad \begin{aligned} \theta(\mathfrak{h})^* &= \theta(\mathfrak{h}), & \kappa(\mathfrak{h})^*\kappa(\mathfrak{h}) &= \theta(\mathfrak{h}),\\ \kappa(\mathfrak{h})\theta(\mathfrak{h}) &= \kappa(\mathfrak{h}), & \kappa(\mathfrak{h})\kappa(\mathfrak{h})^* &= 1. \end{aligned}$$

If $\mathfrak{h}_1 \le \mathfrak{h}_2$, then $\mathfrak{h}_1$ induces a partition of $M(\mathfrak{h}_2)$ and hence defines a polymorphism $\kappa(\mathfrak{h}_2/\mathfrak{h}_1)\colon M(\mathfrak{h}_1)$. It is easy to see that

$$\kappa(\mathfrak{h}_1) = \kappa(\mathfrak{h}_2/\mathfrak{h}_1)\kappa(\mathfrak{h}_2).$$

Consider in $\mathrm{End}(M)$ the subsemigroup $\Gamma(\mathfrak{h})$ of all morphisms of the form $\theta(\mathfrak{h})\pi\theta(\mathfrak{h})$. Let $\alpha \in \mathrm{End}(M(\mathfrak{h}))$. Then $s(\alpha) = \kappa(\mathfrak{h})^*\alpha\kappa(\mathfrak{h}) \in \Gamma(\mathfrak{h})$, and it is easy to see that the mapping $\alpha \mapsto s(\alpha)$ establishes an isomorphism of $\mathrm{End}(M(\mathfrak{h}))$ and $\Gamma(\mathfrak{h})$. We thus see that the semigroup $\mathrm{End}\, M(\mathfrak{h})$ can be embedded canonically into $\mathrm{End}(M)$. If, moreover, $\mathfrak{h}_1 \le \mathfrak{h}_2$, then $\mathrm{End}\, M(\mathfrak{h}_1)$ can be embedded canonically into $\mathrm{End}\, M((\mathfrak{h}_2)$.

We turn directly to the proof of the lemma. We start with (a). Let (T, τ) be a *-representation of $\mathscr{P}$. Let $\mathfrak{h} = \mathfrak{h}_1 \le \mathfrak{h}_2 \le \cdots$ be a basic sequence of partitions of M. The operator $\tau(\kappa(\mathfrak{h}_j/\mathfrak{h}_k))^*$, where $j \ge k$, maps $T(M(\mathfrak{h}_k))$ to $T(M(\mathfrak{h}_j))$, and $\tau(\kappa(\mathfrak{h}_k)^*)$ maps $T(M(\mathfrak{h}_k))$ to $T(M)$. Therefore, $T(M)$ contains as a subspace the inductive limit $\varinjlim T(M(\mathfrak{h}_k))$. The polymorphisms $\theta(\mathfrak{h}_k)$ approximate the identity of $\mathrm{End}(M)$, and hence $\varinjlim T(M(\mathfrak{h}_k))$ is dense in $T(M)$. Finally, $\theta(\mathfrak{h}_k)\pi\theta(\mathfrak{h}_k) \to \pi$ for any $\pi \in \mathrm{End}(M)$. Thus, $\tau(\pi)$ is completely determined by the operators $\tau(\theta(\mathfrak{h}_k)\pi\theta(\mathfrak{h}_k))$, which proves (a).

We prove (b). Let $\pi \in \mathrm{Mor}(M, N)$. Let $\mathfrak{m} = \mathfrak{m}_1 \le \mathfrak{m}_2 \le \cdots$ be a basic sequence of partitions of M and $\mathfrak{n} = \mathfrak{n}_1 \le \mathfrak{n}_2 \le \cdots$ a basic sequence of partitions of N. Let (T, τ) be a *-representation. Put $T(N) = \varinjlim T(N(\mathfrak{n}_j))$ and $T(M) = \varinjlim T(M(\mathfrak{m}_j))$. Let P_j be the projection onto $T(M(\mathfrak{m}_j))$ in $T(M)$. Define the operator $\tau(\pi)\colon T(M) \to T(N)$ as the weak limit of the operators $\tau(\kappa(\mathfrak{m})^*\pi\kappa(\mathfrak{n}))P_n$ (it is here that we use the condition $\|\tau(\varphi)\| \le 1$). We have still to verify the equality $\tau(\varphi \circ \psi) = \tau(\varphi)\tau(\psi)$. It holds on a dense subset (the inductive limit) and hence everywhere. The lemma is proved.

Let (T_0, τ_0) be the simplest representation of $\mathscr{P}$ (see §2.1) and SW_λ the Schur-Weyl functors (see §0.5).

Proposition 2.1. *The representations $\mathrm{SW}_\lambda \circ T_0$ are irreducible, pairwise distinct, and exhaust all the irreducible *-representation of the category $\mathscr{P}$.*

Proof. We shall of course prove the proposition for the category $\mathscr{P}_0$.

Let us first study the representations of the semigroup $\mathrm{End}_{\mathscr{P}_0}(M)$, where M is an $(n+1)$-point object of $\mathscr{P}_0$. This semigroup acts faithfully on $L^2(M)$ and hence is a subsemigroup of $\mathrm{Mat}_{n+1}(\mathbf{R})$. Let $\mathrm{End}^0(M) \hookrightarrow \mathrm{End}(M)$ be the subsemigroup consisting of invertible matrices. Then $\mathrm{End}^0(M)$ leaves invariant the vector $(1, \dots, 1)$ and also the orthogonal complement $v^\perp$ to v ($v^\perp$ is the subspace of functions in $L^2(M)$ with zero mean value). Therefore, $\mathrm{End}^0(M)$ embeds into the group $G_n = \mathrm{GL}(n, \mathbf{R})$ of all operators leaving v and $v^\perp$ invariant. The center of $\mathrm{End}^0(M)$ consists of matrices $A(h)$ with entries $a^h_{ij} = h\delta_{ij} + (1-h)m_i m_j$, where $0 < h \leq 1$ and m_j are the measures of the points of M; they satisfy $A(h)A(t) = A(ht)$. It is easy to see that, for any $g \in G_n$, there is an $\varepsilon > 0$ such that $gA_\varepsilon \in \mathrm{End}^0(M)$.

So let (Σ, ξ) be an irreducible *-representation of the category $\mathscr{P}_0$. The operators $\xi(A(h))$ are (real) scalars, and hence the representation $\xi(g)$ of the semigroup $\mathrm{End}^0(M)$ extends in a unique way to a representation of the group $\mathrm{GL}^+_n(\mathbf{R})$ (the group of real matrices with positive determinant). Moreover, there is an involution on $\mathrm{GL}^+_n(\mathbf{R}) \supset \mathrm{End}^0(M)$ which is an extension of the involution on $\mathrm{End}^0(M)$; with a suitable choice of a basis this involution turns out to be the ordinary transposition.

Lemma 2.2. *Let ξ be an irreducible representation of $\mathrm{GL}^+_n(\mathbf{R})$ on a Hilbert space H and let $\xi(g^t) = \xi(g)^*$. Then ξ is finite-dimensional.*

Proof. Choose a basis $X_1, \dots, X_p$, $Y_1, \dots, Y_q$ of the Lie algebra $\mathfrak{gl}_n(\mathbf{R})$ so that $X_j^t = -X_j$ and $Y_k^t = Y_k$. The operators $\exp(t\xi(Y_k)) = \xi(\exp(tY_k))$ are bounded and selfadjoint (we denote the representation of the Lie algebra by the same letter as the corresponding representation of the Lie group). Therefore, the operators $\xi(Y_k)$ are also bounded and selfadjoint.

The Lie algebra spanned by the X_j is the Lie algebra of the group $\mathrm{SO}(n)$; hence the set of common analytic vectors for $\xi(X_j)$ is dense in H. On the other hand, any vector in H is an analytic vector for $i\xi(Y_k)$ (since these operators are bounded). Thus, the set of common analytic vectors for the operators $\xi(X_j)$ and $i\xi(Y_k)$ is dense in H. Therefore (see, for instance, [1], 11.6), the representation $X \mapsto \xi(X_j)$, $iY_k \mapsto i\xi(Y_k)$ of the Lie algebra $\mathrm{u}(n)$ can be integrated to an (irreducible) representation of the group $\mathrm{U}(n)$, and all irreducible representations of $\mathrm{U}(n)$ are finite-dimensional. The lemma is proved.

Corollary. *Any *-representation of the semigroup $\mathrm{End}^0(M)$ has the form*

$$\rho_{\lambda, s}(g) = \sigma w_\lambda(g) \otimes \det(g)^s, \tag{2.2}$$

where λ is a representation of some group S_p.

A representation ρ can be written in the form (2.2) not uniquely; the point is that the representation $\sigma w_\lambda(g) \otimes \det(g)$, where $\lambda \in \widehat{S}_p$, can itself be written as $\sigma w_{\lambda'}(g)$, where $\lambda' \in \widehat{S}_{p+n}$. We call λ *minimal* if $\sigma w_\lambda(g)$ cannot be represented in the form $\sigma w_{\lambda''}(g) \otimes \det(g)$.

So let ξ have the form (2.2) with λ minimal. Let us discuss what a continuous extension of ξ to $\mathrm{End}(M)$ looks like. First of all, let us note that, for $s < 0$, such an extension does not exist, so this case need not be considered. For $s \geq 0$, the extension has inevitably the form (2.2). Therefore, for $s > 0$, the extension vanishes on $\mathrm{End}(M) \setminus \mathrm{End}^0(M)$.

So we have examined the representations of the individual semigroups $\mathrm{End}_{\mathscr{P}_0}(M)$. Let us turn now to the representations of the category $\mathscr{P}_0$.

For each fixed M, the representation ξ has the form (2.2) with (λ, s) depending a priori on M, and in a multivalued way. Let $\mathfrak{h}$ be a partition of M into two-point and one-point sets. We saw in the proof of the approximation lemma that $\mathrm{End}(M(\mathfrak{h}))$ embeds canonically into $\mathrm{End}(M)$, the image being contained in $\mathrm{End}(M)\setminus\mathrm{End}^0(M)$. Thus, for $s>0$ we have $\Sigma(M(\mathfrak{h}))=0$, and for $s=0$ we have $\xi(g)=\sigma w_\lambda(g)$ on $\mathrm{End}(M(\mathfrak{h}))$. Now the existence of a universal λ such that $\xi(g)=\sigma w_\lambda(g)$ for all $g\in\mathrm{End}(M)$ and for all M becomes obvious.

Now let $\pi\in\mathrm{Mor}(K, L)$. There is an M such that K has the form $M(\mathfrak{h})$ and L has the form $M(\mathfrak{n})$. It is easy to see that π can be written as $\pi=\kappa^*(\mathfrak{n})\sigma\kappa(\mathfrak{h})$, where $\sigma\in\mathrm{End}(M)$. Thus, the groupoid of morphisms of the category $\mathscr{P}$ is generated by the semigroups $\mathrm{End}(M)$ and morphisms of the form $\kappa(\mathfrak{m})$ and $\kappa(\mathfrak{m})^*$. We already know what the representation ξ looks like on $\mathrm{End}(M)$. We must show that the operators $\kappa(\mathfrak{m})$ are defined unambiguously. It suffices to do that in the case where $\mathfrak{m}$ is a partition of M into two-point and one-point sets. The operator $\xi(\kappa(\mathfrak{m}))$ is $\mathrm{End}(M(\mathfrak{m}))$-intertwining and the restriction of an irreducible representation of $\mathrm{SL}_{n+1}(\mathbf{R})$ to $\mathrm{SL}_n(\mathbf{R})$ is without multiplicities, so $\xi(\kappa(\mathfrak{m}))$ is defined unambiguously.

The assertions about irreducibility and pairwise nonequivalence of our representations follow from irreducibility and pairwise nonequivalence of the representations σw_λ for each group $\mathrm{GL}_{n+1}(\mathbf{R})$.

2.3. ***G*-polymorphisms.** Let G be a Lie group. Objects of the category $G\mathscr{P}$ of G-polymorphisms are probability spaces. A morphism $(M, \mu)\to(N, \nu)$ is a measure π on $M\times N\times G$ such that the projection of π on M is μ and the projection of π on N is ν. We require also that the following technical condition should hold: The projection of π on G is contained in a compact set. The polymorphism π may be regarded as a vector-valued measure $\tilde{\pi}$ on $M\times N$ taking values in positive measures on the group G.

Objects of the subcategory $G\mathscr{P}_0$ of $G\mathscr{P}$ are finite probability spaces. Then the morphisms may be regarded as matrices whose entries are measures on G. If $\tilde{\pi}\colon M\to N$ and $\tilde{\theta}\colon N\to K$ are polymorphisms, computing their composition $\tilde{\theta}\circ\tilde{\pi}$ reduces to multiplying matrices according to the formula $\tilde{\theta}\circ\tilde{\pi}=\tilde{\theta}\Lambda^{-1}(N)\tilde{\pi}$, where $\Lambda(N)$ is the same matrix as in §2.1 (addition of measures is the ordinary addition and multiplication is the ordinary convolution).

Now let K, M, and N be arbitrary objects of $G\mathscr{P}$, and $\pi\colon M\to N$ and $\theta\colon N\to K$ morphisms. We define their composition $\psi=\theta\circ\pi$ in terms of conditional measures:

$$\int_{G\times N} f(g, n)\,d\psi_k(g, n)=\int_{G\times M}\int_{G\times N} f(gh, n)\,d\theta_m(h, n)\,d\pi_k(g, m).$$

We define convergence of G-polymorphisms. Let π_n, $\pi\in\mathrm{Mor}(M, N)$. The sequence π_n converges to π if 1) the union of supports of the projections of π_n on G is compact, and 2) for any measurable sets $A\subset N$ and $B\subset M$ the sequence of measures $\pi_n(B\times A)$ converges weakly to $\pi(B\times A)$ in the sense of the weak convergence of measures on G.

Let $\pi\colon M\to N$ and $\theta\colon N\to K$ be G-polymorphisms. Then, by definition, $\theta^*=\pi$ if the mapping $(m, n, g)\mapsto(n, m, g^{-1})$ (from $M\times N\times G$ to $N\times M\times G$) maps π into θ.

Let M be a space with a continuous measure. It is easy to see that $\mathrm{Aut}_{G\mathscr{P}}(M)\cong\mathfrak{B}(G)$.

Proposition 2.2. *The group* $\mathfrak{B}(G)$ *is dense in the semigroup* $\mathrm{End}_{G\mathscr{P}}(M)$.

The proof is fairly easy, and we omit it. For more subtle results of this kind, see §4; see also Proposition 4.1.

2.4. **Representations of the category $G\mathscr{P}$.** Let ρ be a unitary irreducible representation of G on H. We define a representation (T_ρ, τ_ρ) of the category $G\mathscr{P}$. Put $T_\rho(M) = L^2(M, H)$ and

$$\tau_\rho(\varphi)f(n) = \int_{G\times M} \rho(g)f(m)\,d\varphi_n(g, m).$$

It is easy to see that these representations are irreducible except for the trivial case $\rho(g) \equiv 1$. In that case, the operator $\tau_\rho(\varphi)$ depends only on the projection of φ on $M\times N$. We then denote by (T_0, τ_0) the subrepresentation such that $T_0(M) \subset L^2(M)$ consists of function with zero mean value.

The representations T_ρ and T_0 will be said to be the simplest.

Now let G act on the Hilbert space H by affine transformations according to the formula (1.2). Then $\mathfrak{B}(G)$ acts by affine transformations on $L^2(M, H)$ and this action extends in its turn to an embedding $\pi \mapsto [A(\pi), b(\pi), c(\pi)]$ of the category $G\mathscr{P}$ into the category Aff (see §1.1). Let $\pi \in \mathrm{Mor}_{G\mathscr{P}}(M, N)$. Then

$$A(\pi)f(n) = \int_{G\times M} \rho(g)f(m)\,d\varphi_n(g, m),$$
$$b(\pi) = \int_{G\times M} \gamma(g)\,d\varphi_n(g, m),$$
$$c(\pi) = \int_{G\times N} \gamma(g)\,d\varphi^*_m(g, n).$$

Restricting the Fock representation of Aff to $G\mathscr{P}$, we obtain a projective (in general) representation of $G\mathscr{P}$.

Let G act on a real Hilbert space K according to (1.2). Then G acts also on the complexification $H = K_{\mathbf{C}}$ of K. Application of our construction gives a projective representation of $G\mathscr{P}$ which, however, can be linearized by means of the following substitution:

$$\varphi_{\rho,\gamma}(\pi) = \exp\left[\int_{M\times N\times G} \|\gamma(g)\|^2\,d\pi(m, n, g)\right] \varphi([A(\pi), b(\pi), c(\pi)]).$$

Such representations of $G\mathscr{P}$ will be called *Fock representations* and will be denoted by $(F_{\rho,\gamma}, \varphi_{\rho,\gamma})$.

2.5. **An attempt to classify the representations.**

Theorem 2.1. *Let the group G be compact. Then each irreducible representation of the category $G\mathscr{P}$ has the form*

$$\bigotimes_{i=1}^{n}[\mathrm{SW}_{\lambda_i} \circ T^{(i)}], \tag{2.3}$$

where $T^{(i)}$ are pairwise distinct simplest representations of $G\mathscr{P}$ (i.e. $T^{(i)} = T_\rho$ or T_0), and λ_i are irreducible representations of the symmetric groups S_{k_i}. All representations of the form (2.3) are irreducible, and no two of them are equivalent.

Proof. Let (M, μ) be a measure space and $\mathfrak{h}$ its finite partition $M = \bigcup M_i$. As before, let $M(\mathfrak{h})$ be the space consisting of the points m_i with measures $\mu(M_i)$. We define a G-polymorphism $\kappa^G(\mathfrak{h})\colon M \to M(\mathfrak{h})$ which is an analog of the polymorphism $\kappa(\mathfrak{h})$ of §2.2. Let the measure $\kappa^G(\mathfrak{h})$ be the product of the measure $\kappa(\mathfrak{h})$ on $M \times M(\mathfrak{h})$ and a δ-like measure concentrated at the identity of G. Polymorphisms $\theta^G(\mathfrak{h}) \in \mathrm{Mor}_{G\mathscr{P}}(M)$ are defined in a similar fashion.

The proof of the approximation lemma can be repeated verbatim for all the categories $G\mathscr{P}$ (not only for the case of compact G), so we can confine ourselves to the category $G\mathscr{P}_0$.

First of all we must examine representations of the semigroups $\mathrm{End}_{G\mathscr{P}_0}(M)$. A function on G will be said to be simple if it can be represented as a finite sum of matrix elements of irreducible representations of G. Consider the dense subsemigroup $\mathrm{End}^*(M) \hookrightarrow \mathrm{End}_{G\mathscr{P}_0}(M)$ consisting of matrices with entries of the form $\pi_{ij} = \varphi_{ij}(g)\,dg$, where φ_{ij} is a simple function and dg is Haar measure.

We are interested in the structure of $\mathrm{End}^*(M)$. Let $\rho_1, \rho_2, \dots$ be a collection of irreducible representations of G such that, for any irreducible representation μ, exactly one of the representations μ and $\overline{\mu}$ is on the list $\rho_1, \rho_2, \dots$ (by $\overline{\mu}$ we denote the contragredient (complex conjugate) representation). Let $T^{(j)} = T_\rho$ or T_0 be the corresponding simplest representations of $G\mathscr{P}$ and let V_j be the representation space $T^{(j)}(M)$. Let $\mathrm{Mat}(V_j)$ be the space of all operators on V_j.

We define the Fourier transform $\mathscr{F}$ on $\mathrm{End}^*(M)$ by

$$\mathscr{F}(\pi) = (T^{(1)}(\pi), T^{(2)}(\pi), \dots) \in \bigoplus_{j=0}^{\infty} \mathrm{Mat}(V_j).$$

Let Γ be the image of $\mathrm{End}^*(M)$ under the Fourier transform. It is clear that all the elements of Γ are sequences that have only a finite number of nonzero elements. Let the subsemigroup $\Gamma_n \subset \Gamma$ consist of the sequences having zeros in the positions starting from $n+1$. Let the subsemigroup $\Gamma_n^0 \subset \Gamma_n$ consist of sequences $(A_1, \dots, A_n, 0, \dots) \in \Gamma_n$ with A_j invertible.

The semigroup Γ_n has n-dimensional center Z_n. It consists of the images under the Fourier transform of measures on $M \times M \times G$ of the form $\nu \times \chi$, where ν is a central element of the semigroup of ordinary polymorphisms and χ is a measure on G whose density with respect to Haar measure is a linear combination of characters of the representations $\rho_j \oplus \overline{\rho}_j$. Let $a = (A_1, \dots, A_n, 0, \dots)$ with $A_j \in \mathrm{GL}(V_j)$, which in the case where $\rho_j = \overline{\rho}_j$ satisfy $A_j = \overline{A}_j$ and $\det A_j > 0$. It is not hard to verify that there is an $\alpha = (q_1E, q_2E, \dots) \in Z_n$ such that $\alpha a \in \Gamma_0$ (one should take for χ a linear combination of characters close to 1 and for ν a measure close to $\mu \times \mu$).

Now, as in the proof of Proposition 2.1, the problem of describing *-representations of Γ_n^0 reduces to the problem of describing *-representations of some group of the form $\bigoplus_{j=1}^n \mathrm{GL}(\dim V_j, \mathbf{K}_j)$, where $\mathbf{K}_j = \mathbf{R}$ if $\overline{\rho}_j = \rho_j$ and $\mathbf{K}_j = \mathbf{C}$ otherwise. Reasoning as in the proof of Proposition 2.1, we conclude that each representation of Γ_n has the form

$$\text{(2.4)}\qquad \begin{aligned} \xi(A_1, \dots, A_n, 0) = {}& \left[\bigotimes_{j\,:\,\rho_j = \overline{\rho}_j} (\sigma w_{\lambda j}(A_j) \otimes \det(A_j)^{s_j})\right] \\ &\otimes \left[\bigotimes_{j\,:\,\rho_j \neq \overline{\rho}_j} (\sigma w_{\mu_j}(A_j) \otimes \sigma w_{\mu'_j}(\overline{A}_j) \otimes \det(A_j)^{t_j} \otimes \det(\overline{A}_j)^{t'_j})\right], \end{aligned}$$

where s_j, t_j, $t'_j \in \mathbf{R}$ and $t_j - t'_j \in \mathbf{Z}$.

Now note that $\Gamma = \bigcup \Gamma_n$. The restriction of the representation (2.4) to Γ_{n-1} is different from zero only when the factor corresponding to A_n in (2.4) is the trivial one-dimensional representation of $\mathrm{GL}(\dim V_n, \mathbf{K}_n)$. Therefore each representation ξ of Γ depends only on a finite part of the sequence $(A_1, A_2, \dots)$, and so we know all the representations of Γ.

The proof is completed by the same argument as in the proof of Proposition 2.1.

Conjecture. Let G be a type I group. Then all irreducible *-representations of the category $G\mathscr{P}$ have the form

$$\bigotimes_j [\mathrm{SW}_{\lambda_j} \circ T^{(j)}] \otimes \left[\bigotimes_k F_{\rho_k, \gamma_k}\right],$$

where $T^{(j)} = T_{\rho_j}$ or T_0 are the simplest representations and F_{ρ_k, γ_k} the Fock representations of $G\mathscr{P}$ (the tensor product in the first factor may even be countable if the ρ_j converge sufficiently fast to the trivial one-dimensional representation).

2.6. **The category $G_\chi\mathscr{P}$.** Let G be a Lie group and χ a homomorphism from G to $\mathbf{R}^*$, where $\mathbf{R}^*$ denotes the multiplicative group of positive real numbers. Objects of the category $G_\chi\mathscr{P}$ are probability spaces. A morphism $(M, \mu) \to (N, \nu)$ is a measure $\pi(m, n, g)$ on $M \times N \times G$ such that the projection of π on M is μ and the projection of $\chi(g)\tau(m, n, g)$ on N is ν. Composition is exactly the same as in the category of G-polymorphisms. An involution is given by the formula $\pi^*(m, n, g) = \chi(g)\pi(n, m, g^{-1})$.

Each unitary representation ρ of G on H gives rise to a representation (T_ρ, τ_ρ) of the category $G_\chi\mathscr{P}$: the representation space $T_\rho(M) = L^2(M, H)$, and

$$\tau_\rho(\pi)f(n) = \int_{G\times M} \chi(g)^{1/2}\rho(g)f(m)\, d\pi_n(g, m).$$

Presumably all *-representations of the category $G_\chi\mathscr{P}$ occur in the decomposition of tensor products of the representations (T_ρ, τ_ρ).

2.7. **The one-point object.** Here we merely want to remark that the semigroup $\mathrm{End}_{G\mathscr{P}}(M)$ is quite an interesting object even if the space M consists of one point; namely, $\mathrm{End}_{G\mathscr{P}}(M)$ is then the semigroup $\mathscr{M}(G)$ of probability measures on G. There is a vast literature devoted to this semigroup, especially in the case $G = \mathbf{R}$ (this is simply an essential part of probability theory), but several books have also been published about the noncommutative case (see, for instance, [26]). Our considerations show that the semigroup $\mathscr{M}(G)$ has an interesting theory of representations which does not reduce (at least in a broad sense of the word) to the theory of representations of the group G.

3. Trains

3.1. **Abstract construction of a train for $(\mathfrak{G}, \mathfrak{K})$-pairs.** Let $\mathfrak{G}$ be an infinite-dimensional group and $\mathfrak{K}$ a subgroup of $\mathfrak{G}$. Let $\mathfrak{K}(\alpha)$ be a family of subgroups of $\mathfrak{K}$. Let τ be a unitary representation of $\mathfrak{G}$ on H. Let $H(\alpha)$ be the set of vectors in H which are fixed by $\mathfrak{K}(\alpha)$, and let $P(a)$ be the projection on $H(\alpha)$. Let a be a double coset in $\mathfrak{K}(\beta) \setminus \mathfrak{G}/\mathfrak{K}(\alpha)$. Then an operator $\tau(a)\colon H(\alpha) \to H(\beta)$ is correctly defined by the formula

(3.1) $$\tau(a)v = P(\beta)\tau(g)v,$$

where $v \in H(\alpha)$ and g is a representative of the class a. We shall say that the family of subgroups $\mathfrak{K}(\alpha)$ satisfies the multiplicativity theorem if, for any unitary representation τ of $\mathfrak{G}$, the following conditions hold.

(a) $\bigcup_\alpha H(\alpha)$ is dense in H for any τ.

(b) For any α, β, γ and any $a \in \mathfrak{K}(\alpha) \setminus \mathfrak{G}/\mathfrak{K}(\beta)$, $b \in \mathfrak{K}(\beta) \setminus \mathfrak{G}/\mathfrak{K}(\gamma)$, there is a $c \in \mathfrak{K}(\alpha) \setminus \mathfrak{G}/\mathfrak{K}(\gamma)$ such that, for any τ,

(3.2) $$\tau(c) = \tau(a)\tau(b).$$

This gives rise to a category $\mathscr{K}$ (the train of the group $\mathfrak{G}$) whose objects are labeled by α, and

$$\mathrm{Mor}(\alpha, \beta) = \mathfrak{K}(\beta) \setminus \mathfrak{G}/\mathfrak{K}(\alpha)$$

with composition of morphisms defined by (3.2). Each unitary representation τ of $\mathfrak{G}$ gives rise to a canonically defined representation (T, τ) of the category $\mathscr{K}$; namely, $T(\alpha) = H(\alpha)$ and $\tau(a)$ is defined by (3.1).

Below, we give five examples of applying this scheme.

3.2. **The train of** $\mathrm{Aut}(M)$. Let M be a space with a continuous probability measure. Let $\mathfrak{G} = \mathrm{Aut}(M) = \mathfrak{K}$. Let $\mathfrak{h}$ be a finite partition $M = \bigcup M_i$ and $K(\mathfrak{h})$ the group of all automorphisms of M which map each M_i into itself. The double cosets in $K(\mathfrak{h}_1) \setminus \mathfrak{G}/K(\mathfrak{h}_2)$ are parametrized by intersection matrices (see §0.2), and intersection matrices may be regarded as morphisms of the category $\mathscr{P}_0$ acting from $M(\mathfrak{h}_2)$ to $M(\mathfrak{h}_1)$ (recall that the space $M(\mathfrak{h})$ consists of points m_i with measures $\mu(M_i)$).

Theorem 3.1. *The family of subgroups* $K(\mathfrak{h})$ *satisfies the multiplicativity theorem with double cosets being multiplied as morphisms of the category* $\mathscr{P}_0$.

We see that unitary representations of the group $\mathrm{Aut}(M)$ and *-representations of the category $\mathscr{P}_0$ are in a sense the same objects.

Corollary. *All irreducible unitary representations of* $\mathrm{Aut}(M)$ *are exhausted by representations of the form* $\sigma w_\lambda \circ T_0$, *where* T_0 *is the simplest representation.*

3.3. **The proof of Theorem 3.1.**

Lemma 3.1 (see [6] and [19]). *Let* G *be a discrete group,* ρ *its unitary representation on a Hilbert space* H, *and* P *the projection on the subspace of vectors fixed by the operators* $\rho(g)$ $(g \in G)$. *Then, for any* $\xi \in H$ *and any* $\varepsilon > 0$, *there are* $g_1, \dots, g_n \in G$ *and* $\alpha_1, \dots, \alpha_n$ *such that* $\alpha_j > 0$, $\sum \alpha_j = 1$, *and*

$$\left\| \sum \alpha_i \rho(g_i)\xi - P\xi \right\| < \varepsilon. \tag{3.3}$$

Proof. Any closed convex subset V of a Hilbert space contains a unique vector v with minimal norm. Let V be the convex hull of the set of $\rho(g)\xi$, where $g \in G$. It is clear that $v = P\xi$. The lemma is proved.

Let $\mathfrak{h} = \mathfrak{h}_0 \leq \mathfrak{h}_1 \leq \cdots$ be a basic sequence of partitions of M. For any neighborhood U of the identity in $\mathrm{Aut}(M)$, we can find a k such that $K(\mathfrak{h}_k) \subset U$. Let $g_i^{(j)} \in K(\mathfrak{h}_j)$. Then the convex combinations $\sum \alpha_i^{(j)} g_i^{(j)} \xi$ converge to ξ as $i \to \infty$ (by viture of the continuity of ρ). Therefore, $\bigcup H(\mathfrak{h}_j)$ is dense in H.

Consider the partition $\mathfrak{h}_k/\mathfrak{h}$ of $M(\mathfrak{h}_k)$ induced by $\mathfrak{h}$. Consider the polynomial $\theta(\mathfrak{h}_k/\mathfrak{h})$ (see §2.2) of $M(\mathfrak{h}_k)$. It defines a double coset in $K(\mathfrak{h}_k) \setminus \mathrm{Aut}(M)/M(\mathfrak{h}_k)$.

Lemma 3.2. *Let* $g_k \in \theta(\mathfrak{h}_k/\mathfrak{h})$. *Then* $\tau(g_k)$ *converges weakly to the projection* $P(\mathfrak{h})$ *(onto* $H(\mathfrak{h})$*).*

Proof. It suffices to check that $P(\mathfrak{h}_j)\pi(g_j)P(\mathfrak{h}_j) = P(\mathfrak{h})$. To do that, in turn, it suffices to check that, for any $\eta \in H$ and $\varepsilon > 0$, we have

$$\|P(\mathfrak{h}_j)\tau(g_j)P(\mathfrak{h}_j)\eta - P(\mathfrak{h})\eta\| \leq \varepsilon. \tag{3.4}$$

We apply Lemma 3.1 to the subgroup $K(\mathfrak{h})$ of $\mathrm{Aut}(M)$ and the vector $P(\mathfrak{h}_j)\eta$. The inequality (3.3) remains valid if g_i is replaced by $g_i g$. But g can be chosen so that all $g_i g$ lie in the coset $\theta(\mathfrak{h}_j/\mathfrak{h})$. The inequality

$$\left\|\sum \alpha_i \tau(g_i g) P(\mathfrak{h}_j)\eta - P(\mathfrak{h})\eta\right\| \leq \varepsilon$$

implies

$$\left\|P(\mathfrak{h}_j)\left(\sum \alpha_i \tau(g_i g) P(\mathfrak{h}_j)\eta - P(\mathfrak{h})\eta\right)\right\| \leq \varepsilon .$$

But $P(\mathfrak{h}_j)\tau(hg)P(\mathfrak{h}_j)$ depends only on the corresponding double coset in $K(\mathfrak{h}_j)\setminus \mathrm{Aut}(M)/K(\mathfrak{h}_j)$. The lemma is proved.

Lemma 3.3. *Let $\varphi \in K(\mathfrak{n})\setminus\mathrm{Aut}(M)/K(\mathfrak{h})$ and $\psi \in K(\mathfrak{h})\setminus\mathrm{Aut}(M)/K(\mathfrak{m})$. Let $g_1 \in \varphi$ and $g_2 \in \psi$. Let $\mathfrak{h} = \mathfrak{h}_0 \leq \mathfrak{h}_1 \leq \mathfrak{h}_2 \leq \cdots$ be a basic sequence of partitions of M. Then, for each k, there is an $h_k \in \theta(\mathfrak{h}_k/\mathfrak{h})$ such that $g_2 h_k g_1 \in \psi \circ \varphi$.*

Proof. This is obvious.

We turn directly to the proof of the theorem. Let φ, g, and h be as in Lemma 3.3. Then

$$\tau(\psi \circ \varphi) = \tau(g_2 h_k g_1) = P(\mathfrak{m})\tau(g_2)\tau(h_k)\tau(g_1)|_{H(\mathfrak{n})},$$

and as $K \to \infty$ the latter expression converges weakly to

$$P(\mathfrak{m})\tau(g_2)P(\mathfrak{h})\tau(g_1)|_{H(\mathfrak{n})} = \tau(\psi)\tau(\varphi)$$

as required.

3.4. **The train of $\mathfrak{B}(G)$.** We shall now apply our scheme to the group $\mathfrak{B}(G) = \mathrm{Aut}(M) \times \mathscr{F}(M, G)$. Let $K(\mathfrak{h}) \subset \mathrm{Aut}(M)$ be the same family of subgroups as above. The double cosets of $K(\mathfrak{h}_1)\setminus\mathfrak{B}(G)/K(\mathfrak{h}_2)$ are in one-to-one correspondence with G-polymorphisms $M(\mathfrak{h}_1) \to M(\mathfrak{h}_2)$.

Theorem 3.2. (a) *The family of subgroups $K(\mathfrak{h}) \subset \mathfrak{B}(G)$ satisfies the multiplicativity theorem with multiplication of double cosets corresponding to multiplication of G-polymorphisms.*

(b) *The quotient topology on $K(\mathfrak{h}_1)\setminus\mathfrak{B}(G)/K(\mathfrak{h}_2)$ coincides with the natural topology on $\mathrm{Mor}_{G\mathscr{P}}(M(\mathfrak{h}_1), M(\mathfrak{h}_2))$.*

Corollary. *Let G be a compact group. Then all irreducible unitary representations of the group $\mathfrak{B}(G)$ have the form $\bigotimes[\sigma w_{\lambda_j} \circ T^{(j)}]$, where $T^{(j)}$ are elementary the simplest representations of $\mathfrak{B}(G)$.*

Proof. (a) contains nothing new as compared with Theorem 3.1. To prove (b), it suffices to verify the following statement.

Lemma 3.4. *Let K be parallelepiped in $\mathbf{R}^n$ and $\kappa_n \to \kappa$ a weakly convergent sequence of probability measures on K. Let (M, μ) be a space with a continuous probability measure. Then there exists a sequence of measurable functions $f_n: M \to K$, converging almost everywhere, such that the image of the measure μ under f_n is κ_n and its image under f is κ.*

Proof. We shall carry out the proof in the case where K is a rectangle, a generalization of higher dimensions being easy. Without loss of generality we may assume that (M, μ) is a countable product $\{0, 1\}^\infty$ of the two-point spaces $\{0, 1\}$ (the measure of the points 0 and 1 being $1/2$). We fix a measure ν on K and construct, with the help of ν, a descending sequence $\mathfrak{h}_n$ of partitions of K into 2^n rectangles; these rectangles, $K_{s_1,\ldots,s_n}$, are labeled by sequences $(s_1, \ldots, s_n)$ of zeros and ones. We also construct a sequence of measures $\nu_{s_1,\ldots,s_n}$ defined on $K_{s_1,\ldots,s_n}$. We take a horizontal partition $y = y_0$ in K so that

$$\nu\{(x, y) \in K : y > y_0\} = \nu\{(x, y) \in K : y < y_0\}.$$

Put $K_0 = \{(x, y) \in K, y \leq y_0\}$ and $K_1 = \{(x, y) \in K, y \geq y_0\}$. We introduce on the rectangle K_0 a measure ν_0 such that

$$\nu_0(A) = \nu(A \setminus \{y = y_0\}) + \tfrac{1}{2}\nu(A \cap \{y = y_0\}).$$

We introduce a similar measure ν_1 on K_1. Further, we divide each of the rectangles K_0 and K_1 by vertical partitions, and so on. We define a function $f_\nu \colon \{0, 1\}^\infty \to K$ by

$$f_\nu(s_1, s_2, \dots) = \bigcap_{j=1}^{\infty} K_{s_1, \dots, s_j}.$$

It is easy to see that the measure ν is the image of the Bernoulli measure μ under f_ν. Next, the weak convergence of the measures ν implies convergence of the partitions between the rectangles $K_{s_1, \dots, s_{n-1}, 0}$ and $K_{s_1, \dots, s_n, 1}$. This, in turn, implies pointwise convergence of the functions f_ν. The lemma is proved.

3.5. **The train for** $\mathrm{Gut}(M)$. Let $\mathrm{Gut}(M)$ be the group of transformations of a space M with a continuous probability measure μ that leave μ quasi-invariant and have bounded Radon-Nikodým derivative.

Let $\chi \colon \mathbf{R}^* \to \mathbf{R}^*$ be the identity mapping. We are going to show that the theory of representations of the group $\mathrm{Gut}(M)$ is more or less the same thing as the theory of representations of the category $\mathbf{R}^*_\chi\mathscr{P}$. Let $K(\mathfrak{h})$ be the same family of subgroups of $\mathrm{Aut}(M)$ as above. We describe the double cosets of $K(\mathfrak{h}_1) \setminus \mathrm{Gut}(M)/K(\mathfrak{h}_2)$. Let $g \in \mathrm{Gut}(M)$; let M_i be the elements of the partition $\mathfrak{h}_1$ and M'_j the elements of the partition $\mathfrak{h}_2$. Define a measure φ_{ij} on $\mathbf{R}^*$ by

$$\varphi_{ij}(A) = \mu\{m \in M_i \cap g^{-1}(M'_j) : g'(m) \in A\},$$

where $g'(m)$ denotes the Radon-Nikodým derivative at m. It is clear that φ_{ij} depends only on the double coset in $K(\mathfrak{h}_1) \setminus \mathrm{Gut}(M)/K(\mathfrak{h}_2)$ containing g. Hence

$$K(\mathfrak{h}_1)\setminus \mathrm{Gut}(M)/K(\mathfrak{h}_2) \cong \mathrm{Mor}_{\mathbf{R}^*_\chi\mathscr{P}}(M(\mathfrak{h}_1), M(\mathfrak{h}_2)).$$

Theorem 3.3. *The family of subgroups* $K(\mathfrak{h}) \subset \mathrm{Gut}(M)$ *satisfies the multiplicativity theorem with multiplication of double cosets corresponding to multiplication of morphisms of the category* $\mathbf{R}^*_\chi\mathscr{P}$.

(The proof of this theorem and the next two contains nothing new as compared with Theorems 3.1 and 3.2, and so we omit it.)

3.6. **The train for** $\mathrm{Aut}_\infty(M)$. Let (M, μ) be a space with an infinite continuous measure and $\mathrm{Aut}_\infty(M)$ its automorphism group. A sequence $g_i \in \mathrm{Aut}_\infty(M)$ converges to g if for arbitrary measurable sets A, $B \subset M$ of finite measure we have $\mu(g_i A \cap B) \to \mu(g(A) \cap B)$.

By a finite partition $\mathfrak{h}$ of M we mean a partition of the form $M = (\bigcup_{i=1}^m M_i) \cup M_*$, where $\mu(M_i) < \infty$ and $\mu(M_*) = \infty$. By $K^0(\mathfrak{h})$ we denote the subgroup of $\mathrm{Aut}_\infty(M)$ consisting of the automorphisms that preserve the partition $\mathfrak{h}$. Let $\mathfrak{h}'$ be another partition $M = (\bigcup M'_j) \cup M'_*$. Then the double cosets in $K^0(\mathfrak{h}) \setminus \mathrm{Aut}_\infty(M)/K^0(\mathfrak{h}')$ are labeled by numerical matrices $\varphi_{ij} = \mu(M_i \cap g^{-1}M'_j)$. It is easy to see that

$$\sum_i \varphi_{ij} \le \mu(M_i), \qquad \sum_j \varphi_{ij} \le \mu(M'_j).$$

Theorem 3.4. *The family of subgroups $K^0(\mathfrak{h})$ satisfies the multiplicativity theorem. Multiplication of double cosets*

$$\varphi \in K^0(\mathfrak{h}) \setminus \mathrm{Aut}_\infty(M)/K^0(\mathfrak{h}') \quad \textit{and} \quad \psi \in K^0(\mathfrak{h}') \setminus \mathrm{Aut}_\infty(M)/K^0(\mathfrak{h}'')$$

*corresponds to *-multiplication of matrices according to the formula*

$$\psi \circ \varphi = \psi \Lambda^{-1}(\mathfrak{h}) \varphi,$$

where $\Lambda(\mathfrak{h})$ is the diagonal matrix with eigenvalues $\mu(M_i)$.

Representations of the resulting category are easy to classify (the proof repeats the proof of Proposition 2.1).

Corollary. *All unitary representations of* $\mathrm{Aut}_\infty(M)$ *are exhausted by the representations of the form $\sigma w_\lambda \circ T_0$, where T_0 is the natural representation of* $\mathrm{Aut}_\infty(M)$ *on* $L^2(M)$.

3.7. **The train for** $\mathrm{Gut}_\infty(M)$. Let M be a space with a continuous infinite measure μ. Let $\mathrm{Gut}_\infty(M)$ be the group of transformation of M that leave μ quasi-invariant, Radon-Nikodým derivative of these transformations being bounded and vanishing outside a set of finite measure.

Remark. Apparently the same theory of representations is obtained with the condition $(g'(m)^{1/2} - 1) \in L^1(M)$ on the Radon-Nikodým derivative.

We shall construct some representations of $\mathrm{Gut}_\infty(M)$.

The simplest representations τ_s are realized on $L^2(M)$ according to the formula

$$\tau_s(g) = f(gm)g'(m)^{1/2+is}.$$

Let $q_1, \dots, q_n$ be a collection of real numbers and $\lambda > 0$. Let $M^{(n)} = M \times \cdots \times M$ (n times). Let $\mathscr{L}$ be the subspace of $L^2(M^{(n)})$ consisting of functions $f(m_1, \dots, m_n)$ that are invariant under any transposition $m_\alpha \leftrightarrow m_\beta$ such that $q_\alpha = q_\beta$. Let $\mathrm{Gut}_\infty(M)$ act on $\mathscr{L}$ by affine transformations according to the formula

$$A(g)f(m_1, \dots, m_n) = [f(gm_1, \dots, gm_n) + \lambda] \prod_j g'(m_j)^{1/2+iq_j} - \lambda.$$

Composing the Fock representation of $\mathrm{Isom}(\mathscr{L})$ with the embedding

$$A\colon \mathrm{Gut}_\infty(M) \to \mathrm{Isom}(\mathscr{L}),$$

we get a series of representations $\varphi^\lambda_{q_1, \dots, q_n}$ of $\mathrm{Gut}_\infty(M)$. By construction, these representations are bound to be projective; in fact they turn out to be linear.

Remark. The representations $\varphi^\lambda_{q_1, \dots, q_n}$ admit another very beautiful realization on L^2 with the Poisson measure (see [28], and also the survey [4]).

Conjecture. Each irreducible unitary representation of $\mathrm{Gut}_\infty(M)$ has the form

$$\bigotimes_{j=1}^N [\sigma w_{\mu_j} \circ \tau_{s_j}] \otimes \left[\bigotimes_k \varphi^{\lambda_k}_{q_{k1}, \dots, q_{kn_k}}\right],$$

where the unordered collections $Q_k = (q_{k1}, \dots, q_{kn_k})$ are pairwise distinct.

Now let $K^0(\mathfrak{h}) \subset \mathrm{Aut}_\infty(M)$ be the same family of subgroups as above. Let $\mathfrak{h}\colon M = (\bigcup_{i=1}^n M_i) \cup M_*$ and $\mathfrak{h}'\colon M = (\bigcup_{j=1}^m M'_j) \cup M'_*$ be two finite partitions of M. Assign to each double coset $\theta \in K(\mathfrak{h}) \setminus \mathrm{Gut}(M)/K(\mathfrak{h}')$ and $(n+1)\times(m+1)$ block matrix $\left(\begin{smallmatrix} A & b \\ c & d \end{smallmatrix}\right)$, whose entries are measures on $\mathbf{R}^*$, according to the rule

$$a_{ij}(S) = \mu\{\omega \in M_i \cap g^{-1}(M'_j) : g'(\omega) \in S\},$$
$$b_j(S) = \mu\{\omega \in M_* \cap g^{-1}(M'_j) : g'(\omega) \in S\},$$
$$c_i(S) = \mu\{\omega \in M_i \cap g^{-1}(M'_*) : g'(\omega) \in S\},$$
$$d(S) = \mu\{\omega \in M_* \cap g^{-1}(M'_*) : g'(\omega) \in S\},$$

where $g \in \theta$ and $S \subset \mathbf{R}^*$.

Remark. The measure d of the point 1 equals ∞.

Theorem 3.5. *The family of subgroups $K^0(\mathfrak{h}) \subset \mathrm{Gut}_\infty(M)$ satisfies the multiplicativity theorem with multiplication of double cosets*

$$\varphi \in K^0(\mathfrak{h}) \setminus \mathrm{Gut}_\infty(M)/K^0(\mathfrak{h}'), \qquad \psi \in K^0(\mathfrak{h}') \setminus \mathrm{Gut}_\infty(M)/K^0(h'')$$

corresponding to $\circ$-multiplication of matrices according to the formula

$$\begin{pmatrix} A_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \circ \begin{pmatrix} A_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} A_1\Lambda^{-1}A_2 & A_1\Lambda^{-1}b_2 + b_1 \\ c\Lambda^{-1}A_2 + c_2 & c_1\Lambda^{-1}b_2 + d_1 + d_2 \end{pmatrix},$$

where Λ is the diagonal matrix with eigenvalues $\mu(M'_i)$.

3.8. **Comments.** The theory of representations of the groups $\mathrm{Aut}(M)$ and $\mathrm{Aut}_\infty(M)$ is similar to the theory of representations of the full symmetric group S_∞ (see [22] and [25]) and the full unitary group $\mathrm{U}(\infty)$ (and also $\mathrm{Sp}(\infty)$ and $\mathrm{O}(\infty, \mathbf{R})$; see [8], [17]).

The other groups we have considered, $\mathfrak{B}(G)$, $\mathrm{Gut}(M)$, and $\mathrm{Gut}_\infty(M)$, are typical examples of (G, K)-pairs (see [18] and [19]). Proofs of multiplicativity theorems for various (G, K)-pairs are similar; the idea of the proof goes back to [6].

In general, the topology introduced on an infinite-dimensional group substantially affects the supply of representations. It is important to note that we have used in an essential way only the topology in $\mathrm{Aut}(M)$ and $\mathrm{Aut}_\infty(M)$. In the case of $\mathrm{Aut}(M)$ there seems to be no freedom of choice of the topology, and hence the assertions of §§3.2 and 3.3 remain valid (with slight changes) for various changes of topologies in $\mathfrak{B}(G)$ and $\mathrm{Gut}(M)$. In the case of $\mathrm{Aut}_\infty(M)$ the situation is a bit more complicated.

4. Completions of diffeomorphism groups

4.1. **Enveloping semigroups.** Let G be an infinite-dimensional group and $\Gamma \supset G$ a semigroup. We shall say that Γ is an enveloping semigroup for G if each unitary representation ρ of G can be extended in a canonical way to a representation $\tilde{\rho}$ of Γ so that the set of operators $\rho(G)$ is dense in $\tilde{\rho}(\Gamma)$ in the weak operator topology.

Remark. Of course, G is dense in Γ. However, we knowingly avoid such a formulation, because natural topologies on enveloping semigroups may be quite subtle. We also remark that our definition implicity requires a construction of canonical extensions of representations, whereas if we said that G is dense in Γ we could have avoided this trouble.

Proposition 4.1. (a) $\mathrm{Mor}_{G\mathscr{P}}(M)$ *is an enveloping semigroup for* $\mathfrak{B}(G)$.

(b) $\mathrm{Mor}_{\mathbf{R}^*_\chi\mathscr{P}}(M)$ *is an enveloping semigroup for* $\mathrm{Gut}(M)$.

In both cases the canonical extension of a representation from the group to the semigroup is continuous.

This is a fairly simple consequence of the multiplicativity theorems and the approximation lemma.

Remark. The proposition shows that the notion of enveloping semigroup is substantial (which is not clear directly from the definition). For all (G, K)-pairs enveloping semigroups are constructed in the same way (multiplicativity + approximation), but not all infinite-dimensional groups are (G, K)-pairs.

4.2. **Relative enveloping semigroups.** Let S be a family of representations of an infinite-dimensional group G. We say that a semigroup $\Gamma \supset G$ is an S-enveloping semigroup for G if each $\rho \in S$ can canonically be extended to Γ with $\rho(G)$ being weakly dense in $\rho(\Gamma)$.

4.3. **The classes $\mathscr{K}_p$** [9]. Let M^n be a compact n-dimensional manifold. Let $\mathrm{Diff}(M^n)$ be the group of C^∞ diffeomorphisms of M^n, $\mathrm{Diff}(M^n, \Omega)$ the group of C^∞ diffeomorphisms leaving invariant a fixed volume form Ω, and $\mathrm{Diff}(M^n, \omega)$ the group of C^∞ diffeomorphisms leaving invariant a symplectic form ω.

Let $\mathfrak{d}$ be one of these groups. Consider a point $x \in M^n$. Let $\mathfrak{d}(x)$ be the stabilizer of x. Let $\mathfrak{d}_p(x) \subset \mathfrak{d}(x)$ be the subgroup consisting of diffeomorphisms having the same derivatives of order $\leq p$ at x as the identity. Let $J_p = \mathfrak{d}(x)/\mathfrak{d}_p(x)$ be the group of jets of order $\leq p$. We shall denote it by GJ_p, SJ_p, or $\mathrm{Sy}\,J_p$ according to whether $\mathfrak{d} = \mathrm{Diff}(M^n)$, $\mathrm{Diff}(M^n, \Omega)$, or $\mathrm{Diff}(M^n, \omega)$.

Consider a unitary representation ρ of J_p on H (note that J_p is an ordinary Lie group) and extend it to a representation $\tilde{\rho}$ of $\mathfrak{d}(x)$ assuming it is trivial on $\mathfrak{d}_p(x)$. Denote by P_ρ the representation of $\mathfrak{d}$ induced by $\tilde{\rho}$. We shall say that a representation T of $\mathfrak{d}$ belongs to the class $\mathscr{K}_p$ if it occurs as a subrepresentation in some tensor product of representations of the form P_ρ.

Remove from M^n a union S of a finite number of submanifolds so that $M^n \setminus S$ is diffeomorphic to a domain $Q^n \subset \mathbf{R}^n$, the corresponding structure being preserved. The representation P_ρ can be realized on $L^2(Q^n, H)$ according to the formula

$$P_\rho(q)f(x) = [Dq(x)/Dx]^{1/2}\rho(q^{(p)}(x))f(q(x)), \tag{4.1}$$

where $Dq(x)/Dx$ is the Jacobian of the mapping q at x (this factor occurs only in the case $\mathfrak{d} = \mathrm{Diff}(M^n)$) and $q^{(p)}(x)$ is the jet of order p of q at x.

4.4. **Enveloping semigroups for the class $\mathscr{K}_p$.** The group $\mathfrak{d}$ can be embedded in a natural way into the group $\mathrm{Gut}(Q^n) \times \mathscr{F}(Q^n, J_p)$; namely, to each diffeomorphism $q(x)$ we associate the pair $(q(x), q^{(p)}(x)) \in \mathrm{Gut}(Q^n) \times \mathscr{F}(Q^n, J_p)$.

The representation P_ρ extends in a natural way to the semigroup of J_p-polymorphisms of Q^n (i.e. to $\mathrm{Mor}_{(GJ_p)_\chi \mathscr{P}}(Q^n)$ in the case $\mathfrak{d} = \mathrm{Diff}(M^n)$, to $\mathrm{Mor}_{SJ_p \mathscr{P}}(Q^n)$ in the case $\mathfrak{d} = \mathrm{Diff}(M^n, \Omega)$, and to $\mathrm{Mor}_{\mathrm{Sy}J_p \mathscr{P}}(Q^n)$ in the case $\mathfrak{d} = \mathrm{Diff}(M^n, \omega)$, the homomorphism $\chi\colon GJ_p \to \mathbf{R}^*$ being the Jacobian).

Theorem 4.1. *Let $n > 1$. The semigroups* $\mathrm{Mor}_{(GJ_p)_\chi \mathscr{P}}(Q^n)$, $\mathrm{Mor}_{SJ_p \mathscr{P}}(Q^n)$ *and* $\mathrm{Mor}_{\mathrm{Sy}\,J_p \mathscr{P}}(Q^n)$ *are $\mathscr{K}_p$-enveloping semigroups for the groups* $\mathfrak{d} = \mathrm{Diff}(M^n)$, $\mathrm{Diff}(M^n, \Omega)$, *and* $\mathrm{Diff}(M^n, \omega)$, *respectively.*

The case $n = 1$, i.e. the case where M^n is the circle S^1, is slightly, different. Let the subsemigroup Δ_p of $\mathrm{End}_{GJ_p \mathscr{P}}(S^1)$ consist of measures on $S^1 \times S^2 \times GJ_p$ whose projection on $S^1 \times S^1$ is the graph of a diffeomorphism (so that Δ_p is a subsemigroup of the semidirect product of $\mathrm{Diff}(S^1)$ and the semigroup of functions on S^1 taking values in measures on the group GJ_p).

Theorem 4.2. *Δ_p is a $\mathscr{K}_p$-enveloping semigroup for* $\mathrm{Diff}(S^1)$.

The proofs of these theorems occupy §§4.5–4.8.

4.5. **The square rotation.** Assume for simplicity that $n = 2$. Let Γ be the weak closure of the set of all operators $P_\rho(g)$, where $g \in \mathfrak{d}$; see (4.1).

Lemma 4.1. *Let q be a mapping $Q^2 \to Q^2$ which is piecewise smooth, not necessarily continuous, but leaving invariant the form Ω in the case $\mathfrak{d} = \mathrm{Diff}(M^2, \Omega)$. Then $P_\rho(g) \in \Gamma$.*

Consider a square $D \subset Q^2$. By a *square rotation* g_D we mean a mapping which on $Q^2 \setminus D$ is the identity and on the square D is the central inversion with respect to its center.

Lemma 4.2. *There exists a family of diffeomorphisms $g_\varepsilon = g_{\varepsilon, D} \in \mathrm{Diff}(Q^2, \Omega)$ such that $P_\rho(g_{\varepsilon,D}) \to P_\rho(g_D)$ weakly.*

Proof of Lemma 4.2. Let $D' \subset D$ be a square with the same center and with sides parallel to those of D and shorter by ε. We introduce in D polar coordinates (r, φ). We consider a smooth curve $r = \lambda(\varphi)$ in the interior of $D \setminus D'$ and construct a family of curves l_h given by the equations $r^2 - \lambda^2(\varphi) = h$, where $0 \le h \le h_0$ and $h_0 = O(\varepsilon)$ is so small that l_{h_0} is contained in the interior of $D \setminus D'$. Let $s = r^2 - \lambda^2(\varphi)$. Let

$$g_\varepsilon(r, \varphi) = \begin{cases} (r, \varphi) & \text{for } s \ge h_0, \\ (r, \varphi + \pi) & \text{for } s \le 0, \\ (\sqrt{\lambda(\varphi + \tau(s))^2 + s}, \varphi + \tau(s)) & \text{for } 0 \le s \le h_0, \end{cases}$$

where $\tau(h)$ is a monotone C^∞ function such that $\tau(h) = \pi$ for $h \le 0$ and $\tau(h) = 0$ for $h \ge h_0$. It is obvious that $P_\rho(g_{\varepsilon,D})$ converges to $P_\rho(g_D)$. The lemma is proved.

Remark 1. The cumbersome formula for g_ε means the following. Each curve l_h slides along itself under the "rotation". The explicit form of the curves l_h guarantees the invariance of Lebesgue measure.

Remark 2. The "rotations" g_ε can be chosen so that first-order partial derivatives of g_ε have growth order $O(\varepsilon^{-1})$. This will be essential in the proof of Theorem 4.3.

Now consider the rectangle $E = \{0 < x < 2a, 0 < y < a\} \subset Q^2$. It consists of two squares E_1 and E_2 separated by the partition $x = a$. Let $g_E: Q^2 \to Q^2$ be a mapping which on $Q^2 \setminus E$ is the identity and on the rectangle E is the central inversion with respect to its center. A rectangle can be obtained by contraction from a square, so by Lemma 4.2 we have $P_\rho(g_E) \in \Gamma$. Next we perform square rotations in each of the squares E_1 and E_2. We get a mapping p_E which takes E_1 to E_2, takes E_2 to E_1, and leaves all the points of $Q^2 \setminus E$ fixed. We see that $P_\rho(p_E) \in \Gamma$.

Lemma 4.3. *Let $p \in \mathrm{Aut}(Q^2)$. Then the operator $Z(p)f(x) = f(p(x))$ lies in Γ.*

Proof. Take any square D in Q^2 and divide it into small squares K_i. For any permutation σ of two neighboring small squares, we have $Z(\sigma) \in \Gamma$. Therefore, for any permutation r of the small squares K_i, we have $Z(r) \in \Gamma$. Now the lemma is obvious.

Lemma 4.1 follows easily from Lemma 4.3.

4.6. **Proof of the theorem in the case $n = 2$.** Let $D \subset Q^2$ be a square and π a J_ρ-polymorphism of D. We show how to approximate the operator $P_\rho(\pi)$ by the operators $P_\rho(p)$, where p is a piecewise smooth (not necessarily continuous) mapping $D \to D$.

We divide D into small squares $A_1, \dots, A_N$ and we let π_{ij} be the measure on J_p corresponding to $A_i \times A_j \subset D \times D$. We divide each of the small squares A_i into rectangles in two ways: $A_i = \bigcup B_{ij}$ and $A_i = \bigcup C_{ij}$ so that

$$\mu(B_{ij}) = \pi_{ij}(J_p), \qquad \mu(C_{ij}) = \int_{J_p} \chi(h)\, d\pi_{ji}.$$

We shall construct a mapping $p: D \to D$ which takes B_{ij} to C_{ji}. We approximate each measure π_{ij} on J_p by a measure with finite support consisting of the points $a_1, \dots \in J_p$. Let the measures of these points be $m_1, \dots,$ and let the jet corresponding to a_j be

$$(4.2)\qquad \begin{aligned} x' = a_j^{(1)}(x, y) &= \sum_{\alpha>0,\, \beta>0,\, \alpha+\beta\le p} v_{j\alpha\beta}^{(1)} x^\alpha y^\beta, \\ y' = a_j^{(2)}(x, y) &= \sum_{\alpha>0,\, \beta>0,\, \alpha+\beta\le p} v_{j\alpha\beta}^{(2)} x^\alpha y^\beta. \end{aligned}$$

We divide B_{ij} and C_{ij} once more into rectangles $P_{ij\beta}$ and $Q_{ij\beta}$ so that $\mu(P_{ij\beta}) = m\beta$ and $\mu(Q_{ij\beta}) = \chi(a_j)m_\beta$, where $\chi(a_j)$ is the Jacobian of the jet (4.2) at $(x, y) = (0, 0)$. Now, in each of the rectangles $P_{ij\beta}$, we choose a slightly smaller rectangle $P'_{ij\beta}$ and divide it (for the last time) into many small (i.e. with small sides) identical rectangles. We construct an injective mapping $p: P'_{ij\beta} \to Q_{ij\beta}$ which on each small rectangle with center (x_0, y_0) has the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} c_1 + \sum v_{j\alpha\beta}^{(1)} (x - x_0)^\alpha (y - y_0)^\beta \\ c_2 + \sum v_{j\alpha\beta}^{(2)} (x - x_0)^\alpha (y - y_0)^\beta \end{pmatrix}.$$

The image of each small rectangle will be a curvilinear quadrangle which differs very little from a parallelogram, and almost the whole area of $Q_{ij\beta}$ can be tiled by identical parallelograms. Therefore, the mapping $p: P_{ij\beta} \to Q_{ij\beta}$ can be made injective.

Finally, we can extend our mapping to $P_{ij\beta} \setminus P'_{ij\beta}$ as we like, provided it will be piecewise smooth. We have thus constructed the required approximation.

4.7. **The case** $n > 2$. Here, a small modification of the construction is needed only in the symplectic case.

In Lemma 4.3 we take (for simplicity) the product $Q^4 = D^2 \times D^2$ of two squares with the standard symplectic structure $(dx \wedge dy)$. We divide each of them into small squares A_i and A'_i. For any permutation p of all cubes of the form $A_i \times A'_j$, where j is fixed and i is arbitrary, we have $P_\rho(p) \in \Gamma$. Similarly, we can make all kinds of permutations of the cubes $A_i \times A'_j$ for a fixed i. Now Lemma 4.3 becomes obvious.

One more difficulty may arise in the last stage of the proof, since it is not clear if there exists a piecewise smooth symplectomorphism $P_{ij\beta} \setminus P'_{ij\beta} \to Q_{ij\beta} \setminus p(P'_{ij\beta})$. But we can replace the symplectomorphism by any measure-preserving mapping (see Lemma 4.3).

4.8. **The case** $n = 1$. The considerations of §4.4 are of course unnecessary here and the considerations of §4.5 greatly simplify. The obstruction to the existence of polymorphisms is the monotonicity of diffeomorphisms.

4.9. **Comments.** The group $\mathrm{Diff}(S^1)$ is known (see [11]) to have three classes of representations: representations of class $\mathscr{K}_p$ (including some generalizations, see [7]), highest weight representations, and representations connected with almost invariant structures. The semigroup connected with the highest weight representations has been constructed by the author in [10] (see also [12]). It does not resemble at all the semigroups Δ_p from §4.3, and it might seem that there is no relation whatever between these objects. However, this is not the case. One knows (see [11]) many series of unitary representations of $\mathrm{Diff}(S^1)$ one end of which lies in $\mathscr{K}_p$ and the other in the class of highest weight representations. The question of a semigroup connected with these series remains open.

4.10. **New representations of** $\mathrm{Diff}(M^2, \Omega)$. Let M^2 be a two-dimensional compact manifold with a fixed volume form Ω. The group of jets SJ_1 coincides with $\mathrm{SL}_2(\mathbf{R})$, and hence $\mathrm{Diff}(M^2, \Omega)$ can be embedded into the group of $\mathrm{SL}_2(\mathbf{R})$-polymorphisms of $Q^2 = M^2 \setminus S$; see §4.3.

Theorem 4.3. *The restriction of the Fock representation* φ *of the semigroup* $\mathrm{End}_{\mathrm{SL}_2(\mathbf{R})\mathscr{P}}(Q^2)$ *to* $\mathrm{Diff}(M^2, \Omega)$ *is irreducible.*

Proof. Of course, the assertion is based on the fact that $\mathrm{Diff}(M^2, \Omega)$ is dense in the semigroup Γ of all $\mathrm{SL}_2(\mathbf{R})$-polymorphisms. However, for the topology of §2.3 this statement is not true; we shall prove instead that the set of operators $\varphi(\mathrm{Diff}(M^2, \Omega))$ is weakly dense in $\varphi(\Gamma)$.

Let π_j and π be measures on $M \times M \times \mathrm{SL}_2(\mathbf{R})$. It is not hard to check that the following conditions imply the weak convergence $\varphi(\pi_j) \to \varphi(\pi)$.

1) $\pi_j \to \pi$ weakly.

2) $\int_{M\times M\times \mathrm{SL}_2(\mathbf{R})} \|\gamma(g)\|^2 \, d\pi_j$ is uniformly bounded in j.

In §§4.5 and 4.6, for each polymorphism $\pi \in \Gamma$, a sequence of diffeomorphisms p_j was constructed which converges to π in some sense. We want to make sure that this sequence converges in the sense we need (i.e. 1) and 2) hold).

Everything is all right with condition 1), and the only place where condition 2) might fail is the convergence of $g_{\varepsilon, D}$ to a square rotation (see the proof of Lemma 4.2).

Let $g(x) \in \mathrm{Diff}(M^2, \Omega)$ have the 1-jet corresponding to the matrix

$$\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R}) \qquad (|\alpha|^2 - |\beta|^2 = 1).$$

Then a direct calculation shows that $\|\gamma(g(x))\|^2 = \ln(1 - |\beta/\alpha|)$. Thus, $\|\gamma(g)\|^2 = O(\ln T)$, where T is the maximum of first-order partial derivatives of g; as we saw, $T = O(\varepsilon^{-1})$ (see Remark 2 in §4.5). On the other hand, the domain in M where the partial derivatives are greater than 1 has measure order $O(\varepsilon)$; hence the numbers $\int_M \|\gamma(g(x))\|^2$ are uniformly bounded, as required.

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