Sbornik: Mathematics **188**:11 1587–1616

Matematicheskiĭ Sbornik **188**:11 19–50

UDC 519.46+517.4

On the correspondence between boson Fock space and the L^2 space with respect to Poisson measure

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Abstract. The properties of the integral transformation carrying boson Fock space into the L^2 space with respect to Poisson measure are investigated. An explicit formula is obtained for point configuration functions corresponding to Gaussian vectors of boson Fock space.

Bibliography: 20 titles.

According to Fock's original definition, boson Fock space is the direct sum of the symmetric powers of a Hilbert space. In the 1950's I. Segal showed that Fock space can be regarded as the L^2 space with respect to a Gaussian measure on an infinite-dimensional space (see the detailed description of this model in [1]; see also [2] about applications in representation theory). At the beginning of the 1960's Bargmann, Segal, and Berezin introduced into circulation a holomorphic model of boson Fock space (see [3], and also [4] and § 2 below). There was also much discussion of the canonical isomorphism between boson and fermion Fock space (the boson–fermion correspondence; see [5]), as well as the isomorphism between Fock space and the space of symmetric functions (see [5]). Along with these models there are a number of other models that have not attracted much attention: for example, the space of holomorphic functionals on the space of schlicht functions (see [6]), and the space of functionals on a Γ-process (see [7]).

The existence of a canonical isomorphism between boson Fock space and the L^2 space with respect to Poisson measure was discovered in independent work of Vershik, Gel'fand, and Graev [8] and of Ismagilov ([9], [10]); see the expositions of these constructions in [11] and [4]. Our goal in the present article is to transfer the natural structures of boson Fock space into the L^2 space with respect to Poisson measure (the question of the reverse transfer may be no less interesting). The main result is an explicit formula for the point configuration functions corresponding to Gaussian vectors of Fock space. This formula has the form

$$\mathfrak{R}(x) = \sum \left[\prod K(x_{\sigma_j}, x_{\delta_j}) \cdot \prod \alpha(x_k)\right]. \qquad (0.1)$$

This work was carried out with the support of the Russian Foundation for Fundamental Research (grant no. 95-01-00814) and the Russian Programme for Support of Leading Scientific Schools (grant no. 96-01-96249).

AMS 1991 *Mathematics Subject Classification*. Primary 47N50, 46N50; Secondary 47B38, 81R10, 81S05, 47B10, 32A30, 22E70.

The argument $x = (x_1, x_2, \dots)$ of the function $\mathfrak{R}(x)$ is an unordered denumerable (locally finite) subset of a continuum set (for example, $\mathbb{R}^n$), and the summation is over all partitions of the set $x_1, x_2, \dots$ into two-point and one-point subsets, with the number of two-point subsets finite. The product is over the two-point $\{x_{\sigma_j}, x_{\delta_j}\}$ and one-point $\{x_k\}$ subsets of the partition.

Although there is a broad literature (see [12]) about so-called symmetric functions (that is, symmetric expressions of infinitely many variables), I have not previously encountered series of the form (0.1). These series are themselves the natural limits of the coefficients of the polynomials

$$\frac{1}{p!}\left(\frac{1}{2}\sum_{i,j=1}^{N} \varkappa_{ij} z_i z_j + \sum \alpha_i z_i\right)^p \tag{0.2}$$

as $p \to \infty$ and $N \to \infty$ (here the set the indices i run through becomes infinite (a continuum or countable)).

The coefficients of the polynomials (0.2), written out explicitly (see the proof of Theorem 4.1 below), can be seen as not very pleasant conglomerations of factorials, while the formula (0.1) is one of the simplest expressions of point configurations that one could imagine.

The first section contains preliminary information about Poisson measures ([13] is a standard source on this subject), while the second section contains information about boson Fock space (see [3], [4] for details). The boson–Poisson correspondence is introduced in § 3, and the same section contains some of the simplest properties of this correspondence. In § 4 we formulate Theorem 4.2 on the form of Gaussian vectors and we discuss certain explicitly computable integrals of functions of the form (0.1). The proofs are contained in § 5. Some comments about the operators corresponding to integral operators with Gaussian kernels are contained in § 6.

I thank R. S. Ismagilov for a discussion of this subject.

§ 1. Poisson measures

1.1. Configurations. Let $M = (M, \mu)$ be a *Lebesgue space* with finite or σ-finite measure μ. In other words, as a measure space M is isomorphic to the union of an interval (finite, empty, or infinite) of the line $\mathbb{R}$ and a collection (finite, empty, or countable) of points with non-zero measure.

A *configuration* in M is defined to be a denumerable, finite, or empty unordered collection of points in M for which positive integer multiplicities are specified.

Remark. Suppose that a point m appears in a configuration ω with some multiplicity $k > 1$. In this case it is convenient to think that there are k indistinguishable points of the configuration at the point m of M (these points are also indistinguishable from all the remaining points of the configuration).

The set of all configurations in M will be denoted by $\Omega(M)$. On the space $\Omega(M)$ we now introduce a certain probability measure $\nu = \nu_M$.

1.2. The case of finite spaces. Let M be a space consisting of points $m_1, \dots, m_N$ with measures $a_1, \dots, a_N$. To each configuration $\omega \in \Omega(M)$ we associate the collection of numbers

$$(p_1, \dots, p_N),$$

where p_j is the multiplicity with which m_j appears in the configuration (if m_j does not appear in the configuration, then we let $p_j = 0$). We can now identify $\Omega(M)$ with the set $\mathbb{Z}_+^N$ (where $\mathbb{Z}_+$ denotes the set of non-negative integers), or, what is the same, with the set of functions $M \to \mathbb{Z}_+$.

We now define the *Poisson measure* $\nu = \nu_M$ on the (countable) set $\Omega(M)$ by the condition that the measure of a point $p = (p_1, \dots, p_N) \in \mathbb{Z}_+^N$ be given by

$$\nu(p) = \nu_M(p) = \prod_{j=1}^{N} \left(\frac{a_j^{p_j}}{p_j!} e^{-a_j} \right).$$

It is easily seen that

$$\sum_{p \in \mathbb{Z}_+^N} \nu(p) = 1,$$

that is, ν is a probability measure.

1.3. The case of infinite spaces. Suppose now that the space M is arbitrary (infinite, in general). Let $M_1, \dots, M_N$ be disjoint (measurable) subsets of finite measure in M. We denote by

$$\Omega(M_1, \dots, M_N; p_1, \dots, p_N)$$

the set of all configurations $\omega \in \Omega(M)$ such that the intersection of ω with M_j consists of precisely p_j points (counting multiplicity). The *Poisson measure* $\nu = \nu_M$ on $\Omega(M)$ is determined from the condition

$$\nu\big[\Omega(M_1, \dots, M_N; p_1, \dots, p_N)\big] = \prod_{j=1}^{N} \left(\frac{\mu(M_j)^{p_j}}{p_j!} e^{-\mu(M_j)} \right). \tag{1.1}$$

It is easy to show (with the help of Kolmogorov's theorem on projective limits) that the construction is correct, and that ν_M is a probability measure on $\Omega(M)$ (that is, $\nu(\Omega(M)) = 1$).

Remark 1. Suppose that M has finite measure. Then

$$\nu\big(\Omega(M, p)\big) = \frac{\mu(M)^p}{p!} e^{-\mu(M)} \tag{1.2}$$

is the probability that a configuration ω consists of p points. The sum of the numbers (1.2) is equal to 1, and the probability is thereby 1 that an $\omega \in \Omega(M)$ contains only finitely many points.

Remark 2. Suppose that the measure of M is infinite. Let $L \subset M$ be a subset of finite measure. Then the probability that an $\omega \in \Omega(M)$ contains $\geqslant p$ points in L is equal to

$$\sum_{j \geqslant p} \nu\big[\Omega(L; j)\big] = \sum_{j \geqslant p} \frac{\mu(L)^j}{j!} e^{-\mu(L)}.$$

This quantity tends to 1 as $\mu(L) \to \infty$. Therefore, with probability 1 a configuration $\omega \in \Omega(M)$ consists of infinitely many points. We underscore that with probability 1 any set $L \subset M$ of finite measure contains only finitely many points of a configuration.

Remark 3. Suppose that the measure μ is *continuous*, that is, the measure of any point is 0. Then with probability 1 all the points of a configuration $\omega \in \Omega(M)$ have multiplicity 1. To see this we consider in M a subset L of finite measure. Let us partition L into k pieces of equal measure. The probability that each of the pieces contains at most one point of the configuration is equal to

$$\left[\left(1+\frac{\mu(L)/k}{1!}\right)e^{-\mu(L)/k}\right]^k.$$

As $k \to \infty$ this quantity tends to 1, therefore the probability that L contains a multiple point of the configuration ω is equal to 0.

1.4. Cylindrical functions. We fix subsets $M_1, \dots, M_N$ of finite measure in M. Let A be a finite space with points of measure $\mu(M_1), \dots, \mu(M_N)$. We consider the map

$$\pi = \pi_{[M_1,\dots,M_N]} \colon \Omega(M) \to \Omega(A)$$

assigning to each configuration $\omega \in \Omega(M)$ the collection of numbers

$$p_j = \#(\omega \cap M_j)$$

(where the symbol # denotes the number of points in a set).

Remark. The image of the measure ν_M under the map π obviously coincides with the measure ν_A; see § 1.2.

We next consider the space $L^2(\Omega(A), \nu_A)$. This is none other than the space of functions

$$h \colon \mathbb{Z}_+^N \to \mathbb{C}$$

such that

$$\sum_{p \in \mathbb{Z}_+^N} |h(p)|^2 \nu_A(p) < \infty.$$

To each function $h \in L^2(\Omega(A))$ we associate the function

$$J_{[M_1,\dots,M_N]} h = h \circ \pi_{[M_1,\dots,M_N]}$$

on $\Omega(M)$. This is simply the function whose value on $\Omega(M_1, \dots, M_N; p_1, \dots, p_N)$ is equal to $h(p_1, \dots, p_N)$. In other words, we get an isometric embedding

$$J_{[M_1,\dots,M_N]} \colon L^2(\Omega(A), \nu_A) \to L^2(\Omega(M), \nu_M).$$

Its image, that is, the set of L^2-functions that are constant on the sets

$$\Omega(M_1, \dots, M_N; p_1, \dots, p_N),$$

will be denoted by

$$L^2(\Omega(M) \mid M_1, \ldots, M_N) = L^2(\Omega(M) \mid \{M_i\}).$$

It is easy to see that

$$L^2(\Omega(M) \mid M_1, \ldots, M_\alpha) + L^2(\Omega(M) \mid S_1, \ldots, S_\beta) \subset L^2(\Omega(M) \mid \{M_i \cap S_j\}).$$

The union of all possible subspaces $L^2(\Omega(M) \mid \{M_j\})$ over all possible collections $\{M_j\}$ will be called the space of *cylindrical functions.*

It is easy to see that the space of cylindrical functions is dense in $L^2(\Omega(M))$. Computations with cylindrical functions reduce to computations with the sets $\Omega(A)$ for finite measure spaces A.

1.5. Independence. We consider a partition of the space M into a union of two disjoint subsets:

$$M = M_1 \cup M_2.$$

To each configuration $\omega \in \Omega(M)$ we associate the pair

$$\omega_1 = \omega \cap M_1, \qquad \omega_2 = \omega \cap M_2$$

of configurations.

We obtain a map $\omega \mapsto (\omega_1, \omega_2)$ that is easily seen to establish an isomorphism of measure spaces:

$$\Omega(M) \simeq \Omega(M_1) \times \Omega(M_2). \tag{1.3}$$

This implies the existence of a canonical isomorphism (see [14], II.4)

$$L^2(\Omega(M)) \simeq L^2(\Omega(M_1)) \otimes L^2(\Omega(M_2)). \tag{1.4}$$

Namely, if $f_1 \in L^2(\Omega(M_1))$ and $f_2 \in L^2(\Omega(M_2))$, then corresponding to the element $f_1 \otimes f_2$ of the tensor product is the function

$$g(\omega) = f_1(\omega \cap M_1) \cdot f_2(\omega \cap M_2).$$

We note the (obvious) equality

$$\int_{\Omega(M)} g(\omega)\, d\nu_M(\omega) = \int_{\Omega(M_1)} f_1(\omega_1)\, d\nu_{M_1}(\omega_1) \cdot \int_{\Omega(M_2)} f_2(\omega_2)\, d\nu_{M_2}(\omega_2). \tag{1.5}$$

Suppose now that M_1 and M_2 are Lebesgue measure spaces. We denote their disjoint union by $M_1 \sqcup M_2$.

We consider a Hilbert–Schmidt operator

$$S\colon L^2(\Omega(M_2)) \to L^2(\Omega(M_1)),$$

which is given by a formula

$$Sf(\omega_1) = \int_{\Omega(M_2)} K(\omega_1, \omega_2) f(\omega_2)\, d\nu_{M_2}(\omega_2). \tag{1.6}$$

Let us recall (see [14], II.10) that K lies in $L^2(\Omega(M_1) \times \Omega(M_2))$. Thus, to any Hilbert–Schmidt operator S there corresponds a function $K \in L^2(\Omega(M_1 \sqcup M_2))$.

For operators more general than Hilbert–Schmidt operators the question of the correspondence between operators $L^2(\Omega(M_2)) \to L^2(\Omega(M_1))$ and functions on $\Omega(M_1 \sqcup M_2)$ encounters the usual difficulties.

One of the cases when there are no such difficulties is the case of finite spaces M_1 and M_2. Then any bounded operator $S\colon L^2(\Omega(M_2)) \to L^2(\Omega(M_1))$ can be written in the form (1.6), where $K(\cdot, \cdot)$ is a function on $\Omega(M_1 \sqcup M_2)$ (we have actually claimed only that any operator acting in l_2 is given by a matrix).

1.6. Campbell's formula (see [13], [4]). Let $\psi \in L^1(M)$. We consider the function

$$\mathcal{E}_\psi(\omega) = \prod_{m_j \in \omega} (1 + \psi(m_j)) \tag{1.7}$$

on the space $\Omega(M)$. If a point m_j occurs in a configuration ω with multiplicity > 1, then we write the factor $1 + \psi(m_j)$ as many times as the multiplicity of m_j.

Theorem 1.1 (Campbell's formula). *If $\psi \in L^1(M)$, then the product* (1.7) *converges almost everywhere on $\Omega(M)$, $\mathcal{E}_\psi \in L^1(\Omega(M))$, and*

$$\int_{\Omega(M)} \mathcal{E}_\psi(\omega)\, d\nu_M(\omega) = \exp\left(\int_M \psi(m)\, d\mu(m)\right). \tag{1.8}$$

Remark. For a one-point space M, Campbell's formula can be verified by direct computation. The case of a finite space can be reduced to the case of a one-point space with the help of the isomorphism (1.3) and the formula (1.5). Arbitrary spaces can be reduced to finite spaces by passing to the limit.

1.7. Remark: symmetric functions. The question naturally arises (as always in analysis involving infinitely many variables) of how to write functions on $\Omega(M)$. I present an example that is to some degree an answer.

Let us consider a function ψ on M. We set

$$G_\psi^{(r)}(\omega) = \sum_{\{m_1,\dots,m_r\} \in \omega} \psi(m_1)\psi(m_2)\cdots\psi(m_r),$$

where the summation is over all unordered collections of r distinct points in ω. Campbell's formula is a standard and quite convenient tool for working with this kind of 'polynomial' expression. Indeed,

$$\mathcal{E}_{t\psi}(\omega) = \sum_{r=0}^{\infty} t^r G_\psi^{(r)}(\omega).$$

Therefore,

$$\int_{\Omega(M)} G_\psi^{(r)}(\omega)\, d\nu(\omega) = \frac{1}{r!}\left(\int_M \psi(m)\, d\mu(m)\right)^r.$$

Applying Campbell's formula to the function

$$\widetilde{\psi}(m) = -1 + \prod_{s=1}^{k} (1 + t_s \psi_s(m)),$$

we obtain

$$\int_{\Omega(M)} \prod_{s=1}^{k} \mathcal{E}_{t_s\psi_s}(\omega)\, d\nu(\omega) = \exp\left\{\int_M \left(\prod_{s=1}^{k}(1 + t_s\psi_s(m)) - 1\right) d\mu(m)\right\}.$$

Expanding the left-hand and right-hand sides in a series of powers of $t_1, \dots, t_k$, we get that

$$\int_{\Omega(M)} \prod_{s=1}^{k} G_{\psi_s}^{(r_s)}(\omega)\, d\nu(\omega) = \sum \left[\prod \frac{1}{a_{i_1 \dots i_k}!} \left(\int_M \psi_1^{i_1}(m) \cdots \psi_k^{i_k}(m)\, d\mu(m) \right)^{a_{i_1 \dots i_k}} \right].$$

In this formula the product is taken over all collections $i_1, \dots, i_k$, where the i_α take the values 0 and 1 and are not all zero at the same time. The summation is over all collections $a_{i_1 \dots i_k}$ of non-negative integers satisfying the system of equalities

$$\left\{ \sum_{i_1, \dots, i_k :\, i_\alpha = 1} a_{i_1 \dots i_k} = r_\alpha. \right.$$

1.8. The symmetry group. Suppose that M is a space with an infinite continuous measure. We denote by $\mathrm{Gms}_\infty(M)$ the group of transformations g of M leaving the measure quasi-invariant and satisfying the following condition on the Radon–Nikodým derivative:

$$\int_M |g'(m) - 1|\, d\mu(m) < \infty.$$

The transformation g obviously induces the transformation

$$g \colon \omega = (m_1, m_2, \dots) \mapsto g\omega = (gm_1, gm_2, \dots)$$

of the space $\Omega(M)$.

Theorem 1.2. *Let $g \in \mathrm{Gms}_\infty$. Then the transformation $\omega \mapsto g\omega$ leaves the measure ν_M quasi-invariant, and the Radon–Nikodým derivative is equal to*

$$\left[\prod_{m_j \in \omega} g'(m_j) \right] \exp \left(- \int_M (g'(m) - 1)\, d\mu(m) \right).$$

Remark. The theorem is a simple exercise on the Campbell form. It was obtained for smooth transformations with compact support in [8]–[10], and in the above form in [4].

§ 2. Standard structures in boson Fock space

It is mainly §§ 2.1–2.4 that are important for us. The material in §§ 2.5–2.10 will be used only in §§ 3.5–3.6 and § 6.

2.1. Boson Fock space. We consider a so-called *Hilbert space with involution*, that is, a Hilbert space on which an antilinear isometric operator I is fixed that satisfies the condition $I^2 = E$. We shall be interested in the space $H = L^2(M)$; in this case $If = \bar{f}$.

Thus, let H be a Hilbert space with involution. The *boson Fock space* $F(H)$ is a Hilbert space in which a system of vectors (an '*overcomplete basis*') Φ_h is fixed that is labelled by the vectors $h \in H$ and satisfies the following conditions:

(1) $\langle \Phi_h, \Phi_{h'} \rangle_{F(H)} = \exp(\langle h', h \rangle_H)$; (2.1)

(2) the set of linear combinations of the vectors Φ_h is dense in $F(H)$.

The space $F(H)$ is unique in the following sense. Suppose that F and $\widetilde{F}$ are two spaces with the respective total systems of vectors Φ_h and $\widetilde{\Phi}_h$ satisfying (2.1). Then there exists a unique unitary operator $U\colon F \to \widetilde{F}$ for which $U\Phi_h = \widetilde{\Phi}_h$.

The existence of the space $F(H)$ is not entirely obvious, but follows from the holomorphic model described in the next subsection.

2.2. The Bargmann–Segal–Berezin holomorphic model (see [3], [4]). We associate with each vector $w \in F(H)$ the function $Jw(h)$ on H given by the formula

$$Jw(h) := \langle w, \Phi_h \rangle. \tag{2.2}$$

In this way we have realized the space $F(H)$ as a space of holomorphic functions on H. We now give a precise description of the image of the operator J.

Let $e_1, e_2, \dots$ be an orthonormal basis in H. We define the Hilbert space $\widetilde{F}(H)$ as the space of formal series

$$f(z_i e_i) = \sum_{j_1 \geqslant 0,\, j_2 \geqslant 0,\, \dots} c_{j_1 j_2 \dots} z_1^{j_1} z_2^{j_2} \cdots \tag{2.3}$$

(where the summation is over all collections $j_1, j_2, \dots$ with only finitely many non-zero elements) satisfying the condition

$$\sum |c_{j_1 j_2 \dots}|^2 j_1! j_2! \cdots < \infty.$$

The inner product in $\widetilde{F}(H)$ is introduced by the condition that the vectors

$$\frac{z_1^{j_1} \cdots z_N^{j_N}}{\sqrt{j_1! \cdots j_N!}}$$

form an orthonormal basis in $\widetilde{F}(H)$.

It is not hard to verify that the series (2.3) converges if $f \in \widetilde{F}(H)$ and $\sum z_i e_i \in H$ (that is, $\sum |z_i|^2 < \infty$).

The system of functions $\widetilde{\Phi}_h(z)$ is given by the formula

$$\widetilde{\Phi}_h(z) = \exp\Bigl\{\sum z_i \overline{h}_i\Bigr\}.$$

It is easy to see that operator J given by (2.2) is an isometry $F(H) \to \widetilde{F}(H)$, and $J\Phi_h = \widetilde{\Phi}_h$.

We note also the following formula (the *reproducing property*):

$$f(h) = \langle f, \widetilde{\Phi}_h \rangle_{\widetilde{F}(H)}. \tag{2.4}$$

Henceforth we shall not distinguish between the spaces $F(H)$ and $\widetilde{F}(H)$.

2.3. The finite-dimensional case. We consider a finite measure space M with points of measure $a_1,\dots,a_N$. Then $L^2(M)$ is the space of vectors $v=(v_1,\dots,v_N)$, with the inner product

$$\big\langle (v_1,\dots,v_N);(w_1,\dots,w_N)\big\rangle=\sum_{j=1}^{N} v_j\overline{w}_j a_j.$$

The space $F(L^2(M))$ can be identified with the space of holomorphic functions on $L^2(M)=\mathbb{C}^N$ with the inner product

$$\langle f_1,f_2\rangle=\int_{L^2(M)} f_1(v)\overline{f_2(v)}\exp\{-\|v\|^2\}\prod_{j=1}^{N}\Big(\frac{a_j}{\pi}\,dv_j\,d\overline{v}_j\Big). \tag{2.5}$$

The system of vectors Φ_v is given by

$$\Phi_v(z)=\exp\Big\{\sum z_j\overline{v}_j a_j\Big\}.$$

Remark. The formula (2.5) admits a rigorous interpretation also in the case of an infinite-dimensional space H (see [1], [3], [4]), but we do not need this.

2.4. Gaussian vectors. Let $K\colon H\to H$ be a Hilbert–Schmidt operator with $\|K\|<1$ and let $\varkappa\in H$. We consider the function $b[K\mid\varkappa](z)$ on H given by the formula

$$b[K\mid\varkappa](z)=\exp\Big\{\frac{1}{2}\langle Kz,\overline{z}\rangle+\langle z,\overline{\varkappa}\rangle\Big\}. \tag{2.6}$$

Proposition 2.1 (see, for example, [4], § 6.2). *The vector $b[K\mid\varkappa]$ lies in $F(H)$, and*

$$\begin{aligned}&\big\langle b[M\mid m],b[A\mid\alpha]\big\rangle_{F(H)}\\&\qquad=\det\big[(1-M\bar{A})^{-\frac12}\big]\exp\Big\{\frac12(\overline{\alpha}m)\begin{pmatrix}-\bar{A}&E\\E&-M\end{pmatrix}^{-1}\begin{pmatrix}\overline{\alpha}^t\\m^t\end{pmatrix}\Big\}.\end{aligned} \tag{2.7}$$

Remark. It is useful to have in view the formula

$$\begin{pmatrix}-\bar{A}&E\\E&-M\end{pmatrix}^{-1}=\begin{pmatrix}M(E-\bar{A}M)^{-1}&(E-M\bar{A})^{-1}\\(E-\bar{A}M)^{-1}&\bar{A}(E-M\bar{A})^{-1}\end{pmatrix}.$$

2.5. Symbols of operators. Let H_1 and H_2 be Hilbert spaces with involutions, and let A be a bounded operator $F(H_2)\to F(H_1)$. We associate with each such operator a function $K_A(z,u)$ on $H_1\oplus H_2$ (the *symbol* or *kernel of the operator*; see [3], and also [4]) in accordance with the formula

$$K_A(z,u)=\langle A\Phi_u,\Phi_z\rangle_{F(H_1)}, \tag{2.8}$$

where $z\in H_1$, $u\in H_2$, $\Phi_z\in F(H_1)$, and $\Phi_u\in F(H_2)$.

See [3] and [4] on recovering the operator from the symbol. If $H_1=L^2(M_1)$ and $H_2=L^2(M_2)$, where M_1 and M_2 are finite, then

$$Af(z)=\int_{L^2(M)} K_A(z,u)f(u)\exp\Big\{-\sum_j a_j|u_j|^2\Big\}\prod_j\Big(\frac{a_j}{\pi}\,du_j\,d\overline{u}_j\Big).$$

2.6. Gaussian operators. Let H_1 and H_2 be Hilbert spaces with involutions. We consider an operator

$$S = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} : H_1 \oplus H_2 \to H_1 \oplus H_2.$$

Here the sign t indicates transposition ($S^t := \overline{S}^*$). Suppose that this operator satisfies the conditions

1*. $S = S^t$,
2*. $\|S\| \leqslant 1$,
3*. $\|K\| < 1$ and $\|M\| < 1$,
4*. K and M are Hilbert–Schmidt operators.

Let us consider the operator

$$B[S] \colon F(H_2) \to F(H_1)$$

with the symbol

$$K(z,u) = \exp\left\{\frac{1}{2}\langle Kz, \overline{z}\rangle + \langle Lz, u\rangle + \frac{1}{2}\langle M\overline{u}, u\rangle\right\}. \tag{2.9}$$

We write $K(z,u)$ also in the form

$$\exp\left\{\frac{1}{2}(z, \overline{u}) \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} \begin{pmatrix} z^t \\ \overline{u}^t \end{pmatrix}\right\}, \tag{2.10}$$

where z denotes the row $(z_1 \quad z_2 \quad \dots)$ and $\overline{u}$ the row $(\overline{u}_1 \quad \overline{u}_2 \quad \dots)$, and z^t and $\overline{u}^t$ denote the transpose matrices of z and $\overline{u}$.

Remark. The matrix expression (2.10) is convenient to use, but we should remember that it is literally true only in the case when $H = l_2$ (or, what is the same, when an orthonormal basis is fixed in H).

Theorem 2.2 ([15], [16]; also [4], [17], [18]).

(a) *The conditions* 1*–4* *are necessary for the boundedness of the operator with symbol* (2.9).

(b) *If* 1*–4* *hold and if* $\|S\| < 1$, *then the operator* $B[S]$ *is bounded.*

(c) *If* 1*–4* *hold and if the matrices* K *and* M *are trace-class (or nuclear) operators, then* $B[S]$ *is bounded.*

(d) *Consider the bounded operators*

$$B[S_1] \colon F(H_2) \to F(H_1), \qquad B[S_2] \colon F(H_3) \to F(H_2),$$

where

$$S_1 = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}, \qquad S_2 = \begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix}.$$

Then

$$B[S_1]B[S_2] = \det\left[(1 - MP)^{-\frac{1}{2}}\right] B[S_1 * S_2], \tag{2.11}$$

where the $*$-multiplication of the matrices is defined by the formula

$$S_1 * S_2 = \begin{pmatrix} K + LP(E-MP)^{-1}L^t & L(E-PM)^{-1}Q \\ Q^t(E-MP)^{-1}L^t & R + Q^t(E-MP)^{-1}MQ \end{pmatrix}. \tag{2.12}$$

Behind the awkward formula (2.12) is a simple algebraic structure, which we now describe.

As before, suppose that H is a Hilbert space with involution. We consider the Hilbert space

$$V(H) := H \oplus H$$

and the subspaces

$$H^+ = H \oplus 0, \qquad H^- = 0 \oplus H$$

of it. Let $\Gamma(S)$ be the graph of the operator S as an operator

$$S\colon H_1^+ \oplus H_2^- \to H_1^- \oplus H_2^+.$$

It turns out that the subspaces $\Gamma(S)$ multiply like linear relations (correspondences).

Theorem 2.3. *The set $\Gamma(S_1 * S_2) \subset V(H_1) \oplus V(H_3)$ consists of all the (v_1, v_3) in $V(H_1) \oplus V(H_3)$ for which there exists a $v_2 \in V(H_2)$ such that $(v_1, v_2) \in \Gamma(S_1)$ and $(v_2, v_3) \in \Gamma(S_2)$.*

This assertion was obtained in [15] and [16]; see [4], [18] for more details; see also [17], [19] for the finite-dimensional case.

2.7. Remark: the harmonic representation of the symplectic group. We consider the group G of all operators on $V(H)$ of the form

$$g = \begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix}$$

such that Ψ is a Hilbert–Schmidt operator and such that

$$g^t \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} g = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix},$$

that is, such that the skew-symmetric form $\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ is preserved. The group G is the so-called *automorphism group of the canonical commutation relations.* If the dimension of H is finite and equal to n, then G is isomorphic to the real symplectic group $\mathrm{Sp}(2n, \mathbb{R})$.

We associate with a matrix $g \in G$ the operator (see [3])

$$\rho(g) = \det{}^{-\frac{1}{2}}(\Phi^*\Phi) B \begin{bmatrix} \overline{\Psi}\Phi^{-1} & \Phi^{-1} \\ \Phi^{t-1} & \Phi^{-1}\Psi \end{bmatrix} \tag{2.13}$$

acting in $F(H)$. The formulae (2.11) and (2.12) imply that

$$\rho(g_1)\rho(g_2) = \lambda(g_1, g_2)\rho(g_1 g_2),$$

where $\lambda(g_1, g_2) \in \mathbb{C}$ and $|\lambda(\cdot,\cdot)| = 1$. In other words, the operators $\rho(g)$ form a projective unitary representation of the group G. This representation is called the harmonic representation, as well as the Weil representation, the Segal–Shale–Weil representation, and the Friedrichs–Segan–Berezin–Shale–Weil oscillator representation; for more details see [3], [4], [18].

2.8. Remark: the Hilbert–Schmidt condition for Gaussian operators.

Proposition 2.4. *Suppose that the matrix* $S = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ *is such that* $\|S\| < 1$, *the blocks* K *and* M *are trace-class operators, and the block* L *is a Hilbert–Schmidt operator. Then* $B[S]$ *is a Hilbert–Schmidt operator.*

Proof. Let us compute $B[S]^*B[S]$ by the formulae (2.11) and (2.12). We obtain an operator of the form $\sigma \cdot B[T]$, where T is a trace-class matrix, and $\|T\| < 1$. According to § 6.3 in [4], the operator $B[T]$ can be reduced to the form $B\begin{bmatrix} 0 & R \\ \overline{R} & 0 \end{bmatrix}$ by conjugation with an operator of the form (2.13) in the symplectic group, where $\|R\| < 1$ and R is a selfadjoint trace-class matrix. We reduce R to the canonical form $R = U\Lambda U^{-1}$, where U is a unitary matrix and Λ is a diagonal matrix with eigenvalues $\lambda_1, \lambda_2, \dots$ ($|\lambda_j| < 1$, $\sum |\lambda_j| < \infty$). Let us consider the operator $T(U)$ acting in $F(H)$ according to the formula

$$T(U)f(z) = f(Uz).$$

Then

$$T(U)B\begin{bmatrix} 0 & R \\ \overline{R} & 0 \end{bmatrix} T(U)^{-1} = B\begin{bmatrix} 0 & \Lambda \\ \Lambda & 0 \end{bmatrix}.$$

It is obvious that the last operator is of trace class (this is the operator $f(z) \mapsto f(\Lambda z)$ and its eigenvalues are $\lambda_1^{\alpha_1} \dots \lambda_k^{\alpha_k}$).

2.9. Remark: non-homogeneous Gaussian operators. In addition to the Gaussian operators $B[S]$ we often encounter operators with symbols of the form

$$\exp\left\{\frac{1}{2}\begin{pmatrix} z & \overline{u} \end{pmatrix}\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}\begin{pmatrix} z^t \\ \overline{u}^t \end{pmatrix} + z \cdot \varkappa^t + \overline{u} \cdot \lambda^t\right\}, \tag{2.14}$$

where the matrix $S = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ is the same as before, $\varkappa \in H_1$ and $\lambda \in H_2$. We denote the operator with symbol (2.14) by

$$B[S \mid \sigma^t] = B\left[\begin{matrix} K & L \\ L^t & M \end{matrix}\,\middle|\,\begin{matrix} \varkappa^t \\ \lambda^t \end{matrix}\right].$$

A product of such operators is computed according to the formula

$$\begin{aligned} &B\left[\begin{matrix} K & L \\ L^t & M \end{matrix}\,\middle|\,\begin{matrix} \varkappa^t \\ \lambda^t \end{matrix}\right] \cdot B\left[\begin{matrix} P & Q \\ Q^t & R \end{matrix}\,\middle|\,\begin{matrix} \alpha^t \\ \sigma^t \end{matrix}\right] \\ &\quad = \lambda \cdot B\left[\begin{pmatrix} K & L \\ L^t & M \end{pmatrix} * \begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix}\,\middle|\,\begin{matrix} (1-KR)^{-1}(K\sigma^t + \varkappa^t) + \alpha^t \\ (1-RK)^{-1}(R\varkappa^t + \sigma^t) + \lambda^t \end{matrix}\right], \end{aligned} \tag{2.15}$$

where the $*$-product is computed according to the formula (2.12) and

$$\lambda = \det\left[(1-MP)^{-\frac{1}{2}}\right] \exp\left\{\frac{1}{2}\begin{pmatrix} \lambda & \alpha \end{pmatrix}\begin{pmatrix} -M & E \\ E & -P \end{pmatrix}^{-1}\begin{pmatrix} \lambda^t \\ \alpha^t \end{pmatrix}\right\}. \tag{2.16}$$

Behind these formulae there is also a simple algebraic structure: the operators $B[S \mid \sigma^t]$ are labelled by affine subspaces of $V(H_1) \oplus V(H_2)$; see [4], [16].

2.10. The action of the group of affine isometries. An *affine isometry* of a Hilbert space H is defined to be a transformation of the form

$$Ah = Uh + b, \tag{2.17}$$

where U is a unitary operator and $b \in H$. The group of all affine isometries of H is denoted by $\mathrm{Isom}(H)$.

We associate with an affine isometry (2.17) the unitary operator

$$T(A) = T(U, b) \tag{2.18}$$

on $F(H)$ given by the formula

$$T(U,b)f(z) = \lambda \cdot f(Uz + b)\exp\{-\langle z, U^*b\rangle\},$$

where

$$\lambda = \exp\Big\{-\frac{1}{2}\langle b, b\rangle\Big\}. \tag{2.19}$$

A direct check shows that

$$T(\widetilde{U}, \tilde{b})T(U,b) = \varkappa \cdot T(U\widetilde{U}, U\tilde{b} + b), \tag{2.20}$$

where

$$\varkappa = \exp\{i \,\mathrm{Im}\langle \tilde{b}, U^{-1}b\rangle\}, \tag{2.21}$$

that is, we get a unitary projective representation of the group $\mathrm{Isom}(H)$.

Computing the symbol (kernel) of the operator according to the formula (2.8), we get that

$$T(U,b) = \lambda \cdot B\left[\begin{matrix} 0 & U \\ U^t & 0 \end{matrix}\middle| \begin{matrix} \overline{-U^{-1}b} \\ b \end{matrix}\right]. \tag{2.22}$$

The group $\mathrm{Isom}(H)$ contains, in particular, the additive group of H acting on H by parallel translations. The operators

$$T(1,b) = \exp\Big\{-\frac{1}{2}\|b\|^2\Big\}\, B\left[\begin{matrix} 0 & E \\ E & 0 \end{matrix}\middle| \begin{matrix} -\bar{b} \\ b \end{matrix}\right] \tag{2.23}$$

form a projective unitary representation of the additive group of H or a linear representation of the so-called Heisenberg group (see, for example, [3], [4]).

§ 3. The simplest properties of the boson–Poisson correspondence

3.1. Definition of the correspondence (see [4] for more details). Suppose that M is a Lebesgue space with a finite or σ-finite measure. We consider the Fock space $F(L^2(M))$ and the system of functions

$$\Phi_h \in F(L^2(M)),$$

where $h \in L^2(M)$. Let us recall that by definition

$$\langle \Phi_h, \Phi_{h'} \rangle_{F(L^2(M))} = \exp\left\{ \int_M h'(m)\overline{h(m)}\, d\mu(m) \right\}.$$

In $L^2(\Omega(M))$ we consider the system of functions

$$\Psi_h(\omega) = \prod_{m_j \in \omega} (1 + \overline{h}(m_j)) \exp\left\{ - \int_M \overline{h}(m)\, d\mu(m) \right\}. \tag{3.1}$$

The formal use of Campbell's formula gives us that

$$\langle \Psi_h, \Psi_{h'} \rangle_{L^2(\Omega(M))} = \exp\left\{ \int_M h'(m)\overline{h(m)}\, d\mu(m) \right\}. \tag{3.2}$$

We get a canonical unitary isomorphism between $F(L^2(M))$ and $L^2(\Omega(M))$: corresponding to each function $\Phi_h \in F(L^2(M))$ is the function $\Psi_h \in L^2(\Omega(M))$.

Remark. Here it is necessary to exercise some caution. The product (3.1) converges if $h \in L^1(M)$, and (3.2) follows from Campbell's formula if $h, h' \in L^1 \cap L^2(M)$. However, (3.2) implies (see [4], § 10.4) that the map $h \mapsto \Psi_h$, defined on $L^1 \cap L^2(M)$, is continuous in the topology of $L^2(M)$ and it extends by continuity to a map $L^2(M) \to L^2(\Omega(M))$. Therefore, we can assume that Ψ_h is defined for all h in $L^2(M)$ and we can regard the expression (3.2) as formal.

3.2. Formal integral expressions. Suppose that $f^{[p]} \in L^2(\Omega(M))$. Then in view of (2.2) the corresponding function $f^{[b]} \in F(L^2(M))$ is given by the formula

$$\begin{aligned} f^{[b]}(h) &= \langle f^{[p]}, \Psi_h \rangle_{L^2(\Omega(M))} \\ &= \int_{\Omega(M)} f^{[p]}(\omega) \left[\prod_{m_j \in \omega} (1 + h(m_j)) \exp\left\{ - \int_M h(m)\, d\mu(m) \right\} \right] d\nu(\omega). \end{aligned} \tag{3.3}$$

The formula (3.3) appears outwardly to be more constructive than the definition in § 3.1. But the definition with a correspondence of the overcomplete bases is actually more convenient.

Suppose now that M is a finite space and let $f^{[b]} \in F(L^2(M))$. Then by the reproducing property (2.4) and the integral expression (2.5) for the inner product, the function $f^{[p]} \in L^2(\Omega(M))$ corresponding to $f^{[b]}$ can be written as the integral

$$\begin{aligned} f^{[p]}(\omega) = \int_{L^2(M)} f^{[b]}(h) &\left[\prod_{m_j \in \omega} (1 + h(m_j)) \exp\left\{ - \int_M f(m)\, d\mu(m) \right\} \right] \\ &\times \exp(-\|h\|^2) \prod_j \left(\frac{a_j}{\pi}\, dh_j\, d\overline{h}_j \right). \end{aligned} \tag{3.4}$$

It is natural to think that in the case of infinite M the integration must be carried out over a canonical extension of the space $L^2(M)$ (the 'abstract Wiener space'); see, for example, [1] and [4], § 6.1. However, in this case h is a generalized function on M, and then it is not clear what is meant by the infinite product in (3.4). In any case we do not know of a nice interpretation of (3.4) for infinite M.

3.3. The case of a one-point space M. Suppose that M consists of a single point with measure a.

Then $F(L^2(M))$ is the space of holomorphic functions on $\mathbb{C}$ with the inner product

$$\langle f_1, f_2\rangle = \int_{\mathbb{C}} f_1(z)\overline{f_2(z)}\exp(-a|z|^2)\left(\frac{a}{\pi}\,dz\,d\overline{z}\right).$$

Corresponding to the function $h = \text{const}$ on M is the vector

$$\Phi_h(z) = \exp(a\overline{h}z) \in F(L^2(M)).$$

The space $L^2(\Omega(M))$ is the space of sequences $v(p)$, $p = 0, 1, 2, \ldots$, with the inner product

$$\langle v, w\rangle_{L^2(\Omega(M))} = \sum_{p=0}^{\infty} v(p)\overline{w(p)}\,\frac{a^p}{p!}\,e^{-a}. \tag{3.5}$$

The functions (sequences) Ψ_h are given by the formula

$$\Psi_h(p) = (1+\overline{h})^p e^{-a\overline{h}}.$$

Let us see what in $L^2(\Omega(M))$ corresponds to the standard orthogonal basis (non-normalized) $z^n \in F(L^2(M))$. We expand $\Phi_{\overline{h}}$ and $\Psi_{\overline{h}}$ in a series with respect to h:

$$\Phi_{\overline{h}}(z) = \sum_{n=0}^{\infty} \frac{a^n z^n}{n!}\overline{h}^n,$$

$$\Psi_{\overline{h}}(p) = \sum_{n=0}^{\infty} h^n\left\{\sum_{j=0}^{p} \frac{p(p-1)\cdots(p-j+1)}{j!(n-j)!}(-a)^{n-j}\right\}.$$

We see that corresponding to the vector $z^n \in F(L^2(M))$ is the Charlier polynomial

$$S_n(p) = S_n(a \mid p) = \sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}a^{-j}p(p-1)\cdots(p-j+1). \tag{3.6}$$

This was to be expected, because the Charlier polynomials form a standard orthogonal basis in the space of sequences with the inner product (3.5) (see, for example, [20]).

Let us next consider the sequence $\delta_\alpha(p) \in L^2(\Omega(M))$ (that is, there is a 1 at the position with index α and 0's elsewhere).

With the help of (2.4) we define a corresponding function $e_\alpha(z)$ in $F(L^2(M))$:

$$\begin{aligned} e_\alpha(z) &= \langle \delta_\alpha, \Psi_z\rangle_{L^2(\Omega(M))} = \Psi_z(\alpha)\frac{a^\alpha}{\alpha!}e^{-a} \\ &= (1+z)^\alpha e^{-az}\frac{a^\alpha}{\alpha!}e^{-a} = \frac{[a(1+z)]^\alpha}{\alpha!}e^{-a(1+z)}, \end{aligned}$$

that is, we obtain the correspondence

$$\delta_\alpha \longleftrightarrow e_\alpha(z) = \frac{[a(1+z)]^\alpha}{\alpha!}e^{-a(1+z)}. \tag{3.7}$$

Further, we consider in $F(L^2(M))$ the operators

$$R^{[b]}f(z) := zf(z), \qquad Q^{[b]}f(z) := \frac{\partial}{\partial z}f(z)$$

(the so-called creation and annihilation operators; see [3], [6]).

Let $R^{[p]}$ and $Q^{[p]}$ be the corresponding operators on $L^2(\Omega(M))$. It is easy to see that

$$(R^{[b]}+E)e_\alpha = \frac{\alpha+1}{a}e_{\alpha+1}, \qquad (Q^{[b]}+a)e_\alpha = ae_{\alpha-1}.$$

Hence,

$$R^{[p]}\delta_\alpha = -\delta_\alpha + \frac{\alpha+1}{a}\delta_{\alpha+1},$$
$$Q^{[p]}\delta_\alpha = -a\delta_\alpha + a\delta_{\alpha-1}.$$

3.4. The case of a finite space M. Assume now that M is a union of one-point spaces (points) $M_1,\dots,M_N$ with measures $a_1,\dots,a_N$. Then

$$F(L^2(M)) = \bigotimes_{j=1}^N F(L^2(M_j)),$$
$$L^2(\Omega(M)) = \bigotimes_{j=1}^N L^2(\Omega(M_j)).$$

Therefore, the correspondences in § 3.3 immediately imply the correspondences

$$\prod_{j=1}^N z_j^{n_j} \longleftrightarrow \prod_{j=1}^N S_{n_j}(a_j \mid p_j),$$
$$\prod_{j=1}^N \frac{[a_j(1+z_j)]^{\alpha_j}}{\alpha_j!}e^{-a_j(1+z_j)} \longleftrightarrow \prod_{j=1}^N \delta_{\alpha_j}(p_j). \tag{3.8}$$

3.5. Multiplication by a function. Let M be an arbitrary measure space and let $\sigma(m)$ be a function on M satisfying the conditions

1) $|\sigma(m)| \leqslant 1$,
2) $(\sigma(m)-1) \in L^1(M)$.

We consider the operator $C(\sigma)$ acting in $L^2(\Omega(M))$ according to the formula

$$C(\sigma)f(\omega) = \Bigl(\prod_{m_j\in\omega}\sigma(m_j)\Bigr)f(\omega).$$

It is easy to see that $\|C(\sigma)\| \leqslant 1$; if $|\sigma(m)| = 1$, then $C(\sigma)$ is unitary. Obviously,

$$C(\sigma_1\sigma_2) = C(\sigma_1)C(\sigma_2).$$

A simple computation shows that

$$C(\sigma)\Psi_h = \Psi_{\overline{\sigma}h+\overline{\sigma}-1} \cdot \exp\left\{-\int_M \big(\sigma(m)-1\big)\big(h(m)+1\big)\,d\mu(m)\right\}.$$

Let $C^{[b]}(\sigma)$ be the corresponding operator on $F(H)$. By using (2.8) it is easy to get a formula for the symbol (kernel) of $C^{[b]}(\sigma)$:

$$\exp\left\{\int_M \big[\sigma z u + z(\sigma-1) - (\sigma-1)\overline{u} - (\sigma-1)\big]\,d\mu(u)\right\},$$

where $z, u \in L^2(M)$. We remark that we have obtained operators of the form $B[S \mid \sigma^t]$ (see § 2.9).

Let us now consider the (multiplicative) group $\mathcal{A}$ consisting of functions σ on M such that $|\sigma(m)| = 1$ and $\sigma(m) - 1 \in L^1(M)$. The group $\mathcal{A}$ acts on the Hilbert space $L^2(M)$ as affine isometries of the form

$$A(\sigma)z(m) = \sigma(m)z(m) + \sigma(m) - 1.$$

Restricting the representation T of the group $\mathrm{Isom}(L^2(M))$ (see § 2.10) to the group $\mathcal{A}$, we obtain a projective unitary representation $T(A(\sigma))$ of $\mathcal{A}$.

Comparison of the symbols shows that

$$C^{[b]}(\sigma) = \exp\left\{-i\cdot \mathrm{Im}\int_M \big(\sigma(m)-1\big)\,d\mu(m)\right\} T(A(\sigma)), \tag{3.9}$$

that is, the operators $C^{[b]}(\sigma)$ and $T(A(\sigma))$ coincide up to a factor. Thanks to this factor the projective representation $T(A(\sigma))$ of $\mathcal{A}$ 'is straightened' into a linear representation $C^{[b]}(\sigma)$.

3.6. The action of the group $\mathbf{Gms}_\infty$. Let M be a measure space with an infinite continuous measure (see § 1.8). We consider the group Gms_∞ and the unitary representation $R(g)$ of it on $L^2(\Omega(M))$ given by the formula

$$R(g)f(\omega) = f(g\omega)\left[\prod_{m_j\in\omega} g'(m_j)^{1/2}\right]\exp\left\{-\frac{1}{2}\int_M \big(g'(m)-1\big)\,d\mu(m)\right\}.$$

It is easy to see that

$$\begin{aligned} R(g)\Psi_h &= \Psi_{S(g)h} \\ &\times \exp\left\{\int_M \left[h(g(m))g'(m)^{1/2} + g'(m)^{1/2} - \frac{1}{2}g'(m) + \frac{1}{2} - \overline{h}(m)\right] d\mu(m)\right\}, \end{aligned}$$

where

$$S(g)h(m) = h(g(m))g'(m)^{1/2} + g'(m)^{1/2} - 1. \tag{3.10}$$

This enables us to write out the symbol (kernel) of the operator $R^{[b]}(g)$ on $F(L^2(M))$ corresponding to the operator $R(g)$ (we omit the formula).

We now observe that the formula (3.10) determines an affine isometric action of the group Gms_∞ on the space $L^2(M)$. Restricting the representation T of the group $\mathrm{Isom}(L^2(M))$ to the subgroup Gms_∞, we obtain a unitary representation $T(S(g))$ of Gms_∞ on $F(L^2(M))$.

Convolution of the symbols (kernels) shows that

$$R^{[b]}(g) = T(S(g)).$$

In other words, the boson–Poisson correspondence is an operator intertwining the representations $R(g)$ and $T(S(g))$ of Gms_∞. It was revealed in this form (to within some slight stipulations) in [8]–[10].

Remark. The construction of the preceding subsection can also be regarded as the motivation for the boson–Poisson correspondence. We have a representation of the Abelian group $\mathcal{A}$ on $F(L^2(M))$ given by the formula $T(A(\sigma))$ (more precisely, the formula (3.9)). It is natural to expect that this representation can be realized on some space $L^2(X)$ by operators of multiplication by functions. The space $\Omega(M)$ is exactly this X (see the discussion of a similar construction in [7]).

§ 4. Gaussian vectors and the functions $\mathfrak{R}[K,\theta](\omega)$

4.1. The functions $\mathfrak{R}[K,\theta](\omega)$. We consider a configuration $\omega \in \Omega(M)$. A *tangle* R in ω is defined to be a finite (possibly empty) collection $\{\delta_1, \sigma_1\}, \dots, \{\delta_q, \sigma_q\}$ of unordered two-point subsets of ω (we write $\{\delta_j, \sigma_j\} \in R$). We let $[\omega]_R$ be the set of all elements of ω not appearing in the tangle R. The set of all tangles in ω will be denoted by $\tan(\omega)$.

Remark. To avoid ambiguity we must specify what the word "tangle" means in the case of configurations with multiple points. We suppose that k points of a configuration are concentrated at a point of multiplicity k, and that any of them can occur on an equal footing in a tangle. For example, the total number of tangles on an n-point configuration (multiplicity is taken into account for the points) is equal to

$$\sum_{j:\, j \geqslant 0,\, 2j \leqslant n} \frac{n!}{2^j \cdot j!\,(n-2j)!}$$

and this number does not depend on whether the configuration has multiple points.

Let us fix a function $K(m, m')$ on $M \times M$ satisfying the condition $K(m,m') = K(m',m)$, and a function $\theta(m)$ on M. We consider the function on $\Omega(M)$ given by the formula

$$\mathfrak{R}[K,\theta](\omega) = \sum_{R \in \tan(\omega)} \Bigl[\prod_{\{\delta_j, \sigma_j\} \in R} K(\delta_j, \sigma_j) \cdot \prod_{m_j \in [\omega]_R} (1 + \theta(m_j)) \Bigr] \tag{4.1}$$

(provided that this formula makes sense). These functions are the basic subject of our paper.

4.2. The case of a finite space M. Suppose that the space M is finite and consists of points $m_1, \dots, m_N$ with measures $a_1, \dots, a_N$. Then the function $K(m, m')$ can be regarded as the matrix

$$\varkappa_{ij} = K(m_i, m_j),$$

and θ as the column vector $\theta_i = \theta(m_i)$.

Theorem 4.1. *Suppose that the quadratic form*

$$\frac{1}{2} \sum \varkappa_{ij} x_i x_j$$

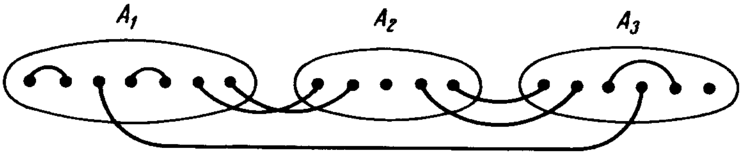
is strictly less than 1 *on the ellipsoid* $\sum a_j x_j^2 \leqslant 1$. *Then*

$$\mathfrak{R}[K, \theta](\omega) \in L^2(\Omega(M))$$

and the corresponding vector in $F(L^2(M))$ *is given by the formula*

$$\exp\left\{ \frac{1}{2} \sum_{i,j} \varkappa_{ij}(z_i + 1)(z_j + 1) a_i a_j + \sum_j \theta_j a_j (z_j + 1) \right\}. \tag{4.2}$$

Proof. Let $\omega \in \Omega(M)$ be a configuration. We recall that ω can be regarded as a point $(p_1, \dots, p_N) \in \mathbb{Z}_+^N$. It is convenient to represent the tangles in ω as pictures of the form



Here $N = 3$, the closed ovals represent the points $m_1, m_2, \dots$, and p_j points of the configuration are concentrated at the point m_j. The elements of the two-point sets of the tangle are joined by arcs.

Let s_{ij} be the number of arcs leading from the ith oval to the jth oval ($s_{ij} = s_{ji}$) and let q_i be the number of points in the ith oval that are not end-points of arcs. We note the obvious equalities

$$p_i = \sum_{j \neq i} s_{ij} + q_i + 2s_{ii}. \tag{4.3}$$

Then for the given configuration the number of tangles with given s_{ij} and q_i (of course, the equalities (4.3) are assumed) is equal to

$$\left[\prod_i \frac{p_i!}{\prod_{j \neq i} s_{ij}! \, (2s_{ii})! \, q_i!} \right] \times \prod_{i<j} (s_{ij})! \times \prod_i \frac{(2s_{ii})!}{s_{ii}! \, 2^{s_{ii}}}. \tag{4.4}$$

The first factor of this product corresponds to partitions of the sets A_i encircled by the ovals into subsets of s_{ij}, q_i and $2s_{ii}$ elements. The second factor is the number of bijections between the distinguished subsets of A_i and A_j. The last factor corresponds to partitions of the $2s_{ii}$-element sets into pairs.

The term

$$\prod K(\sigma_j, \delta_j) \prod (1 + \theta(m_i))$$

in the sum (4.1) corresponding to a tangle with a given collection s_{ij}, q_i is equal to

$$\prod_{i \leqslant j} \varkappa_{ij}^{s_{ij}} \cdot \prod_i (1 + \theta_i)^{q_i}.$$

Therefore,

$$\begin{aligned} \mathfrak{R}[K,\theta](p_1, \dots, p_N) &= \mathfrak{R}[K,\theta](\omega) \\ &= \sum \left[\frac{\prod p_i!}{\prod_{i \leqslant j} s_{ij}! \prod_i q_i! \prod_i 2^{s_{ii}}} \cdot \prod_{i \leqslant j} \varkappa_{ij}^{s_{ij}} \prod_i (1 + \theta_i)^{q_i} \right], \end{aligned} \tag{4.5}$$

where the summation is over all the collections s_{ij}, q_i satisfying (4.3) (we have made the obvious cancellations in (4.4)). We now use (3.8) to compute the image of the function $\mathfrak{R}[K,\theta]$ in boson Fock space. This image is equal to

$$\begin{aligned} \sum_{(p_1, \dots, p_N) \in \mathbb{Z}_+^N} \Bigg\{ &\left[\sum \frac{\prod p_i!}{\prod_{i \leqslant j} s_{ij}! \prod_i q_i! \prod_i 2^{s_{ii}}} \prod_{i \leqslant j} \varkappa_{ij}^{s_{ij}} \prod_i (1 + \theta_i)^{q_i} \right] \\ &\times \prod_i \left(\frac{(a_i(z_i + 1))^{p_i}}{p_i!} \exp(-a_i(z_i + 1)) \right) \Bigg\}, \end{aligned} \tag{4.6}$$

where the summation in the square brackets is over all collections q_i, s_{ij} satisfying (4.3). Substituting (4.3) in place of p_i, we transform our expression to the form

$$\begin{aligned} \sum_{s_{ij}, q_i \in \mathbb{Z}_+} &\left\{ \prod_{i<j} \frac{\left[\varkappa_{ij} a_i a_j (z_i + 1)(z_j + 1) \right]^{s_{ij}}}{s_{ij}!} \times \prod_i \frac{\left[\varkappa_{ii} a_i^2 (z_i + 1)^2 / 2 \right]^{s_{ii}}}{s_{ii}!} \right. \\ &\left. \times \prod_i \frac{\left[(1 + \theta_i) a_i (z_i + 1) \right]^{q_i}}{q_i!} \right\} \times \prod_i \exp(-a_i(z_i + 1)) \\ = &\prod_{i<j} \exp\{ \varkappa_{ij} a_i a_j (z_i + 1)(z_j + 1) \} \times \prod_i \exp\left\{ \frac{1}{2} \varkappa_{ii} a_i (z_i + 1)^2 \right\} \\ &\times \prod_i \exp\{ (1 + \theta_i) a_i (z_i + 1) \} \times \prod_i \exp\{ -a_i (z_i + 1) \}, \end{aligned} \tag{4.7}$$

which coincides with the desired expression.

We have gone through the computations without considering the convergence of the series in the Hilbert spaces $F(L^2(M))$ and $L^2(\Omega(M)) \simeq l_2(\mathbb{Z}_+^N)$; we have only checked for coordinatewise convergence in $l_2(\mathbb{Z}_+^N)$ and pointwise convergence in the space of holomorphic functions on $\mathbb{C}^N$. However, we performed the computation while identifying orthogonal bases in $F(L^2(M))$ and $L^2(\Omega(M)$. Therefore, in the case of $\mathfrak{R} \in L^2(\Omega(M))$ we must get the right answer in $F(L^2(M))$, while in the case when (4.2) lies in $F(L^2(M))$ the expression (4.6) is its expansion in the orthogonal basis, so that (4.5) lies in $\Omega(L^2(M))$.

Remark. The condition of convergence of the series in the Hilbert space played an insignificant role in our proof. We see below in § 6 that the correspondence formula (4.2) can remain in force even when the convergence condition is not satisfied.

Theorem 4.1′. *Suppose that $\varkappa_{ij} = K(m_i, m_j)$ satisfies the same conditions as in Theorem* 4.1. *Then corresponding to the function*

$$\exp\left\{\frac{1}{2}\sum \varkappa_{ij}a_i a_j z_i z_j + \sum \theta_j a_j z_j\right\} \in F(L^2(M))$$

is the function

$$\mathfrak{R}^*[K,\theta](\omega) := \exp\left\{\frac{1}{2}\sum \varkappa_{ij}a_i a_j - \sum \theta_j a_j\right\}\mathfrak{R}[K,\tilde\theta](\omega) \in L^2(\Omega(M)),$$

where

$$\tilde\theta_j = \theta_j - \sum_i \varkappa_{ij}a_i.$$

Proof. This is a reformulation of Theorem 4.1.

Remark. Theorem 4.1′ is not hard to prove directly without referring to Theorem 4.1. To do this it suffices to expand the Gaussian vector $b[K \mid \theta^t]$ with respect to the orthogonal system $\prod e_{\alpha_j}(z_j)$ (see (3.9)), and this is not hard (it is convenient first to perform a Heisenberg shift corresponding to the function $-1 \in L^2(M)$ (see § 2.10)).

4.3. Some 'integrals'. Suppose, as before, that the space M is finite, with the measures of the points of M equal to $a_1, \dots, a_N$. Let K and L be functions on $M \times M$ satisfying the conditions of Theorem 4.1. We regard these functions as $N \times N$ matrices. Let θ and λ be functions on M, regarded as row matrices. We denote by 1 both the function equal to 1 on M and the corresponding row matrix. Let

$$D = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{pmatrix}.$$

We introduce the notation

$$\tilde\theta = \theta \cdot D^{1/2}, \qquad \tilde\lambda = \lambda \cdot D^{1/2}, \qquad \tilde 1 = 1 \cdot D^{1/2},$$
$$\widetilde K = D^{1/2} K D^{1/2}, \qquad \widetilde L = D^{1/2} L D^{1/2}.$$

Using the formula (2.7), we get that

$$\int_{\Omega(M)} \mathfrak{R}^*[K,\theta](\omega)\overline{\mathfrak{R}^*[L,\lambda](\omega)}\, d\nu(\omega)$$

$$= \det\left[\left(1 - \widetilde K\overline{\widetilde L}\right)^{-1/2}\right]\exp\left\{\frac{1}{2}\begin{pmatrix}\tilde\theta & \bar{\tilde\lambda}\end{pmatrix}\begin{pmatrix} -\widetilde K & E \\ E & -\overline{\widetilde L}\end{pmatrix}^{-1}\begin{pmatrix}\tilde\theta^t \\ \bar{\tilde\lambda}^t\end{pmatrix}\right\}. \tag{4.8}$$

Similarly,

$$\int_{\Omega(M)} \mathfrak{N}[K,\theta](\omega)\overline{\mathfrak{N}[L,\lambda](\omega)}\,d\nu(\omega) = \det\left[\left(1-\widetilde{K}\overline{\widetilde{L}}\right)^{-1/2}\right]$$
$$\times \exp\left\{-\sum a_j + \frac{1}{2}\begin{pmatrix}\tilde\theta+\tilde 1 & \bar{\tilde\lambda}+\tilde 1\end{pmatrix}\begin{pmatrix}-\widetilde{K} & E\\ E & -\overline{\widetilde{L}}\end{pmatrix}^{-1}\begin{pmatrix}(\tilde\theta+\tilde 1)^t\\ (\bar{\tilde\lambda}+\tilde 1)^t\end{pmatrix}\right\} \tag{4.9}$$

(to see this formula we must compute the inner product of two vectors in $F(L^2(M))$ of the form (4.2); for this we make the change of variables $z' = z+1$ and use (2.7)).

Remark. Theorem 4.1 is precisely equivalent to the formula (4.9) with $L = 0$. Campbell's formula is obtained if we set $K = L = 0$ and $\lambda = 0$ in (4.9). We note also the special case of these formulae with $L = 0$ and $\lambda = 0$:

$$\int_{\Omega(M)} \mathfrak{N}^*[K,\theta](\omega)\,d\nu(\omega) = 1,$$
$$\int_{\Omega(M)} \mathfrak{N}[K,\theta](\omega)\,d\nu(\omega) = \exp\left\{\frac{1}{2}\sum \varkappa_{ij}a_ia_j + \sum \theta_j a_j\right\}.$$

Remark. A natural question is whether it is possible to compute the integrals (4.8) and (4.9) (of course, these are not integrals but multiple series) directly, without recourse to the boson–Poisson correspondence. The corresponding computation is quite long and complicated; in the end the series turn into Gaussian integrals.

4.4. The case of infinite spaces. Suppose now that the Lebesgue space M is arbitrary. For a function $K(m,m') \in L^2(M\times M)$ with $K(m,m') = K(m',m)$ we consider the integral operator

$$\mathcal{K}f(m) = \int_M K(m,m')f(m')\,d\mu(m'). \tag{4.10}$$

Assume that K is such that the norm of the operator $\mathcal{K}$ on $L^2(M)$ is less than 1 and let $\theta \in L^2(M)$. We take the function

$$\mathfrak{N}[K,\theta](\omega).$$

One would like to think (see Theorem 4.1) that the corresponding element of the space $F(L^2(M))$ is given by the formula

$$\beta(z) = \exp\left\{\frac{1}{2}\langle\mathcal{K}(z+1),\overline{(z+1)}\rangle_{L^2(M)} + \langle\theta,\overline{(z+1)}\rangle_{L^2(M)}\right\}, \tag{4.11}$$

or, in expanded form,

$$\beta(z) = \exp\left\{\frac{1}{2}\int_M\int_M K(m,m')\big(z(m)+1\big)\big(z(m')+1\big)\,d\mu(m)\,d\mu(m')\right.$$
$$\left.+\int_M \theta(M)\big(z(m)+1\big)\,d\mu(m)\right\}, \tag{4.12}$$

where $z(m) \in L^2(M)$.

Similarly, we consider the following Gaussian vector in $F(L^2(M))$:

$$b[\mathcal{K} \mid \theta](z) = \exp\left\{\frac{1}{2}\langle \mathcal{K}z, \overline{z}\rangle + \langle \theta, \overline{z}\rangle\right\}$$
$$= \exp\left\{\frac{1}{2}\int_M\int_M K(m,m')z(m)z(m')\,d\mu(m)\,d\mu(m') + \int_M \theta(m)z(m)\,d\mu(m)\right\}. \tag{4.13}$$

One would like to think (see Theorem 4.1′) that corresponding to this vector is the function in $L^2(\Omega(M))$ given by

$$\mathfrak{R}^*[K,\theta](\omega) = \mathfrak{R}[K,\tilde{\theta}](\omega) \times \exp\left\{\frac{1}{2}\int_M\int_M K(m,m')\,d\mu(m)\,d\mu(m') - \int_M \theta(m)\,d\mu(m)\right\}, \tag{4.14}$$

where

$$\tilde{\theta}(m) = \theta(m) - \int_M K(m,m')\,d\mu(m'). \tag{4.15}$$

However, it is a striking fact that the integrals in (4.12), (4.14), and (4.15) are divergent in general. We prove the desired result under slight additional restrictions on the functions $K(m,m')$ and $\theta(m)$.

Theorem 4.2. *Suppose that $K(m,m') \in L^2 \cap L^1(M \times M)$, $\theta(m) \in L^2 \cap L^1(M)$, and $\|\mathcal{K}\| < 1$, and assume that the function*

$$\sigma_K(m) = \int_M K(m,m')\,d\mu(m') \tag{4.16}$$

lies in $L^2(M)$. Then:

(a) *the expression*

$$\mathfrak{R}[K,\theta](\omega) = \sum_{R\in\tan(\omega)}\left(\prod_{\{\sigma_j,\delta_j\}\in R} K(\sigma_j,\delta_j)\cdot\prod_{m_j\in[\omega]_R}(1+\theta(m_j))\right) \tag{4.17}$$

defining the function $\mathfrak{R}[K,\theta](\omega)$ converges absolutely almost everywhere on $\Omega(M)$. Furthermore, $\mathfrak{R}[K,\theta](\omega)$ lies in $L^2(\Omega(M))$ and corresponds to the Gaussian vector $\beta(z)$ given by (4.12);

(b) *the function $\mathfrak{R}^*[K,\theta](\omega)$ given by (4.14) corresponds to the Gaussian vector $b[K \mid \theta^t]$ given by (4.13).*

Remark. Theorem 4.2 looks outwardly like a theorem on pointwise convergence of the function series (4.17). However, the summation set in (4.17) depends on ω, and hence the individual terms of the series are not functions of ω. A kind of summation of all values of a multivalued function at the given point is being carried out. In reality it is possible to apply some 'force' and 'cut' the multivalued function into branches. To do this it suffices to assume that M is the half-line with a Stieltjes measure. Then almost all configurations become ordered sets equivalent to $\mathbb{Z}_+$, and we can speak of summation over tangles in $\mathbb{Z}_+$. Next, we could try to apply the monotone convergence theorem to the series (4.17). I do not feel that this path can prove to be simpler than the proof presented below.

Remark. The functions $\mathfrak{R}^*[K,\theta](\omega)$ make sense for any $\theta \in L^2(M)$ and K in $L^2(M \times M)$. This is so because they are images of Gaussian vectors under the boson–Poisson correspondence. They can be defined also as L^2-limits of cylindrical functions of the form $\mathfrak{R}^*[K_j,\theta_j]$ (see § 5.2 below).

Remark. Let us take a function $\theta > 0$ such that $\theta \in L^2 \setminus L^1$. It can be shown that $\mathfrak{R}[0,\theta](\omega) = +\infty$ a.e. This shows that the functions $\mathfrak{R}[K,\theta](\omega)$ can fail to exist for some $K,\theta \in L^2$. Therefore, the formula (4.14) for computing the function $\mathfrak{R}^*[K,\theta](\omega)$ is not always true.

Proposition 4.3. *Suppose that the functions $K(m,m')$, $\theta(m)$, $L(m,m')$, and $\lambda(m)$ satisfy the conditions of Theorem 4.2. Then:*

(a)

$$\int_{\Omega(M)} \mathfrak{R}[K,\theta](\omega)\overline{\mathfrak{R}[L,\lambda]}(\omega)\,d\nu(\omega) = \det\left[(E-\mathcal{K}\overline{\mathcal{L}})^{-1/2}\right]$$
$$\times \exp\left\{\frac{1}{2}\begin{pmatrix}\theta & \bar\lambda\end{pmatrix}\begin{pmatrix}-\mathcal{K} & E\\ E & -\overline{\mathcal{L}}\end{pmatrix}^{-1}\begin{pmatrix}\theta^t\\ \bar\lambda^t\end{pmatrix} + \begin{pmatrix}\theta & \lambda\end{pmatrix}\begin{pmatrix}-\mathcal{K} & E\\ E & -\overline{\mathcal{L}}\end{pmatrix}^{-1}\begin{pmatrix}1^t\\ 1^t\end{pmatrix}\right.$$
$$\left. + \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\left[\begin{pmatrix}-\mathcal{K} & E\\ E & -\overline{\mathcal{L}}\end{pmatrix}^{-1} - \begin{pmatrix}0 & E\\ E & 0\end{pmatrix}\right]\begin{pmatrix}1^t\\ 1^t\end{pmatrix}\right\}; \tag{4.18}$$

(b) *the inner products of $\mathfrak{R}^*[K,\theta]$ and $\mathfrak{R}^*[L,\lambda]$ are computed by* (2.7).

Remark. To avoid the possibility of ambiguity we underscore that in the expression (4.18) we are using notation of the type (2.9)–(2.10). The symbol 1 denotes the function on M identically equal to 1.

Remark. The formula (4.18) is the formula (4.9) rewritten in a worse form. However, thanks to this worsening of the expression the two (divergent, in general) terms $(-\mu(M))$ and $(+\mu(M))$ in (4.9) cancel.

§ 5. Proofs of Theorem 4.2 and Proposition 4.3

5.1. Continuity of the formula (4.18). First of all, we introduce on the space of functions K, θ a topology that is natural from the point of view of the statement of the theorem. On the set of functions θ we introduce the weakest topology in which the norms of the spaces $L^1(M)$ and $L^2(M)$ are continuous. On the set of functions K we take the weakest topology in which the norms of $L^1(M \times M)$ and $L^2(M \times M)$ are continuous together with the map into $L^2(M)$ given by the formula (4.16) (that is, $K \mapsto \sigma_K$).

We denote the right-hand side of the formula (4.18) by

$$\mathfrak{A}(K,\theta \mid L,\lambda).$$

Lemma 5.1. *The function $\mathfrak{A}(K,\theta \mid L,\lambda)$ is jointly continuous with respect to its variables.*

Proof. This is more or less obvious but we go through it exactly. The map $(\mathcal{K},\mathcal{L}) \mapsto \mathcal{K}\mathcal{L}$ from the space of Hilbert–Schmidt operators to the space of trace-class operators is jointly continuous (see [14], Problem 6.28). The determinant is continuous with respect to the trace-class topology, and this implies continuity of the factor $\det(\,\cdot\,)$.

The first term in the exponential in (4.18) has the form

$$\langle Ax, y\rangle, \tag{5.1}$$

where x and y are vectors in a Hilbert space (in the present case it is the space $L^2(M)\oplus L^2(M)$) and A is a bounded operator depending on $\mathcal{K}$ and $\mathcal{L}$. The expression (5.1) is jointly continuous in the variables x, y, A if the usual Hilbert space topology is introduced on x, y and the uniform topology on the set of operators. Thus, the continuity of this term is obvious.

We write the second term in the exponential by using the identities

$$\begin{pmatrix} -\mathcal{K} & E \\ E & -\overline{\mathcal{L}} \end{pmatrix}^{-1} = \begin{pmatrix} \overline{\mathcal{L}}(E-\mathcal{K}\overline{\mathcal{L}})^{-1} & (E-\overline{\mathcal{L}}\mathcal{K})^{-1} \\ (E-\mathcal{K}\overline{\mathcal{L}})^{-1} & \mathcal{K}(E-\overline{\mathcal{L}}\mathcal{K})^{-1} \end{pmatrix},$$
$$\mathcal{K}(E-\overline{\mathcal{L}}\mathcal{K})^{-1} = (E-\mathcal{K}\overline{\mathcal{L}})^{-1}\mathcal{K}, \quad (E-\mathcal{K}\mathcal{L})^{-1} = E + \mathcal{K}\overline{\mathcal{L}}(E-\mathcal{K}\overline{\mathcal{L}})^{-1},$$

which are true for $\|\mathcal{K}\|<1$, $\|\mathcal{L}\|<1$. We get that

$$\begin{aligned} &\big\langle (E-\mathcal{K}\overline{\mathcal{L}})^{-1}\theta, \mathcal{L}\cdot 1\big\rangle + \big(\big\langle \mathcal{K}(E-\overline{\mathcal{L}}\mathcal{K})^{-1}\theta, \mathcal{L}\cdot 1\big\rangle + \langle\theta, 1\rangle\big) \\ &\quad + \big(\langle\bar\lambda, 1\rangle + \big\langle \overline{\mathcal{L}}(E-\mathcal{K}\overline{\mathcal{L}})^{-1}\bar\lambda, \overline{\mathcal{K}}\cdot 1\big\rangle\big) + \big\langle (E-\overline{\mathcal{L}}\mathcal{K})^{-1}\bar\lambda, \overline{\mathcal{K}}\cdot 1\big\rangle, \end{aligned}$$

where $\mathcal{K}\cdot 1 = \sigma_K$ and $\langle\theta,1\rangle = \int\theta\,d\mu$. In this expansion we see four terms of the form (5.1) (here $x,y=\lambda,\theta$ and $\overline{\mathcal{K}}\cdot 1, \mathcal{L}\cdot 1 \in L^2(M)$) and the two convergent integrals $\int\theta\,d\mu$ and $\int\bar\lambda\,d\mu$.

Finally, the third term in the exponential is equal to

$$\begin{aligned} &\big(\langle\overline{\mathcal{L}}\cdot 1, 1\rangle + \big\langle (E-\mathcal{K}\overline{\mathcal{L}})^{-1}\mathcal{K}\cdot[\overline{\mathcal{L}}\cdot 1], \mathcal{L}\cdot 1\big\rangle\big) \\ &\quad + 2\big\langle (E-\mathcal{K}\overline{\mathcal{L}})^{-1}[\mathcal{K}\cdot 1], \overline{\mathcal{L}}\cdot 1\big\rangle \\ &\qquad + \big(\langle\mathcal{K}\cdot 1, 1\rangle + \big\langle (E-\overline{\mathcal{L}}\mathcal{K})^{-1}\overline{\mathcal{L}}\cdot[\mathcal{K}\cdot 1], \overline{\mathcal{K}}\cdot 1\big\rangle\big). \end{aligned}$$

We see three terms of the form (5.1) and the two terms

$$\iint K(m,m')\,d\mu(m)\,d\mu(m'), \qquad \iint L(m,m')\,d\mu(m)\,d\mu(m').$$

The lemma is proved.

5.2. Cylindrical functions. We now show that the right-hand side of (4.18) is not only continuous but also sometimes equal to the left-hand side.

We say that θ is a *step function* if it takes only finitely many values and is equal to 0 outside a set of finite measure. We say that $K(m,m')$ is a *step function* if there exist subsets $M_1,\dots,M_\alpha \subset M$ of finite measure such that

(1) K is constant on the sets $M_i\times M_j$,
(2) K is equal to 0 outside the set $\big(\bigcup M_j\big)\times\big(\bigcup M_j\big)$.

If K and θ are step functions, then the function $\mathfrak{R}[K,\theta]$ is a cylindrical function. The following statement is simply a reformulation of the formula (4.9).

Lemma 5.2. *The formula* (4.18) *is true if K, θ, L, and λ are step functions.*

5.3. Pointwise limits of step functions. Suppose now that $K(m,m') \geqslant 0$ and $\theta(m) \geqslant 0$. We consider (not strictly) increasing sequences $K_j(m,m')$ and $\theta_j(m)$ of step functions converging to K and θ pointwise.

Obviously, the sequence $\mathfrak{R}[K_j,\theta_j](\omega)$ is monotonically increasing for each ω.

Proposition 5.3. *The sequence $\mathfrak{R}[K_j,\theta_j](\omega)$ converges pointwise to some function $T(\omega) \in L^2(\Omega(M))$.*

Lemma 5.4. *$K_j \to K$ and $\theta_j \to \theta$ in the sense of the topology in* § 5.1.

Proof of the lemma. In view of the Lebesgue dominated convergence theorem, $\int(K-K_j) \to 0$ and $\int(K-K_j)^2 \to 0$. By the same theorem,

$$\int K_j(m,m')\,d\mu(m') \to \int K(m,m')\,d\mu(m')$$

for almost all m, that is, $\sigma_{K_j} \to \sigma_K$ pointwise. Next, we apply the dominated convergence theorem to $\int(\sigma_K - \sigma_{K_j})$.

Proof of the proposition.

$$\int \mathfrak{R}[K_j,\theta_j](\omega)\,d\nu(\omega) = \mathfrak{A}(K_j,\theta_j \mid 0,0) \to \mathfrak{A}(K,\theta \mid 0,0).$$

The sequence of integrals is bounded, and therefore $T(\omega) = \lim_{j\to\infty} \mathfrak{R}[K_j,\theta_j](\omega)$ almost everywhere by Beppo Levi's monotone convergence theorem. Applying the monotone convergence theorem to the sequence of integrals

$$\int \left|\mathfrak{R}[K_j,\theta_j](\omega)\right|^2 d\nu(\omega) = \mathfrak{A}(K_j,\theta_j \mid K_j,\theta_j) \to \mathfrak{A}(K,\theta \mid K,\theta),$$

we get that

$$\int |T(\omega)|^2\,d\nu(\omega) = \mathfrak{A}(K,\theta \mid K,\theta).$$

5.4. Convergence of the series $\mathfrak{R}[K,\theta](\omega)$.

Proposition 5.5. *Suppose that K, θ, K_j, and θ_j are the same as above. Then the series* (4.17) *defining the function $\mathfrak{R}[K,\theta](\omega)$ converges almost everywhere and*

$$\mathfrak{R}[K,\theta](\omega) = T(\omega) = \lim_{j\to\infty} \mathfrak{R}[K_j,\theta_j](\omega)$$

almost everywhere.

Proof. We fix an ω such that the sequence $\mathfrak{R}[K_j,\theta_j](\omega)$ has a finite limit for it.

Let us fix a tangle $R \in \tan(\omega)$. Applying the monotone convergence theorem to the sequence of series

$$\sum_{m_\alpha \in [\omega]_R} \ln(1+\theta_j(m_\alpha))$$

(which depends on j), we get that

$$\lim_{j\to\infty} \prod_{m_\alpha \in [\omega]_R} (1+\theta_j(m_\alpha)) = \prod_{m_\alpha \in [\omega]_R} (1+\theta(m_\alpha)).$$

The desired result is now obtained by applying the monotone convergence theorem to the sequence of series $\mathfrak{R}[K_j,\theta_j](\omega)$ (this time the summation is over all tangles).

Proposition 5.6. *Suppose that K and θ are arbitrary functions satisfying the conditions of the theorem. Then for almost all ω:*

(1) *all the products in* (4.17) *converge absolutely;*
(2) *the series over $R \in \tan(\omega)$ converges absolutely.*

Proof. We compare the series defining the functions

$$\mathfrak{R}[K,\theta](\omega) \quad \text{and} \quad \mathfrak{R}\big[|K|,|\theta|\big](\omega).$$

In the right-hand expression everything converges absolutely. In particular, the convergence of any infinite product in the right-hand series implies the convergence of the corresponding product in the left-hand series. Moreover,

$$\Big|\prod K(\sigma_\beta,\delta_\beta)\cdot\prod(1+\theta_j(m_\alpha))\Big| \leqslant \prod|K(\sigma_\beta,\delta_\beta)|\cdot\prod(1+|\theta_j(m_\alpha)|),$$

which implies the absolute convergence of the series $\mathfrak{R}[K,\theta](\omega)$.

5.5. Once again, pointwise limits of cylindrical functions. Suppose, as before, that K and θ are arbitrary functions satisfying the conditions of the theorem. We consider sequences K_j and θ_j of step functions such that

(1) $|K_j|$ and $|\theta_j|$ are monotonically increasing (not strictly),
(2) K_j and θ_j converge pointwise to K and θ.

Proposition 5.7. *The sequence $\mathfrak{R}[K_j,\theta_j](\omega)$ converges to $\mathfrak{R}[K,\theta](\omega)$ almost everywhere.*

Remark. This statement has a certain independent interest in comparison with Theorem 4.2: namely, it gives another way of computing the functions $\mathfrak{R}[K,\theta](\omega)$.

Proof. First of all, we have the pointwise (for a fixed ω) convergence of the terms

$$\prod_{m_\alpha \in [\omega]_R} (1+\theta_j(m_\alpha)) \to \prod_{m_\alpha \in [\omega]_R} (1+\theta(m_\alpha)).$$

This convergence is ensured by the fact that the corresponding series of logarithms is majorized by the convergent series

$$\sum_{m_\alpha \in [\omega]_R} \ln\bigl(1 + |\theta(m_\alpha)|\bigr).$$

Further, the series defining $\mathfrak{R}[K_j, \theta_j](\omega)$ are majorized by the series defining $\mathfrak{R}\bigl[|K|, |\theta|\bigr](\omega)$, and the dominated convergence theorem yields the desired result.

5.6. L^2-convergence of the sequence $\mathfrak{R}[K_j, \theta_j](\omega)$.

Lemma 5.8. *The sequence $\mathfrak{R}[K_j, \theta_j](\omega)$ converges to $\mathfrak{R}[K, \theta](\omega)$ in the sense of L^2.*

Proof. We show that $\mathfrak{R}[K_j, \theta_j](\omega)$ is a Cauchy sequence. Indeed, the quantity

$$\begin{aligned}\int \bigl|\mathfrak{R}[K_j, \theta_j](\omega) &- \mathfrak{R}[K_i, \theta_i](\omega)\bigr|^2 \, d\nu(\omega) \\ &= \mathfrak{A}(K_j, \theta_j \mid K_j, \theta_j) + \mathfrak{A}(K_i, \theta_i \mid K_i, \theta_i) \\ &\quad - \mathfrak{A}(K_j, \theta_j \mid K_i, \theta_i) - \mathfrak{A}(K_i, \theta_i \mid K_j, \theta_j)\end{aligned}$$

tends to 0 as $i, j \to \infty$. Thus, the sequence $\mathfrak{R}[K_j, \theta_j](\omega)$ converges in L^2 and we have seen that it converges pointwise to $\mathfrak{R}[K, \theta](\omega)$. The lemma is proved, but the proof used the following statement:

Lemma 5.9. *The convergences $K_j \to K$ and $\theta_j \to \theta$ described in* § 5.5 *imply convergences in the sense of the topology in* § 5.1.

Proof. We consider the set Σ_+ where $K > 0$, and the set Σ_- where $K < 0$. Then we restrict K to each of these sets and repeat the arguments in the proof of Lemma 5.4.

5.7. Completion of the proofs. We have seen (Lemma 5.2) that the formula (4.18) is true if K, L, θ, and λ are step functions. Lemma 5.8 and Lemma 5.1 imply that it is always true.

We next substitute $L = 0$ and $\lambda = z$ in the formula (4.18). This gives us an explicit formula for

$$\bigl\langle \mathfrak{R}[K, \theta], \Psi_z \bigr\rangle_{L^2(\Omega(M))},$$

that is, for the image of the function $\mathfrak{R}[K, \theta]$ in boson Fock space. The rest is obvious.

§ 6. Gaussian operators

6.1. Hilbert–Schmidt operators. We consider a Gaussian operator

$$B[S \mid \sigma^t] \colon F(L^2(M_2)) \to F(L^2(M_1)). \tag{6.1}$$

Suppose that S satisfies the conditions of Proposition 2.4. Then $B[S \mid \sigma^t]$ is a Hilbert–Schmidt operator (to see this, use the simple arguments in § 6.4 of [4] in conjunction with Proposition 2.4 (see also [16])). We identify the matrix S with the kernel of the corresponding integral operator in $L^2(M_1 \sqcup M_2)$. By the remarks in § 1.5, the corresponding operator

$$B^{[p]}[S \mid \sigma^t]\colon L^2(\Omega(M_2)) \to L^2(\Omega(M_1))$$

is given by the formula

$$B^{[p]}[S \mid \sigma^t]f(\omega_1) = \int_{\Omega(M_2)} \mathfrak{R}^*[S,\sigma](\omega_1,\omega_2)f(\omega_2)\,d\nu(\omega_2), \tag{6.2}$$

where $\omega_1 \in \Omega(M_1)$ and $\omega_2 \in \Omega(M_2)$, and hence $(\omega_1,\omega_2) \in \Omega(M_1 \sqcup M_2)$.

It is an interesting question as to whether the operators $B^{[p]}[S \mid \sigma^t]$ corresponding to Gaussian operators that are not Hilbert–Schmidt can be described explicitly (examples of such correspondences were given above in §§ 3.5 and 3.6).

6.2. The case of finite spaces.

Proposition 6.1. *Suppose that the spaces M_1 and M_2 are finite. Then the operator $B^{[p]}[S \mid \sigma^t]\colon L^2(\Omega(M_2)) \to L^2(\Omega(M_1))$ corresponding to an arbitrary bounded operator of the form $B[S \mid \sigma^t]$ is given by the formula* (6.2).

Remark. The spaces $\Omega(M_1)$ and $\Omega(M_2)$ are discrete, so, properly speaking, we have obtained a formula for a matrix operator.

Proof. We need to see that

$$\langle B^{[p]}[S \mid \sigma^t]\Psi_{h_1},\Psi_{h_2}\rangle_{L^2(\Omega(M_1))} = \langle B[S \mid \sigma^t]\Phi_{h_1},\Phi_{h_2}\rangle_{F(L^2(M_2))}.$$

If we write out the left-hand side of the equality, then we get precisely an expression of the same form as in the proof of Theorem 4.1.

Remark. In particular, we obtain formulae for actions on $L^2(\Omega(M))$ of the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$, the group $\mathrm{Isom}(L^2(M))$, and the finite-dimensional Heisenberg group (see §§ 2.7 and 2.10 above). Fairly amusing formulae are obtained in the last two cases.

6.3. Some integrals. Let us consider the function $\mathfrak{R}[S,\sigma](\omega,\widetilde{\omega})$ on $\Omega(M \sqcup \widetilde{M})$. We fix the argument $\widetilde{\omega}$. The resulting function on $\Omega(M)$ does not have the form $\mathfrak{R}[K,\theta](\omega)$ in general, that is, we obtain a broader family of 'special functions' on $\Omega(M)$.

Suppose now that M', M, M'' are measure spaces, with M' and M'' finite and M arbitrary. Then the formula (2.15) can be rewritten in the form

$$\begin{aligned}\int_{\Omega(M)} &\mathfrak{R}^*[S_1,\sigma_1](\omega',\omega)\cdot\mathfrak{R}^*[S_2,\sigma_2](\omega,\omega'')\,d\nu(\omega)\\ &= \lambda\cdot\mathfrak{R}^*\bigl[(S_1 \mid \sigma_1^t)\circ(S_2 \mid \sigma_2^t)\bigr](\omega',\omega''),\end{aligned}$$

where λ is taken from (2.16), and $(S_1 \mid \sigma_1^t)\circ(S_2 \mid \sigma_2^t)$ from the right-hand side of (2.15).

This formula holds for any fixed $\omega' \in \Omega(M')$ and $\omega'' \in \Omega(M'')$ under slight restrictions on S_1,σ_1 and S_2,σ_2, but we shall not go through the restrictions.

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Received 11/MAR/97
Translated by H. H. McFADEN

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX