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The group of diffeomorphisms of the half-line, and random Cantor sets

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Abstract. A certain one-parameter family of measures is constructed on the space of closed totally disconnected subsets of the half-line without isolated points. It is shown that these measures are quasi-invariant with respect to the group of smooth diffeomorphisms of the half-line, and the Radon–Nikodým derivatives are explicitly computed.

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In the past few years probability-theoretic methods have gradually become more important in the theory of infinite-dimensional groups and their representations (see [9], [12], [15], [20], [22]). It was remarked in [16] that all the known constructions of representations of the group of diffeomorphisms of the circle and of loop groups are closely connected with diffusion of fractional order. In particular, this opens the possibility of using the well-developed theory of ordinary Brownian motion (see [7], [5], [21]) in the theory of representations of infinite-dimensional groups. In the same paper many quasi-invariant actions were constructed for the group of diffeomorphisms of the circle and for loop groups on diverse measure spaces, including examples of quasi-invariant measures on the space of Cantor subsets of the circle.

The question of the existence of such measures has been familiar in the theory of representations since about 1970 (see [4], [1]): it is interesting that the very measures that turned out to be quasi-invariant had long been well known and had been extensively studied in probability theory, although the question of their quasi-invariance and of their use in representation theory had not been raised. We are concerned with the following standard construction ([5], [13], [19]), which goes back to the book [7] of Lévy. Let us consider a random process with continuous sample paths, and associate with each sample path its set of zeros. Thus, we get a map of the space of sample paths (which is endowed with a measure by definition) onto the space of closed subsets of an interval, and this map gives us a measure on the space of closed subsets of the interval. In the case when the random process is a fractional diffusion (see, for example, [14] concerning fractional diffusion)

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the measure obtained in this way turns out to be quasi-invariant with respect to smooth time changes (see [16]). However, in this form the construction does not yet give the possibility of writing out explicit formulae for the representations.

In this paper we discuss a different construction of measures on the space of Cantor sets (see [7], §§47, 48, and [2]). We consider an increasing stable process, and we associate with each sample path its set of values. Thus, a measure is obtained on the space of subsets of the half-line. We show that these measures are quasi-invariant with respect to the group of diffeomorphisms of the half-line, and we compute the Radon–Nikodým derivative explicitly.

Our main assertions are formulated in §1. The construction of a measure on the space of Cantor subsets is discussed in greater detail in §2, and the quasi-invariance theorem is proved in §3.

§1. Formulation of the theorem

1.0. The space $\mathrm{Cant}(\mathbb{R}_+)$. We denote by $\mathbb{R}_+$ the positive half-line $u \geqslant 0$. By a Cantor subset of $\mathbb{R}_+$ we mean a closed subset of $\mathbb{R}_+$ of measure zero without isolated points and containing the point 0. The space of all Cantor subsets of $\mathbb{R}_+$ will be denoted by $\mathrm{Cant}(\mathbb{R}_+)$.

We want to define on $\mathrm{Cant}(\mathbb{R}_+)$ a one-parameter family of probability measures $\varkappa_\alpha$. The first question arising is: in what language can a measure on $\mathrm{Cant}(\mathbb{R}_+)$ be described? As is well known, the complement of a closed subset of $\mathbb{R}$ is a finite or countable family of intervals. We give a description of the measures $\varkappa_\alpha$ in terms of the distribution of the end-points of the complementary intervals.

Let $a > 0$. All our measures $\varkappa_\alpha$ will have the following property: For any a the probability is 1 that an element $X \in \mathrm{Cant}(\mathbb{R}_+)$ does not contain a. Thus, with probability 1 there exists a complementary interval (u, v) to X that contains a.

Suppose now that A is a finite subset of $\mathbb{R}_+$, let

$$a_1 < a_2 < \cdots < a_n$$

be its elements, and let $a_1 > 0$. For $X \in \mathrm{Cant}(\mathbb{R}_+)$ we consider the minimal collection of complementary intervals to X

$$(u_1, v_1), \dots, (u_s, v_s) \tag{1.1}$$

that covers all the points a_j; we assume for definiteness that

$$u_1 < v_1 < u_2 < v_2 <$$

To specify a measure on $\mathrm{Cant}(\mathbb{R}_+)$ it suffices to describe the joint distribution of the points u_j, v_j for all possible finite subsets $A \subset \mathbb{R}_+$.

First of all we note that the points a_j can be distributed in various ways over the intervals (u_k, v_k). To take this into account we choose from the $(n-1)$st elementary set $\{1, \dots, n-1\}$ an arbitrary subset I consisting of the elements

$$i_1 < i_2 < \cdots < i_s$$

(both the set $\{1,\dots,n-1\}$ itself and the empty set are allowed as I). We consider next the polyhedron

$$Q(A \mid I) = Q(a_1,\dots,a_n \mid i_1,\dots,i_s)$$

in $\mathbb{R}^{2(s+1)}$ given by the inequalities

$$\begin{cases} 0 < u_1 < a_1, \\ a_{i_k} < v_k < u_{k+1} < a_{i_k+1} \quad \forall\, k = 1,2,\dots,s, \\ a_n < v_{s+1}. \end{cases}$$

With almost every point $X \in \mathrm{Cant}(\mathbb{R}_+)$ there is associated a point in one of the 2^{n-1} polyhedra $Q(A \mid I)$, namely, the end-points of the intervals (1.1) are associated with X. Thus, to define the probability measures $\varkappa_\alpha$ on $\mathrm{Cant}(\mathbb{R}_+)$ it suffices to write out their projections on all possible sets

$$R(A) := \bigcup_{I \subset \{1,2,\dots,n-1\}} Q(A \mid I).$$

1.1. Drawing lots for Cantor sets. Let $0 < \alpha < 1$. We define the probability measure $\varkappa_\alpha$ on $\mathrm{Cant}(\mathbb{R}_+)$.

For $a > 0$ the end-points of the complementary interval (u,v) of $X \in \mathrm{Cant}(\mathbb{R}_+)$ covering a are distributed according to the 'generalized arcsine law'

$$\frac{\sin \pi\alpha}{\pi} \cdot \frac{du\,dv}{u^{1-\alpha}(v-u)^{1+\alpha}}, \qquad u < a < v. \tag{1.2}$$

The measure on $Q(A \quad I)$ corresponding to the measure $\varkappa_\alpha$ on $\mathrm{Cant}(\mathbb{R}_+)$ has the form

$$\left(\frac{\sin \pi\alpha}{\pi}\right)^{s+1} \prod_{j=1}^{s+1} \frac{du_j\,dv_j}{(u_j - v_{j-1})^{1-\alpha}(v_j - u_j)^{1+\alpha}} \tag{1.3}$$

(where v_0 is understood to be 0). This equality determines $\varkappa_\alpha$ on $\mathrm{Cant}(\mathbb{R}_+)$.

Here, of course, there arise questions about whether this definition is unambiguous. In §2 we shall give another description of $\varkappa_\alpha$, less convenient to work with but more transparent in certain respects. For the present we only make some remarks.

Remark. The formula (1.2) for the distribution of the end-points of a random interval (or of 'a jump short' and 'a jump over') is actually very natural (see Dynkin's paper [2] and Feller's discussion of it in §XIV.3 of [11]).

Remark. The meaning of the formula (1.3) is as follows. Let the points $a_1 < a_2 < \cdots$ be given, and let (u_1, v_1) be the complementary interval containing a_1. Its end-points are distributed according to the law (1.2). The interval (u_1, v_1) may cover some of the points $a_2, a_3, \dots$; let a_l be the first of the points not covered. Assume that (u_2, v_2) is the interval covering a_l. Then the numbers

$$\widetilde{u} = u_2 - v_1, \quad \widetilde{v} = v_2 - v_1$$

are also distributed according to the law (1.2). In other words, after the interval (u_1, v_1) is 'drawn', the 'drawing' of the interval containing a_l is carried out like the drawing of the interval (u_1, v_1), except that we regard v_1 as the initial point of the half-line instead of 0. For a thorough treatment of the case $\alpha = 1/2$ see §§47, 48 in Lévy's book [7], and also [5].

1.2. The group $\mathrm{Diff}^0(\mathbb{R}_+)$. We denote by $\mathrm{Diff}^0(\mathbb{R}_+)$ the group of diffeomorphisms p of the half-line $u \geqslant 0$ satisfying the conditions

(1) the second derivative of p is continuous,

(2) the limit

$$\lim_{u\to+\infty} p'(u)$$

exists (denote it by $p'(\infty)$) and is finite and non-zero,

(3)

$$\int_0^\infty \left|\frac{p''(u)}{p'(u)}\right|$$

(in other words, the function $\ln p'(u)$ has bounded variation).

Remark. The conditions (2) and (3) mean that the diffeomorphism p does not differ greatly from a linear function $\alpha u + \beta$ at infinity, so the two conditions can be regarded as a kind of requirement that p be differentiable at the point $u = \infty$. We note also that if $p''(u)$ has constant sign for sufficiently large u, then the condition (3) follows from (2). Indeed,

$$\int_C^\infty \frac{p''(u)}{p'(u)}\,du = \ln p'(\infty) - \ln p'(C).$$

1.3. The theorem on quasi-invariance.

Theorem. *Let $p \in \mathrm{Diff}^0(\mathbb{R}_+)$. Then the map*

$$\mathrm{Cant}(\mathbb{R}_+) \to \mathrm{Cant}(\mathbb{R}_+)$$

determined by the diffeomorphism p constitutes a measure $\varkappa_\alpha$ on $\mathrm{Cant}(\mathbb{R}_+)$ that is quasi-invariant, and the Radon–Nikodým derivative of this transformation is given by the formula

$$\gamma_\alpha(p,X) = \frac{p'(\infty)}{p'(0)} \prod \left(\frac{p'(u_j)(v_j-u_j)}{p(v_j)-p(u_j)}\right)^{1+\alpha} \tag{1.4}$$

where the product is over all intervals complementary to the set $X \in \mathrm{Cant}(\mathbb{R}_+)$ This product converges absolutely, and

$$|\ln \gamma_\alpha(p,X)| \leqslant (1+\alpha)\int_0^\infty \left|\frac{p''(u)}{p'(u)}\right|$$

for almost all $X \in \mathrm{Cant}(\mathbb{R}_+)$

1.4. Representations and the problem of constructing the enveloping semigroup. For any $\alpha \in (0,1)$ and any $s \in \mathbb{R}$ we define a unitary representation $\rho_{\alpha,s}$ of the group $\mathrm{Diff}^0(\mathbb{R}_+)$ on $L^2(\mathrm{Cant}(\mathbb{R}_+), \varkappa_\alpha)$ by the formula

$$\rho_{\alpha,s}(p)f(X) = f(p\cdot X)\gamma_\alpha(p,X)^{1/2+is}$$

Nothing is known at present about the properties of these representations. In connection with the representations $\rho_{\alpha,s}$ I think it is a very interesting problem to find the enveloping semigroup of the group $\mathrm{Diff}^0(\mathbb{R}_+)$ in the sense of Olshanskiĭ (see [18], [17]). In our case the problem can be reformulated as follows.

We consider the space

$$Z = \mathrm{Cant}(\mathbb{R}_+) \times \mathrm{Cant}(\mathbb{R}_+) \times \mathbb{R}_+.$$

With each element $p \in \mathrm{Diff}^0(\mathbb{R}_+)$ we associate the map

$$\theta_p^{(\alpha)}\colon \mathrm{Cant}(\mathbb{R}_+) \to Z$$

defined by the formula

$$\theta_p^{(\alpha)}(X) = (X, p\cdot X, \gamma_\alpha(p, X)).$$

Let h_p be the image of the measure $\varkappa_\alpha$ under the map $\theta_p^{(\alpha)}$. We have obtained a family h_p of probability measures on Z indexed by the diffeomorphisms p in $\mathrm{Diff}^0(\mathbb{R}_+)$. It is required to describe the closure $\overline{\mathrm{Diff}^0(\mathbb{R}_+)}$ of the family h_p in the space of probability measures on Z with respect to weak convergence (see [9]). According to [9], $\overline{\mathrm{Diff}^0(\mathbb{R}_+)}$ is equipped with a natural operation of multiplication that extends the usual multiplication in $\mathrm{Diff}^0(\mathbb{R}_+)$.

1.5. The local time. For almost every (with respect to $\varkappa_\alpha$) $X \in \mathrm{Cant}(\mathbb{R}_+)$ we now define a canonical measure λ_X on the set X, called the *local time* on X. We can also assume that λ_X is a singular measure on $\mathbb{R}_+$ with support X.

Let $b > a \geqslant 0$. Then the measure λ_X of the closed interval $[a,b]$ is defined to be

$$\frac{1}{1-\alpha}\lim_{\varepsilon\to 0}\varepsilon^{\alpha-1}\cdot\Sigma', \tag{1.5}$$

where Σ' is the sum of the lengths of the complementary intervals $(\alpha_j, \beta_j) \subset [a,b]$ such that $\beta_j - \alpha_j < \varepsilon$.

The same quantity $\lambda_X([a,b])$ is equal to

$$\frac{1}{\alpha}\lim_{\varepsilon\to 0}\varepsilon^{\alpha}\cdot\Sigma'', \tag{1.6}$$

where Σ'' is the number of complementary intervals $(\alpha_j, \beta_j) \subset [a,b]$ such that $\beta_j - \alpha_j > \varepsilon$.

Of course, we must show that the limits here exist, and that they coincide (see §2.5 below).

Let α be fixed, and let $X \in \mathrm{Cant}(\mathbb{R}_+)$ and $p \in \mathrm{Diff}^0(\mathbb{R}_+)$. Then on the set $p\cdot X$ we have two measures: first, the local time $\lambda_{p\cdot X}$ of the set $p\cdot X$, and second, the image $p\cdot\lambda_X$ of the local time λ_X under the map p. It follows from the formulae (1.5) and (1.6) that $\lambda_{p\cdot X}$ is absolutely continuous with respect to $p\cdot\lambda_X$, and the density is equal to

$$p'(u)^{-\alpha}.$$

Remark. For $\alpha = 1/2$ this local time is the usual local time of Lévy.

Remark. It seems likely that the measures $\varkappa_\alpha$ are equivalent to the measures arising in the consideration of the zero level sets for diffusions of fractional order (see the introduction to the paper).

§ 2. Stable processes and random Cantor sets

2.1. Poisson measures. Let L be a space with a continuous σ-finite measure λ, let Ω be the space of all unordered countable subsets of L, and let $A \subset L$ be a measurable subset of finite measure. Denote by $\Omega(A, n)$ the set of all points $\omega \in \Omega$ such that $\omega \cap A$ consists of n points. We define a probability measure σ on Ω by the following rule:

(1) $\sigma(\Omega(A,n)) = \frac{\lambda(A)^n}{n!} e^{-\lambda(A)}$;

(2) if $A_1, \dots, A_k$ are disjoint sets of finite measure, then the events $\Omega(A_j, n_j)$ are independent, that is,

$$\sigma\Big(\bigcap_{j=1}^{k} \Omega(A_j, n_j)\Big) = \prod_{j=1}^{k} \sigma(\Omega(A_j, n_j))$$

It is well known ([6], [1]) that this definition is unambiguous (as follows from the Kolmogorov theorem on projective limits). The measure obtained in this way on Ω is called a *Poisson measure*.

Remark. For any measurable subset $B \subset A$ of infinite measure the set of all $\omega \in \Omega$ with $\omega \cap B$ infinite has full measure in Ω. If $B \subset A$ has finite measure, then $\omega \cap B$ is finite with probability 1.

2.2. Random discrete measures. Denote by $\mathcal{M}(\mathbb{R}_+)$ the set of all positive discrete measures on $\mathbb{R}_+$. Any such measure can be represented as a sum

$$h = \sum u_j \delta_{t_j},$$

where δ_t denotes the unit measure on $\mathbb{R}_+$ concentrated at a point $t \in \mathbb{R}_+$.

We fix an α such that $0 < \alpha < 1$, and construct a probability measure μ_α on the space $\mathcal{M}(\mathbb{R}_+)$ according to the following rule. We consider the quadrant $L = \mathbb{R}_+ \times \mathbb{R}_+$ with coordinates (t, u), and the measure λ on L having density

$$\frac{1}{u^{\alpha+1}}$$

with respect to Lebesgue measure $dt\,du$ on L. Let Ω be the set of countable subsets of L equipped with the Poisson measure σ constructed in the preceding subsection. With each point $h = \sum u_j \delta_{t_j} \in \mathcal{M}(\mathbb{R}_+)$ we associate the countable subset $\omega \in \Omega$ consisting of the points of the form (t_j, u_j). Thus, we have obtained a bijection $\Omega \to \mathcal{M}(\mathbb{R}_+)$, and to the measure σ on Ω there corresponds a measure μ_α on $\mathcal{M}(\mathbb{R}_+)$

Remark. For very small intervals $(t, t+\Delta t)$ and $(u, u+\Delta u)$ it is easy to show that the mean number of terms in the sum $h = \sum u_j \delta_{t_j}$ such that

$$t < t_j < t + \Delta t, \quad u < u_j < u + \Delta u$$

is approximately equal to

$$\frac{\Delta t \Delta u}{u^{\alpha+1}}$$

(this property is usually taken as the definition of the measure μ_α)

2.3. Remarks about the measures μ_α. The remark at the end of subsection 2.1 has the following two consequences (A) and (B).

(A) For given $v_2 > v_1 > 0$ the μ_α-probability is 1 that the expression $h = \sum u_j \delta_{t_j}$ contains infinitely many terms such that $v_1 < u_j < v_2$. In particular, $\sum u_j = \infty$ with probability 1.

(B) For given $s_2 > s_1 > 0$ the μ_α-probability is 1 that the expression $h = \sum u_j \delta_{t_j}$ contains infinitely many terms such that $s_1 < t_j < s_2$. In particular, the μ_α-probability is 1 that the points t_j are dense on the half-line $t \geqslant 0$.

The following three statements are also very simple and are well known.

(C) For given $s_2 > s_1 > 0$ the quantity

$$\sum_{j: s_1 < t_j < s_2} u_j$$

is finite with probability 1; its mean value is infinite.

(D) Let $s_2 > s_1 > 0$. We consider the quantity

$$\xi_\varepsilon(h) = \sum_{j: u_j < \varepsilon,\, s_1 < t_j < s_2} u_j.$$

Then for almost all h

$$\frac{1}{1-\alpha} \lim_{\varepsilon \to +0} \frac{\xi_\varepsilon(h)}{\varepsilon^{1-\alpha}} = s_2 - s_1.$$

(E) Let $s_2 > s_1 > 0$, and let N_ε be the number of terms in the sum $h = \sum u_j \delta_{t_j}$ such that $u_j > \varepsilon$ and $s_1 < t_j < s_2$. Then for almost all h

$$\frac{1}{\alpha} \lim_{\varepsilon \to +0} \varepsilon^\alpha N_\varepsilon = s_2 - s_1$$

2.4. Measures on the space of monotone functions. Let $h \in \mathcal{M}(\mathbb{R}_+)$. We consider the function

$$\varphi_h(s) = \int_0^s h = \sum_{j: 0 \leqslant t_j \leqslant s} u_j.$$

By the property (C) in subsection 1.3, the probability is 1 that this integral is finite for all s. It is also obvious that $\varphi_h(s)$ is monotonically increasing, and $\varphi_h(0) = 0$ for almost all h. It is clear that the measure h can be uniquely recovered from the function φ_h.

We remark further that φ_h is a jump function, that is,

$$\varphi_h(a) = \sum (\varphi(t_j + 0) - \varphi(t_j - 0))$$

for any point of continuity a of the function φ_h, where the summation is over all the points t_j of discontinuity of φ with $t_j < a$.

Let $J(\mathbb{R}_+)$ be the space of all monotonically increasing jump functions f on the half-line $t \geqslant 0$ such that $f(0) = 0$. The map $h \mapsto \varphi_h$ carries $\mathcal{M}(\mathbb{R}_+)$ into $J(\mathbb{R}_+)$; let ν_α be the image of the measure μ_α under this map.

The measure ν_α on $J(\mathbb{R}_+)$ gives one of the stable random processes (see, for example, [7]).

The statements (A)–(E) in subsection 2.3 are easily carried over to the language of monotone functions. For instance, (B) means that the points of discontinuity of a function $f \in J(\mathbb{R}_+)$ (for ν_α-almost all functions f) are dense in $\mathbb{R}_+$.

Let $a \in \mathbb{R}$. We consider the random variable

$$\eta_a(f) = f(a), \quad f \in J(\mathbb{R}_+)$$

(with probability 1 the point a is not a point of discontinuity of f, and hence $f(a)$ is well defined). Let $r_a(u)$ be the distribution function of η_a. With the exception of the case $\alpha = 1/2$ there is no nice analytic expression for the function $r_a(u)$. The properties of the 'special functions' $r_a(u)$ have been extensively studied (see, for example, [3], and [8], §§5.7–5.10). The characteristic function of the distribution $r_a(u)$ is given by the formula

$$\exp\left(-a\left(\frac{x}{i}\right)^\alpha\right), \qquad \operatorname{Im} x \geqslant 0.$$

Here the branch of the function $(x/i)^\alpha$ in the upper half-plane is chosen so that $(x/i)^\alpha$ is real for $x = i\tau$, $\tau > 0$.

2.5. Measures on the space of Cantor sets. Let $f \in J(\mathbb{R}_+)$. We consider the graph $\operatorname{Graph}(f)$ of f, understood in the following sense. By definition, the set $\operatorname{Graph}(f)$ contains all the points of the form $(t, f(t))$, where t is a point of continuity of f, along with all the points of the form

$$(t_j, f(t_j - 0)), \quad (t_j, f(t_j + 0)),$$

where the t_j are the points of discontinuity of f.

With each function $f \in J(\mathbb{R}_+)$ we associate the projection Q_f of $\operatorname{Graph}(f)$ on the vertical semiaxis. It is easy to see that Q_f is closed and does not have isolated points, and it is also easy to see that Q_f has Lebesgue measure zero (since f is a jump function).

Thus, $f \mapsto Q_f$ is a map $J(\mathbb{R}_+) \to \operatorname{Cant}(\mathbb{R}_+)$. We note further that Q_f is equipped with a canonical measure, namely, the image of Lebesgue measure on $\mathbb{R}_+$ under the map $f\colon \mathbb{R}_+ \to \mathbb{R}_+$. This measure is the local time λ of the set Q_f; the definitions (1.5) and (1.6) of the local time are reformulations of the expressions (D)–(E) in subsection 2.3.

Obviously, a function $f \in J(\mathbb{R}_+)$ is uniquely determined by the set Q_f and by the local time λ on Q_f. Indeed, let u be a point of the vertical axis, and let $l(u) := \lambda([0, u])$. Then

$$u = f(l(u)).$$

On the other hand, as we have just seen, the local time on Q_f is uniquely determined by Q_f itself for almost all f. Therefore, the map $f \mapsto Q_f$ is an injective (up to a set of measure 0 in $J(\mathbb{R}_+)$) map $J(\mathbb{R}_+) \to \operatorname{Cant}(\mathbb{R}_+)$. We denote by $\varkappa_\alpha$ the measure on $\operatorname{Cant}(\mathbb{R}_+)$ that is the image of ν_α under the map $f \mapsto Q_f$.

(a) The first factor obviously tends to

$$\frac{p'(\infty)}{p'(0)}.$$

(b) The second factor. We take its logarithm:

$$\begin{aligned}\left|\ln \prod \frac{p'(v_{j-1})}{p'(u_j)}\right| &\leqslant \sum \left|\ln p'(v_{j-1}) - \ln p'(u_j)\right| \\ &\leqslant \sum_j \int_{v_{j-1}}^{u_j} \left|\frac{p''(\omega)}{p'(\omega)}\right| d\omega \\ &\leqslant \sum_j \int_{v_{j-1}}^{u_j} \left|\frac{p''(\omega)}{p'(\omega)}\right| d\omega + \int_{v_{s+1}}^{\infty} \left|\frac{p''(\omega)}{p'(\omega)}\right| d\omega.\end{aligned}$$

Let us consider the set

$$Y(A) = [v_0, u_1] \cup [v_1, u_2] \cup \cdots \cup [v_{s+1}, \infty)$$

The set $Y(A)$ decreases as the set A increases in size, and the intersection of the sets $Y(A)$ is a Cantor set of measure 0. Therefore, since the function $|p''/p'|$ is integrable and the Lebesgue integral is absolutely continuous, the expression

$$\int_{Y(A)} \left|\frac{p''(\omega)}{p'(\omega)}\right|$$

tends to 0 as A increases in size. Thus, the second factor in (3.1) tends to 1.

(c) The third factor. Again we pass to logarithms:

$$\left|\ln \prod \frac{p'(v_{j-1})(u_j - v_{j-1})}{p(u_j) - p(v_j)}\right| \leqslant \sum \left|\ln p'(v_j) + \ln \frac{u_j - v_{j-1}}{p(u_j) - p(v_{j-1})}\right|. \tag{3.3}$$

We next choose $\xi_{j-1} \in (v_{j-1}, u_j)$ such that

$$\frac{p(u_j) - p(v_{j-1})}{u_j - v_{j-1}} = p'(\xi_{j-1})$$

Then the expression (3.3) can be rewritten in the form

$$\sum \left|\ln p'(v_{j-1}) - \ln p'(\xi_{j-1})\right| \leqslant \sum \int_{v_{j-1}}^{\xi_{j-1}} \left|\frac{p''(\omega)}{p'(\omega)}\right| d\omega \leqslant \sum \int_{v_{j-1}}^{u_j} \left|\frac{p''(\omega)}{p'(\omega)}\right| d\omega$$

and this expression tends to 0 as A increases in size for the same reasons as (3.2).

(d) The fourth factor. Here new factors are adjoined to the product as A is augmented, but the factors that were present before remain unchanged. We show that the product stays uniformly bounded as A increases in size:

$$\begin{aligned}\left|\ln\left(\prod \frac{p'(u_j)(v_j - u_j)}{p(v_j) - p(u_j)}\right)\right| &\leqslant \sum \left|\ln p'(u_j) - \ln \frac{p(v_j) - p(u_j)}{v_j - u_j}\right| \\ &\leqslant \sum \left|\ln p'(u_j) - \ln p'(\xi_j)\right|,\end{aligned}$$

where $\xi_j \in (u_j, v_j)$.

Accordingly, we have constructed a family of measures $\varkappa_\alpha$ on $\operatorname{Cant}(\mathbb{R}_+)$ that depends on the parameter $\alpha \in (0,1)$.

It remains to bring the picture described into correspondence with subsection 1.1. Let $f \in J(\mathbb{R}_+)$. For $a > 0$ let t be the moment of time such that

$$f(t-0) \leqslant a \leqslant f(t+0)$$

The problem is to compute the joint distribution of $f(t-0)$ and $f(t+0)$ for a given value of a. The solution was obtained by Dynkin in [2], and this solution is given by the formula (1.2). Since the random process $(J(\mathbb{R}_+), \nu_\alpha)$ has the strong Markov property, the formula (1.3) is a consequence of (1.2) (see, for example, [10], §4.1).

§ 3. Proof of the theorem

Let the set $A = \{a_1, \dots, a_n\}$ be the same as in subsections 1.0 and 1.1, and let $p(A)$ be the set $\{p(a_1), \dots, p(a_n)\}$. The map $\operatorname{Cant}(\mathbb{R}_+) \to \operatorname{Cant}(\mathbb{R}_+)$ given by the diffeomorphism p induces a map of the polyhedra

$$Q(A \mid I) \to Q(p(A) \mid I)$$

given by the formula

$$u_j \mapsto p(u_j), \quad v_j \mapsto p(v_j).$$

Furthermore, the Radon–Nikodým derivative is given by the formula

$$\frac{\left[\prod_{j=1}^{s+1} \dfrac{dp(u_j)\,dp(v_j)}{(p(u_j)-p(v_{j-1}))^{1-\alpha}(p(v_j)-p(u_j))^{1+\alpha}}\right]}{\left[\prod_{j=1}^{s+1} \dfrac{du_j\,dv_j}{(u_j-v_{j-1})^{1-\alpha}(v_j-u_j)^{1+\alpha}}\right]}$$
$$= \prod_{j=1}^{s+1}\left[p'(u_j)p'(v_j)\left(\frac{u_j-v_{j-1}}{p(u_j)-p(v_{j-1})}\right)^{1-\alpha}\left(\frac{v_j-u_j}{p(v_j)-p(u_j)}\right)^{1+\alpha}\right]$$

We rewrite this expression in the form

$$\frac{p'(v_{s+1})}{p'(v_0)}\left[\frac{p'(v_0)p'(v_1)\cdots p'(v_s)}{p'(u_1)p'(u_2)\cdots p'(u_{s+1})}\right]^\alpha$$
$$\times \prod_{j=1}^{s+1}\left(\frac{p'(v_{j-1})(u_j-v_{j-1})}{p(u_j)-p(v_{j-1})}\right)^{1-\alpha} \cdot \prod_{j=1}^{s+1}\left(\frac{p'(u_j)(v_j-u_j)}{p(v_j)-p(u_j)}\right)^{1+\alpha} \tag{3.1}$$

where $v_0 = 0$.

We now begin to augment the finite set A in such a way that a countable dense subset of the half-line is obtained in the limit. We consider how the four factors in the product (3.1) behave.

(a) The first factor obviously tends to

$$\frac{p'(\infty)}{p'(0)}.$$

(b) The second factor. We take its logarithm:

$$\begin{aligned}\left|\ln\prod\frac{p'(v_{j-1})}{p'(u_j)}\right| &\leqslant \sum\left|\ln p'(v_{j-1})-\ln p'(u_j)\right| \\ &\leqslant \sum_j\int_{v_{j-1}}^{u_j}\left|\frac{p''(\omega)}{p'(\omega)}\right|d\omega \\ &\leqslant \sum_j\int_{v_{j-1}}^{u_j}\left|\frac{p''(\omega)}{p'(\omega)}\right|d\omega+\int_{v_{s+1}}^{\infty}\left|\frac{p''(\omega)}{p'(\omega)}\right|\end{aligned}$$

Let us consider the set

$$Y(A)=[v_0,u_1]\cup[v_1,u_2]\cup \quad \cup[v_{s+1},\infty).$$

The set $Y(A)$ decreases as the set A increases in size, and the intersection of the sets $Y(A)$ is a Cantor set of measure 0. Therefore, since the function $|p''/p'|$ is integrable and the Lebesgue integral is absolutely continuous, the expression

$$\int_{Y(A)}\left|\frac{p''(\omega)}{p'(\omega)}\right|$$

tends to 0 as A increases in size. Thus, the second factor in (3.1) tends to 1.

(c) The third factor. Again we pass to logarithms:

$$\left|\ln\prod\frac{p'(v_{j-1})(u_j-v_{j-1})}{p(u_j)-p(v_j)}\right|\leqslant\sum\left|\ln p'(v_j)+\ln\frac{u_j-v_{j-1}}{p(u_j)-p(v_{j-1})}\right.$$

We next choose $\xi_{j-1}\in(v_{j-1},u_j)$ such that

$$\frac{p(u_j)-p(v_{j-1})}{u_j-v_{j-1}}=p'(\xi_{j-1}).$$

Then the expression (3.3) can be rewritten in the form

$$\sum\left|\ln p'(v_{j-1})-\ln p'(\xi_{j-1})\right|\leqslant\sum\int_{v_{j-1}}^{\xi_{j-1}}\left|\frac{p''(\omega)}{p'(\omega)}\right|d\omega\leqslant\sum\int_{v_{j-1}}^{u_j}\left|\frac{p''(\omega)}{p'(\omega)}\right|d\omega,$$

and this expression tends to 0 as A increases in size for the same reasons as (3.2).

(d) The fourth factor. Here new factors are adjoined to the product as A is augmented, but the factors that were present before remain unchanged. We show that the product stays uniformly bounded as A increases in size:

$$\begin{aligned}\left|\ln\left(\prod\frac{p'(u_j)(v_j-u_j)}{p(v_j)-p(u_j)}\right)\right| &\leqslant\sum\left|\ln p'(u_j)-\ln\frac{p(v_j)-p(u_j)}{v_j-u_j}\right. \\ &\leqslant\sum\left|\ln p'(u_j)-\ln p'(\xi_j)\right|,\end{aligned}$$

where $\xi_j\in(u_j,v_j)$

This quantity is bounded above by the variation of the function $\ln p'(\omega)$, that is, by the integral

$$\int_0^\infty \left|\frac{p''(\omega)}{p'(\omega)}\right| d\omega \tag{3.4}$$

We emphasize that this upper bound is independent of the choice of the sequence of sets A and of the point $X \in \mathrm{Cant}(\mathbb{R}_+)$. Thus, we have shown that the family of functions (3.1) converges pointwise to the desired expression (1.4) as A increases in size.

Moreover, the functions (3.1) remain uniformly bounded (the moduli of the logarithms of the second and third factors are bounded above by the same integral (3.4)). We can now use the Lebesgue dominated convergence theorem. As a result we get that the family of functions (3.1) converges on $\mathrm{Cant}(\mathbb{R}_+)$ to (1.4) in the L^1-sense with respect to the measure $\varkappa_\alpha$, and this concludes the proof of the theorem.

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