Алгебра и анализ
Том 3 (1991), вып. 1

St. Petersburg Math. J.
Vol. 3 (1992), No. 1

EXTENSION OF REPRESENTATIONS OF THE CLASSICAL GROUPS TO REPRESENTATIONS OF CATEGORIES

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ABSTRACT. It is shown that with each series A_n, B_n, C_n, D_n of the complex classical Lie groups a certain category GA, B, C, D is connected in a natural fashion. A finite-dimensional representation of any classical group extends to a representation of the corresponding category. It is proved that the representations of the categories GA, B, C, D can be indexed by a set of numerical markers on infinite Dynkin diagrams. Similar assertions are obtained for infinite-dimensional representations with highest weight of the groups $U(p, q)$, $\mathrm{Sp}(2n, \mathbf{R})$, and $\mathrm{SO}^*(2n)$. There is also a definition given for a morphism of noncompact symmetric spaces, and natural categories are constructed that are connected with all the series of real classical groups.

We lay out in this paper a new point of view for certain classical objects in representation theory (finite-dimensional representations of the classical groups and the symmetric groups, infinite-dimensional representations of Lie groups with highest weight). Specifically, we show that the classical groups of any one series (A_n, B_n, C_n, D_n) but of different rank combine in a natural fashion into certain categories, and the theory of representations of all the groups of any one series becomes the theory of representations of the corresponding category. It is curious that these facts, which it would seem could have been discovered 50 years ago, have been noticed only recently (see [1]–[3]), from the "eminence" of representation theory for infinite-dimensional groups.

In [4] and [5] the point of view is expressed that the representations of an infinite-dimensional group are in fact representations of a considerably larger, and to the naked eye invisible, semigroup. More recently [1]–[3] it has become clear that in many interesting cases these semigroups (which have proved to be a very effective instrument for dealing with the original groups corresponding to them) consist of linear relations (i.e., of operators that are possibly not everywhere defined, and possibly multivalued, but are in all other respects linear). Furthermore, it has turned out that the formulas for the operators in a representation can be written so as to apply also to linear relations between various spaces; i.e., it has turned out that we are dealing with representations of categories.

When it became clear that linear operators and linear relations are of more or less equal status as far as concerns representation theory for infinite-dimensional groups, the question naturally arose of the finite-dimensional case. This is the question explored in the present paper. In §§2 and 3 we consider the finite-dimensional representations of the classical groups; in §4, the representations of the groups $U(p, q)$ with highest weight; in §5, the categories connected with the real classical groups; and in §6, those connected with finite groups. In §1 we illustrate, by exhibiting the simplest

1991 *Mathematics Subject Classification*. Primary 22E46; Secondary 18B10.
Key words and phrases. Classical group, representations with highest weight, infinite-dimensional group, semigroup, representation of a category.

example, the general scheme of all the proofs. An Appendix at the end of the paper is devoted to certain questions concerning the geometry of symmetric spaces.

Finally, we note here four topics in representation theory that involve the category ideology.

1. **Representations of the Virasoro algebra and of affine algebras** with highest weight; see [1] and [3]. The theory of representations of the category Shtan connected with these is, in the opinion of various authors (M. L. Kontsevich, G. B. Segal [6]; see also [7]), a mathematical reformulation of conformal quantum field theory. We note that five of the categories defined below (GA, B, C, GD, and Sp; more precisely, not these themselves, but their "infinite-dimensional completions") are used in [1] and [3] to construct the representations of Shtan.

2. **Polymorphisms.** The objects of the category are spaces with probability measure; the morphisms are "mappings" that "spread" each point into the measure (this seems to have been the first topic, in order of discovery, connected with the category ideology; see [8]).

3. **p-adic groups.** See [2]. Here there exist analogues of the results of §§4 and 5 that are sufficiently unlike the real case so as to be of separate interest.

4. **Topological field theory.** See [6].

The author wishes to thank G. I. Ol′shanskiĭ for his helpful discussions touching on this subject.

§1. Example: the category Op

Our paper contains a number of classification theorems for the representation of categories. All the proofs have a generic trivial part. In this section we examine the simplest case, where the proof reduces entirely to this "generic part". The results of this section can hardly be called new, although they seem not to have previously received any explicit formulation.

1.0. **Notation.** Let $\mathscr{K}$ be a category, and let V and W be objects in it. We denote by $\operatorname{Mor}_{\mathscr{K}}(V, W) = \operatorname{Mor}(V, W)$ the set of morphisms $V \to W$. In addition, by $\Gamma_{\mathscr{K}}(V) = \Gamma(V)$ we mean the semigroup $\operatorname{Mor}(V, V)$, and by $\Gamma^0_{\mathscr{K}}(V)$ the set of invertible elements of $\Gamma_{\mathscr{K}}(V)$.

1.1. **The category** Op **and its representations.** The objects in Op are the finite-dimensional complex linear spaces, and the morphisms are the linear operators. We are interested in the *representations* $T = (T, \tau)$ of this category Op, i.e., the functors from Op to itself. Thus, for each linear space V (i.e., object in Op) we must construct a finite-dimensional linear space $T(V)$, and for each operator $A: V \to W$ an operator $\tau(A): T(V) \to T(W)$ such that $\tau(AB) = \tau(A)\tau(B)$ whenever the product AB is meaningful. We restrict ourselves to holomorphic representations; i.e., we shall suppose that $A \to \tau(A)$ is a holomorphic operator-valued function on $\operatorname{Mor}(V, W)$. The definitions of subrepresentation, irreducible representation, direct sum of representations, and tensor product of representations are entirely obvious, and will be omitted. Let us state, however, the definition of a morphism of representations (an "*intertwining transformation*") $(T, \tau) \to (T', \tau')$. This is a set of operators $S_V: T(V) \to T'(V)$ such that for any $A \in \operatorname{Mor}(V, W)$ the diagram

$$\begin{tikzcd} T(V) \arrow[r, "\tau(A)"] \arrow[d, "S_V"'] & T(W) \arrow[d, "S_W"'] \\ T'(V) \arrow[r, "\tau'(A)"] & T'(W) \end{tikzcd}$$

is commutative.

We give also the definition of *cyclic hull.* Let L be a subset of $T(V)$. In each space $T(W)$ consider the subspace $H(W)$ spanned by all the vectors of the form $\tau(Q)v$, where $v \in L$, $Q \in \mathrm{Mor}(V, W)$. It is easily seen that the set of subspaces $H(W)$ defines a subrepresentation in T, which we call the "cyclic hull of L".

Let $T = (T, \tau)$ be a representation of Op. Then in every space $T(V)$ there is an action of the group $\Gamma^0(V) = \mathrm{GL}(V) \supset \mathrm{SL}(V)$ and, more than that, of the semigroup $\Gamma(V)$ of all operators $V \to V$. These representations of the groups $\mathrm{SL}(V)$ and $\mathrm{GL}(V)$, and of the semigroup $\Gamma(V)$, will be called representations *subordinate* to the representation T of the category Op.

Lemma 1.1. a) *A representation* τ *of the group* $\mathrm{GL}(n, \mathbb{C})$ *has at most one continuous extension* τ' *to a representation of the semigroup* $\Gamma(\mathbb{C}^n)$. *If* τ *is reducible, so is* τ'.

b) *A holomorphic irreducible representation* τ *of the group* $\mathrm{GL}(n, \mathbb{C})$ *extends by continuity to a representation of the semigroup* $\Gamma(\mathbb{C}^n)$ *if and only if* τ *is polynomial.*

We recall that a representation of $\mathrm{GL}(V)$ is called *polynomial* if all its matrix elements are polynomials in the elements of the corresponding matrices $g \in \mathrm{GL}(V)$. Polynomial representations of $\mathrm{GL}(V)$ are completely reducible. All the irreducible polynomial representations of $\mathrm{GL}(V)$ can be realized in the contravariant tensors over the identity representation.

Proof. a) $\mathrm{GL}(n, \mathbb{C})$ is dense in $\Gamma(\mathbb{C}^n)$.

b) If τ is not polynomial, then it cannot be extended by continuity even to the multiplicative semigroup of all diagonal matrices. ●

Lemma 1.2. *Let* (T, τ) *be an irreducible representation of the category* Op. *Then all the subordinate representations* τ_n *of the groups* $\mathrm{GL}(n, \mathbb{C})$ *are irreducible.*

Proof. Let Q be a proper $\mathrm{GL}(n, \mathbb{C})$-invariant subspace in $T(\mathbb{C}^n)$. Then the cyclic hull of the subspace Q is a subrepresentation in T different from T itself. ●

1.2. **Schur-Weyl duality.** Let $L = (L, \lambda)$ be the identity representation of the category Op. Consider its nth tensor power $L^{\otimes n}$. In each space $L(V)^{\otimes n}$ there is defined an action, by permutation of factors, of the symmetric group S_n. Clearly, this action commutes with any operator $\lambda(P)^{\otimes n}$, where P is a morphism in Op; i.e., each element $g \in S_n$ determines an intertwining transformation $\sigma(g)\colon L^{\otimes n} \to L^{\otimes n}$.

Proposition 1.1. a) *Any* S_n*-intertwining operator* $L(V)^{\otimes n} \to L(W)^{\otimes n}$ *is a linear combination of operators of the form* $\lambda(P)$, *where* $P \in \mathrm{Mor}(V, W)$.

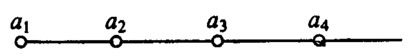
b) *Any intertwining transformation* $L^{\otimes n} \to L^{\otimes n}$ *is a linear combination of intertwining transformations of the form* $\sigma(g)$, *where* $g \in S_n$.

(This is an obvious corollary of the usual Schur-Weyl duality theorem.)

1.3. **The classification theorem.** It is reasonable to expect that all the representations of the category Op are well known, and that they are all realized in tensors over the identity representation. This is indeed the case.

Proposition 1.2. a) *The holomorphic representations of the category* Op *are completely reducible.*

b) *All the irreducible holomorphic representations* $T = (T, \tau)$ *of* Op *can be indexed by diagrams of the form*



where the markers a_j *are nonnegative integers, with only finitely many different from* 0. *Let* a_α *be the rightmost nonzero marker. If* $n \geq \alpha - 1$ (sic!), *then the subordinate* *representation of the group* $\mathrm{SL}(n, \mathbb{C})$ *has the markers* $a_1, \dots, a_n$ *on the Dynkin diagram of type* A_n. *If* $n < \alpha - 1$, *then the subordinate representation of* $\mathrm{SL}(n, \mathbb{C})$ *is zero-dimensional.*

Remark. The representation with the markers a_j is contained in the $(\sum a_j)$th tensor power of the identity representation of Op. More precisely, it is realized in the tensors corresponding to the Young diagram with rows of length $a_1, a_2, \dots$ (see [9]).

Lemma 1.3. *Let* $T = (T, \tau)$ *and* $T' = (T', \tau')$ *be two irreducible representations of the category* Op; *and suppose that for some* V *the corresponding subordinate representations of the group* $\mathrm{GL}(V)$ *are nonzero and equivalent. Then* $T = T'$.

Proof of the lemma. Consider the representation $T \oplus T'$. In $T(V) \oplus T'(V)$ choose an invariant subspace other than the two direct summands, and consider its cyclic hull $H \subset T \oplus T'$. Then it is easily seen that for any W the subspace $H(W)$ is the graph of an operator $S_W\colon T(W) \to T'(W)$. This set of operators determines an equivalence between the representations. ●

Proof of Proposition 1.2. b) Thus, a representation (T, τ) is completely determined by its subordinate representations. To prove Proposition 1.2b) it remains only to verify that no other irreducible representations than those realized in tensors over the identity representation can exist for the category Op. But in fact any representation of any semigroup $\Gamma(V)$ is subordinate to some one of the tensor representations of Op.

We come now to Proposition 1.2a). Let T be a representation of Op. We show first that T contains an irreducible subrepresentation. Choose a minimal n such that the representation of the semigroup $\Gamma(\mathbb{C}^n)$ subordinate to the representation T is nonzero. In the space $T(\mathbb{C}^n)$ pick an irreducible subrepresentation M of $\Gamma(\mathbb{C}^n)$, and consider its "cyclic hull" (S, σ) under the action of Op. We show that the representation S is irreducible. In $S(\mathbb{C}^m)$, where $m > n$, consider the set L of all vectors l such that $\sigma(R)l = 0$ for all $R \in \mathrm{Mor}(\mathbb{C}^m, \mathbb{C}^n)$. Suppose $L = S(\mathbb{C}^m)$. Any element X of the semigroup $\Gamma(\mathbb{C}^n)$ can be written $X = YZU$, where $U \in \mathrm{Mor}(\mathbb{C}^n, \mathbb{C}^m)$, $Z \in \mathrm{Mor}(\mathbb{C}^m, \mathbb{C}^m)$, $Y = \mathrm{Mor}(\mathbb{C}^m, \mathbb{C}^n)$. By our assumption, $\sigma(Y) = 0$, so that $\sigma(X) = 0$ for any $X \in \Gamma(\mathbb{C}^n)$. Contradiction.

Thus, $L \neq S(\mathbb{C}^m)$. Choose any invariant complement N to L in $S(\mathbb{C}^m)$, and consider the cyclic hull of N with respect to Op. Clearly, this cyclic hull contains $M = S(\mathbb{C}^n)$, and so must coincide with S. Hence $N = S(\mathbb{C}^m)$. This proves the existence of irreducible subrepresentations. ●

Now let V be a space of smallest dimension n for which $T(V) \neq 0$. Let S be an irreducible subrepresentation in T. In $T(V)$ choose a subspace $R(V)$ complementary to $S(V)$ and invariant with respect to $\mathrm{GL}(V)$. Let R be the cyclic hull of $R(V)$. Clearly, for any space W we have $R(W) \cap S(W) = 0$ (otherwise S would be reducible). It may be, however, that $R(W) \oplus S(W) \neq T(W)$. Let W be the space of minimal dimension for which this is so. In $T(W)$ choose a subspace $R'(W)$ invariant with respect to $\mathrm{GL}(W)$, complementary to $S(W)$, and containing $R(W)$. Consider the cyclic hull R' of $R'(W)$, etc. We obtain in this fashion an increasing sequence $R, R', R'', R''', \dots$ of subrepresentations in T. Then their union, by construction, is an invariant complement to S.

4. Commentary.

Remark 1. This proof of the classification theorem (Proposition 1.2) has three main elements:

a) We describe all the representations of the semigroup $\Gamma(\mathbb{C}^n)$ (Lemma 1.1).

b) We provide ourselves with a stock of representations of the category Op such that any representation of any semigroup $\Gamma(\mathbb{C}^n)$ is subordinate to at least one of the representations in our stock.

c) We show that any one representation of $\Gamma(\mathbb{C}^n)$ cannot be subordinate to two different representations of Op (Lemma 1.3).

This scheme will recur repeatedly in what follows. For now, we note only that in the case of projective representations the analogues of Lemma 1.3 may not prove to be so simple.

Remark 2. By the *N-kernel* of a representation (T, τ) we mean the set of all morphisms A such that $\tau(A) = 0$ (using the letter N to avoid mistakenly thinking that $\tau(A) = 1$). We call the representation N-faithful if the N-kernel consists of the zero operators only. For a representation with markers $a_1, \dots,$ the N-kernel consists of all operators of rank $< \alpha$, where α is as defined in the statement of Proposition 1.2.

§2. The categories connected with the complex classical groups

2.1. **Linear relations.** Let V and W be finite-dimensional linear spaces. By a *linear relation* $P: V \rightrightarrows W$ we mean an arbitrary subspace of $V \oplus W$. Linear relations can be naturally interpreted as not-everywhere-defined, multivalued linear operators. Such objects, in the words of Mac Lane [10], "occur more frequently than is usually realized", and in certain respects behave no worse than ordinary operators. (See [10], [11], §II.6; [12].)

For any linear relation $P: V \rightrightarrows W$ we make the following definitions:

1) The *kernel* $\mathrm{Ker}(P)$ is the set of all $v \in V$ such that $(v, 0) \in P$.

2) The *image* $\mathrm{Im}(P)$ is the projection of P on W parallel to V.

3) The *domain of definition* $D(P)$ is the projection of P on V parallel to W.

4) The *indefiniteness* $\mathrm{Indef}(P)$ is the set of all $w \in W$ such that $(0, w) \in P$.

5) The *rank*

$$\mathrm{rk}(P) = \dim D(P) - \dim \mathrm{Ker}(P) = \dim \mathrm{Im}(P) - \dim \mathrm{Indef}(P).$$

Finally, given two linear relations $P: V \rightrightarrows W$ and $Q: W \rightrightarrows Y$, we define their *product* $QP: V \rightrightarrows Y$; namely, $(v, y) \in QP$ if there exists a vector $w \in W$ such that $(v, w) \in P$ and $(w, y) \in Q$.

2.2. **The category** GA. We want to define a category of which the objects are linear spaces and the morphisms are linear relations. For our purposes the multiplication defined above for linear relations will not quite do. The set $m(V, W)$ of all linear relations $V \rightrightarrows W$ has a natural representation as the disjoint union of the grassmannians $\mathrm{Gr}_k(V \oplus W)$, where $\mathrm{Gr}_k(V \oplus W)$ means the set of all k-dimensional subspaces in $V \oplus W$. But it is obvious that for two linear relations $P: V \rightrightarrows W$ and $Q: W \rightrightarrows Y$ the number $\dim(QP)$ is not determined solely by the numbers $\dim(P)$ and $\dim(Q)$. Thus, our multiplication is not even separately continuous.

The objects of the category GA are the finite-dimensional complex linear spaces. The set $\mathrm{Mor}(V, W)$ consists of all linear relations $V \rightrightarrows W$, together with a formal "null" element null $= \mathrm{null}_{V,W}$ (not to be identified with any linear relation). If $P: V \rightrightarrows W$ and $Q: W \rightrightarrows Y$ are nonnull morphisms, with

$$\mathrm{Ker}(Q) \cap \mathrm{Indef}(P) = 0, \tag{2.1}$$

$$\mathrm{Im}(P) + D(Q) = W, \tag{2.2}$$

then Q and P multiply like linear relations. If one of the equalities (2.1) or (2.2) fails to hold, we put $QP = \mathrm{null}$. In addition, for any Q and P we put

$$Q \cdot \mathrm{null} = \mathrm{null} \cdot P = \mathrm{null} \cdot \mathrm{null} = \mathrm{null}.$$

Lemma 2.1. *The multiplication so constructed is associative.*

Proof. Direct verification. ●

We assign to the set of nonnull morphisms the same topology as before; and in addition we take $\mathrm{null}_{V,W}$ to be contained in the closure of any other point of the set $\mathrm{Mor}(V, W)$. Thus, we have defined on the set $\mathrm{Mor}(V, W)$ a non-Hausdorff topology.

Lemma 2.2. *Suppose* $P \in \mathrm{Mor}(V, W)$ *and* $Q \in \mathrm{Mor}(W, Y)$

a) *If* $QP \neq \mathrm{null}$, *then*

$$\dim QP = \dim P + \dim Q - \dim W.$$

b) *The multiplication* $(P, Q) \to QP$ *is continuous.*

Proof. In $Z = V \oplus W \oplus W \oplus Y$, consider the subspace H consisting of all vectors of the form (v, w, w, y). From condition (2.2) it follows that the sum of the subspaces H and $P \oplus Q$ in Z is all of Z, and therefore

$$\dim(H \cap (P \oplus Q)) = \dim P + \dim Q - \dim W.$$

Project the space $H \cap (P \oplus Q) \subset Z$ into the subspace $V \oplus Y$ of all vectors of the form $(v, 0, 0, y)$ parallel to the space X of all vectors of the form $(0, w, w, 0)$. If (2.1) holds, this projection mapping is injective. But this projection is precisely QP. This proves part a). From this argument also follows part b). ●

By a *projective representation* $T = (T, \tau)$ of the category GA we mean a functor from GA to the category of finite-dimensional linear spaces and mappings, the latter being specified up to multiplication by a scalar. To avoid possible ambiguity, we state this definition a little more carefully. For every object V in GA we must construct a linear space $T(V)$, and for every morphism $A\colon V \to V'$ an operator $\tau(A)\colon T(V) \to T(V')$ such that:

1) For any morphisms $A\colon V \to V'$ and $B\colon V' \to V''$ we have $\tau(B)\tau(A) = c(A, B)\tau(BA)$ where $c(A, B)$ is a *nonzero* complex number.

2) $\tau(\mathrm{null}) = 0$.

3) The mapping τ is continuous; i.e., for any convergent sequence of morphisms $A_n \to A$ there exist numbers λ_n such that $\lambda_n\tau(A_n) \to \tau(A)$.

4) We consider only holomorphic representations; i.e., we require that the operator-valued functions τ on all the spaces $\mathrm{Mor}(V, V')\backslash\mathrm{null}$ be holomorphic.

Remark. If we had not "rectified" the multiplication of linear relations, we would have $c(A, B) = 0$ at points of discontinuity. Allowing the cocycle $c(A, B)$ to vanish would give a definition of representation that would be satisfied by many objects of little interest. (To see this, it suffices to consider the case when $c(A, B) = 0$ identically.)

All the remaining definitions of §1 carry over to the category GA automatically.

The representation theory for the semigroup $\Gamma_{\mathrm{GA}}(V)$ does not reduce to that for the group $\mathrm{GL}(V)$, since $\mathrm{GL}(V) = \Gamma^0(V)$ is not dense in $\Gamma_{\mathrm{GA}}(V)$. We define the subsemigroup $\Gamma^*(V)$ of $\Gamma_{\mathrm{GA}}(V)$ as consisting of null and all the linear relations $V \rightrightarrows V$ of dimension $\dim V$. In it $\mathrm{GL}(V)$ is dense. As a semigroup, $\Gamma^*(V)$ is generated by the group $\mathrm{GL}(V)$ and two arbitrary relations P and Q subject to the conditions $\dim\mathrm{Ker}\,Q = 0$, $\dim\mathrm{Indef}\,Q = 1$, $\dim\mathrm{Ker}\,P = 1$, and $\dim\mathrm{Indef}\,P = 0$. For any representation τ of $\mathrm{GL}(V)$ there exist at most four extensions of τ to a projective representation of $\Gamma^*(V)$. Indeed, either $\tau(P) = 0$ or $\tau(P)$ is uniquely determined from continuity considerations; and similarly for Q.

Theorem 2.1. a) *The projective holomorphic representations of the category* GA *are completely reducible.*

b) *The irreducible projective holomorphic representations* (T, τ) *of* GA *can be indexed by diagrams of the form*

$$-\!\!-\!\!\overset{a_{-1}}{\circ}\!\!-\!\!-\!\!-\!\!\overset{a_0}{\circ}\!\!-\!\!-\!\!-\!\!\overset{a_1}{\circ}\!\!-\!\!-\!\!-\!\!\overset{a_2}{\circ}$$

where the markers a_j *are nonnegative integers, with only finitely many different from* 0. *Let* a_α *be the leftmost, and* a_β *the rightmost, nonzero numerical marker. If* $\beta - \alpha - 1 > n$, *then the subordinate representation of the group* $\mathrm{SL}(n+1, \mathbb{C})$ *is zero-dimensional. If* $\beta - \alpha - 1 \leq n$, *then the subordinate representation of* $\mathrm{SL}(n+1, \mathbb{C})$ *is the simple direct sum (i.e., each term has multiplicity* 1) *of all the representations* $\tau_{\mu,\nu}$ *of* $\mathrm{SL}(n+1, \mathbb{C})$ *with markers* $a_\mu, a_{\mu+1}, \ldots, a_\nu$ *on the Dynkin diagram of type* A_n, *where* μ *and* ν *are subject to the conditions* $\alpha \leq \mu+1$ *and* $\beta \geq \nu - 1$ $(\nu - \mu = n-1)$.

c) *Let* P *and* Q *be the same generators in* $\Gamma^*(V)$ *as above. Then* $\tau(P) \neq 0$ *if and only if* $\alpha < \mu + 1$, *and* $\tau(Q) = 0$ *if and only if* $\beta > \nu - 1$.

Remark 1. Essentially, part c) gives a description of the subordinate representations of the semigroups.

Remark 2. The N-kernel of (T, τ) (see §1.3) consists of all relations of rank $\beta - \alpha - 1$.

For the proof of the theorem, see §3.

2.3. **The categories** B, C, D. The objects of the category C are the finite-dimensional complex linear spaces, provided with a symplectic (i.e., nondegenerate bilinear skew-symmetric) form $\{\cdot, \cdot\}_V$. Let V and W be two objects in C. On $V \oplus W$ consider the symplectic form

$$\Lambda((v_1, w_1), (v_2, w_2)) = \{v_1, v_2\}_V - \{w_1, w_2\}_W.$$

We say that a linear relation $P\colon V \rightrightarrows W$ *preserves the symplectic form* if P is a Lagrange (i.e., maximal isotropic) subspace in $V \oplus W$.

Example. If $V = W$ and $A\colon V \to V$ is a symplectic operator, then the graph of A preserves the bilinear form. If V and W have different dimensions, then P cannot be the graph of an operator.

Lemma 2.3. *Let* V, W, *and* Y *be objects in the category* C, *and let* $P\colon V \rightrightarrows W$ *and* $Q\colon W \rightrightarrows Y$ *be relations preserving the symplectic form. Then* $QP\colon V \rightrightarrows Y$ *also preserves the symplectic form.*

Proof. Let Z, H, and X be as in Lemma 2.2. Then H is Lagrange, Z is coisotropic, and $X = Z^\perp$ is the skew-orthogonal complement to Z. From this it is easily seen that the image of H in $Z/Z^\perp$ is a Lagrange subspace of $Z/Z^\perp$.

This lovely assertion bears within it the elements of a pathology. Namely, as in §2.2, the multiplication so constructed is not even separately continuous.

So, let V and W be objects in the category C. A morphism from V to W is either null or a linear relation $P\colon V \rightrightarrows W$ preserving the symplectic form. Morphisms multiply as in GA. (Note that conditions (2.1) and (2.2) are in this case equivalent.)

It is easily seen that $\Gamma^0_C(\mathbb{C}^{2n}) \simeq \mathrm{Sp}(2n, \mathbb{C})$.

The categories B and GD are defined in almost the same way. The only difference is that an object in B (resp. GD) is an odd-dimensional (resp. even-dimensional) complex orthogonal space (i.e., a space provided with a nondegenerate bilinear form). It is easily seen that $\Gamma^0_B(\mathbb{C}^{2n+1}) = O(2n+1, \mathbb{C})$ and $\Gamma^0_{\mathrm{GD}}(\mathbb{C}^{2n}) = O(2n, \mathbb{C})$.

Let H be an even-dimensional orthogonal space. Let $\mathrm{Gr}^0(H)$ be the grassmannian of all maximal isotropic subspaces in H. It is easily seen that H consists of two connected components (two subspaces P and Q lie in the same component if and only if $P \cap Q$ has even codimension in P and Q). We would like to construct a category D such that $\Gamma_D^0(\mathbb{C}^{2n}) = \mathrm{SO}(2n, \mathbb{C})$. To do so, we must canonically select one component from each set $\mathrm{Mor}(V, W)$. An object of the category D is an orthogonal space V with one of the two components of the grassmannian $\mathrm{Gr}^0(V)$ fixed. This component we call $\mathrm{Gr}_+^0(V)$. Let V and W be objects in D, and suppose $V_+ \in \mathrm{Gr}_+^0(V)$, $W_+ \in \mathrm{Gr}_+^0(W)$, and $V_- \in \mathrm{Gr}^0(V)$ is transversal to V_+. Then $P \in \mathrm{Mor}_{\mathrm{GD}}(V, W)$ is called a morphism in D if the dimension of $P \cap (V_- \oplus W_+)$ is even. Representations of the categories B, C, D, and GD are defined as in §2.2.

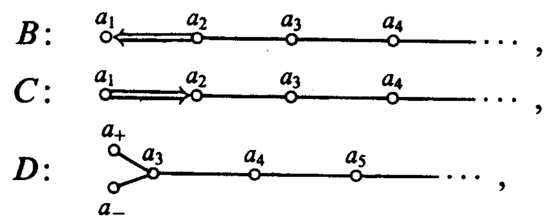
For each of the categories B, C, and D the group $\Gamma^0(V)$ is dense in the semigroup $\Gamma(V)$. Furthermore, as a semigroup, $\Gamma(V)$ is generated by the group $\Gamma^0(V)$ and any one element P such that $\dim \operatorname{Ker} P = \dim \operatorname{Indef} P = 1$. Hence a representation τ of $\Gamma^0(V)$ can be extended to $\Gamma(V)$ in only two ways:

1) The zero extension: $\tau(P) = 0$, and consequently $\tau(Q) = 0$ for any $Q \in \Gamma(V) \backslash \Gamma^0(V)$.

2) A maximal extension: $\tau(P) \neq 0$.

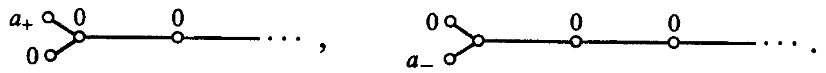
Theorem 2.2. a) *The holomorphic projective representations of the categories* B, C, D *and* GD *are completely reducible.*

b) *The irreducible holomorphic projective representations of* B, C, *and* D *can be indexed by diagrams of the form*



where the markers a_j *are nonnegative integers, with only finitely many different from* 0. *Consider, e.g., the case of* C (*the two other cases are similar*). *Let* a_α *be the rightmost nonzero marker. If* $n < \alpha - 1$, *then the subordinate representation of the group* $\mathrm{Sp}(2n, \mathbb{C})$ *is zero. If* $n \geq \alpha - 1$, *the subordinate representation of* $\Gamma_C^0(\mathbb{C}^{2n})$ *is the irreducible representation with markers* $a_1, \ldots, a_n$ *on a Dynkin diagram of type* C_n. *If* $n = \alpha - 1$, *the subordinate representation of the semigroup* $\Gamma_C(\mathbb{C}^{2n})$ *is the zero extension of the representation of* $\mathrm{Sp}(2n, \mathbb{C})$. *If* $n > \alpha - 1$, *it is the maximal one.*

Remark 1. The N-kernel of an irreducible representation (T, τ) consists of all relations of rank $< \alpha - 1$. An exception is the case of representations of the category D that have diagrams of the forms



These representations take relations of rank 1 into nonzero operators. The set $\mathfrak{n}$ of relations $V \rightrightarrows W$ of rank 0 consists of two connected components. Representations with diagrams of the first form are zero on one of these components of $\mathfrak{n}$; and of the second form, on the other.

Remark 2. We see that in contrast to the category Op the representations of the categories $\mathscr{K} = B, C, D$ are not uniquely determined by the subordinate representation of a fixed group $\Gamma_{\mathscr{K}}^0(V)$ (or even of the semigroup $\Gamma_{\mathscr{K}}(V)$). But if two irreducible representations T and T' of the category $\mathscr{K}$ are such that the corresponding subordinate representations of some semigroup $\Gamma_{\mathscr{K}}(V)$ coincide and are not identically zero on $\Gamma_{\mathscr{K}}(V) \backslash \Gamma_{\mathscr{K}}^0(V)$, then $T = T'$.

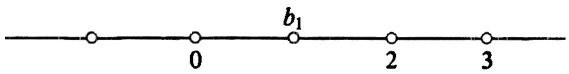
Proof. See §3.

2.4. The categories $A(\lambda)$. The categories defined below will play an ancillary role in this paper: they are used in the proof of Theorem 2.1.

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a sequence of zeros and ones. Let $c_k = \sum_{j=1}^{k} \lambda_j$. The objects in the category $A(\lambda)$ are the same as in the category GA. The set $\operatorname{Mor}(V, W)$ consists of null and of all linear relations $V \rightrightarrows W$ of dimension $n + c_m - c_n$, where $m = \dim W$, $n = \dim V$. The multiplication rules are the same as in GA. The basic virtue of the category $A(\lambda)$ is that the semigroup $\Gamma(V)$ is connected (it coincides with the semigroup $\Gamma^*(V)$ of §2.2).

Theorem 2.3. a) *The holomorphic projective representations of the category* $A(\lambda)$ *are completely reducible.*

b) *Suppose among the numbers* $\lambda_1, \lambda_2, \dots$ *there are infinitely many zeros and infinitely many ones. Then the irreducible holomorphic representations of* $A(\lambda)$ *can be indexed by diagrams of the form*

(2.3) 

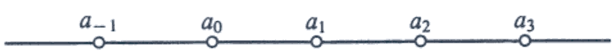
where the position of the marker b_1 *is fixed, while the marker* b_2 *appears to the left of* b_1 *if* $\lambda_2 = 0$ *and to the right if* $\lambda_2 = 1$. *If* $\lambda_3 = 0$, *the marker* b_3 *appears to the left of the segment containing* b_1 *and* b_2; *if* $\lambda_3 = 1$, *then* b_3 *appears to the right; etc. All the* b_j *are nonnegative integers, of which only finitely many are different from* 0. *To obtain the subordinate representation of the group* $A_n = \mathrm{SL}(n+1, \mathbb{C})$, *as we need to cut out from the diagram* (2.3) *the part containing the markers* $b_1, \dots, b_n$. *If the piece cut off on the left is not of the form*

or if the one on the right is not of the form

then the subordinate representation of $\mathrm{SL}(n+1, \mathbb{C})$ *is zero-dimensional. In any other case we consider the part of* (2.3) *of length* n *that has been cut out; then the set of markers so obtained is the set for the subordinate representation of the group* $\mathrm{SL}(n+1, \mathbb{C})$.

Remark. If the sequence $\lambda_1, \lambda_2, \dots$ has only finitely many zeros or finitely many ones, then to every diagram (2.3) correspond two representations of $A(\lambda)$; this is because of the nonuniqueness of the extension of the representation from $\mathrm{SL}(n+1, \mathbb{C})$ to $\Gamma^*(\mathbb{C}^{n+1})$.

Remark. In what follows we shall use a different indexing of markers on diagrams of the $A(\lambda)$ type. Namely, instead of (2.3) our diagram will be



§3. Construction of the fundamental representations of the categories GA, B, C, D, $A(\lambda)$, and proof of the classification theorems

Let $\mathscr{K}$ be one of the categories GA, B, C, D, $A(\lambda)$. By the *fundamental representation* $P^{\alpha}_{\mathscr{K}}$ of $\mathscr{K}$ will be meant the irreducible representation for which the marker a_α on the Dynkin diagram is equal to 1 and all other markers are 0. We note that α can run over the values $1, 2, 3, \dots, +, -$; furthermore, for the category GA the fundamental representation is unique, and we shall denote it by P_{GA}. All the fundamental representations of all these categories will be constructed in §§3.1–3.4, and in §3.5 all the remaining representations. In §3.6 we prove Theorems 2.1–2.3.

3.0. **Retrospection on the fundamental representation of the classical groups.** We denote by $\pi^\alpha(G)$ the fundamental representation of the classical group G for which the marker a_α is equal to 1 and all the other markers at 0. By $\Lambda^k\tau$ we denote the kth exterior power of the representation τ. Let $\mu = \mu_G$ be the identity representation of one of the groups $G = A_n, B_n, C_n, D_n$. We list here the fundamental representations of these groups.

A_n: $\pi^k(A_n) = \Lambda^k\mu$, $k = 1, 2, \dots, n$.

B_n: $\pi^{k+1}(B_n) = \Lambda^{n-k}\mu$, $k = 1, 2, \dots, n-1$.

C_n: To construct all the fundamental representations of the groups C_n, consider the exterior powers $\Lambda^1\mu, \Lambda^2\mu, \dots, \Lambda^n\mu$. Clearly, $\Lambda^2\mu$ has an invariant vector, which we denote by q (it exists because μ has an invariant skew-symmetric form). Exterior multiplication by q is obviously an intertwining operator $\Lambda^s\mu \to \Lambda^{s+2}\mu$. Then $\pi^{k+1}(C_n)$ is the factor representation $\Lambda^{n-k}\mu/q\Lambda^{n-k-2}\mu$, $k = 1, 2, \dots, n$.

D_n: $\pi^{k+2}(D_n) = \Lambda^{n-k}\mu$, $k = 1, 2, \dots, n-2$.

The constructions of the "spin representations" $\pi^1(B_n)$ and $\pi^\pm(D_n)$, omitted from this list, are more complicated. Our immediate objective is to reinterpret the constructions of the fundamental representations of all these groups in categorical terms.

3.1. **The spin representation of the category** GD. Let $\xi_1, \xi_2, \dots, \xi_n$ be a set of pairwise anticommuting variables; i.e.,

$$\xi_i\xi_j = -\xi_j\xi_i$$

for any i and j. Let Λ_n be the algebra of functions in the variables ξ_i. As is known, the spin representation of the group $O(2n, \mathbb{C})$ is realized in Λ_n.

Consider a $2n$-dimensional complex linear space V with a basis $e_1^V, \dots, e_n^V, f_1^V, \dots, f_n^V$ and symmetric bilinear form

$$\{e_i^V, e_j^V\} = \{f_i^V, f_j^V\} = 0, \qquad \{e_i^V, f_j^V\} = \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$. To each $v = \sum v_i^+ e_i + \sum v_i^- f_i \in V$ associate the operator

$$\hat{a}(v) = \sum v_i^+\xi_i + \sum v_i^- \frac{\partial}{\partial \xi_i}$$

in $\Lambda(V)$.

Let W be a $2m$-dimensional complex linear space provided with a similar basis and similar bilinear form, and let $\eta_1, \dots, \eta_m$ be corresponding anticommuting variables.

Theorem 3.1. a) *For any nonnull* $T \in \mathrm{Mor}_{\mathrm{GD}}(V, W)$ *there is an operator* $\mathrm{Spin}(T)$ $\Lambda_n \to \Lambda_m$, *unique up to proportionality, such that*

$$\hat{a}(v)\mathrm{Spin}(T) = \mathrm{Spin}(T)\hat{a}(w) \tag{3.1}$$

for all $(v, w) \in T$.

b) *The mapping* $T \to \mathrm{Spin}(T)$ *is a projective representation of the category* GD.

(For details on the spin representation, see [3].)

Proof. a) We start from the fact that if $W = V$, and $T \in \Gamma^0(O)(2n, \mathbb{C})$, then $\mathrm{Spin}(T)$ is the ordinary spin representation operator for the group $O(2n, \mathbb{C})$. Now let $T: V \rightrightarrows W$ be arbitrary, with $k = \operatorname{rk} T$. Write T as a product $T = PQR$, where $P \in O(2m, \mathbb{C})$, $R \in O(2n, \mathbb{C})$, and $Q \in \operatorname{Mor}(V, W)$ has a basis of the form (e_i^V, e_i^W), (f_i^V, f_i^W), $(e_j^V, 0)$, $(0, f_s^W)$, with $k < j \le n$, $k < s \le m$, $1 \le i \le k$. Put

$$\operatorname{Spin}(Q)\xi_{\alpha_1} \cdots \xi_{\alpha_m} = \begin{cases} \eta_{\alpha_1} \cdots \eta_{\alpha_m} & \text{if } \alpha_\nu \le k, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that this operator satisfies equality (3.1). Now put $\operatorname{Spin}(T) = \operatorname{Spin}(P)\operatorname{Spin}(Q)\operatorname{Spin}(R)$.

b) In essence, this follows from a). The only thing that needs verification is the condition for the product $\operatorname{Spin}(S)\operatorname{Spin}(T)$ to vanish. This is most easily dealt with by reducing the pair of relations S and T to a sufficiently simple canonical form. ●

3.2. **The spin representations of the categories B and D.** The restriction of the representation Spin to the category D is $P_D^+ \oplus P_D^-$. (We observe the formality that by $P_D^\pm$ is meant the representation of D such that the corresponding subordinate representation of $\mathrm{SO}(2n, \mathbb{C})$ coincides with $\pi^\pm(\mathrm{SO}(2n, \mathbb{C}))$. We have so far only constructed $P^+(D)$ and $P^-(D)$, but not proved their uniqueness.) Further, $\pi^1(\mathrm{SO}(2n+1, \mathbb{C}))$ is the restriction of $\pi^\pm(\mathrm{SO}(2n+2, \mathbb{C}))$ to the subgroup $\mathrm{SO}(2n+1, \mathbb{C})$. Similarly, P_B^1 is the restriction of $P_D^\pm$ to the category B.

3.3. **The representation P_{GA}.** Let V be an object in GA. Let V' be the dual space to V. In $F(V) = V \oplus V'$ define the symmetric bilinear form

$$\{(v_1, f_1), (v_2, f_2)\} = f_1(v_2) + f_2(v_1).$$

Thus, $F(V) \in \operatorname{Ob}(\mathrm{GD})$. Now suppose $T \in \operatorname{Mor}_{\mathrm{GA}}(V, W)$. Let T' be the annihilator of T in $V' \oplus W'$. Then the subspace $F(T) = T \oplus T'$ in $F(V) \oplus F(W)$ is a morphism in the category GD. We also put $F(\text{null}) = 0$.

Thus, F is a functor from GA to GD. Restricting the spin representation of GD to GA, we obtain, as is easily seen, a representation P_{GA} (the same stipulations being needed here as in §3.2).

There is a more direct and more transparent description of $P_{\mathrm{GA}} = (P, \pi)$. Let $\Lambda(V)$ be the exterior algebra on the space V. Then $P(V) = \Lambda(V)$. Suppose $T \in \operatorname{Mor}(V, W)$. Then $\pi(T) = HQR$, where the operators H, Q, R are defined as follows.

1. $R: \Lambda(V) \to \Lambda(D(T))$ is the operator of interior multiplication by $f_1 \wedge f_2 \wedge \cdots \wedge f_s$, where $f_1, \dots, f_s$ is a basis in the space of linear functionals on V that annihilate $D(T)$ ($D(T)$ has been defined in §2.1).
2. $Q: \Lambda(D(T)) \to \Lambda(W/\operatorname{Indef}(T))$. For any linear relation T there is a naturally constructed operator $T': D(T) \to W/\operatorname{Indef}(T)$. The operator Q is just the corresponding mapping of exterior algebras.
3. $H: \Lambda(W/\operatorname{Indef}(T)) \to \Lambda(W)$ is the operator of exterior multiplication by $e_1 \wedge e_q$, where $e_1, \dots, e_q$ is a basis in $\operatorname{Indef}(T)$.

The fact that (P, π) is actually a representation of the category GA is most easily seen by verifying that $\pi(T) = \operatorname{Spin}(F(T))$.

3.4. **The remaining fundamental representations.** By construction, all the categories $\mathscr{K} = B, C, D, A(\lambda)$ are contained in GA. All their fundamental representations not yet constructed are contained in the restrictions of the representation P_{GA} to $\mathscr{K}$.

For each fundamental representation we indicate here only the space $P^\alpha(V)$.

1. $P_C^\alpha(\mathbb{C}^{2n}) = \Lambda^{n-\alpha+1}(\mathbb{C}^{2n})/q\Lambda^{n-\alpha-1}(\mathbb{C}^{2n})$, $\alpha = 1, 2, \dots$; the element q has been defined in §3.0.
2. $P_B^\alpha(\mathbb{C}^{2n+1}) = \Lambda^{n-\alpha+1}(\mathbb{C}^{2n+1})$, $\alpha = 2, 3, \dots$.

3. $P_D^\alpha(\mathbb{C}^{2n}) = \Lambda^{n-\alpha+1}(\mathbb{C}^{2n})$, $\alpha = 3, 4, \dots$.
4. $P_{A(\lambda)}^\alpha(\mathbb{C}^n) = \Lambda^{\alpha+h_n}(\mathbb{C}^n)$, $-\infty < \alpha < \infty$, $h_n = \sum_{i=1}^n (2\lambda_i - 1)$.

The description of the operators of the representation is clear from §3.3.

3.5. **Construction of the remaining representations.** We start with the case $\mathscr{K} = B, C, D, A(\lambda)$. Suppose we want to construct an irreducible representation with markers a_μ. For each μ consider the a_μth tensor power $(P_{\mathscr{K}}^\mu)^{\otimes a_\mu}$ of the fundamental representation $P_{\mathscr{K}}^\mu$; take the tensor product of these tensor powers; and in the representation T so obtained for the category $\mathscr{K}$ take in each space $T(V)$ the cyclic hull $S(V)$ of the vector of highest weight. This gives the desired representation.

The validity of this construction is not entirely obvious and needs verification.

Lemma 3.1. *Let $P_{\mathscr{K}}^\mu$ be one of the fundamental representations of the category $\mathscr{K} = B, C, D, A(\lambda)$. Let $V, W \in \mathrm{Ob}(\mathscr{K})$ be such that $P_{\mathscr{K}}^\mu(V) \neq 0$ and $P_{\mathscr{K}}^\mu(W) \neq 0$; and let h be a vector of highest weight in $P_{\mathscr{K}}^\mu(V)$. Then there exists a $Q \in \mathrm{Mor}(V, W)$ such that $\pi_{\mathscr{K}}^\mu(Q)h$ is a vector of highest weight in $P_{\mathscr{K}}^\mu(W)$.*

Proof. Direct verification. ●

It follows from the lemma that the set of subspaces $S(V) \subset T(V)$ does indeed give a subrepresentation in T, and that this subrepresentation is irreducible. It is clear also that $S(V)$ satisfies all the conditions of Theorems 2.2 and 2.3.

Consider now the category $\mathscr{K} = \mathrm{GA}$. The restriction of the fundamental representation P_{GA} to $\mathrm{SL}(n+1, \mathbb{C})$ is $\bigoplus_{k=0}^{n+1} \Lambda^k \mu$, where μ is the identity representation of $\mathrm{SL}(n+1, \mathbb{C})$. Let $v_k^{(n)}$ be a vector of highest weight in $\Lambda^k \mu$. To construct a representation of GA with markers $(\dots, 0, 0, \dots, a_\alpha, \dots, a_\beta, 0, \dots)$ we take the $(\sum a_j)$th tensor power of P_{GA}, choose $n > \beta$, and in the subordinate representation of $\mathrm{SL}(n+1, \mathbb{C})$ take the vector $h = v_\alpha^{\otimes a_\alpha} \otimes \cdots \otimes v_\beta^{\otimes a_\beta}$. The desired representation is the cyclic hull of h with respect to GA. That this is indeed the desired representation, i.e., that it satisfies part b) of Theorem 2.1, will be proved in §3.6.

3.6. **Proof of the classification theorems 2.1–2.3.** We start by proving the completeness of the lists of representations for the category C (the proofs for B, D, and $A(\lambda)$ are similar).

Denote by Γ_n the semigroup of morphisms from a $2n$-dimensional object in the category C to itself. Let $T = (T, \tau)$ be a representation of the category C. Let τ_n be the representation of Γ_n subordinate to the representation T.

Lemma 3.2. a) *Let $T^{(1)}$ and $T^{(2)}$ be two irreducible holomorphic projective representations of the category C. If for some n we have $\tau_n^{(1)} = \tau_n^{(2)}$, then also $\tau_{n-1}^{(1)} = \tau_{n-1}^{(2)}$.*

b) *If in addition $\tau_n^{(1)}$ is identically zero on $\Gamma_n \backslash \mathrm{Sp}(2n, \mathbb{C})$, then $\tau_{n-1}^{(i)}$ is zero-dimensional.*

Proof. a) In the $2n$-dimensional object V_{2n} of the category C, choose a basis $e_1, \dots, e_{2n}$ such that $\{e_{2i-1}, e_{2i}\} = 1$ and the remaining pairs of basis vectors are orthogonal. In the $(2n-2)$-dimensional object V_{2n-2} of C choose a similar basis $e_1', \dots, e_{2n-2}'$. Consider the linear relation $Q: V_{2n} \rightrightarrows V_{2n}$ with basis (e_i, e_i), $(e_{2n-1}, 0)$, $(0, e_{2n})$, where $i = 1, \dots, 2n-2$. It is easily seen that $Q^2 = Q$, so that the operator $\tau^{(i)}(Q)$ is a projection. Consider also the linear relation $P: V_{2n-2} \rightrightarrows V_{2n}$ with basis (e_i', e_i), $(0, e_{2n})$, where $i = 1, 2, \dots, 2n-2$, and the linear relation $R: V_{2n} \rightrightarrows V_{2n-2}$ with basis (e_i, e_i'), $(e_{2n-1}, 0)$, where $i = 1, 2, \dots, 2n-2$. Now observe that all the relations P, Q, R commute with the elements of the group $\mathrm{Sp}(2n-2, \mathbb{C})$ (more than that, with the elements of the semigroup Γ_{n-1}).

Furthermore, it is easily seen that

$$QP = P, \quad RQ = R, \quad RP = E. \tag{3.2}$$

It follows that the $\mathrm{Sp}(2n-2, \mathbb{C})$-intertwining operators

$$\tau^{(i)}(P)\colon T^{(i)}(V_{2n-2}) \to \operatorname{Im}\tau^{(i)}(Q),$$
$$\tau^{(i)}(R)\colon \operatorname{Im}\tau^{(i)}(Q) \to T^{(i)}(V_{2n-2})$$

are inverse to one another. But the representations of the semigroup Γ_{n-1} in the spaces $\operatorname{Im}\tau^{(1)}(Q)$ and $\operatorname{Im}\tau^{(2)}(Q)$ are equivalent (by hypothesis). Hence, so are $\tau^{(1)}_{n-1}$ and $\tau^{(2)}_{n-1}$.

b) By hypothesis, $\tau(Q) = 0$. From (3.2) we find $\tau(P) = 0$, and therefore $0 = E$. ●

We pass directly to the proof of Theorem 2.2. Let $T = (T, \tau)$ be an irreducible representation of the category C. By Lemma 3.2b), there exists a sufficiently large n such that the subordinate representation τ_n of the semigroup Γ_n is nonzero on $\Gamma_n \backslash \mathrm{Sp}(2n, \mathbb{C})$; i.e., in the terminology of §2, τ_n is a maximal extension of the representation of the group $\mathrm{Sp}(2n, \mathbb{C})$. Let $T' = (T', \tau')$ be one of the representations constructed in §3.5 such that $\tau_n = \tau'_n$. By Lemma 3.2a), we have for all $m < n$ the equality $\tau_m = \tau'_m$. Suppose $\tau_k \neq \tau'_k$ for some $k > n$. Choose a representation $T'' = (T'', \tau'')$ out of those constructed in §3.5 such that $\tau''_k = \tau_k$. Then by Lemma 3.2a), $\tau''_n = \tau_n = \tau'_n$. But this implies $\tau''_k = \tau'_k$ (see Remark 2 in §2.3).

Thus, for all l we have $\tau_l = \tau'_l$. We must still verify that $\tau(S) = \tau'(S)$ for all morphisms $S\colon Y \to W$ for which $Y \neq W$. Let R and P be as in the proof of Lemma 3.2. Then $\tau(R)$ and $\tau(P)$ are operators intertwining two irreducible representations. They are therefore uniquely reconstructed from τ_n and τ_{n-1}. But the groupoid of morphisms of the category C is generated by the semigroups Γ_n and all possible operators of the forms R and P (recall that R and P are constructed for any natural number n). Thus, $T = T'$. This proves the theorem. ●

It remains to prove the completeness of the lists in the case of the category GA, and also, to fill the gap left at the end of §3.5. These tasks are both very simple: we need only restrict the irreducible representation of GA to the category $A(\lambda)$ and make use of the classification theorem for $A(\lambda)$ already proved. We must also verify complete reducibility. For $\mathscr{K} = B, C, D, A(\lambda)$ the proof is a verbatim repetition of the proof of Proposition 1.2a). For $\mathscr{K} = \mathrm{GA}$, we can again make use of the category $A(\lambda)$.

§4. The category U and representations of the groups $U(p, q)$ with highest weight

It seems natural to consider categories analogous to the categories GA, B, C, D, but with the complex spaces replaced by real ones (an attempt at this has already been made, to be sure, but for other purposes, in, e.g., [13]). The categories so obtained, however, most likely have no substantive theory of representations. An obstacle, at least on a heuristic level, is the Howe-Moore theorem [14], which asserts that the weak closure of the set of operators for a unitary representation is, as a rule, trivial (so that the representation cannot be extended by continuity from the group to the semigroup, since in this case the group is dense in the semigroup). Nevertheless, it turns out that with any series of real classical simple Lie groups a certain category can still be associated in a natural fashion. These categories and their representations are examined in this and the following section.

4.0. **Motivation of the definition.** It is known that the unitary representations of semisimple Lie groups with highest weight can be extended to the complex domain [15]. In particular, the representations of $U(p, q)$ with highest weight extend to the so-called semigroup of J-contractions. This semigroup $\Gamma_{p,q}$ consists of the operators A that "contract" the pseudo-hermitian form $\Lambda(\cdot, \cdot)$, i.e., that satisfy the condition

$$\Lambda(Ax, Ax) \leq \Lambda(x, x).$$

Clearly, $U(p, q) \subset \Gamma_{p,q} \subset \mathrm{GL}(p+q, \mathbb{C})$. It turns out (see [2] for the case of the groups $\mathrm{Sp}(2n, \mathbb{R})$) that the representations with highest weight extend to a certain larger semigroup $\widetilde{\Gamma}_{p,q} \supset \Gamma_{p,q}$, where $\widetilde{\Gamma}_{p,q} \not\subset \mathrm{GL}(p+q, \mathbb{C})$. This larger semigroup is in fact a semigroup of linear relations.

4.1. **The category** U. An object of the category is a complex linear space V, provided with a nondegenerate hermitian form $\Lambda_V(\cdot, \cdot)$. The positive and negative indices of inertia of this form will be denoted by p_V and q_V. A morphism from V to W is a linear relation $R: V \rightrightarrows W$ satisfying the following conditions:

1) If $(v, w) \in R$, then $\Lambda_V(v, v) \geq \Lambda_W(w, w)$.

2) If $v \in \mathrm{Ker}(R)$, then $\Lambda_V(v, v) > 0$; if $w \in \mathrm{Indef}(R)$, then $\Lambda_W(w, w) < 0$ (note that the nonstrict equality already follows from 1).

3) $\dim R = p_V + q_W$ (i.e., the maximum possible dimension for a relation satisfying 1)).

In each $V \in \mathrm{Ob}(U)$ take a fixed decomposition $V = V_+ \oplus V_-$ into a direct sum such that the form Λ_V is positive definite on V_+ and negative definite on V_-. Then a subspace $R \in \mathrm{Mor}(V, W)$ must, as is easily seen, be the graph of an operator $V_+ \oplus W_- \to V_- \oplus W_+$. Its matrix (called a Potapov-Ginzburg transformation in the terminology of [11])

$$\pi o\tau(R) = \begin{pmatrix} K & L \\ M & N \end{pmatrix}$$

has dimension $(p_V + q_W) \times (p_W + q_V)$ and must satisfy the following conditions:

1*) $\|\pi o\tau(R)\| \leq 1$, where $\|\cdot\|$ denotes the Euclidean norm of the matrix (consequence of 1)).

2*) $\|K\| < 1$, $\|M\| < 1$ (consequence of 2)).

Conversely, any matrix, of appropriate dimension, that satisfies conditions 1*) and 2*) is the Potapov-Ginzburg transformation for some morphism $R: V \to W$. If

$$\pi o\tau(R) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \pi o\tau(Q) = \begin{pmatrix} K & L \\ M & N \end{pmatrix}$$

then

$$\pi o\tau(QR) = \begin{pmatrix} K + LA(1-NA)^{-1}M & L(1-AN)^{-1}B \\ C(1-NA)^{-1}M & D + C(1-NA)^{-1}NB \end{pmatrix} \tag{4.1}$$

(by direct, though not very pleasant, verification).

4.2. **Matrix balls and generalized linear-fractional mappings.** Consider the functor L that assigns to each $V \in \mathrm{Ob}(U)$ the set $L(V) = \mathrm{Mor}(\mathbb{C}^0, V)$, and to each morphism $P: V \to W$ the mapping $l_P: L(V) \to L(W)$ given by the formula $l_P(Q) = PQ$.

The Potapov-Ginzburg transformation for $Q \in \mathrm{Mor}(\mathbb{C}^0, V)$ is a matrix of dimension $q_V \times p_V$ with norm < 1; i.e., the set $L(V)$ gives rise to a so-called matrix ball. Furthermore, the mapping l_P is specified, at the level of Potapov-Ginzburg transformations, by the formula (a special case of (4.1))

$$l_P(Z) = K + LZ(1-NZ)^{-1}M, \tag{4.2}$$

where $\pi \circ \tau(P) = \begin{pmatrix} K & L \\ M & N \end{pmatrix}$. Such mappings (introduced by M. G. Kreĭn) are called generalized linear-fractional.

4.3. **The representations** $L(R_1, R_2, \lambda)$. In this section we construct a set of projective representations $L(R_1, R_2, \lambda)$ that exhausts all irreducible holomorphic representations of the category U. We first define certain auxiliary representations $M(R_1, R_2, \lambda)$. Let (R_1, ρ_1) and (R_2, ρ_2) be two irreducible representations of the category Op; and let $\lambda \in \mathbb{C}$. In each object V of U take a fixed decomposition $V = V_+ \oplus V_-$; let $\mathscr{Z}(V)$ be the corresponding matrix ball. We define the space $M(V) = M_{R_1, R_2, \lambda}(V)$ as the space of all holomorphic functions on $\mathscr{Z}(V)$ with values in $R_1(\mathbb{C}^{p_V}) \otimes R_2(\mathbb{C}^{q_V})$ (recall that p_V and q_V are the indices of inertia of the form). We define operators

$$\mu_{R_1, R_2, \lambda}(P): M_{R_1, R_2, \lambda}(V) \to M_{R_1, R_2, \lambda}(W),$$

where $P \in \mathrm{Mor}(V, W)$, as follows. Suppose $\pi \circ \tau(P) = \begin{pmatrix} K & L \\ M & N \end{pmatrix}$ and $Z \in \mathscr{Z}(W)$. Then

$$\begin{aligned} \mu_{R_1, R_2, \lambda}(P) f(Z) = {} & \det[(1 - NZ)^{-\lambda/2}] \rho_1(L(1 - ZN)^{-1}) \\ & \otimes \rho_2((1 - NZ)^{-1} M) f(K + LZ(1 - NZ)^{-1} M). \end{aligned} \tag{4.3}$$

Now consider the representation $M(R_1, R_2, \lambda, p, q)$ of the group $U(p, q)$ subordinate to the representation $M(R_1, R_2, \lambda)$ of the category U. By a well-known theorem of Harish-Chandra (see, e.g., [16]), the representation $M(R_1, R_2, \lambda, p, q)$ contains a unique irreducible subrepresentation $L(R_1, R_2, \lambda, p, q)$. It is important to observe that $L(R_1, R_2, \lambda, p, q)$ are precisely all the irreducible representations of $U(p, q)$ with highest weight. The set of subrepresentations $L(R_1, R_2, \lambda, p, q) \subset M(R_1, R_2, \lambda, p, q)$ determines a subrepresentation in $M(R_1, R_2, \lambda)$ and it is this that we denote by $L(R_1, R_2, \lambda)$.

4.4. **Classification of the holomorphic representations of the category** U. We start with a definition of the notions of "holomorphic representation" and "equivalent representations". As in the case of nonunitary representations of Lie groups, there are wide possibilities here for arbitrariness in the formulations, but the notions obtained are in fact independent of this arbitrariness.

We require that the spaces $T(V)$ of the representation $T = (T, \tau)$ be Banach (a possibly more natural requirement might be local convexity and completeness), and that the function τ be a uniformly holomorphic operator-valued function on the interior of the domain $\mathrm{Mor}(V, W)$ and weakly continuous up to the boundary. From this it follows rather easily that all the subordinate representations of the groups $U(p, q)$ have highest weight. (Indeed, consider a compact Cartan subgroup H in $U(p, q)$; let $H_{\mathbb{C}} \subset \mathrm{GL}(p + q, \mathbb{C})$ be its complexification; and let Δ be the subgroup consisting of all J-contractions contained in $H_{\mathbb{C}}$. Then any operator $\tau(\delta)$, where $\delta \in \Delta$, must have (in view of its boundedness) a bounded spectrum, which imposes severe restrictions on the set of weights of the representation τ.) Finally, we require that in the equality $\tau(PQ) = c(P, Q)\tau(P)\tau(Q)$ the function $c(P, Q)$ should not vanish. Now consider the definition of equivalence. In each space $T(V)$ there is a distinguished vector $\lambda = \lambda(V)$—the vector of highest weight; and also a distinguished covector l—the linear functional that annihilates all the weight vectors except the one of highest weight. This allows us to construct canonical bases in the spaces of vectors and covectors (distributing λ and l by means of the group $U(p, q)$ or its enveloping algebra). Two representations T and T' can now be called equivalent if multiplication of $\tau_1(P)$ by a suitable scalar function would make the matrices $\tau_1(P)$ and $\tau_2(P)$ coincide identically.

In order for the representations $M(R_1, R_2, \lambda)$ to come under the above definition, the spaces $M(V)$ must be provided with a Banach space structure. For this we require that the spaces $M(V)$ consist of bounded vector-valued holomorphic functions with the natural norm (there are many such norms, but they are all equivalent).

Theorem 4.1. *The representations $L(R_1, R_2, \lambda)$ exhaust all the irreducible holomorphic representations of the cateogry U.*

The proof is similar to that of Theorem 2.2.

4.5. Classification of the holomorphic unitary projective representations of the category U. Let $P \in \operatorname{Mor}(W, V)$. Consider in $V \oplus W$ the form

$$\Lambda_{V\oplus W}((v_1, w_1), (v_2, w_2)) = \Lambda_V(v_1, v_2) - \Lambda_W(w_1, w_2).$$

Let P^* be the orthogonal complement of P with respect to this form. Then P^*, as is easily verified, is also contained in $\operatorname{Mor}(W, V)$. A representation $T = (T, \tau)$ of the category U will be called unitary if all the spaces $T(V)$ are Hilbert and $\tau(P^*) = \tau(P)^*$ for any morphism P. The definition of equivalence for unitary representations is obvious.

Theorem 4.2. *Let R_1 and R_2 be representations of the category* Op, *with markers $a_1, a_2, \dots$ and $b_1, b_2, \dots$. Then a representation $L(R_1, R_2, \lambda)$ is unitary if and only if the number $\lambda - \sum a_i - \sum b_i$ is a nonnegative integer.*

Proof. Necessity. If a representation of the category U is unitary, then all the subordinate representations of the groups $U(p, q)$ are unitary. Since all the unitary representations of the groups $U(p, q)$ with highest weight are known (see [17]), there is no difficulty in verifying necessity.

Sufficiency. In §5 below we define the category $\mathrm{Sp}(\mathbb{R})$. In [2] (see also [1]), a construction is given for the Weil representation of this category. If we enclose the category U into $\mathrm{Sp}(\mathbb{R})$ and restrict the Weil representation to U, we obtain a representation of U, which we denote by T. It can be shown that all the unitary holomorphic representations of U are realized in the tensor powers of the representation T, and we are therefore provided with explicit models for all such representations. (The tensor powers of the Weil representations of the groups $U(p, q)$ are examined in detail in, e.g., [16]. Knowing how they decompose into irreducibles, we can easily decompose also the tensor powers of the representation of the category.) ●

4.6. Classification of the unitary (projective) representations of the category U

Theorem 4.3. *Any unitary projective representation of the category U is the tensor product of a holomorphic unitary representation and an antiholomorphic unitary representation.*

Proof. The theorem is essentially a corollary of a similar assertion proved in [15] for the semigroups $\Gamma_{p,q}(V)$ (see §4.0). We need only duplicate the proof of Theorem 2.2. Lemma 3.2 is also proved analogously, but in addition to the argument of §2 we need to "skim out" those representations of the semigroups $\Gamma_{p,q}$ that cannot be subordinate to representations of the category U. If we know the subordinate representation $\rho_{p,q}$ of the semigroup $\Gamma_{p,q}$, then we know all the subordinate representations $\rho_{r,s}$ of the semigroups $\Gamma_{r,s}$ for $r < p$, $s < q$; i.e., $\rho_{r,s}$ is a function of $\rho_{p,q}$ (notation $\rho_{r,s} := F(\rho_{p,q}) = F(p, q, r, s, \rho_{p,q})$). But it can happen that for a given $\rho_{r,s}$ there is no $\rho_{p,q}$ such that $\rho_{r,s} = F(\rho_{p,q})$ is "skimmed out". ●

§5. Morphisms of symmetric spaces

In this section we give a definition for a morphism of (Riemannian noncompact) symmetric spaces (the author regards this geometric subject as being of interest independent of the theory of representations; see also the Appendix) and carry over the results of §4 to arbitrary real classical Lie groups. Proofs are omitted. We note analogies with work of Kreĭn and Shmul'yan (see, e.g., [12], [18], [19]), Ol'shanskiĭ (properly speaking, this section is the category interpretation of [15] and [20]), and Howe [21].

5.1. **Martix balls.** Let G be one of the real classical groups $G = \mathrm{GL}(n, \mathbb{R}), \mathrm{GL}(n, \mathbb{C})$, $\mathrm{GL}(n, \mathbb{H})$, $\mathrm{SO}^*(2n)$, $O(p, q)$, $U(p, q)$, $\mathrm{Sp}(p, q)$, $O(n, \mathbb{C})$, $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{Sp}(2n, \mathbb{C})$; and let $\mathscr{K}$ be a maximal compact subgroup. It can be shown that all the symmetric spaces of the form G/K can be realized as matrix balls; more precisely, as the set of all matrices of a fixed dimension (they may be real, complex or quaternion, and may in addition satisfy other conditions—of being symmetric, skew-symmetric, Hermitian, or anti-Hermitian) with norm < 1. Table 1 lists these matrix balls. The first column indicates the symmetric space G/K; the second, the division ring $\mathbb{K}$ to which the matrix elements belong; the third, the dimensions of the matrices; the fourth, any additional condition that the matrices Z must satisfy (Z^t means the transposed matrix, Z^* the Hermitian conjugate).

5.2. **Generalized linear-fractional mappings.** We consider a category $\mathscr{K}$ (its name is listed in the sixth column of the table) whose objects are the matrix balls of a fixed type. Suppose $Z_1, Z_2 \in \mathrm{Ob}(\mathscr{K})$ and the dimensions of the matrices in these balls are $p_1 \times q_1$ and $p_2 \times q_2$, respectively. Then by a morphism from Z_1 to Z_2 we mean a matrix $S = \begin{pmatrix} K & L \\ M & N \end{pmatrix}$ of dimension $(p_1 + q_2) \times (p_2 + q_1)$, whose elements belong to the division ring $\mathbb{K}$ (second column) and that satisfies the following conditions:

1) $\|S\| \leq 1$ (as above, we use the Euclidean norm).
2) $\|K\| < 1$, $\|N\| < 1$.
3) S satisfies the same conditions as Z (fourth column).

Morphisms multiply in accordance with formula (4.1).

To every morphism $S\colon Z_1 \to Z_2$ corresponds a generalized linear-fractional mapping $Z_1 \to Z_2$ of the form

$$l_S Z = K + LZ(1 - NZ)^{-1}M. \tag{5.1}$$

It is easily verified that $l_{SH} = l_S \circ l_H$.

The group G is isomorphic to the automorphism group of the ball, and K is the stabilizer of zero. The injective mappings of the form (5.1) form an open subsemigroup in the group $\widetilde{G}$ (fifth column).

5.3. **Theory of representations.** The theory of projective representations of the categories $\mathrm{Sp}(\mathbb{R})$ and SO^* is similar to that of the category U: the representations are given by the same formulas as in §4.3, only $R_1 = R_2$; and the unitary conditions are written in the same way. For the remaining cases the notion of holomorphic representation is undefined, and any irreducible projective unitary representation of the category $\mathscr{K}$ is the restriction of a holomorphic unitary representation of the category $\mathscr{K}_{\mathbb{C}}$ (last column; the explicit form of the imbedding $\mathscr{K} \to \mathscr{K}_{\mathbb{C}}$ is clear in each individual case). Thus, there exists a bijection between the unitary projective representations of $\mathscr{K}$ and the unitary projective holomorphic representations of $\mathscr{K}_{\mathbb{C}}$ (analogue of the Weyl trick; for the case of semigroups, see [15]).

5.4. **Categories of linear relations.** We give here an independent description of the above-constructed categories. In all the cases listed below, morphisms multiply like linear relations.

TABLE 1

Symmetric space G/K	Division ring $\mathbb{K}$	Matrix dimensions	Additional condition	$\widetilde{G}$	$\mathscr{K}$	$\mathscr{K}_{\mathbb{C}}$
$U(p,q)/(U(p)\times U(q))$	$\mathbb{C}$	$p\times q$		$\mathrm{GL}(p+q,\mathbb{C})$	U	$U\times U$
$\mathrm{Sp}(2n,\mathbb{R})/U(n)$	$\mathbb{C}$	$n\times n$	$Z=Z^t$	$\mathrm{Sp}(2n,\mathbb{C})$	$\mathrm{Sp}(\mathbb{R})$	$\mathrm{Sp}(\mathbb{R})\times\mathrm{Sp}(\mathbb{R})$
$\mathrm{SO}^*(2n)/U(n)$	$\mathbb{C}$	$n\times n$	$Z=-Z^t$	$\mathrm{SO}(2n,\mathbb{C})$	SO^*	SO
$O(p,q)/(O(p)\times O(q))$	$\mathbb{R}$	$p\times q$		$\mathrm{GL}(p+q,\mathbb{R})$	$O(\mathbb{R})$	U
$\mathrm{GL}(n,\mathbb{R})/O(n,\mathbb{R})$	$\mathbb{R}$	$n\times n$	$Z=Z^t$	$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{GL}(\mathbb{R})$	$\mathrm{Sp}(\mathbb{R})$
$O(n,\mathbb{C})/O(n,\mathbb{R})$	$\mathbb{R}$	$n\times n$	$Z=-Z^t$	$O(n,n)$	$O(\mathbb{C})$	SO^*
$\mathrm{GL}(n,\mathbb{C})/U(n)$	$\mathbb{C}$	$n\times n$	$Z=Z^*$	$U(n,n)$	$\mathrm{GL}(\mathbb{C})$	U
$\mathrm{GL}(n,\mathbb{H})/\mathrm{Sp}(n)$	$\mathbb{H}$	$n\times n$	$Z=Z^*$	$\mathrm{SO}^*(4n)$	$\mathrm{GL}(\mathbb{H})$	SO^*
$\mathrm{Sp}(2n,\mathbb{C})/\mathrm{Sp}(n)$	$\mathbb{H}$	$n\times n$	$Z=-Z^*$	$\mathrm{Sp}(n,n)$	$\mathrm{Sp}(\mathbb{C})$	$\mathrm{Sp}(\mathbb{R})$
$\mathrm{Sp}(p,q)/(\mathrm{Sp}(p)\times\mathrm{Sp}(q))$	$\mathbb{H}$	$p\times q$		$\mathrm{GL}(p+q,\mathbb{H})$	$\mathrm{Sp}(\mathbb{H})$	U

a) The categories U, $O(\mathbb{R})$, $\mathrm{Sp}(\mathbb{H})$. An object of the category is, respectively, a complex, real or quaternion space, provided with a nondegenerate Hermitian form $\Lambda_V(\cdot, \cdot)$ having indices of inertia p_V, q_V. A morphism from V to W is a linear relation $L\colon V \rightrightarrows W$ satisfying the following conditions:

1) If $(v, w) \in L$, then $\Lambda_V(v, v) \geq \Lambda_W(w, w)$.
2) If $v \in \operatorname{Ker} L$, then $\Lambda_V(v, v) > 0$; if $w \in \operatorname{Indef}(L)$, then $\Lambda_W(w, w) < 0$.
3) $\dim L = p_V + q_W$.

Such linear relations will be called contractions.

b) The categories $\mathrm{GL}(\mathbb{R})$, $\mathrm{GL}(\mathbb{C})$, $\mathrm{GL}(\mathbb{H})$. An object is, respectively, a real, complex, or quaternion space V. Let V^* be the space of antilinear functionals $(f(\lambda v) = \overline{\lambda} f(v))$ on V. Consider on $V \oplus V^*$ the forms

$$\Lambda((v_1, f_1), (v_2, f_2)) = f_1(v_2) + f_2(v_1),$$
$$M((v_1, f_1), (v_2, f_2)) = f_1(v_2) - f_2(v_1).$$

The elements of $\mathrm{Mor}(V, W)$ are the linear relations $V \oplus V^* \rightrightarrows W \oplus W^*$ that preserve the form M and contract the form Λ.

c) The category $\mathrm{Sp}(\mathbb{R})$. Its objects are the real spaces, provided with a nondegenerate symplectic form $\{\cdot, \cdot\}$. Let V be an object in $\mathrm{Sp}(\mathbb{R})$. Extending the form $\{\cdot, \cdot\}$ by bilinearity to the complexification $V_{\mathbb{C}}$ of the space V, we obtain on $V_{\mathbb{C}}$ a symplectic form M. Extending the form $\{\cdot, \cdot\}$ by sesquilinearity, we obtain on $V_{\mathbb{C}}$ a Hermitian form Λ. An element of $\mathrm{Mor}(V, W)$ is a linear relation $V_{\mathbb{C}} \rightrightarrows W_{\mathbb{C}}$ that preserves M and contracts Λ.

d) The category SO^*. An object is a quaternion space V, provided with a nondegenerate anti-Hermitian form $L(\cdot, \cdot)$, i.e., $L(v, w) = \overline{-L(w, v)}$. Write L in the form

$$L(v, w) = i\Lambda(v, w) + jM(v, w)$$

where M and Λ are complex-valued forms. Let $V^{\mathbb{C}}$ be the space V regarded as complex. An element of $\mathrm{Mor}(V, W)$ is a linear relation $V^{\mathbb{C}} \rightrightarrows W^{\mathbb{C}}$ that preserves the symmetric bilinear form M and contracts the hermitian form Λ.

e) The categories $O(\mathbb{C})$, $\mathrm{Sp}(\mathbb{C})$. An object of the category is a complex linear space V, provided respectively with an orthogonal or a symplectic form $L(\cdot, \cdot)$. Let $\overline{V}$ be the same space, but with the conjugate complex structure (if $v \to \tilde{v}$ is the identity mapping $V \to \overline{V}$, then $\widetilde{\lambda v} = \overline{\lambda}\tilde{v}$) and with the same, but no longer bilinear, form $\overline{L}$. Let $V^0 \subset V \oplus \overline{V}$ be the real subspace consisting of all vectors of the form $(v, i\overline{v})$. The set $\mathrm{Mor}(V, W)$ consists of the (real) linear relations $V^0 \rightrightarrows W^0$ that preserve the form $M = \mathrm{Re}(L \oplus \overline{L})$ and contract the form $\Lambda = \mathrm{Im}(L \oplus \overline{L})$.

Equivalence of the definitions of categories in §§5.2 and 5.4 can be verified in the same way as in §4. For the subspaces V_+ and V_- we must take maximal isotropic subspaces relative to the form M, on which the form Λ is respectively positive definite and negative definite.

§6. Discrete analogs

We confine ourselves here just to listing certain categories on which to one degree or another our results can be carried over.

6.1. **Algebraic groups.** Let $\mathbb{F}$ be an algebraically closed field of finite characteristic p. The categories GA, B, C, GD are defined similarly to the categories in §2, with $\mathbb{C}$ replaced by $\mathbb{F}$. Representations are taken over the field $\mathbb{F}$.

6.2. Matrix groups over a finite field: modular representations. Categories are defined in the same way as in §1, with the field $\mathbb{F}$ replaced by the finite field $\mathbb{F}_{p^n}$. Representations are taken over the algebraic closure of $\mathbb{F}_{p^n}$.

6.3. Matrix groups over a finite field: complex representations. Same categories as in §6.2, with complex representations.

6.4. Representations. In §§6.1 and 6.2, existence of representations is obvious: the constructions of the fundamental representations in §3 are independent of the characteristic of the field. As for §6.3, the construction of the Weil representation for the symplectic group over $\mathbb{F}_{p^n}$ (see [22]) can be reinterpreted in categorical terms (as with the spin representation in §2.1).

6.5. The category of partial bijections. The objects of the category are finite sets; the morphisms are partially defined relations. The constructions of the representations of semigroups in [5] can be easily reinterpreted in categorical terms. The classification theorem for representations of this category is a corollary of the classification theorem for semigroups; see [5].

6.6. The Brauer category. An object is a finite set with an even (for definiteness) number of elements. A morphism $\theta: M \to N$ is an arbitrary partition of the set $M \cup N$ into pairs (we assume M and N are disjoint). If $k_1, k_2 \in M \cup N$ lie in the same pair, we write $k_1 \sim_\theta k_2$. Let $\theta: M \to N$ and $\psi: N \to K$ be two morphisms. Their product $\psi\theta: M \to K$ is defined as follows: $l_1 \sim_{\psi\theta} l_2$ $(l_1, l_2 \in M \cup K)$ if there exist $p_1 = l_1, p_2, p_3, \dots, p_\alpha = l_2$ such that for any i either $p_i \sim_\theta p_{i+1}$ or $p_i \sim_\psi p_{i+1}$.

The Brauer duality theorem [23] easily carries over from the Brauer semigroup to the Brauer category. Also easily carried over are the construction and the classification theorem of Kerov [24].

6.7. The braid category. The author would like to call attention to the existence of an obvious "hybrid" of the braid group and the Brauer category.

Appendix. Geometry of symmetric spaces and generalized linear-fractional mappings

Here we examine the question, somewhat to the side of the main theme of the paper, of the connection between generalized linear-fractional mappings and the rich geometry of symmetric spaces. So as not to overload the discussion, we take up only the case of the spaces $G/K = U(p, q)/(U(p) \times U(q)) \overset{\text{def}}{=} \mathscr{Z}_{p,q}$.

A.1. **Geometry of matrix balls.** In §4.2 we described two realizations of $\mathscr{Z}_{p,q}$ as a matrix ball and as an open set $L_{p,q}$ in the Grassmannian of p-dimensional subspaces in $\mathbb{C}^{p+q}$. We can define on $\mathscr{Z}_{p,q}$ the following geometric structures:

a) The *composite distance* (E. Cartan) between matrices $Z_1, Z_2 \in \mathscr{Z}_{p,q}$. This is the set of eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots$ of the matrix

$$(1 - Z_1^* Z_1)^{-1}(1 - Z_1^* Z_2)(1 - Z_2^* Z_2)^{-1}(1 - Z_2^* Z_1).$$

It will be convenient to add to the sequence $\lambda_1, \lambda_2, \dots$ an infinite number of ones. It is known that the set of λ_j is a complete set of invariants of pairs of points with respect to the isometry group $U(p, q)$ of the domain $\mathscr{Z}_{p,q}$. If we interpret $\mathscr{Z}_{p,q}$ as a domain in the Grassmannian, then $\lambda_i = \cosh^{-2} \varphi_i$, where the φ_i are the hyperbolic angles between the subspaces in the sense of the indefinite metric in $\mathbb{C}^{p+q}$. As is known [25], the Riemannian distance in $\mathscr{Z}_{p,q}$ is computed by the formula

$$\rho(Z_1, Z_2) = \frac{1}{4} \sum \ln^2 \frac{1 + \sqrt{r_k}}{1 - \sqrt{r_k}}, \qquad r_k = \frac{\lambda_k - 1}{\lambda_k}.$$

b) *Projective structure.* Consider the Plücker imbedding of the Grassmannian in $\mathbb{C}^{p+q}$ into a C_p^{p+q}-dimensional projective space $\mathbb{P}$. It is found that the image of the Grassmannian in $\mathbb{P}$ contains a certain family of lines. The inverse images of these lines in $\mathscr{Z}_{p,q}$ are called the *Chow lines*; Chow himself, in [26], was considering compact Hermitian symmetric spaces. The Chow lines are 1-dimensional complex submanifolds in $\mathscr{Z}_{p,q}$, conformally equivalent to the disk $|z| < 1$ in the complex plane.

The Chow lines can be represented in a more visual fashion. Suppose $V_1, V_2 \in L_{p,q}$ and the codimension of $V_1 \cap V_2$ in V_1 is 1. Then the set of all subspaces $V \in L_{p,q}$ such that

$$(V_1 \cap V_2) \subset V \subset (V_1 + V_2) \tag{A.1}$$

constitutes a Chow line; and any Chow line is of this form.

It is important to observe that the set $C(V_1, V_2)$ of all subspaces satisfying (A.1) is provided in a natural fashion with the structure of a projective line. The Chow line is an open subset in $C(V_1, V_2)$ projectively equivalent to the disk $|z| < 1$ in the augmented complex plane. Thus, the notion of projective mapping of a Chow line into a Chow line is well defined.

c) The *integral distance* (in the case of compact symmetric spaces a similar, but not directly analogous, definition can be found in [27]) $n(Z_1, Z_2)$ between two points $Z_1, Z_2 \in \mathscr{Z}_{p,q}$ is the minimal length n of a chain $Z_1 = X_0, X_1, \dots, X_n = Z_2$ such that X_j and X_{j+1} lie on the same Chow line. We note that $n(Z_1, Z_2)$ is equal to the number of non-ones in the composite distance and also equal to the rank of the matrix $(Z_1 - Z_2)$.

A.2. **Geometric properties of morphisms.** Let Q be a morphism in the category U, and l_Q the corresponding generalized linear-fractional mapping of matrix balls.

Proposition. a) $\rho(l_Q(Z_1), l_Q(Z_2)) \le \rho(Z_1, Z_2)$.

b) $\lambda_j(l_Q(Z_1), l_Q(Z_2)) \le \lambda_j(Z_1, Z_2)$.

c) $n(l_Q(Z_1), l_Q(Z_2)) \le n(Z_1, Z_2)$.

A.3. **Geometric characterization of morphisms.** We call a mapping f of a matrix ball $\mathscr{Z}_{p,q}$ into a matrix ball $\mathscr{Z}_{r,s}$ projective if:

1°) for any Chow line $C \subset \mathscr{Z}_{p,q}$ the image $f(C)$ is contained in some Chow line $D \subset \mathscr{Z}_{r,s}$;

2°) the mapping $f: C \to D$ is projective.

Of course, it would be desirable to be able to drop the latter requirement. But the same difficulty arises here as in the "fundamental theorem of projective geometry" in dimension 1. Namely, if $f(\mathscr{Z}_{p,q})$ is already contained in a single Chow line, then the requirement 1° is automatically satisfied.

Theorem. *Any projective mapping of matrix balls is either of the form* $Z \to l_Q(Z)$ *or of the form* $Z \to l_Q(Z^t)$, *where* Q *is a morphism in the category* U.

The proof is in the spirit of [28], Chapter III.

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Received 25/SEPT/89

Translated by J. A. ZILBER