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Supercomplete Bases in the Space of Symmetric Functions*

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As is known, various inner products are of importance in the theory of symmetric functions [3]. In particular, in Macdonald's book [3], a two-parameter family of inner products is considered (this family includes, as special cases, the classical inner product and the Hall–Littlewood and Jack inner products). A more general family of inner products that depends on countably many parameters was introduced by Kerov in [2]. In all these cases, the products p_λ of the Newton sums form an orthogonal basis, and the various inner products of this form differ in the normalization of the basis p_λ.

It was noted in [6] that, in connection with the Poisson measures, in the space of symmetric functions there arise inner products that differ from the Kerov products. In this case, the functions p_λ can be nonorthogonal, and the simplest orthogonal (nonnormalized) basis is formed by the monomial symmetric functions m_λ (the symmetrized monomials).

In all these situations, there exists a natural *unitary* isomorphism between the space Symm of symmetric functions and the boson Fock space (for a discussion of this isomorphism in the case of the classical inner product, see [8]). This raises the question of transferring the natural structures from the boson Fock space to the space Symm and vice versa.

The main objective of the paper is to find out what are the images of the Gaussian vectors of the boson Fock space in the space Symm. The answer to this question is given by Theorem 1 in the case of the Kerov inner products and by formula (21) in the situation related to the Poisson measures.

We solve, in fact, a more general problem and consider a family of inner products $\langle\cdot\,,\,\cdot\rangle_{K,\omega}$ in Symm that contains all the above inner products. These inner products are parametrized by a sequence $\omega = (\omega_1, \omega_2, \dots)$ of positive numbers and by a formal series $K(h) = 1 + \sum_{j>0} \varkappa_j h^j$ and are defined as follows. We write

$$\Psi_\alpha(x_1, x_2, \dots) = \prod_j K\Big(\sum_{k>0} \alpha_k x_j^k\Big)$$

set

$$\langle \Psi_\alpha, \Psi_\beta \rangle_{K,\omega} = \exp\Big(\sum_{k>0} \alpha_j \overline{\beta_j} \omega_j\Big)$$

and thus define uniquely an inner product in Symm. The Kerov inner products are obtained here for $K(h) = \exp(h)$ (in particular, the classical case corresponds to $\omega_j = j$), and $K(h) = 1 + h$ relates to the Poisson measures.

In this more general situation, the formula for the Gaussian vectors is given by Theorems 2 and 2′. Hence, in the space Symm (more exactly, in its Hilbert completion), we obtain some supercomplete bases that consist of Gaussian vectors. Recall that by an supercomplete basis in a Hilbert space we mean a total system of vectors that depends holomorphically on a parameter (ranging over a complex variety), and admits an explicit formula for the pairwise inner products. As is known, it is often convenient to develop the analysis in Hilbert spaces using supercomplete bases (or, which is the same, "systems of coherent states") instead of ordinary orthogonal bases.

The question arises as to how the supercomplete bases can be expanded with respect to the various standard bases in Symm. Since there are numerous standard bases [3], we present only some of these expansions (Theorem 3).

Finally, in §3 we discuss the space ASymm of skew-symmetric functions and the problem of transferring the Gaussian fermion vectors to the spaces ASymm and Symm.

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§1. Multiplicative Boson-Symmetric Correspondences

1.1. The spaces $l_2(\omega)$ and $l_2^\circ(\omega)$. Let us choose a sequence

$$\omega = (\omega_1, \omega_2, \dots)$$

of positive numbers. Denote by $l_2(\omega)$ the space of sequences $\alpha = (\alpha_1, \alpha_2, \quad)$ satisfying the condition $\sum_j |\alpha_j|^2 \omega_j < \infty$. The inner product in $l_2(\omega)$ is given by the formula

$$\langle \alpha, \beta \rangle = \sum_{j=1}^{\infty} \alpha_j \overline{\beta_j} \omega_j$$

Denote by $l_2^\circ(\omega)$ the space of sequences $\alpha = (\alpha_1, \alpha_2, \dots)$ satisfying the condition $\sum_j |\alpha_j|^2/\omega_j < \infty$. inner product in $l_2^\circ(\omega)$ is given by the formula

$$\langle \alpha, \beta \rangle = \sum_{j=1}^{\infty} \frac{\alpha_j \overline{\beta_j}}{\omega_j} .$$

It is natural to regard $l_2^\circ(\omega)$ as the dual space of $l_2(\omega)$, where the pairing $l_2^\circ(\omega) \times l_2(\omega) \to \mathbf{C}$ is given by the formula $\{\alpha, \beta\} = \sum \alpha_j \beta_j$.

1.2. The spaces $H^2(\omega)$. Let a sequence ω be of at most exponential growth, i.e., let there exist numbers C and σ such that $\omega_j \le C \exp(\sigma j)$ for all j. Suppose that $\alpha \in l_2(\omega)$. Then the series

$$f(z) = \sum_{j>0} \alpha_j z^j \tag{1}$$

is convergent in the circle $|z| < \rho := \exp(-\sigma)$. Here we have $f(0) = 0$. Thus, the space $l^2(\omega)$ is interpreted as the Hilbert space of the functions (1), which are holomorphic in the circle $|z| < \rho$, that is endowed with the inner product

$$\left\langle \sum_{j>0} \alpha_j z^j, \sum_{j>0} \beta_j z^j \right\rangle = \sum_{j>0} \alpha_j \overline{\beta_j} \omega_j .$$

We denote this space by $H^2(\omega)$.

The *reproducing kernel* of the space $H^2(\omega)$ is expressed by the formula

$$K(z, u) = \sum_{j>0} \frac{z^j \overline{u}^j}{\omega_j} . \tag{2}$$

This means that, for an arbitrary function $f \in H^2(\omega)$ and any a belonging to the circle of convergence, the following *reproducing property* holds:

$$\langle f(z), K(z, a) \rangle_{H^2(\omega)} = f(a)$$

Example 1. Let $\omega_j = j$. Then

$$\langle f(z), g(z) \rangle = \frac{i}{2\pi} \iint_{|z|<1} f'(z) \overline{g'(z)}\, dz\, d\overline{z}$$

and $K(z, u) = -\ln(1 - z\overline{u})$

Example 2. Let $\omega_j = j(1 - q^j)/(1 - t^j)$ for some q, $0 < q < 1$, and t, $0 < t < 1$ Then the inner product in $H^2(\omega)$ is given by the formula

$$\langle f(z), g(z) \rangle = \frac{i}{2\pi} \sum_{k=0}^{\infty} \iint_{qt^k < |z|^2 < t^k} f'(z) \overline{g'(z)}\, dz\, d\overline{z}, \tag{3}$$

and the reproducing kernel of the space $H^2(\omega)$ is

$$K(z,u) = \ln\left[\prod_{n\geq 0}\frac{1-z\bar{u}tq^n}{1-z\bar{u}q^n}\right] \tag{4}$$

Formula (3) for the inner product in the space of holomorphic functions is somewhat unusual. For some other inner products of a similar type, see [7].

1.3. Boson Fock space. Let H be a Hilbert space. A *boson Fock space* $F(H)$ is a Hilbert space in which a system of vectors (an "supercomplete basis") Φ_h indexed by the vectors $h \in H$ is chosen such that

(a) $\langle \Phi_h, \Phi_{h'}\rangle_{F(H)} = \exp(\langle h', h\rangle_H)$;

(b) the linear span of the vectors Φ_h is dense in $F(H)$.

With each vector $q \in F(l_2(\omega))$ we associate a function $f = f_q$ on $l_2^\circ(\omega)$ that depends holomorphically on $z = (z_1, z_2, \dots)$ according to the formula

$$f_q(z_1, z_2, \dots) = \langle q, \Phi_{(z_1/\omega_1, z_2/\omega_2, \dots)}\rangle_{F(l_2(\omega))}\,. \tag{5}$$

For example, to an element $\Phi_{(a_1,a_2,\dots)}$ of the supercomplete basis, the function $\exp(\sum z_j\overline{a_j})$ corresponds.

We identify the boson Fock space $F(l_2(\omega))$ with the space of holomorphic functions $f_q(z_1, z_2, \dots)$ on $l_2^\circ(\omega)$.

The functions $z_1^{k_1}z_2^{k_2}\dots$ form an orthogonal basis in $F(l_2(\omega))$, and we have

$$\|z_1^{k_1}z_2^{k_2}\dots\|^2 = \prod(\omega_j^{k_j}\cdot k_j!)\,. \tag{6}$$

1.4. The space of symmetric functions. Let $x_1, x_2, \dots$ be a countable set of formal variables. By a *symmetric formal series* we mean a formal series in the variables $x_1, x_2, \dots$ that is not changed under arbitrary permutations of the variables. A *symmetric polynomial* (a "symmetric function" in the sense of [3]) is a symmetric formal series such that the degrees of the monomials $x_1^{k_1}x_2^{k_2}\dots$ it involves are bounded; note that the number of terms in a (nonconstant) symmetric polynomial is infinite. We denote the space of symmetric polynomials by Symm or by Symm(x) if it is important to stress that the polynomials in question are symmetric with respect to the variables $x_1, x_2, \dots$.

Recall the following standard notation (see [3, §I.2]):

$$p_k = p_k(x) = x_1^k + x_2^k + \dots, \qquad m_\lambda(x) = m_{\lambda_1\cdots\lambda_s}(x) = \sum x_{j_1}^{\lambda_1}\cdots x_{j_s}^{\lambda_s},$$

where $\lambda_j > 0$. In the second sum, the monomial $x_1^{\lambda_1}x_2^{\lambda_2}\dots$ and all (different) monomials that can be obtained from it by permutations of factors are added together. It is convenient to assume that $\lambda_1 \geq \lambda_2 \geq \dots$.

1.5. Boson-symmetric correspondences. Let $\omega = (\omega_1, \omega_2, \dots)$ be a chosen sequence. Following Kerov [2], we introduce an inner product $\langle\cdot,\cdot\rangle_\omega$ in the space Symm by means of the condition that the vectors $p_1^{k_1}\cdots p_\alpha^{k_\alpha}$ form an orthogonal basis and

$$\|p_1^{k_1}\cdots p_\alpha^{k_\alpha}\|_\omega^2 = \prod k_j!\,\omega_j^{k_j}\,. \tag{7}$$

Remark. (a) The *classical inner product* in Symm [3, §I.4] corresponds to the sequence $\omega_j = j$.

(b) For the inner products corresponding to $\omega_j = j(1-q^j)/(1-t^j)$ (in Sec. 1.2), see [3, §§III.5, VI.6].

Consider the completion Symm$_\omega$ of the space Symm with respect to the norm defined by the inner product $\langle\cdot,\cdot\rangle_\omega$. We introduce the unitary operator $I\colon F(l_2(\omega)) \to$ Symm$_\omega$ under which to any holomorphic function $f(z_1, z_2, \dots) \in F(l_2(\omega))$ (where $z = (z_1, z_2, \dots) \in l_2^\circ(\omega)$) the formal symmetric series $f(p_1(x), p_2(x), \dots)$ corresponds. In other words, we simply perform the substitution $z_j = p_j(x)$ (by formulas (6)–(7), this operator is, in fact, unitary).

For example, to an element

$$\exp\left(\sum \alpha_n z_n\right) \in F(l_2(\omega))$$

of the supercomplete basis, the symmetric formal series

$$\exp\Big(\sum \alpha_n p_n(x)\Big) = \prod_j \exp\Big(\sum_n \alpha_n x_j^n\Big) \in \mathrm{Symm}_\omega$$

corresponds.

1.6. Gaussian vectors. The supercomplete basis $\Phi_h \in F(l_2(\omega))$ can be included in a broader supercomplete basis consisting of the Gaussian vectors $b[Q|r]$. These vectors (or functions on $l_2^\circ(\omega)$) are defined by the formula

$$\exp\Big\{\frac{1}{2}\sum q_{ij}z_iz_j + \sum r_jz_j\Big\} \in F(l_2(\omega)), \tag{8}$$

where $q_{ij} = q_{ji}$. Denote by Ω the diagonal matrix with diagonal elements $\omega_1, \omega_2, \dots$. A vector $b[Q|r]$ is contained in $F(l_2(\Omega))$ if and only if the following conditions hold:

(i) $r \in l_2^\circ(\omega)$,

(ii) $R := \Omega^{-1/2}Q\Omega^{-1/2}$ is a Hilbert–Schmidt matrix (that is, the trace of the matrix R^* is finite),

(iii) the norm of the matrix R (in the sense of the operator norm in the (ordinary) space l_2) is less than unity.

The inner products of the vectors $b[Q|r]$ (provided that they actually belong to $F(l_2(\omega))$) can readily be calculated; for instance, see [5, §6.2].

However, from the viewpoint of the present paper, it is more natural to interpret expression (8) as a formal series, disregarding the above conditions (i)–(iii).

Let us find out what symmetric functions correspond to the Gaussian vectors $b[Q|r]$.

Theorem 1. *The element of* Symm_ω *that corresponds to a vector* $b[Q|r]$ *is given by the formula*

$$\prod_{i<j}\exp\Big(\sum_{m,n>0} q_{mn}x_i^m x_j^n\Big)\prod_i \exp\Big(\sum_{n>0}\Big(r_n + \frac{1}{2}\sum_{\alpha+\beta=n} q_{\alpha\beta}\Big)x_i^n\Big)$$

The proof is obvious. The symmetric formal series corresponding to $b[Q|r]$ is

$$\exp\Big\{\frac{1}{2}\sum q_{mn}p_m(x)p_n(x) + \sum r_n p_n(x)\Big\}$$
$$= \exp\Big\{\frac{1}{2}\sum q_{mn}(x_1^m + x_2^m + \cdots)(x_1^n + x_2^n + \cdots) + \sum r_n(x_1^n + x_2^n + \cdots)\Big\}$$

The further transformations are quite clear.

Remark. Thus, to the Gaussian formal series $b[Q|r]$ of the form (8) precisely all possible products of the form

$$\prod_{i>j} S(x_i, x_j)\cdot\prod_i T(x_i)$$

correspond, where the formal series S and T have the form

$$S(x,y) = 1 + \sum_{m>0,n>0} s_{mn}x^my^n, \quad s_{mn} = s_{nm}, \qquad T(x) = 1 + \sum_{n>0} t_nx^n$$

1.7. Example: the Virasoro algebra and the images of the Gaussian vectors. Let $\omega_j = j$. Choose $\alpha, \beta \in \mathbf{C}$. Let us consider the representations of the Virasoro algebra in the Fock space $F(H^2(\omega))$ by means of the operators L_n (where $n \in \mathbf{Z}$) that are given by the following standard formulas (e.g., see [4, 5]):

$$L_n = \sum_{k\geq 0} z_{n+k}k\frac{\partial}{\partial z_k} + \frac{1}{2}\sum_{\substack{k+m=n\\ k>0,\, m>0}} z_mz_k + (\alpha + i\beta n)z_n \quad \text{for } n > 0, \tag{9}$$

$$L_k := L_{-k}^* \quad \text{for } k < 0, \qquad L_0 = \sum z_kk\frac{\partial}{\partial z_k} + \frac{1}{2}(\alpha^2 + \beta^2). \tag{10}$$

In this case, the relation

$$[L_m, L_n] = (n-m)\,L_{m+n} + \frac{n^3-n}{24}(1+12\beta^2)\,\delta_{m+n,0}$$

holds for all m and n. Consider the projective representation of the group Diff of analytic orientation-preserving diffeomorphisms of a circle, which corresponds to the representation (9)–(10) of the Virasoro algebra (for explicit formulas, see [4]). This group acts in the boson Fock space, and hence in the space Symm_ω as well. The orbit of the vector $f(z)=1$ in the boson Fock space under the action of the group Diff consists of Gaussian vectors; explicit formulas for these vectors were derived in [4]. We shall describe the orbit of the (symmetric) function 1 under the action of the group Diff. The consideration of the presented construction and the one in [4, §4] shows that the orbit consists of symmetric formal series of the form

$$\Xi_\theta(x) = \prod_{i>j}\left(\frac{\theta(x_i)-\theta(x_j)}{x_i-x_j}\right)\cdot\prod_i \theta'(x_i)^{1+\alpha}\cdot\prod_i\left(\frac{\theta(x_i)}{x_i}\right)^\beta$$

where $\theta(z)$ ranges over the functions of the form

$$\theta(z) = z+\theta_2 z^2+\theta_3 z^3+\dots$$

that are one-sheeted in the circle $|z|<1+\varepsilon$ (where $\varepsilon=\varepsilon(\theta)>0$).

1.8. Operators. We consider the boson Fock spaces $F(l_2(\omega))$ and $F(l_2(\nu))$ and the formal series (the *kernel*)

$$L(z,u) = \sum l^{j_1 j_2\dots}_{i_1 i_2\dots} z_1^{i_1} z_2^{i_2}\dots \overline{u}_1{}^{j_1}\overline{u}_2{}^{j_2}\dots .$$

Set $l_z(u) := \overline{L(z,u)}$. Furthermore, let $\widehat{L}\colon F(l_2(\omega))\to F(l_2(\nu))$ be the operator defined by the formula

$$\widehat{L}f(z) = \langle f, l_z\rangle_{F(l_2(\omega))}.$$

Recall (e.g., see [5, §6.1]) that any bounded operator from $F(l_2(\omega))$ into $F(l_2(\nu))$ can, in fact, be represented in this form.

We now describe how the corresponding operator

$$\widehat{\mathscr{L}}\colon \mathrm{Symm}_\omega(y)\to\mathrm{Symm}_\nu(x)$$

can be defined. Consider a formal series $\mathscr{L}(x_1,x_2,\dots,y_1,y_2,\dots)$ (a *bisymmetric kernel*) that is symmetric with respect to the set of variables $x_1,x_2,\dots$ and also with respect to the set of variables $y_1,y_2,\dots$ and is given by the formula

$$\mathscr{L}(x,y) = L(p_1(x),p_2(x),\dots;p_1(y),p_2(y),\dots) = \sum l^{j_1j_2\dots}_{i_1i_2\dots} p_1(x)^{i_1}p_2(x)^{i_2}\dots p_1(y)^{j_1}p_2(y)^{j_2}\dots .$$

In this case, we have

$$\widehat{\mathscr{L}}g(x) = \langle g, \overline{\mathscr{L}(x,y)}\rangle_{\mathrm{Symm}_\omega(y)}.$$

1.9. Gaussian operators. A Gaussian operator from $F(l_2(\omega))$ to $F(l_2(\nu))$ is an operator with a kernel of the form

$$L(z,u) = \exp\left\{\frac12\begin{pmatrix} z & \overline{u}\end{pmatrix}\begin{pmatrix} K & L\\ L^t & M\end{pmatrix}\begin{pmatrix} z^t\\ \overline{u}^t\end{pmatrix}\right\} = \exp\left\{\frac12\sum k_{ij}z_iz_j+\sum l_{ij}z_i\overline{u}_j+\frac12\sum m_{ij}\overline{u}_i\overline{u}_j\right\}$$

The corresponding bisymmetric kernel $\mathscr{L}(x,y)$ is

$$\prod_{\alpha>\beta}\exp\left\{\sum_{i,j}k_{ij}x_\alpha^i x_\beta^j\right\}\prod_{\alpha,\beta}\exp\left\{\sum_{i,j}l_{ij}x_\alpha^i y_\beta^j\right\}\cdot\prod_{\alpha>\beta}\exp\left\{\sum_{i,j}m_{ij}y_\alpha^i y_\beta^j\right\}$$
$$\times\prod_\alpha\exp\left\{\frac12\sum_{i,j}k_{ij}x_\alpha^{i+j}\right\}\prod_\alpha\exp\left\{\frac12\sum_{i,j}m_{ij}y_\alpha^{i+j}\right\}$$

1.10. Example: the bisymmetric kernel of the identity operator. The kernel of the identity operator in $F(l_2(\omega))$ is given by the formula

$$L(z,u)=\exp\left\{\sum\frac{z_i\overline{u}_i}{\omega_i}\right\}$$

Note that the argument of the exponential function is the reproducing kernel of the space $H^2(\omega)$ (see Sec. 1.2). The corresponding bisymmetric kernel $\mathscr{L}$ is

$$\mathscr{L}(x,y)=\prod_{\alpha,\beta}\exp\left\{\sum\frac{x_\alpha^i y_\beta^i}{\omega_i}\right\} \tag{11}$$

In particular, in the classical case $\omega_j=j$

$$\mathscr{L}(x,y)=\prod_{\alpha,\beta}(1-x_\alpha y_\beta)^{-1}$$

and, by (4), for $\omega_j=j\,(1-q^j)/(1-t^j)$ we have

$$\mathscr{L}(x,y)=\prod_{\alpha,\beta}\prod_{n\geq 0}\frac{1-x_\alpha y_\beta t q^n}{1-x_\alpha y_\beta q^n}$$

We can now compare what has been said with some known theorems. Given an inner product $\langle\cdot,\cdot\rangle$ in the space Symm such that the subspaces of homogeneous polynomials are pairwise orthogonal, consider a homogeneous orthonormal basis $u_\xi(x)$ in the space Symm. The *reproducing kernel* for the inner product $\langle\cdot,\cdot\rangle$ is defined by the formula

$$\sum_\xi u_\xi(x)\overline{u_\xi(y)}\,.$$

We can readily see that the operator in Symm_ω that corresponds to the bisymmetric kernel (14) is exactly the identity operator (in particular, this means that expression (14) does not depend on the choice of the basis u_ξ). Therefore, (11)–(13) are formulas for the reproducing kernels. Formula (12) is classical [3, I.4]. Formula (13) belongs to Macdonald [3, VI.2] (also see [3, §III.4]; the formula for the reproducing kernel in the case of the Jack polynomials can be derived in a similar way (see [3, VI.10])). Formula (11) was discovered by Kerov [2].

§2. Nonmultiplicative Boson-Symmetric Correspondences

2.1. The bases $\mathfrak{q}(x)$. Choose a formal series

$$K(h)=1+\sum_{j>0}\varkappa_j h^j\,, \tag{15}$$

where $\varkappa_1\neq 0$. Consider the symmetric formal series

$$\Psi_\alpha(x)=\prod_\xi K\Big(\sum_{j>0}\alpha_j x_\xi^j\Big) \tag{16}$$

where the product is taken over all variables $x_\xi=x_1,x_2,\dots$.

Let us expand the expression Ψ_α in a series with respect to the variables $\alpha_1,\alpha_2,\dots$

$$\Psi_\alpha(x)=\sum_{n_1,n_2,\dots}\alpha_1^{n_1}\alpha_2^{n_2}\cdots\,\mathfrak{q}_{n_1,n_2,\dots}(x)$$

We can readily see that the symmetric polynomials $\mathfrak{q}_{n_1,n_2,\dots}(x)=\mathfrak{q}^K_{n_1,n_2,\dots}(x)$ form a homogeneous basis in Symm.

Example. If $K(h) = \exp(h)$, then

$$\mathfrak{q}_{n_1,n_2,\dots}(x) = \prod \frac{p_j^{n_j}}{n_j!}.$$

If $K(h) = 1 + h$ (see Sec. 2.3), then the basis $\mathfrak{q}$ consists of the monomial symmetric functions m_λ; see Sec. 1.4.

2.2. Boson-symmetric correspondences. Let us choose a formal series $K(x)$ (see (15)) and a sequence ω (see 1.1). Introduce the inner product $\langle\cdot,\cdot\rangle_{K,\omega}$ in Symm by means of the condition that the polynomials $\mathfrak{q}^K_{n_1,n_2,\dots}(x)$ form an orthogonal basis and

$$\|\mathfrak{q}^K_{n_1,n_2,\dots}\|^2 = \prod \frac{\omega_j^{n_j}}{n_j!}.$$

We also give an equivalent (but more convenient) definition of the inner product $\langle\cdot,\cdot\rangle_{K,\omega}$ by introducing it on the elements of the supercomplete basis Ψ_α in the following way:

$$\langle\Psi_\alpha,\Psi_\beta\rangle_{K,\omega} = \left\langle \prod_\xi K\Big(\sum_{j>0}\alpha_j x_\xi^j\Big), \prod_\xi K\Big(\sum_{j>0}\beta_j x_\xi^j\Big)\right\rangle_{K,\omega} = \exp\Big(\sum \alpha_j\overline{\beta}_j\omega_j\Big). \tag{17}$$

Denote by $\mathrm{Symm}_{K,\omega}$ the completion of the space Symm with respect to this inner product (it is clear that the elements of $\mathrm{Symm}_{K,\omega}$ can be realized as formal symmetric series).

A unitary isomorphism between $F(l_2(\omega))$ and $\mathrm{Symm}_{K,\omega}$ is established as follows: with an element $\Phi_\alpha \in F(l_2(\omega))$ (see 1.3) we associate the element $\Psi_\alpha \in \mathrm{Symm}_{K,\omega}$. By condition (a) in Sec. 1.3 and formula (17), this correspondence defines, in fact, a unitary operator.

Consider an arbitrary function

$$f(z_1,z_2,\dots) = \sum c_{n_1,n_2,\dots} \prod \frac{z_j^{n_j}}{n_j!}$$

in $F(l_2(\omega))$. We can readily see that the symmetric formal series corresponding to this function is

$$\mathscr{F}(x) = \sum c_{n_1,n_2,\dots}\mathfrak{q}^K_{n_1,n_2,\dots}(x).$$

2.3. Example: $K(h) = 1 + h$ (see [6]). Consider a measure μ on $\mathbb{C}^* = \mathbb{C}\setminus 0$ that is invariant with respect to the rotations. We assume that μ satisfies the following condition: all the expressions

$$\omega_j = \int_{\mathbb{C}^*} |z|^{2j}\,d\mu(z), \qquad j = 1,2,\dots, \tag{18}$$

are finite. In particular, the measure of any disk $|z| \geq \varepsilon$ is finite (however, the measure of the entire space $\mathbb{C}^*$ can be infinite).

Denote by Ω the set of all at most countable subsets of $\mathbb{C}^*$. Let us introduce the *Poisson measure* ν on Ω. Recall its definition. Suppose that $M_1,\dots,M_k$ are pairwise disjoint subsets of finite measure in $\mathbb{C}^*$. Then the probability that M_j contains exactly p_j points (for all j) is

$$\prod_j \left[\frac{\mu(M_j)^{p_j}}{p_j!}\exp(-\mu(M_j))\right]. \tag{19}$$

We now state this assertion in a somewhat different manner. Consider the set $\mathscr{A}[M_1,\dots,M_k;p_1,\dots,p_k]$ consisting of all $\zeta = \{\zeta_1,\zeta_2,\dots\}\in\Omega$ such that the intersection of ζ and M_j is comprised of exactly j points. Then the measure $\nu(\mathscr{A}[M_1,\dots,M_k;p_1,\dots,p_k])$ of this set is given by formula (19). If $\mu(\mathbb{C}^*) < \infty$, then the measure ν is supported by the set of finite subsets in $\mathbb{C}^*$. If $\mu(\mathbb{C}^*) = \infty$, then the support of ν is the space of sequences convergent to 0.

With a symmetric polynomial $f(x_1,x_2,\dots)$ we associate an ordinary function (defined almost everywhere) on Ω according to the following rule. Let $\zeta = \{\zeta_1,\zeta_2,\dots\}\in\Omega$. Then the function in question is obtained by the substitution $x_1 = \zeta_1$, $x_2 = \zeta_2$, ... The resulting series are, in fact, convergent in $L^2(\Omega, \nu)$.

Thus, the space of symmetric polynomials can be embedded in the space $L^2(\Omega, \nu)$, and hence an inner product is induced on the space Symm (note that Symm is not dense in L^2). It turns out that this inner product corresponds to the function $K(h) = 1 + h$ and the sequence ω_j in formula (18).

2.4. The images of the Gaussian vectors. By a *graph* we mean a nonoriented graph with finitely many edges (the set of vertices is assumed to be infinite), and we admit multiple edges (i.e., there can be several edges from one vertex to another) and loops (i.e., edges from a vertex to itself). By a *rigged graph* we mean a graph whose vertices are some variables $x_1, x_2, \dots$ and to each of whose vertices x_i, a nonnegative integer w_i is assigned (we call it a *makeweight*), and all makeweights w_i, except for finitely many of them, must be equal to 0.

For a rigged graph Γ we introduce the following notation:

- s_{ij} is the number of edges from x_i to x_j; in particular, s_{ii} is the number of loops with the beginning and end at x_i;
- m_i (the weight of a vertex) is the number of edges issuing from x_i plus the makeweight w_i; in other words, $m_i = w_i + 2s_{ii} + \sum_{j \neq i} s_{ij}$;
- $\operatorname{edge}(\Gamma)$ is the set of edges of the graph Γ;
- $u(\gamma)$ and $v(\gamma)$ are the endpoints of an edge γ.

We next consider a Gaussian vector $b[Q|r] \in F(l_2(\omega))$. Set

$$\tilde{Q}(x, y) = \sum q_{ij} x^i y^j, \qquad \tilde{r}(x) = \sum r_j x^j.$$

Theorem 2. *To the vector $b[Q|r] \in F(l_2(\omega))$ there corresponds an element $\mathfrak{B}^K[Q|r]$ of the space $\mathrm{Symm}_{K,\omega}$ that is defined by the formula*

$$\sum_{\Gamma} \left\{ \frac{\prod_i m_i!}{\prod_{i>j} s_{ij}! \cdot \prod_i [s_{ii}! \cdot 2^{s_{ii}} \cdot w_i!]} \cdot \prod_i \varkappa_{m_i} \prod_{\gamma \in \operatorname{edge}(\Gamma)} \tilde{Q}(x_{u(\gamma)}, x_{v(\gamma)}) \cdot \prod_i \tilde{r}(x_i)^{w_i} \right\}$$

where the summation extends over all rigged graphs Γ.

The factorial coefficient in this formula can be deleted by means of a linguistic transformation that is performed in Secs. 2.5–2.6.

Example. Let $K(h) = 1 + h$. Denote by $\mathscr{R}$ the set of all partitions of the set $1, 2, 3, \dots$ of positive integers into two- and one-point subsets among which there are only finitely many two-point subsets. Then relation (20) can be rewritten in the form

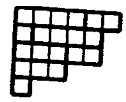
$$\sum_{S \in \mathscr{R}} \left(\prod_{\{\alpha,\beta\} \in S} \tilde{Q}(x_\alpha, x_\beta) \cdot \prod_{\{\sigma\} \in S} (1 + \tilde{r}(x_\sigma)) \right)$$

where the summation extends over all partitions of the set of positive integers (or, equivalently, over all partitions of the set of variables $x_1, x_2, \dots$) satisfying the above condition and $\{\alpha, \beta\}$ and σ ranges over all two-point and one-point subsets of the partition $S \in \mathscr{R}$, respectively.

2.5. Tangles. Let N be a countable set. In what follows, N will coincide with the set $\mathbf{N}$ of positive integers or with the set $\mathscr{X}$ of symmetric variables $x_1, x_2, \dots$. With every element $a \in N$ we associate a nonnegative integer l_a so that all the numbers l_a, except for finitely many of them, are equal to 0. This set of data will be called a *configuration* over N. It is convenient to imagine that we choose points from the set N, and l_a is the number of occurrences of the point a. To avoid any possible ambiguity below, we state a more formal definition.

By a *configuration* over N we mean a pair (L, π), where L is a finite set and $\pi\colon L \to N$ is a mapping. Two configurations (L, π) and (L', π') over N are said to *coincide* if there exists a bijection $\theta\colon L \to L'$ such that $\pi = \pi' \circ \theta$.

2.8. Expansion of Gaussian vectors with respect to the bases $\mathfrak{q}_{n_1,n_2,\dots}$. Consider two formal series $K(h)=\sum \varkappa_j h^j$ and $M(h)=\sum \mu_j h^j$, where $\varkappa_0=\mu_0=1$, $\varkappa_1\neq 0$, and $\mu_1 \neq 0$, and also a sequence ω, which has no influence now. Let $b[Q|r]\in F(l_2(\omega))$ be a Gaussian vector and let $\mathfrak{B}^M[Q|r](x)$ be its image in $\mathrm{Symm}_{M,\omega}$. We are interested in the expansion of the function $\mathfrak{B}^M[Q|r](x)$ with respect to the basis $\mathfrak{q}^K_{n_1,n_2,\dots}$. To this end, we introduce the Young diagram $\Sigma=(1^{n_1}2^{n_2}\dots)$, i.e., the diagram with exactly n_j rows of length j (for all j). By a *row cut* we mean the cut of the Young diagram into horizontal bands of the form shown in the figure below.



In other words, to make a row cut means to take an ordered partition for each row of the diagram.

With every piece of a row cut we associate a positive integer equal to the length of this piece. Then the set of pieces of a row cut T can be regarded as a configuration over $\mathbb{N}$. We denote this configuration by $\mathscr{W}_T$. Furthermore, for a given row cut T, denote by $\zeta_s=\zeta_s(T)$ the number of pieces into which the row with index s is cut.

Theorem 3. *Let $M(h)=K(\sum_{j>0}\sigma_j h^j)$. Then*

$$\mathfrak{B}^M[Q|r]=\sum_{n_1,n_2,\dots} A_{n_1,n_2,\dots}\mathfrak{q}^K_{n_1,n_2,}$$

where the coefficients $A_{n_1,n_2,\dots}$ are calculated by the formula

$$A_{n_1,n_2,\dots}=\sum_T\Big(\prod_s \sigma_{\zeta_s}\cdot \mathfrak{N}[Q|r](\mathscr{W}_T)\Big) \tag{26}$$

In this formula, the summation and multiplication extend over all row cuts and all rows of the diagram $\Sigma=(1^{n_1}2^{n_2}\dots)$, respectively, and the functions $\mathfrak{N}$ are defined in the previous subsection.

Proof. The proof is reduced to the direct calculation of the expression

$$\mathfrak{B}^M[Q|r](x)=\exp\Big\{\sum q_{ij}\frac{\partial^2}{\partial\alpha_i\partial\alpha_j}+\sum r_j\frac{\partial}{\partial\alpha_j}\Big\}\cdot\prod_\xi K\Big(\sum_{n>0}\sigma_n\Big(\sum_{j>0}\alpha_j x_\xi^j\Big)^n\Big)$$

We can write down the factor $\exp\{\quad\}$ using formula (24) and then directly calculate the derivatives, after which the numerous factorials are canceled out.

Remark. In particular, we obtain formulas for the expansion of the functions $\mathfrak{B}^K$ with respect to the monomial symmetric functions m_λ and also with respect to the functions $p_1^{k_1}p_2^{k_2}\dots$ (which makes sense in the case of $K(h)=\exp(h)$ or $K(h)=1+h$). Curiously, the form of the expansions relative to the other standard bases (e_λ, h_λ, and the Schur functions; see [3]) closely resembles that of (26) (we omit these formulas).

Remark. Consider two operators $I_1\colon F(l_2(\omega))\to \mathrm{Symm}_{M,\omega}$ and $I_2\colon F(l_2(\nu))\to\mathrm{Symm}_{K,\nu}$. We have

$$(I_2^{-1}I_1 b[Q|r])(z)=\sum A_{n_1,n_2,\dots}\frac{z_1^{n_1}z_2^{n_2}\dots}{n_1!\,n_2!\dots}.$$

§3. Space of Skew-Symmetric Functions

3.1. Fermion Fock space. Let $\dots,\xi_{-1},\xi_0,\xi_1,\xi_2,\dots$ be a set of anticommuting variables (i.e., $\xi_i\xi_j=-\xi_j\xi_i$). Consider the space Λ whose basis is formed by all monomials

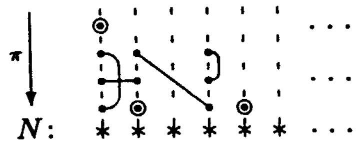
$$\xi_{i_0}\xi_{i_1}\xi_{i_2}\xi_{i_3}\dots, \tag{27}$$

where $i_0>i_1>i_2>\dots$ and $i_k=-k$ for sufficiently large k. The space Λ is one of the versions of the fermion Fock space (this version seems to be for the first time mentioned in [1]); see [8; 5, 4.1].

The *inner product* in Λ is defined by the condition that the vectors (27) form an orthonormal basis.

Consider a configuration (L, π) over N. For any $a \in N$, we denote by l_a the number of points in the preimage of the point a. This number is called the *multiplicity* of the point a in the configuration (L, π). Obviously, two configurations coincide provided that the multiplicities of all points in the set N coincide. Thus, we return to the definition of a configuration in the beginning of this subsection.

Let (L, π) be a configuration over N. By a *tangle* on (L, π) we mean a partition of the set L into two- and one-point subsets:



The asterisks, the heavy points, and the points in the circles in the figure mark the elements of the set N, the elements of L, and the one-point sets, respectively, and the arcs join the elements of the two-point sets.

Denote by $\operatorname{tng}(L,\pi)$ the set of all tangles on a configuration (L, π), by $\operatorname{one}(S)$ the set of all one-point subsets of a tangle $S \in \operatorname{tng}(L, \pi)$, and by $\operatorname{two}(S)$ the set of its two-point subsets.

2.6. Another formula for Gaussian vectors. Denote by $\mathscr{X}$ the set of all variables $x_1, x_2, \dots$ and by $\operatorname{Conf}(\mathscr{X})$ the set of all configurations over $\mathscr{X}$. Then Theorem 2 can be stated as follows.

Theorem 2′. *To a Gaussian vector $b[Q|r] \in F(l_2(\omega))$ there corresponds an element $\mathfrak{B}^K[Q|r]$ of $\mathrm{Symm}_{K,\omega}$ that is determined by the formula*

$$\mathfrak{B}^K[Q|r](x) = \sum_{L \in \operatorname{Conf}(\mathscr{X})} \Bigl(\prod_i \varkappa_{l_i} \sum_{S \in \operatorname{tng}(L)} \prod_{\{x_{\alpha_i}, x_{\beta_i}\} \in \operatorname{two}(S)} \tilde{Q}(x_{\alpha_i}, x_{\beta_i}) \prod_{\{x_{\tau_j}\} \in \operatorname{one}(S)} \tilde{r}(x_{\tau_j})\Bigr)$$

where l_i is the multiplicity of the variable x_i in the configuration L.

Proof. We must prove that the coefficients of the expansion of the right-hand side of formula (22) with respect to the functions m_λ coincide with those in formula (26) below. The equivalence of Theorems 2 and 2′ follows from formula (25).

2.7. The functions $\mathfrak{R}[Q|r]$. Let $Q = Q(n, \tilde{n}) = q_{n\tilde{n}}$ be a function on $N \times N$ such that $Q(n, \tilde{n}) = Q(\tilde{n}, n)$ and let $r = r(n) = r_n$ be a function on N. Define a function $\mathfrak{R}[Q|r]$ on the set of configurations by the formula

$$\mathfrak{R}[Q|r](L, \pi) = \mathfrak{R}[Q|r](l_1, l_2, \dots) = \sum_{S \in \operatorname{tng}(L, \pi)} \Bigl(\prod_{\{\sigma_i, \delta_i\} \in \operatorname{two}(S)} Q(\sigma_i, \delta_i) \cdot \prod_{\{\tau_\alpha\} \in \operatorname{one}(S)} r(\tau_\alpha)\Bigr)$$

where l_n is the multiplicity of the point $n \in N$ in the configuration (L, π).

We state two more definitions of $\mathfrak{R}[Q|r](l_1, l_2, \dots)$. First,

$$\exp\Bigl(\frac{1}{2}\sum q_{ij} z_i z_j + \sum r_j z_j\Bigr) = \sum_{l_1, l_2, \dots} \mathfrak{R}[Q|r](l_1, l_2, \ \ .) \prod_j \frac{z_j^{l_j}}{l_j!}$$

and, second,

$$\frac{1}{\prod l_j!} \mathfrak{R}[Q|r](l_1, l_2, \dots) = \sum \Bigl\{\prod_{i>j} \frac{q_{ij}^{s_{ij}}}{s_{ij}!} \prod_i \frac{q_{ii}^{s_{ii}}}{2^{s_{ii}} s_{ii}!} \prod_i \frac{r_i^{t_i}}{t_i!}\Bigr\} \tag{25}$$

where the summation extends over all sets of nonnegative integers s_{ij} and r_i such that $s_{ij} = s_{ji}$ and

$$r_j + \sum_{i \neq j} s_{ij} + 2 s_{jj} = l_j$$

for all j

3.2. The standard supercomplete basis. Nonformally speaking, the standard supercomplete basis in Λ consists of the products $l_1(\xi)l_2(\xi)\dots$, where $l_j(\xi)$ are linear forms. Since these products can be divergent, this definition must be stated more accurately.

First method. Let $C = \{c_{ij}\}$, $i = 1, 2, \dots$, $j = 0, 1, 2, \dots$, be a Hilbert–Schmidt matrix (that is, $\operatorname{tr} C^*C < \infty$). We set

$$\Xi(C) = \prod_{j=\infty}^{0} (\xi_{-j} + c_{j1}\xi_1 + c_{j2}\xi_2 + \dots) := \Big(\xi_0 + \sum_{k>0} c_{0k}\xi_k\Big)\Big(\xi_{-1} + \sum_{k>0} c_{1k}\xi_k\Big) \tag{28}$$

Then

$$\langle \Xi(C), \Xi(D)\rangle_\Lambda = \det(1 + CD^*)$$

(and, in particular, the norm $\|\Xi(C)\|_\Lambda$ is finite).

Second method. Let $R = \{r_{ij}\}$ be a matrix in which i and j range over the integers $0, -1, -2, \dots$ and over the set $\mathbf{Z}$, respectively, so that $r_{ij} = 0$ for $j < i$. Suppose that $r_{ii} = 1$ for all i with sufficiently large absolute values. We set

$$\Xi\{R\} = \prod_{i=-\infty}^{0} \Big(\sum_{j=i}^{\infty} r_{ij}\xi_j\Big) \tag{29}$$

If the parentheses were formally removed, then there would appear uncountably many terms in this expression. However, we require that only the first term be taken from each of the parentheses except for finitely many of them.

In any case, expression (29) makes sense as a formal series with respect to the basis (27). However, the norm $\|\Xi(R)\|_\Lambda$ may happen to be infinite.

Remark. Expression (28) has the form (29). Moreover, it can be represented in this form in many ways. At the same time, expression (27), which has the form (29), cannot be represented in the form (28).

3.3. The space of skew-symmetric functions. Let us introduce the formal symbol ∞. By a *pseudomonomial* we mean an expression of the form

$$x_0^{\infty+a_0} x_1^{\infty+a_1} \dots,$$

where it is assumed that $a_j = -j$ for all sufficiently large j. A *pseudopolynomial* is defined as a (generally, infinite) formal linear combination of pseudomonomials.

By a *skew-symmetric formal series* we mean a pseudopolynomial that is skew-symmetric with respect to the group S_∞ of all finite permutations of the variables $x_1, x_2, \dots$. Denote by $\widehat{\mathrm{ASymm}}$ the space of all skew-symmetric formal series.

Consider the skew-symmetric formal series

$$S_{\alpha_0,\alpha_1,\dots} := \sum_{\sigma\in S_\infty} (-1)^\sigma x_{\sigma(0)}^{\infty+\alpha_0} x_{\sigma(1)}^{\infty+\alpha_1} \tag{30}$$

(where $\alpha_j = -j$ for all large j). By a *skew-symmetric polynomial* we mean a finite linear combination of the formal series $S_{\alpha_0,\alpha_1,\dots}$. Denote by ASymm the space of all skew-symmetric polynomials. The *inner product* in ASymm is defined by the condition that the polynomials $S_{\alpha_0,\alpha_1,\dots}$ form an orthonormal basis in ASymm.

Remark. Let us denote by ASymm_n the space of polynomials in n variables that are skew-symmetric with respect to the permutations of the variables. Any of these polynomials has the form

$$g(x_0, \dots, x_{n-1}) \cdot \prod_{0\le i<j\le n-1} (x_i - x_j),$$

where $g(x_0, \dots, x_{n-1})$ is a symmetric polynomial. The canonical projection $\pi_{n+1}\colon \mathrm{ASymm}_{n+1} \to \mathrm{ASymm}_n$ is defined by the formula

$$(\pi_{n+1}f)(x_0, \dots, x_{n-1}) = f(x_0, \dots, x_{n-1}, 0) \cdot (x_0 x_1 \dots x_{n-1})^{-1}$$

In this case, ASymm is the inverse limit of the spaces ASymm_n in the category of $\mathbb{Z}$-graded spaces (we assume that the degree of homogeneity of the series $S_{\alpha_0,\alpha_1,\dots}$ is equal to $\sum_j(\alpha_j + j)$).

Remark. Let $f(x)$ be a skew-symmetric formal series. Then

$$f(x) = (x_0^{\infty} x_1^{\infty-1} x_2^{\infty-2} \quad .) g(x),$$

where $g(x)$ is an ordinary formal series in positive and negative powers of x_i (i.e., an infinite linear combination of monomials of the form $x_0^{\beta_0} x_1^{\beta_1} \dots x_s^{\beta_s}$, where $\beta_j \in \mathbf{Z}$). Let $P_{ij}g(x)$ be the expression obtained from $g(x)$ by transposing x_i and x_j (where $i > j$). Then

$$P_{ij} g(x) = -\frac{x_i}{x_j} g(x).$$

Conversely, if $g(x)$ satisfies this condition, then expression (31) is a skew-symmetric series.

3.4. The fermion-skew-symmetric correspondence. The natural isomorphism between the space of skew-symmetric functions and the fermion Fock space is defined as follows: to a vector $\xi_{\alpha_0}\xi_{\alpha_1}\cdots$ we put into correspondence the skew-symmetric function $S_{\alpha_0,\alpha_1,\dots}$ defined by formula (30).

3.5. The supercomplete basis in the space of skew-symmetric functions. Consider the system of functions

$$q_j(x) = \sum_{k \geq 0} a_k^{(j)} x^k, \qquad j = 0, 1, 2, \dots,$$

where $a_0^{(j)} = 1$ for all sufficiently large j, and the skew-symmetric function

$$\Xi[q_0, q_1, \dots \;:= (x_0^{\infty} x_1^{\infty-1} x_2^{\infty-2} \quad) \cdot \det \begin{array}{ccc} q_0(x_0) & x_1^{-1} q_0(x_1) & x_2^{-2} q_0(x_2) \\ x_0 q_1(x_0) & q_1(x_1) & x_2^{-1} q_1(x_2) \\ x_0^2 q_2(x_0) & x_1 q_2(x_1) & q_2(x_2) \\ \dots\dots & \dots\dots & \dots\dots \end{array}$$

(in this determinant, we take products that differ from $q_0(x_0)q_1(x_1)\dots$ only in finitely many factors).

We can readily see that the skew-symmetric function $\Xi[q_1, q_2, \dots]$ corresponds to the vector $\Xi\{R\} \in \Lambda$ given by formula (29) with $r_{ij} = a_{j-i+1}^{(-i+1)}$.

3.6. The correspondence between skew-symmetric and symmetric functions. Denote by Δ the skew-symmetric function

$$S_{0,-1,-2,\dots} = \sum_{\sigma \in S_\infty} (-1)^\sigma x_{\sigma(0)}^{\infty} x_{\sigma(1)}^{\infty-1} x_{\sigma(2)}^{\infty-2} \dots = \prod_{j \geq 0} x_j^{\infty-j} \prod_{\alpha<\beta} \left(1 - \frac{x_\beta}{x_\alpha}\right)$$

It is convenient to imagine or to assume, by definition (this is a kind of normalization for the symbol ∞), that

$$\Delta = \prod_{0 \leq i < j < \infty} (x_i - x_j).$$

Let f be a symmetric function. Then $f \cdot \Delta$ is a skew-symmetric function and $f \mapsto f \cdot \Delta$ is a unitary operator from the space Symm of symmetric functions (depending on $x_0, x_1, \dots$) *that is endowed with the classical inner product* into the space ASymm of skew-symmetric functions.

Remark. The operator $f \mapsto f \cdot \Delta$ is none other than the *boson-fermion correspondence* (see [8]).

3.7. Examples. (a) The functions $S_{\alpha_1,\alpha_2\dots} \in$ ASymm correspond to the Schur functions (see [3]) in Symm (see [8]).

(b) *Supercomplete boson basis.* Let $q(t) = 1 + \sum_{k>0} a_k t^k$. Consider the symmetric function $\prod_{j=0}^{\infty} q(x_j)$. We can readily see that the related skew-symmetric function is $\Xi[q, q, \dots]$, and the related element of the fermion Fock space is $\Xi(C(q))$ (see (28)), where the entries $c_{ij}(q)$ of the matrix $C(q)$ are found from the relation

$$\sum_{i,j\geq 0} c_{ij}(q)\, z^i u^j = \frac{q(z)/q(u) - 1}{z - u}.$$

(c) *Images of fermion vectors in the boson space.* Let $c(\xi)\in\Lambda$. Then, by (5), the corresponding element $g \in F$ is

$$g(z_1, z_2, \ldots) = \langle c(\xi), \Xi[C(q_z)]\rangle_\Lambda,$$

where

$$q_z(t) = \exp\left(\sum \frac{z_j t^j}{j}\right)$$

In particular, the function corresponding to the vector $\Xi(A)$ is $\det(1 + AC(q_z))$.

(d) The *Virasoro algebra.* Let θ be the same function as in Sec. 1.7. Then, in the notation of (32), we have

$$\Delta \cdot \prod_{i>j}\left(\frac{\theta(x_i) - \theta(x_j)}{x_i - x_j}\right) = \prod_{i>j}(\theta(x_i) - \theta(x_j)) = \Xi[\theta, \theta^2, \theta^3, \ldots].$$

References

1. F. A. Berezin, "Some remarks on the representation of the commutation relations," Usp. Mat. Nauk, **24**, No. 4, 65–88 (1969).
2. S. V. Kerov, "Generalized Hall–Littlewood symmetric functions and orthogonal polynomials," Adv. Sov. Math., **9**, 67–94 (1992).
3. I. G. Macdonald, Symmetric Functions and Hall Polynomials. Second edition, Clarendon Press, Oxford, 1995.
4. Yu. A. Neretin, "Holomorphic continuations of representations of the group of diffeomorphisms of the circle," Mat. Sb., **180**, No. 5, 635–657 (1989).
5. Yu. A. Neretin, Categories of Symmetries and Infinite-Dimensional Groups, Clarendon Press, Oxford, 1996.
6. Yu. A. Neretin, "On the correspondence between the boson Fock space and the L^2 space with respect to the Poisson measure," Mat. Sb., **188**, No. 11, 1997.
7. A. Odzijevicz, Quantum Algebras and q-Special Functions Related to Coherent States on the Disc, Preprint, Warsaw University.
8. A. Pressley and G. Segal, Loop Groups, Clarendon Press, Oxford, 1988.

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