$$\Pi_1 = \prod_{u\in Z_+} (1 + efX^{2n} + efX^{4n} + efX^{6n} + \ldots) = \prod_{n\in Z_+} [1 - (1 - ef)X^{2n}](1 - X^{2n}),$$

which proves (2).

The author is grateful to A. V. Zelevinskii and D. B. Fuks for lively interest in the paper and inestimable help in writing it. I thank L. A. Kaluzhina in whose seminar in Kiev at KSU I first reported this work (September, 1984).

LITERATURE CITED

1. E. M. Wright, J. London Math. Soc., 40, 55-57 (1965).
2. C. Sudler, Jr., Proc. Edinburgh Math. Soc., 15, Part 1, 67-71 (1966).
3. J. Zolnowsky, Discrete Math., 9, No. 3, 293-298 (1974).
4. R. P. Lewis, Am. Math. Mon., 91, No. 7, 420-423 (1984).

REPRESENTATIONS OF COMPLEMENTARY SERIES ENTERING DISCRETELY IN TENSOR PRODUCTS OF UNITARY REPRESENTATIONS

Yu. A. Neretin

UDC 519.46

The problem of decomposing the tensor product of two representations of the complementary series of a simple Lie group into irreducible representations was solved only for the groups $SL_2(\mathbf{R})$ (L. Pukanszky, V. F. Molchanov) and $SL_2(\mathbf{C})$ (M. A. Naimark) [1]. (Concerning the application of these results to representations of infinite-dimensional groups see [2, 3]). Here we investigate the problem of decomposing tensor products for a number of complementary series of representations, induced from maximal parabolic subgroups.

1. Notations. Let P be a parabolic subgroup of the semisimple Lie group G. We denote by $\mathrm{Ind}_P(\rho)$ the representation of G induced by the representation of P equal to representation ρ on the reductive part of P and to the trivial representation on the nilradical of P (induction is considered in the ordinary nonunitary sense; see [1]). Let Reg denote the set of representations of G that are weakly contained in the regular representation (see [11, 18]).

2. Tensor Products of Representations of the Spherical Series of the Groups SO(p, 1), SU(p, 1), Sp(p, 1). For simplicity we consider the case $G = SO_0(n+1, 1)$ [the cases SU(p, 1) and Sp(p, 1) are treated similarly]. Let G act on S^n by conformal transformations. We denote by k(g, x) the dilation coefficient of the transformation g at the point x. Let P be the stabilizer of the point x_0. Then the reductive part of P is isomorphic to $SO(n) \times R$. Let χ_λ denote the character of P, defined by $\chi(a) = k(a, x_0)^{n-\lambda/2}$. The spherical representations $D_\lambda = \mathrm{Ind}_P(\chi_\lambda)$ of the groups $SO_0(n+1, 1)$ are defined by the formula

$$D_\lambda(g) f(x) = f(gx) k(g, x)^{n-\lambda/2}, \tag{1}$$

where $f \in C^\infty(S^n)$. If $\lambda = n + is, s \in \mathbf{R}$, then the operators (1) are unitary in the metric of $L^2(S^n)$. Such representations form the spherical basic series, denoted here by T_s. If $0 < \lambda < 2n$, $\lambda \neq n$, then the operators (1) are unitary in the space H_λ with inner product

$$\langle f_1; f_2\rangle = \iint \| x_1 - x_2\|^{-\lambda} f_1(x_1) \overline{f_1(x_2)}\, dx_1\, dx_2,$$

where $x_1, x_2 \in \mathbf{R}^{n+1}, \|x_1\| = \|x_2\| = 1$ (for $\lambda > n$ the integral is defined as the analytic continuation in λ). These representations form the spherical complementary series D_λ. We recall that $T_s \simeq T_{-s}$, and $D_\lambda \simeq D_{2n-\lambda}$.

LEMMA 1. a) $T_s \otimes T_t$ does not depend on t, s; b) $T_s \otimes T_t \in \mathrm{Reg}$.

Proof. a) The equivalence of $T_s \otimes T_t$ and $T_{s+\alpha} \oplus T_{t+\alpha}$ is established with the help of the unitary intertwining operator $A_{i\alpha}$ of multiplication by $\|x_1 - x_2\|^{i\alpha}$. On the other hand, $T_s \otimes T_t \simeq T_s \otimes T_{-t}$; see also [1, 9, 10].

Moscow Institute of Electronic Engineering. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 20, No. 1, pp. 79-80, January-March, 1986. Original article submitted September 17, 1984.

 0016-2663/86/2001-0068$12.50

b) If B is an arbitrary representation of G, then $X \in \mathrm{Reg}$ implies $X \otimes B \in \mathrm{Reg}$ (see [11], 18.3.5 or 13.11.3).

Let $0 < \lambda, \mu < n$. Let θ_k denote the representation of SO(n) with the highest weight (k, 0,...,0).

THEOREM.

$$D_\lambda \otimes D_\mu = \Big(\bigoplus_{k,j} \operatorname{Ind}_P(\theta_{k-2j} \otimes \chi_{\lambda+\mu+2k})\Big) \oplus (T_0 \otimes T_0),$$

where the sum is taken over all k, j such that $\lambda + \mu + 2k < n$, $k - 2j \geqslant 0$.

Proof. Let Δ denote the diagonal of $S^n \times S^n$. We let $\mathscr{F}_p$ denote the space of all derivatives of order $\leqslant p$ of δ-functions supported at points of Δ. The space $\mathscr{F}_p$ is contained in $H_\lambda \otimes H_\mu$ if and only if $p < 1/2(n - \lambda - \mu)$. The normal bundle to Δ is isomorphic to the tangent bundle. Hence, the representation of $SO_0(n + 1, 1)$ in $\mathscr{F}_q/\mathscr{F}_{q-1}$ is isomorphic to $\operatorname{Ind}_P \times (S^q\theta_1 \otimes \chi_{\lambda+\mu+2q})$. But $S^q(\theta_1) \simeq \bigoplus_{0\leqslant 2j\leqslant q} \theta_{q-2j}$.

Let $\lambda \leqslant \mu$, $\alpha = n - \mu$. Consider the operator A_α of multiplication by the function $\|x_1 - x_2\|^\alpha$. This is a closed (unbounded !) operator intertwining $D_\lambda \otimes D_\mu$ and $T_0 \otimes D_{\lambda+\alpha}$. Its kernel consists of distributions supported on Δ. It remains to show that for every $0 < \nu < 2n$ we have $T_0 \otimes D_{n-2\nu} \simeq D_{n+\nu} \otimes D_{n-\nu} \simeq D_{n-\nu} \otimes D_{n-\nu} \simeq T_0 \otimes T_0$.

Remark (the dual model). Let $\mu, \lambda > n$. We denote by $\mathscr{F}'_p$ the subspace of $H_\lambda \otimes H_\mu$, consisting of functions that have a zero of order $>p$ on Δ. Then the "discrete" part of $D_\lambda \otimes D_\mu$ is realized in the factors of the filtration $\mathscr{F}'_0 \supset \mathscr{F}'_1 \supset \dots$ (see also [5]).

3. Tensor Products for the Groups $G = SO_0(p, 1)$, $U(p, 1)$, $Sp(p, 1)$. The arguments given in Sec. 2 are readily carried over to tensor products of representations of arbitrary complementary series of G. Let $T_1 = \operatorname{Ind}_P(\rho_1)$, $T_2 = \operatorname{Ind}_P(\rho_2)$ be unitary representations of G belonging to the complementary series. Let ρ be the representation of the reductive part of P in the tangent space to G/P. Then the problem of decomposing the representation of G in $\mathscr{F}_p/\mathscr{F}_{p-1}$ into irreducible subrepresentations reduces to that of decomposing the finite dimensional representation $\rho_1 \otimes \rho_2 \otimes S^p\rho$.

4. α. Tensor Representations for the Groups SO(p, q), U(p, q), S(p, q), p < q. Let **K** = **R**, **C**, or **H**. We consider in $\mathbf{K}^{p+q}$ the Grassmanian M of maximal isotropic subspaces. Let P be the stabilizer of a point of M; then the reductive part of P is GL(p, **K**) × U(p − q, **K**). Let ρ be an irreducible representation of P whose restriction to SL(p, **K**) is trivial. Then statements analogous to those of Sec. 2 hold true for the complementary series of the form $\operatorname{Ind}_P(\rho)$ (except for Lemma 1, b).

β. (the notations are those of Sec. 4. α) There exist unitary representations of the form $\operatorname{Ind}_P(\rho)$, where ρ is a finite dimensional representation of P whose restriction to SL(p, **K**) is trivial. (These representations were discovered in the theory of representations of the groups $O(p, \infty)$, $U(p, \infty)$, $Sp(p, \infty)$ as "prelimit" representations; see [4]. Some of them can be constructed as "discrete" components in Sec. 4. α.) The arguments of Sec. 2 permit us to guess the discrete spectrum in this case.

5. α. For the Stein representation of SL(2n, R) and SL(2n, C) (see [1]), and also for the Molchanov representations of $O(p, q)$ (see [8]), we reach the conclusion that a tensor product of representations of the complementary series are equivalent to tensor product of representations of the fundamental series (here the complementary series lie "too close" to the fundamental one).

β. Let G be the universal covering of $SL_2(R)$. Then the set of irreducible representations $T_{\alpha,s}$ of G which do not lie in Reg is parametrized by the points of the triangle $-\alpha \leqslant s \leqslant \alpha$, $0 \leqslant s < 1$ (where (0, 0) corresponds to the one-dimensional representations, $s = \pm\alpha$ to highest and lowest weight representations, and the remaining points to the complementary series (see [6]); α and s are connected with the parameters of [6] by: $\alpha = 2\tau - 1$, $q = s^2 + 1/4$, $l = s$; to $SL_2(R)$ correspond integral values of α). Then for $s + t < 1$ we have $t_{\alpha,s} \otimes T_{\beta,t} = T_{\alpha+\beta,s+t} \pmod{\mathrm{Reg}}$, while for $s + t > 1$, $T_{\alpha,s} \otimes T_{\beta,t} \in \mathrm{Reg}$.

γ. The arguments of Sec. 2 carry over to tensor products of representations of indefinite complementary series and products of representations of the fundamental series by representations of the indefinite complementary series. Here additional interesting effects occur, similar to those discussed in [7]; see also [8].

δ. The arguments of Sec. 2 are applicable to problems concerning restriction to a subgroup (see also [8]).

ε. The theorem of Sec. 2, in conjunction with results of the works [9, 10], gives the decomposition of tensor products in the case of arbitrary complementary series of the de Sitter group $SO_0(4, 1)$, as well as in the case of the spherical series of $SO_0(p, 1)$.

As M. I. Graev informed the author, the imbedding of $D_{\lambda+\mu}$ in $D_\lambda \otimes D_\mu$ for the groups SO(p, 1), SU(p, 1) (see the theorem) was discovered by A. M. Vershik, M. I. Gel'fand, and M. I. Graev.

LITERATURE CITED

1. D. P. Zhelobenko and A. I. Stern, Representations of Lie Groups [in Russian], Nauka, Moscow (1983).
2. A. M. Vershik, I. M. Gel'fand, and M. I. Graev, Usp. Mat. Nauk, 28, No. 5, 83-128 (1973).
3. Yu. A. Neretin, Dokl. Akad. Nauk SSSR, 272, No. 3, 528-531 (1983).
4. G. I. Ol'shanskii, Funkts. Anal. Prilozhen., 18, No. 1, 28-42 (1984).
5. H. P. Jacobsen and M. Vergne, J. Funct. Anal., 34, No. 1, 29-53 (1979).
6. L. Pukanszky, Math. Ann., 156, No. 1, 96-143 (1974).
7. Sh. Sh. Sultanov, Funkts. Anal. Prilozhen., 11, No. 4, 92-93 (1967).
8. V. F. Molchanov, Dokl. Akad. Nauk SSSR, 237, No. 4, 782-785 (1977).
9. R. P. Martin, Trans. Am. Math. Soc., 284, No. 2, 795-814 (1984).
10. V. K. Dobrev, Lect. Notes Phys., Vol. 63, Springer-Verlag (1977).
11. J. Diximier, Les C*-Algebres et leurs Representation, Gauthier-Villars, Paris (1969).

FREE BANACH SPACES AND REPRESENTATIONS OF TOPOLOGICAL GROUPS

V. G. Pestov

UDC 513.83+513.88

We introduce below the notion of free Banach space B(X, *) over a metric space X with marked point *. We use it to show that every Hausdorff topological group G admits a complete system of representations in Banach spaces. We denote by K the fields **R**, **C**, and **H**.

Let X = (X, ρ) be a metric space and * a point of X. We define a free Banach space over K of the metric space X with marked point * as a Banach space (X, *) together with a fixed isometric embedding of X in (B, *), which takes * into 0, such that: 1) (B, *) is the closed linear span of X; 2) for every Banach space E and every nonexpanding map f:X → E [i.e., such that $\|fx - fy\| \leqslant \rho(x, y)$ for all $x, y \in X$] taking * into 0_E, there is a linear operator $\tilde{f}$:B(X, *)→ E of norm not exceeding 1, whose restriction to X is f.

LEMMA. If A is a subset of a metric space (X, ρ), then the map x → ρ(x, A) is nonexpanding.

Proof. Suppose that for $x, y \in X$ we have ρ(y, A) − ρ(x, A) > ρ(x, y). Pick $a \in A$, such that ρ(x, *a*) < ρ(x, A) + ε, where ε = ρ(y, A) − ρ(x, A) − ρ(x, y) > 0. Then ρ(y, A) ⩽ ρ(y, *a*) ⩽ ρ(x, y) + ρ(x, *a*) < ρ(y, A); contradiction.

THEOREM 1. For every metric space (X, ρ) with marked point * its free Banach space over K exists and is unique up to an isometric isomorphism.

Proof. Let $\mathscr{F}$ denote the family of all nonexpanding maps of X into K that take * into 0_K. For $f \in \mathscr{F}$ let $\bar{f}$ denote the linear extension of f to the linear space sp X containing * as the zero element and X \ {*} as a Hamel basis. For $x \in \mathrm{sp}\, X$ we put $\|x\| = \sup\{|\bar{f}(x)| : f \in \mathscr{F}\}$. The correctness of the definition follows from the fact that for arbitrary $\lambda_i \in K$ and $x_i \in X$ we have $\|\sum_{i=1}^n \lambda_i x_i\| \leqslant \sum_{i=1}^n |\lambda_i| \sup |f(x_i)| < \infty$, since $|f(x_i)| \leqslant \rho(x_i, *)$. Function $\|\cdot\|$ is obviously a prenorm. Let $x = \sum_{i=1}^n \lambda_i x_i$ be a nonzero element of sp X; we may assume that $\lambda_i \neq 0$, $x_i \neq x_j$ for i, j = 1,..., n, i ≠ j. By the lemma, the map $f_1 : x \to \rho(x, \{x_2, \ldots, x_n, *\})$ belongs to $\mathscr{F}$. Therefore,

Tomsk State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 20, No. 1, pp. 81-82, January-March, 1986. Original article submitted May 29, 1984.

0016-2663/86/2001-0070$12.50