Differential operators on Lie groups

Yu.A. Neretin

In this note we study the structure of the algebra of differential operators on a Lie group generated by all left and right Lie derivations.

1. Let G be a Lie group, $\mathfrak{G}$ its Lie algebra, $U(\mathfrak{G})$ a universal enveloping algebra for $\mathfrak{G}, Z$ the centre of $U(\mathfrak{G})$, $U(\mathfrak{G})^T$ the opposite algebra to $U(\mathfrak{G})$, $C^\infty(G)$ the space of smooth functions on G, and $C(G, e)$ the space of generalized functions on G with carrier e (e is the identity of G). The algebra $\mathfrak{G}$ acts on $C^\infty(G)$ by left and right Lie derivations, which we denote by L_x and R_x, respectively ($x \in \mathfrak{G}$)

$$(L_x f)(g) = \frac{d}{dt} f(\exp(tx)\, g)\,|_{t=0}\ 5\ (R_x f)(g) = \frac{d}{dt} f(g \exp(tx))\,|_{t=0}.$$

We consider the algebra S of differential operators on G generated by all the operators of the form L_x and R_x. The algebra generated by R_x alone is isomorphic to $U(\mathfrak{G})$ and that generated by L_x to $U(\mathfrak{G})^T$ (since L acts on the right). Because the left and right Lie derivations commute, S is a homomorphic image of $U(\mathfrak{G})^T \otimes U(\mathfrak{G})$. Let us find the kernel of this homomorphism.

2. Now $U(\mathfrak{G})^T \otimes U(\mathfrak{G})$ acts on $C^\infty(G)$.

Lemma. The annihilators of $C^\infty(G)$ and $C(G, e)$ in $U(\mathfrak{G})^T \otimes U(\mathfrak{G})$ are the same.

Thus, it suffices to find the annihilator of $C(G, e)$. As is known (see, for example, [2]), there is an isomorphism between $U(\mathfrak{G})$ and $C(G, e)$. Let $x_1, \ldots, x_k \in \mathfrak{G}$. Then to the element $x_1 x_2 \ldots x_k$ of $U(\mathfrak{G})$ there corresponds the generalized function $\theta(x_1 \ldots x_k)$ defined by the formula

$$\langle \theta(x_1 \ldots x_k), f\rangle = (R_{x_1} \ldots R_{x_k} f)(e).$$

We transfer the action of $U(\mathfrak{G})^T \otimes U(\mathfrak{G})$ from $C(G, e)$ to $U(\mathfrak{G})$. It can be verified directly that $U(\mathfrak{G})^T \otimes U(\mathfrak{G})$ acts on $U(\mathfrak{G})$ by left and right multiplication. Since left and right multiplications by central elements are the same, our action is, in effect, an action of the algebra $U(\mathfrak{G})^T \otimes_Z U(\mathfrak{G})$, which is a factor algebra of $U(\mathfrak{G})^T \otimes U(\mathfrak{G})$.

The mapping from $U(\mathfrak{G})^T \otimes_Z U(\mathfrak{G})$ into S may have a non-zero kernel. The simplest example of this is the 3-dimensional soluble Lie algebra with basis x, y, z satisfying $[x, y] = y$, $[x, z] = z$, and $[y, z] = 0$. It can easily be verified that $y \otimes z$ and $z \otimes y$ act on $U(\mathfrak{G})$ in the same way, whereas the centre of $U(\mathfrak{G})$ is trivial.

3. Let K be the field of fractions of $U(\mathfrak{G})$ (see, for example, [1]), C the centre of K, and K^T the opposite field to K. We extend the action of $U(\mathfrak{G})^T \otimes U(\mathfrak{G})$ on $U(\mathfrak{G})$ to an action of $K^T \otimes K$ on K. The resulting action is, in fact, an action of $K^T \otimes_C K$.

***Theorem* 1.** *The annihilator $U(\mathfrak{G})$ in $K^T \otimes_C K$ is zero.*

Proof. It is easy to see that 1) the annihilator is a right ideal in $K^T \otimes_C K$; 2) it is closed under left multiplication by elements of the form $u \otimes_C v$, where $u \in U(\mathfrak{G})^T$, and $v \in U(\mathfrak{G})$.

Let us assume that there is a set $I \subset K^T \otimes_C K$ having the properties 1) and 2), other than 0 and the whole algebra $K^T \otimes_C K$. Let $v \in K^T \otimes_C K$. We consider all of its representations in the form $\sum_{i=1}^{n} a_i \otimes b_i$. The smallest possible n is called the length of v. We take in I a non-zero element l of smallest possible length k ($k > 1$, for otherwise $I = K^T \otimes_C K$). Let $\sum a_i \otimes_C b_i$ be its shortest form. We multiply l on the right by $a_1^{-1} \otimes_C 1$ and obtain an element of the form

$$1 \otimes_C b_1 + c_2 \otimes_C b_2 + \ldots + c_k \otimes_C b_k \in I.$$

Using 1) and 2) we find that for any h in $U(\mathfrak{G})$

$$[h, c_2] \otimes_C b_2 + \ldots + [h, c_k] \otimes_C b_k \in I.$$

But the length of this element is less than k. Consequently, it is zero. But the b_i are linearly independent over C, and so all $[h, c_i] = 0$ for any h in $U(\mathfrak{G})$, that is, all the c_i are in C. But then l has length 1, which is a contradiction and proves the theorem.

Hence, it is sufficient to find the kernel of the natural composition mapping γ

$$U(\mathfrak{G})^T \otimes U(\mathfrak{G}) \xrightarrow{\alpha} K^T \otimes K \xrightarrow{\beta} K^T \otimes_C K.$$

It is obvious that α is an embedding. Then, we have to find $\mathrm{Im}\,\alpha \cap \mathrm{Ker}\,\beta$.

Definition. Let $\lambda \in \mathfrak{G}^*$ ($\mathfrak{G}^*$ is the conjugate space to $\mathfrak{G}$). By $U(\mathfrak{G})_\lambda$ we denote the set consisting of all $v \in U(\mathfrak{G})$ such that $[v, x] = \lambda(x)v$ for any $x \in \mathfrak{G}$. The union of all subspaces $U(\mathfrak{G})_\lambda$ is called the semicentre of $U(\mathfrak{G})$.

The kernel of β is the two-sided ideal in $K^T \otimes K$, generated by all the elements of the form $C \otimes 1 - 1 \otimes C$, where $C \in c$. It is known that for any C in c there are a $\lambda \in \mathfrak{G}^*$ and $u, v \in U(\mathfrak{G})_\lambda$, such that $c = uv^{-1}$ (see [3]). Then $u \otimes v - v \otimes u \in \mathrm{Im}\,\alpha \cap \mathrm{Ker}\,\beta$. Let us consider the two-sided ideal N in $U(\mathfrak{G})^T \otimes U(\mathfrak{G})$ generated by all the elements of this form. It is not hard to show that $\mathrm{Ker}\,\gamma$ consists of all $p \in U(\mathfrak{G})^T \otimes U(\mathfrak{G})$ for which there is a $\mu \in \mathfrak{G}^*$ and an $\omega \neq 0$ in $U(\mathfrak{G})_\mu$ for which $p(\omega \otimes \omega) \in N$. Obviously, the multiplication by $\omega \otimes \omega$ is superfluous, that is, $\mathrm{Ker}\,\gamma = N$.

***Theorem* 2.** *If $\mathfrak{G}$ is semisimple, then the algebra S is isomorphic to $U(\mathfrak{G}) \otimes_Z U(\mathfrak{G})$.*

Proof. In this case C coincides with the field of fractions of Z (see [1]). Now the injectivity of the mapping $U(\mathfrak{G})^T \otimes_Z U(\mathfrak{G}) \to K^T \otimes_C K$ follows easily from the theorem of Kostant (that $U(\mathfrak{G})$ is a free Z-module for semisimple $\mathfrak{G}$; see [1], 8.2).

In conclusion, I should like to thank A.A. Kirillov for posing this problem and also V.A. Ginzburg for useful discussion and advice.

References

[1] J. Dixmier, Enveloping Algebras, North Holland Publ. Co., Amsterdam-New York-Oxford 1977. MR 58 # 16803a, b, c.
Translation: *Universal'nye obertyvayushchie algebry,* Mir, Moscow 1978.

[2] A.A. Kirillov, *Elementy teorii predstavlenii,* Nauka, Moscow 1972. MR 53 # 10985.
Translation: Elements of the theory of representations, Springer-Verlag, Berlin-Heidelberg-New York 1976. MR 54 # 447.

[3] R. Rentshler and M. Vergne, Sur le semi-centre du corps enveloppant d'une algèbre de Lie, Ann. Sci. École Norm. Sup. (4) 6 (1973), 380–405. MR 50 # 13177.

Moscow State University

Received by the Editors 30 May 1980