In conclusion, the author conveys thanks to V. I. Arnol'd for a useful discussion.

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A SPINOR REPRESENTATION OF AN INFINITE-DIMENSIONAL ORTHOGONAL SEMIGROUP AND THE VIRASORO ALGEBRA

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Most infinite-dimensional representations of Lie groups can be easily realized by means of operators which are products of the change in variables and the multiplication by a function. In the case of infinite-diminsional groups, two very special classes of operators, acting in the boson and fermion Fock space are almost as important; this means that representations of infinite-dimensional groups have a habit of "passing through" the Weyl representation and the spinor representation (see, e.g., [3, 8, 9, 14]).

A spinor representation of the automorphism group of the canonical anticommutation relations (CAR) has been constructed by Berezin in [1]. The aim of our paper is to extend this representation onto as large a domain as possible; this domain is a semigroup (which is not surprising, cf. [11]), containing some linear transformations of CAT, in general unbounded (there are many more bounded transformation CAR than had been usually assumed, see Sec. 2.3). Speaking of unbounded operators, it is natural to use the language of their graphs, in other words, our semigroup consists of linear relations between CAR. Notice that even in the finite-dimensional case our construction does not coincide with the standard sources on spinor representations [4, 1, 15, 2].

The considered construction (a part of it has been announced in [8]) implies a number of corollaries for the theory of representations of infinite-dimensional groups. In Sec. 3, we show that any irreducible representation of the Virasoro algebra with the highest-order weight, no necessarily unitary, can be integrated to a projective representation of the group Diff of diffeomorphisms of the circle which, in turn, extends onto the complex extension of the group Diff constructed in [10]. Further, we consider a problem arising in conformal quantum field theory concerning the construction of an operator with respect to an arbitrary Riemannian surface in such a way that the operators should multiply by each other when the Riemannian surfaces are patched together (notice that recently there appeared a number of articles in which the patching of Riemannian surfaces and the Virasoro algebra are considered, cf. [5-7, 10, 16]). Some other applications of the construction (in which only the group part of our subgroup has been used) have been considered in [8] and [9, Sec. 9].

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anticommuting variables:

$$\xi_k \xi_l = -\xi_l \xi_k, \quad \bar{\xi}_k \xi_l = -\xi_l \bar{\xi}_k, \quad \xi_k \bar{\xi}_l = -\bar{\xi}_l \xi_k, \quad \bar{\xi}_k \bar{\xi}_l = \bar{\xi}_l \bar{\xi}_k.$$

Let Λ_0 be the space of polynomials of the variables $\xi_1, \xi_2, \ldots$. We will introduce in Λ_0 the left differentiation as

$$\frac{\partial}{\partial \xi_k} f(\xi) = 0, \quad \frac{\partial}{\partial \xi_k} (\xi_k f(\xi)) = f(\xi),$$

if $f(\xi)$ does not depend on ξ_k. We will define also a formal integral: $\int \prod_{j=1}^{k} (\xi_{\alpha_j} \bar{\xi}_{\alpha_j}) d\xi d\bar{\xi} = 1$, the integral of the remaining terms is equal to 0. Let $\bar{\Lambda}$ be the completion of Λ_0 with respect to the scalar product

$$\langle f(\xi), g(\xi) \rangle = \int f(\xi) \overline{g(\xi)}\, d\xi\, d\bar{\xi}.$$

This formula, actually, is a formal notation of the fact that vectors of the form $\xi_{i_1}, \ldots, \xi_{i_k}$, where $i_1 < \ldots < i_k$, form an orthonormal basis in $\bar{\Lambda}$. Notice that $\bar{\Lambda}$ is a Hilbert direct sum of subspaces Λ_k, where Λ_k is the space of homogeneous forms of degree k, $k \geq 0$. Let $f \in \bar{\Lambda}$, $f = \sum f_k$, where $f_k \in \Lambda_k$. Denote by Λ the set of all $f \in \Lambda$, such that for any $C > 0$, there exists a number $A = A(f, C)$, such that $\|f_k\| < A \exp(-Ck)$. We will introduce in Λ a family of seminorms $\|f\|_C = \sup_k (\|f_k\| \exp(Ck))$. Then Λ becomes a Frechet space (a complete countably normed space).

Definition. The space $\bar{\Lambda}$ will be called the Hilbert fermion Fock space and the space Λ, the polynormed fermion Fock space.

Example. The space Λ contains all the vectors of the form $\exp(\sum a_{ij} \xi_i \xi_j)$, where $\sum |a_{ij}|^2 < \infty$ (this is a special case of Lemma 1.4).

Let now H be a Hilbert space of dimension n, where n = 0, 1, 2, ..., ∞. We choose in it a basis $e_1, e_2, \ldots$, with each basis vector e_j we will associate the variable ξ_j and construct over the variables ξ_j the space Λ which will be denoted by $\Lambda(H)$. Obviously, $\Lambda_k(H) = \Lambda_k$ is canonically isomorphic to the k-th outer power of the space H. Thus, the construction of $\Lambda(H)$ does not depend on the choice of a basis in H. Analogously, we define space $\bar{\Lambda}(H)$.

Let $(a, b) = (a_1, a_2, \ldots, b_1, b_2, \ldots) \in \ell_2 \oplus \ell_2$. The creation—annihilation operator $\hat{A}(a, b)$ in Λ is defined by the formula

$$\hat{A}(a, b) f(\xi) = \left(\sum a_i \xi_i + \sum b_j \frac{\partial}{\partial \xi_j} \right) f(\xi).$$

Denote now by $\{P, Q\} = PQ + QP$ the anticommutator of operators. Then

$$\{\hat{A}(a, b), \hat{A}(a', b')\} = \sum (a_j b'_j + a'_j b_j) E. \tag{1.2}$$

1.2. Symbols. We will consider a polynormed Fock space Λ_ξ of functions of the variables $\xi_1, \xi_2, \ldots$ and the space Λ_η of functions of $\eta_1, \eta_2, \ldots$. an operator $\hat{K}$ from Λ_η into Λ_ξ can be conveniently written in the form

$$\hat{K} f(\xi) = \int K(\xi, \bar{\eta}) f(\eta)\, d\eta\, d\bar{\eta},$$

where the symbol $K(\xi, \bar{\eta})$ of the operator $\hat{K}$ is a formal series in $\xi, \bar{\eta}$.

It is easy to verify that the symbol of $\hat{A}(a, b)\hat{K}$ equals

$$\sum \left(a_j \xi_j + b_j \frac{\partial}{\partial \xi_j} \right) K(\xi, \bar{\eta}), \tag{1.3}$$

and

$$\sum \left(b_j \bar{\eta}_j + a_j \frac{\partial}{\partial \bar{\eta}_j} \right. \tag{1.4}$$

1.3. Berezin Operators. Let $\hat{T}_i^\xi = \xi_i + (\partial/\partial\xi_i)$. Let A, B, C be matrices, a_{ij}, b_{ij}, c_{ij} be their matrix coefficients. Let ξ denote the symbolic matrix row $(\xi_1, \xi_2, \ldots)$, and η be symbolic matrix row $(\eta_1, \eta_2, \ldots)$.

Definition. An operator $\hat{Q}$ from Λ_η into Λ_ξ is called a Berezin operator if it satisfies one of the two equivalent conditions:

α) The symbol of $\hat{Q}$ has the form:

$$\lambda\Big(\prod_{j=1}^{l} f_j(\xi, \bar{\eta})\Big)\exp\left\{\frac{1}{2}(\xi\,\bar{\eta})\begin{pmatrix} A & B \\ -B^t & \end{pmatrix}\begin{pmatrix} \xi^t \\ \bar{\eta}^t \end{pmatrix}\right\}, \tag{1.5}$$

where f_j has the form $\sum_i \mu_{ij}\xi_i + \sum_i \nu_{ij}\bar{\eta}_i$, $\sum (|\mu_{ij}|^2 + |\nu_{ij}|^2) < \infty$. $A = -A^t$, $C = -C^t$, B is a bounded operator, and A and C are Hilbert–Schmidt operators, $\lambda \in \mathbb{C}$.

β) $Q = \hat{T}^\xi_{i_1} \ldots \hat{T}^\xi_{i_k} \hat{R} \hat{T}^\eta_{j_1} \ldots \hat{T}^\eta_{j_l}$, (1.6)

where $\hat{R}$ is an operator with the symbol

$$\lambda\exp\left\{\frac{1}{2}(\xi\,\bar{\eta})\begin{pmatrix} A & B \\ -B^t & C \end{pmatrix}\begin{pmatrix} \xi^t \\ \bar{\eta}^t \end{pmatrix}\right\}, \tag{1.7}$$

and where A, B, C are the same as in α), and $\lambda \in \mathbb{C}$.

Proposition 1.1. Conditions α and β are equivalent.

LEMMA 1.1. Let $\hat{Q}$ have the symbol of the form (1.5). Then $\hat{T}_{i_1}{}^\xi, \ldots, \hat{T}_{i_k}{}^\xi \hat{Q} \hat{T}_{j_1}{}^\eta, \ldots, \hat{T}_{j_\ell}{}^\eta$ has the symbol of the form (1.5) too.

Proof. We will verify, for example, that the symbol of $\hat{T}_1{}^\xi\hat{Q}$ has the form (1.5). With the help of (1.3) we obtain that this symbol is equal to

$$\lambda\left[\xi_1 f_1 \cdots f_l + \sum_{j=1}^{l}(-1)^j \frac{\partial f_j}{\partial \xi_1} f_1 \cdots f_{j-1} f_{j+1} \cdots f_l + f_1 \cdots f_l \frac{\partial K}{\partial \xi_1}\right]\exp(K(\xi, \bar{\eta})),$$

where $K(\xi, \bar{\eta}) = 1/2(\xi\bar{\eta})\begin{pmatrix} A & B \\ -B^t & C \end{pmatrix}\begin{pmatrix} \xi^t \\ \bar{\eta}^t \end{pmatrix}$. If all $(\partial f_j/\partial\xi_1) = 0$, then our result is obvious. Otherwise, the expression

$$N = \sum (-1)^j \frac{\partial f_j}{\partial \xi_1} f_1 \cdots f_{j-1} f_{j+1} \cdots f_l$$

expands into a product of $\ell - 1$ linear forms of the form $\sum \chi_i f_i$, and $f_1, \ldots, f_\ell$ can be divided by N, i.e., there exists a linear form g of the form $\sum \theta_i f_i$, such that $f_1, \ldots, f_\ell = gN$. Thus, the expression in (1.8) in square brackets is reduced to the form $N(1 + g \times (\xi_1 + (\partial K/\partial\xi_1)) = N \exp(g(\xi_1 + (\partial K/\partial\xi_1)))$. The lemma has been proved.

Proof of Proposition 1.1. From Lemma 1.1, it immediately follows that a Berezin operator in the sense β) is a Berezin operator in the sense of α). Let now the formal series of the form (1.5) for the symbol of operator $\hat{Q}$ contain the components $\hat{\xi}_{i_1}, \ldots, \hat{\xi}_{i_k}\hat{\eta}_{j_1}, \ldots, \hat{\eta}_{j_\ell}$. Then, $\hat{Q}' = \hat{T}_{i_1}{}^\xi, \ldots, \hat{T}_{i_k}{}^\xi\hat{Q}\hat{T}_{j_1}{}^\eta, \ldots, \hat{T}_{j_\ell}{}^\eta$ by Lemma 1.1 is a Berezin operator in the sense of α). But in the formal series for the symbol of $\hat{Q}'$ the free term is not null and, therefore, in (1.5) we have $\ell = 0$, i.e., the symbol $\hat{Q}'$ has the form of (1.7); consequently, $\hat{Q}'$ and, that is also $\hat{Q}$, is a Berezin operator in the sense of β).

Remark. The representation of a Berezin operator in the form α) as well as in the sense β) is not unique. We will discuss this nonuniqueness in detail.

α. Let Q be the symbol of a Berezin operator $\hat{Q}$, let $Q = \sum_{m=0}^{\infty} Q_k$, where Q_k are homogeneous forms of the degree k in ξ, $\bar{\eta}$. Let Q_r be the first nonnull component of this sense. Then $\lambda f_1, \ldots, f_\ell = Q_r$. Thus, although $\lambda, f_1, \ldots, f_\ell$ can be chosen in various manners, their product is uniquely determined. Moreover, the quadratic form $\Omega(\xi, \bar{\eta})$, occurring in (1.5) in braces, is determined uniquely up to the transformations $\Omega(\xi, \bar{\eta}) \to \Omega(\xi, \bar{\eta}) + \sum_j f_j(\sum \alpha_{ij}\xi_i + \sum \beta_{ij}\bar{\eta}_j)$.

β. It is clear from the proof of Lemma 1.1 that operator $\hat{Q}$ can be represented in the form (1.6) with given $i_1, \ldots, i_k, j_1, \ldots, j_\ell$ if and only if $\langle \xi_{i_1}, \ldots, \xi_{i_k}, \hat{Q}\eta_{j_1}, \ldots, \eta_{j_\ell}\rangle \neq 0$.

1.4. THEOREM 1. a) All Berezin operators from $\Lambda(H)$ into $\Lambda(K)$ are bounded.

b) A product of Berezin operators is a Berezin operator.

Statement b) is a corollary to Theorem 3 (cf. Sec. 2); the proof of a) takes the rest of this section.

LEMMA 1.2. Let r be an analytic Hilbert–Schmidt operator, such that $\langle x, Ry\rangle = -\langle y, Rx\rangle$; then there exists an orthonormal basis, with respect to which the form $\langle x, Ry\rangle$ reduces to $\sum \lambda_k(x_{2k-1}y_{2k} - y_{2k-1}x_{2k})$, $\lambda > 0$.

The proof is completely standard.

LEMMA 1.3. Let $A = \sum|a_{ij}|^2 < \infty$. Let $P_r^{(s)}$ be an operator from Λ_s into Λ_{s+2r} mapping f on $\frac{1}{r!}\left(\sum a_{ij}\xi_i\xi_j\right)^r f$. Then $\|P_r^{(s)}\|^2 \leq (1/r!)\ a^{r+s/2}$, where $a = 2\max(A, 1)$.

Proof. By Lemma 1.2 one can assume that $\sum a_{ij}\xi_i\xi_j$ has the form $\sum \lambda_k\xi_{2k-1}\xi_{2k}$. Let

$$b = \sum b_{i_1\ldots i_s}\xi_{i_1}\ldots\xi_{i_s} \in \Lambda_s.$$

Then

$$P_r^{(s)}b = \sum \pm b_{i_1\ldots i_s}\lambda_{\alpha_1}\ldots\lambda_{\alpha_r}\xi_{i_1}\ldots\xi_{i_s}\prod_{i\leq r}(\xi_{2\alpha_i-1}\xi_{2\alpha_i}),$$

where among the indices $i_1, \ldots, i_s, 2\alpha_1, \ldots, 2\alpha_r, 2\alpha_1 - 1, \ldots, 2\alpha_r - 1$ there are no repetitions. Using the inequality

$$(x_1 + \ldots + x_n)^2 \leqslant n\,(x_1^2 + \ldots + x_n^2),$$

we obtain

$$\|P_r^{(s)}b\|^2 \leqslant C_{r+[s/2]}\left(\sum|b_{i_1\ldots i_s}|^2\lambda_{\alpha_1}^2\cdot\ \ \lambda_{\alpha_r}\right) \leqslant \left(\sum|b_{i_1\ldots i_s}|^2\right)\left(\sum\lambda_{\alpha_1}^2\ldots\lambda_{\alpha_r}^2\right) < \left(\sum|b_{i_1\ldots i_s}|^2\right)\frac{(\sum\lambda_j^2)^r}{r!};$$

consequently,

$$\|P_r^{(s)}\|^2 \leqslant \frac{1}{r!}C_{r+\left[\frac{s}{2}\right]}^r\left(\sum\lambda_j^2\right)^r,$$

follows from the desired estimation.

LEMMA 1.4. Operator L of multiplication by $\exp(\sum a_{ij}\ \xi_i\xi_j)$, where $\sum|a_{ij}|^2 < \infty$ is bounded in Λ.

Proof. Let $f = \sum f_k$, $Lf = \sum (Lf)_k$, where $f_k \in \Lambda_k$, $(LF)_k \in \Lambda_k$. Let $\|f_j\| < \exp(-Cj)$ Let $P_r^{(s)}$ and a be such as in Lemma 1.3. Then

$$\|(Lf)_n\|^2 = \sum_{2j\leqslant n}\|P_j^{(n-2j)}f_{n-2j}\|^2 \leqslant \sum_{2j\leqslant n}\|P_j\|^2\|f_{n-2j}\|^2 \leqslant \sum_{2j\leqslant n}\frac{1}{j!}a^{n/2}e^{-C(n-2j)} = e^{-Cn}a^{n/2}\sum_{2j\leqslant n}\frac{e^{2Cj}}{j!} < e^{-Cn}a^{n/2}e^{e^{2C}}.$$

Hence, $\|f\|_{C-1/2\ln\alpha} < \|f\|_C\cdot\text{const}$, and from this the lemma follows.

LEMMA 1.5. Let B be a bounded operator from a Hilbert space H into a Hilbert space K. Then the operator $\lambda[B]$: $\Lambda(H) \to \Lambda(K)$, acting on $\Lambda_k(H)$ as the k-th outer power of B, is bounded.

The proof is obvious.

LEMMA 1.6. Let $\sum|a_{ij}|^2 < \infty$. Then the operator $N = \exp\left(\sum a_{ij}\frac{\partial}{\partial\xi_i}\frac{\partial}{\partial\xi_j}\right)$ is bounded in Λ

Proof. Let us consider operator $Q_r^{(s+2r)}$, acting from Λ_{s+2r} into Λ_s, as $\frac{1}{r!}\left(\sum a_{ij}\frac{\partial}{\partial\xi_i}\frac{\partial}{\partial\xi_j}\right)^r$. It is easy to see that $Q_r^{(s+2r)} = P_r^{(s)})^*$, where $P_r^{(s)}$ has been introduced in Lemma 1.3. Let $f = \sum f_k$, where $f_k \in \Lambda_k$. Let $Nf = \sum (Nf)_k$, where $(NF)_k \in \Lambda_k$. Then

$$\|(Nf)_n\|^2 = \sum_{r\geqslant 0}\|Q_r^{(n+2r)} f_{n+2r}\|^2 \leqslant \sum \|Q_r^{(n+2r)}\|^2\,\|f_{n+2r}\|^2 = \tag{1.8}$$

$$= \sum_{r\geqslant 0}\|P_r^{(n)}\|^2\,\|f_{n+2r}\|^2 \leqslant \sum_{r\geqslant 0}\frac{1}{r!}\,a^{\frac{n}{2}+r}e^{-C(n+2j)} = a^{n/2}e^{-Cn}\exp\left(ae^{-Cn}\right).$$

Hence $\|Nf\|_{C-1/2\ln a} \leq \|f\|_C\cdot\mathrm{const}$, from which the boundedness of n follows.

Proof of Theorem 1a). We will apply the definition β of a Berezin operator. It is enough to verify the boundedness of an operator with the symbol of the form (1.7). But this operator is equal to

$$\mu\exp\left(\sum a_{ij}\xi_i\xi_j\right)\circ\lambda[B]\circ\exp\left(\sum c_{ij}\frac{\partial}{\partial\xi_i}\,\frac{\partial}{\partial\xi_j}\right) \tag{1.9}$$

where $\mu \in \mathbf{C}$, and $\lambda[B]$ have been introduced in Lemma 1.5.

1.5. Berezin Operators in the Hilbert Space $\bar{\Lambda}$.

THEOREM 2. Let $\hat{Q}$ be a Berezin operator from $\bar{\Lambda}(H)$ to $\bar{\Lambda}(K)$.

a) The necessary condition for the boundedness of $\hat{Q}$ is that the matrix B have the form $L(1 + S)$, where $\|L\| \leq 1$ and S is a Hilbert–Schmidt operator.

b) The sufficient condition for the boundedness of the operator $\hat{Q}$ is the B have the form $L(1 + S)$, where $\|L\| < 1$ and S is a nuclear operator.

c) If A and C are nuclear operators, then $\hat{Q}$ is bounded if and only if B is of the form $L(1 - S)$, where $\|L\| \leq 1$ and operator S is nuclear.

LEMMA 1.7. a) Let K be the operator of multiplication by $\exp(\lambda\xi_1\xi_2)$ or the operator $\exp(\lambda(\partial/\partial\xi_1)(\partial/\partial\xi_2))$ in $\Lambda(\mathbf{C}^2)$. Then there exists a constant θ, such that

$$1-\frac{|\lambda|}{2}-\theta|\lambda|^2\leqslant\langle Kx, Kx\rangle\leqslant 1+\frac{|\lambda|}{2}+\theta|\lambda|^2 \quad \text{for} \quad \|x\|=1.$$

b) Let L be an operator in $\Lambda(\mathbf{C}^2)$ with the symbol $\exp(\mu\xi_1\eta_1 + \mu\xi_2\eta_2 + \chi\xi_1\xi_2)$, where μ is a constant number, $0 < \mu < 1$. Then for $|\lambda| \to 0$, $\|L\| = 1 + C(\mu)|\lambda|^2 + o(|\lambda^2|)$ holds.

Proof. It is an easy calculation.

Proof of the Theorem. Without loss of generality, we can assume that H and K are infinite-dimensional, $H = K = \ell_2$, the operator $\hat{Q}$ has the symbol of the form (1.7), and that the matrices A and C are represented as in Lemma 1.2. Finally, in the computations of norms we should take into account that $H \to \Lambda(H)$ is a functional mapping direct sums into tensor products.

c) From Lemma 1.7a and the nuclearity of A and C it follows that the operator of multiplicity by $\exp(\sum a_{ij}\xi_i\xi_j)$ and the operator $\exp\left(\sum c_{ij}\frac{\partial}{\partial\xi_i}\frac{\partial}{\partial\xi_j}\right)$ are bounded together with their inverses. Therefore, one can assume that $A = C = 0$ [see (1.9)]. Further, for self-adjoint B's the problem of boundedness is easy to solve (this is the problem of uniform boundedness of outer powers of B) and, in the general case, we will take the polar decomposition of B.

b) It is enough to show the result for the obvious case $A = C = 0$ and for operators with symbols of the form

$$\exp\left\{\sum \varkappa_k\xi_{2k-1}\xi_{2k} + (1-\varepsilon)\sum_k \xi_k\bar{\eta}_k\right\},$$

$$\exp\left\{(1-\varepsilon)\sum_k \xi_k\bar{\eta}_k + \sum \varkappa_k\bar{\eta}_{2k-1}\eta_{2k}\right\},$$

where $\varepsilon > 0$, which can be easily done with the help of Lemma 1.7b.

LEMMA 1.8. Let M be a subset of N. Let H be a subspace in ℓ_2, spanned by e_i, $i \in M$. Let P_H be the projection in $\Lambda = \Lambda(\ell_2)$ onto $\Lambda(H)$. Let $f(\xi, \bar{\eta})$ be the symbol of the operator $\hat{Q}$. then the symbol of operator $P_H\hat{Q}P_H$ can be obtained if in $f(\xi, \bar{\eta})$ we put $\xi_j = 0$, $\bar{\eta}_j = 0$ for all $j \notin M$.

Proof. A direct verification.

Proof. 1) Without loss of generality we can assume that B is self-adjoint (otherwise we can take the polar decomposition of B), and that the operator $B - 1$ is positive (otherwise we can choose the corresponding spectral subspace). Suppose that for some $\varepsilon > 0$ the spectral subspace of B, corresponding to $[\varepsilon, \infty)$, is infinite-dimensional. Then from the very beginning we can assume that $B - 1 - \varepsilon \geq 0$. Let us consider an arbitrary n-dimensional subspace $H \subset \ell_2$ and the operator $P_H\hat{Q}P_H$ (see Lemma 1.8). We will represent it in the form (1.9). Then the norm of the middle factor is not less than $(1 + \varepsilon)^n$, the first factor cannot diminish the length of a vector more than $\prod_{i=1}^{n} (1 + |\lambda_i|/2 + \theta|\lambda_i|^2)$ times, where λ_i are the eigenvalues of the matrix A (see Lemma 1.7a), and the third factor cannot shorten the length of a vector more than $\prod_{i=1}^{n} (1 + |\mu_i|/2 + \theta|\mu_i|^2)$ times, where μ_i are the eigenvalues of the matrix C. Letting n tend to infinity, we obtain that the number $\|P_H\hat{Q}P_H\|$ can be arbitrarily great. A contradiction.

Thus, $B - 1$ is a positive compact operator. It remains to show that it is of the Hilbert–Schmidt type. For this end, we have to repeat only the just-presented reasoning, but now H should be spanned by a eigenvectors of $B - 1$ with the greatest eigenvalues.

2. Spinor Representation

2.1. An Object of the Category Or is a complex Hilbert space V of the dimension 2n (where $n = 0, 1, \ldots, \infty$), in which:

1. There are fixed vector subspaces V_+ and V_-, with $V = V_+ \oplus V_-$.
2. There is given an antilinear invertible isometry $L\colon V_- \to V_+$.
3. There is given a symmetric bilinear form

$$\{(v_+, v_-), (w_+, w_-)\} = \langle v_+, Lv_-\rangle + \langle w_+, Lv_-\rangle,$$

where $\langle\cdot, \cdot\rangle$ is a scalar product in V_+.

Notice that $V_\pm$ are maximal isotropic with respect to the form $\{\cdot, \cdot\}$ subspaces in V. Let us fix an orthonormal basis e_j^V in V_+, and an orthonormal basis $f_j^V = Le_j^V$ and an orthonormal basis $f_j^V = Le_j^V$ in V_-. Then $\{e_i, f_j\} = \delta_{ij}$.

By V_R we will denote a real subspace in $V = V_+ \oplus V_-$ consisting of all vectors of the form (Lv, v).

2.2. Orthogonal Relations. Let V, W be objects of the category Or. We will introduce in $V \oplus W$ an orthogonal form $\{(v_1, w), (v_2, w_2)\}' = \{v_1, v_2\} - \{w_1, w_2\}$. An orthogonal relation $P\colon V \rightrightarrows W$ will be called the maximal isotropic with respect to the form $\{\cdot, \cdot\}'$ subspace of $V \oplus W$.

Example. Let $V = W$, and let A be an orthogonal operator from V into itself. Then the set of pairs of the form (v, Av) is a maximal orthogonal relation $V \rightrightarrows V$.

Let $P\colon V_1 \rightrightarrows V_2$ and $Q\colon V_2 \rightrightarrows V_3$ be orthogonal relations. Then their superposition $QP\colon V_1 \rightrightarrows V_3$ is the set of all pairs $(v_1, v_3) \in V_1 \oplus V_3$, for which there exists a v_2, such that $(v_1, v_2) \in P$, $(v_2, v_3) \in Q$.

2.3. Morphisms of the Category Or. Let V and W be objects of the category Or. We will say that an orthogonal relation T lies in set $\mathrm{mor}_0(V, W)$, if T is a graph of an operator from $V_+ \oplus W_-$ into $V_- \oplus W_+$, while the matrix of this operator $\Omega_T = \begin{pmatrix} K & L \\ -L^t & M \end{pmatrix}$ satisfies the conditions:

1. $\|\Omega_L\| < \infty$.
2. $K = -K^t$, $M = -M^t$ (it follows from the fact that the relation T is orthogonal).
3. K and M are Hilbert–Schmidt operators.

We will say that matrix Ω_T is the Potapov–Ginzburg transformation of relation T.

We will define now the subset $\mathrm{mor}(V, W)$ of the set of all orthogonal relations from V into W. Namely, $L \in \mathrm{mor}(V, W)$ if there exists a relation $L' \in \mathrm{mor}(V, W)$, such that the codimension of the subspace $L \cap L'$ in L is finite. Finally, the set $\mathrm{Mor}(V, W)$ of all morphisms from V into W will be defined as the set $\mathrm{mor}(V, W)$, to which the formal "null" element $\mathrm{null}_{V,W}$ has been adjoined.

2.4. Multiplication of Morphisms. It can be shown (it is far from obvious), that for any $p \in \mathrm{Mor}(V, W)$, $Q \in \mathrm{mor}(W, Y)$ their product QP as the product of orthogonal relations is in $\mathrm{Mor}(V, Y)$. The product of morphisms of the category Or could be defined just in this way; however, the resulting multiplication, even in the case of finite-dimensional spaces V, W, and Y is not continuous. The following definition leads to the same theory of representations of the category Or; however, from the technical point of view, it appears to be much more useful.

Let $P \in \mathrm{mor}(V, W)$, $Q \in \mathrm{mor}(W, Y)$. If there exists a nonnull vector $w \in W$, such that $(0, w) \in P$, $(w, 0) \in Q$, then $QP = \mathrm{null}_{V,Y}$; if, however, there is no vector like that, then the product QP is defined as the product of orthogonal relations. Finally, the product of the null morphism with any other equals to the null morphism.

2.5. Semigroup $\Gamma O(V)$. Let V be an object of the category Or. The orthogonal semigroup $\Gamma O(V)$ will be defined as $\mathrm{Mor}(V, V)$. The group $O(V)$ of the invertible elements of the semigroup $\Gamma O(V)$ consists of all invertible orthogonal operators in V whose matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$: $V_+ \oplus V_- \to V_+ \oplus V_-$ satisfies the condition: B and C are Hilbert–Schmidt operators (a spinor representation of this group has been constructed in [8]). Finally, the classical automorphism group of the caninical anticommutation relations consists of operators $P \in O(V)$, preserving the subspace V_R.

2.6. Spinor Representation. Let $P \in \mathrm{Mor}(V, W)$. Let $\Lambda(V_+)$ and $\Lambda(W_+)$ be polynormed Fock spaces. Let at first $P \in \mathrm{mor}_0(V, W)$, and let $\begin{pmatrix} K & L \\ -L^t & M \end{pmatrix}$ be the Potapov–Ginzburg transformation of the relation P. Then the operator $\mathrm{Spin}(P)$: $\Lambda(V_+) \to \Lambda(W_+)$ will be defined as an operator with the symbol

$$\exp\left\{\frac{1}{2}(\xi\ \bar\eta)\begin{pmatrix} K & L \\ -L^t & M \end{pmatrix}\begin{pmatrix} \xi \\ \bar\eta \end{pmatrix}\right\}.$$

Let now $P \in \mathrm{mor}(V, W)$. Let $S = P \cap (V_- \oplus W_+)$, and let $P' \in \mathrm{mor}_0(V, W)$ be such that $P \cap P'$ is the complementary subspace to the subspace S in P. Let $\begin{pmatrix} K & L \\ -L^t & M \end{pmatrix}$ be the Potapov–Ginzburg transformation of P'. Let $s_1, \ldots, s_k$ be a basis in S, with $s_m = \sum_\alpha p_\alpha^m e_\alpha^V + \sum_\beta q_\beta^m f_\beta^W$. Then the operator $\mathrm{Spin}(P)$: $\Lambda(V_+) \to \Lambda(W_+)$ has the symbol

$$\prod_{m=1}^{k}\Big(\sum_\alpha p_\alpha^m \xi_\alpha + \sum_\beta q_\beta^m \bar\eta_\beta\Big)\exp\left\{(\xi\bar\eta)\begin{pmatrix} K & L \\ -L^t & M \end{pmatrix}\begin{pmatrix} \xi \\ \bar\eta \end{pmatrix}\right\}$$

At last $\mathrm{Spin}(\mathrm{null}_{V,W}) = 0$.

THEOREM 3. a) $\mathrm{Spin}(QP) = c(Q, P)\,\mathrm{Spin}(Q)\,\mathrm{Spin}(P)$, where $c(Q, P) \in \mathbb{C} \setminus 0$.
b) Let $(v, w) \in P$. Let $v = (v_+, v_-) \in V_+ \oplus V_-$, $w = (w_+, w_-) \in W_+ \oplus W_-$. Then

$$\hat A(w)\,\mathrm{Spin}(P) = \mathrm{Spin}(P)\,\hat A(v).$$

c) Any Berezin operator T from $\Lambda(V_+)$ into $\Lambda(W_+)$ has the form $T = \mathrm{Spin}(Q)$.

Proof. The statement c) is obvious; the statement b) can be verified by a direct computation; the statement a) follows from b), except for the fact that $c(Q, P) \neq 0$. The easiest way to verify the latter is to compute the vectors $\mathrm{Spin}(Q)\,\mathrm{Spin}(P)h$ and $\mathrm{Spin}(QP)h$, where h runs over all simple spinors (see Sec. 2.8).

2.7. Another Form for Spin P. Let V be an object of the category Or. Let D_j^V be an operator in V, defined by the equalities

$$D_j^V e_j^V = f_j^V,\quad D_j^V f_j^V = e_j^V,\quad D_j^V e_i^V = e_i^V,\quad D_j f_i^V = f_i^V$$

for $i \neq j$. Let $P \in \mathrm{mor}(V, W)$ and let i_α and j_β be such that $P_0 = D_{i_1}^V \ldots D_{i_k}^V P D_{j_\ell}^W \in \mathrm{mor}_0(V, W)$. Then

$$\mathrm{Spin}(P) = \hat T_{i_1}^{\xi} \ldots \hat T_{i_k}^{\xi}\,\mathrm{Spin}(P_0)\,\hat T_{j_1}^{\bar\eta} \ldots \hat T_{j_l}^{\bar\eta}$$

(for the definition of T^ξ, see Sec. 1.3).

2.8. Simple Spinors. Let N be a null-dimensional object of Or. Let $P \in \mathrm{mor}(N, V)$. Then the image of the operator Spin(P) is a one-dimensional space in $\Lambda(V)$. The elements of such one-dimensional subspaces, following [4], will be called simple spinors.

3. Overlapping of Riemannian Surfaces and the Complexification of the Group of Diffeomorphisms of the Circle

3.0. Representations of Categories. Let K be a category, Ob(K) be its objects, $\mathrm{Mor}_K(A, B)$ be the morphisms from A into B. We will say that there is given a representation (= projective representation) of K, if for any $A \in \mathrm{Ob}(K)$ there is constructed a linear space H(A), and for any $q \in \mathrm{Mor}_K(A, B)$ an operator $T(q)$: $H(A) \to H(B)$, so that $T(pq) = \lambda(p, q)T(p)T(q)$, where $\lambda(p, q) \in \mathbb{C} \setminus 0$.

3.1. Category $\Gamma\Lambda$. An object of the category is the direct sum of two copies of the same Hilbert space $V = V_1 \oplus V_2$. A morphism from V into W is either the null morphism or a linear subspace $Q \subset V \oplus W$, for which there exists a subspace $R \in V \oplus W$, such that:

1. The codimensions of $Q \cap R$ in q and R are finite (and they might not coincide).

2. R is the graph of the bounded operator $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$: $V_1 \oplus W_2 \to V_2 \oplus W_1$, with A and D being Hilbert–Schmidt operators. We will define the product of morphisms. The product of the null morphism with any other morphism is the null morphism. Let P: $V \to W$, Q: $W \to Y$ be nonnull morphisms. Then P and Q are multiplied as linear relations with the exception of the following two cases, when their product is the null morphism: 1) the subspaces $P \cap W$ and $Q \cap W$ in W have a nonempty intersection; 2) the sum (no matter which, algebraic or topological) of the projection P onto W parallel to V and the projection Q onto W parallel to Y does not coincide with W.

3.2. Embedding of Category $\Gamma\Lambda$ in Category Or. Let H' denote the space adjoint to H. Let $V \in \mathrm{Ob}(\Gamma\Lambda)$, let $\tilde{V} = V \oplus V'$. We will introduce on $\tilde{V}$ the structure of an object of category Or, putting (see Sec. 2.1) $\tilde{V}_+ = V$, $\tilde{V}_- = V'$, $\{(x_1, f_1), x_2, f_2)\} = f_1(x_2) + f_2(x_1)$.

Let $V, W \in \mathrm{Ob}(\Gamma\Lambda)$, $Q \subset V \oplus W$ be a morphism of category $\Gamma\Lambda$. Let $R \subset (V \oplus W)' = V' \oplus W'$ be the annihilator of Q. Then $Q \oplus R \subset (V \oplus V') \oplus (W \oplus W') = \tilde{V} \oplus \tilde{W}$ is a morphism from $\tilde{V}$ into $\tilde{W}$ in category Or.

Hence, we have embedded $\Gamma\Lambda$ into Or. Restricting the spinor representation of Or to $\Gamma\Lambda$, we get a representation of $\Gamma\Lambda$, which we will also call a spinor representation.

3.3. Category Shtan. An object of category Shtan is a nonnegative integer. A morphism from m to n is the collection (R, r_j^+, r_j^-), where

1. R is a compact complex Riemann surface with a boundary, such that the boundary consists of m + n enumerated components.

2. $r_1^+, \ldots, r_m^+, r_1^-, \ldots, r_n^-$: $[0, 2\pi] \to R$ are fixed analytic parametrization of, respectively, 1, 2, ..., m + n components of the boundary, so directed that going along $r_i^+(\varphi$ the surface remains on the left, and along $r_i^-(\varphi)$ on the right side of the contour.

Let (R, r_i^+, r_j^-), (Q, q_i^+, q_i^-) be morphisms from m to n. We will consider them to coincide if there exists a biholomorphic mapping τ: $R \to Q$, such that $q_\alpha^{\pm} = \tau \circ r_\alpha^{\pm}$.

Let (R, r_i^+, r_j^-) be a morphism from m to n, and (P, p_j^+, p_k^-) a morphism from n to ℓ. Then their product is the collection (S, r_i^+, p_k^-), where the Riemannian surface S has been obtained from a nonconnected union of R and P, and pasting of the points $r_j^-(\varphi)$ and $p_j^+(\varphi)$, where $j = 1, \ldots, n$; $\varphi \in [0, 2\pi]$.

3.4. Embedding of Category Shtan into Category $\Gamma\Lambda$. Let A^λ be the space of forms of the weight $\lambda \in \mathbb{Z}$ on the circle S^1: $|z| = 1$ with the scalar product

$$\langle f(z)(dz)^\lambda, g(z)(dz)^\lambda\rangle = \int^{2\pi} f(e^{i\varphi})\overline{g(e^{i\varphi})}\,d\varphi.$$

We will introduce on A^λ the structure of an object of the category $\Gamma\Lambda$ putting $A^\lambda = A_1{}^\lambda \oplus A_2{}^\lambda$, where $A_1{}^\lambda$ consists of forms, holomorphically extendable into the interior of the circle, and $A_2{}^\lambda$ is the orthogonal complement to $A_1{}^\lambda$. Let B_n be the direct sum of n copies of A_λ.

Let $(R, r_i{}^+, r_j{}^-) \in \mathrm{Mor}_{\mathrm{Shtan}}(m, n)$. We will construct for it the subspace $L = L_\lambda \times (R, r_i{}^+, r_j{}^-) \subset B_m \oplus B_n$. Namely, $(b_1{}^+, \ldots, b_m{}^+, b_1{}^-, \ldots, b_n{}^-) \in L$, if there exists a holomorphic form F of the weight λ on R, such that the boundary values of F on the curve $r_i{}^\pm(\varphi)$ is the direct image of the form $b_i{}^\pm$ under the mapping $e^{i\varphi} \to r(\varphi)$ from S^1 into **R**.

THEOREM 4. $L = L_\lambda(R, r_i{}^+, r_j{}^-) \in \mathrm{Mor}_{\Gamma\Delta}(B_m, B_n)$.

Now, restricting the spinor representation of $\Gamma\Lambda$ to Shtan, we obtain a series of representations of Shtan depending on n. It turns out that the representation operators are bounded not only in the sense of the polynormed Fock space Λ, but also in the sense of the Hilbert space $\bar{\Lambda}$.

3.5. Complexification Γ of the Group Diff of Analytic Diffeomorphisms of the Circle, Preserving the Orientation (cf. [11]). Semigroup Γ consists of all elements of $(R, r^+, r^-) \in \mathrm{Mor}_{\mathrm{Shtan}}$ (1.1), for which R is homeomorphic to a ring. The limit elements of Γ correspond to the group Diff: one needs to take a ring degenerating to a circle, then $(r^+)^{-1} r^- \in$ Diff.

Remark. The group Diff does not have any group complexification [because there are no complexifications of n-fold enveloping groups of $SL_2(\mathbf{R})$ contained in it]. Let us explain why Γ is an existing subsemigroup in a nonexisting group Diff_c. It is natural to take as a neighborhood of Diff in Diff_c a local group, consisting of mappings ρ of the circle S^1: $|z| = 1$ into its small neighborhood. Let us consider in this local group a local subsemigroup of all ρ, for which $|\rho(e^{i\varphi})| < 1$. Then R is a ring between the contours S^1 and $\rho(S^1)$, and $r_+(\varphi) = e^{i\varphi}$, $r_-(\varphi) = \rho(e^{i\varphi})$.

3.6. Proof of Theorem 4. Any object of category Shtan can be "cut" into elementary objects of the following 4 types:

1, 2, 3. A domain on the plane bordered by one, two, or three circles with parametrizations of the form $\varphi \to a + be^{\pm i\varphi}$.

4. Elements of Diff.

The conditions of Sec. 3.1 can be easily verified for 1, 2, and 3, and they have been verified for Diff in [9, Secs. 4.3, 9.4].

3.7. Representations of Γ.

THEOREM 5. Any irreducible (not necessarily unitary) representation of the Virasoro algebra with older weight (cf. [10]) is integrable to a projective representation of Γ by bounded operators in a Hilbert space.

Remark. It is interesting that extending this representation onto the "skeleton" Diff of the semigroup Γ we obtain operators which are, generally speaking, unbounded in the sense of Hilbert space.

Proof. Among the subfactors of representations of Γ, constructed in Sec. 3.8, are all representations of the Virasoro algebra with highest-order weight.

3.8. Embeddings of Γ into Morphisms of the Category $\Gamma\Lambda$. Let us consider enveloping $\tilde{\Gamma}$ over Γ, which is a set of quadruples of the form (R, θ, r^+, r^-), where

1. R is a domain, holomorphically equivalent to a circle;

2. θ is a hyperbolic automorphism of R;

3. $r^\pm$ are analytic diffeomorphisms from **R** into R, such that $r^\pm(x + 2\pi) = \theta(r^\pm(x))$. By approach along the curve $r^+(x)$ the domain should remain on the left side, and along $r^-(x)$ on the right side.

Remark. It is convenient to assume that R is a strip, and θ is a shift.

The product of (R, θ, r^+, r^-) by (Q, ψ, q^+, q^-) is, as before, carried out by pasting together points $r^-(x) \in R$ and $q^+(x) \in Q$.

Let now $A_\alpha^\lambda(\lambda, \alpha \in \mathbf{C})$ be the space of forms on $\mathbf{R}$ of the form $f(x)(dx)^\lambda$ satisfying the conditions $f(x + 2\pi) = e^{2\pi i\alpha}f(x)$ with the scalar product

$$\langle f(x)(dx)^\lambda, g(x)(dx)^\lambda\rangle = \int_0^{2\pi} f(x)\overline{g(x)}\,dx.$$

We will introduce on A_α^λ the structure of category $\Gamma\Lambda$, assuming that $(A_\alpha^\lambda)_1$ are forms which are holomorphically extendable into the lower halfplane, $(A_\alpha^\lambda)_2$ is the orthogonal complement to $(A_\alpha^\lambda)_1$. Two elements $b_+, b_- \in A_\alpha^\lambda$ are connected by the linear relation $L = L_{\lambda,\alpha}(R, \theta, r^+, r^-)$, if there exists a holomorphic form μ of the weight λ on R, satisfying the conditions: 1) under the mapping θ the form μ is multiplied by $e^{2\pi i\alpha}$; 2) the boundary values of μ on the curves $r^\pm(x)$ are direct images of the forms $b_\pm$ under the mappings $r^\pm$.

It turns out that $L \in \mathrm{Mor}_{\Gamma\Lambda}(A_\alpha^\lambda, A_\alpha^\lambda)$ and, consequently, we have a projective representation of Γ.

Remark. On the group Diff the construction from [9, Sec. 5.1.3]; the remaining constructions of [9, Sec. 5.1] can be extended on Γ too.

3.9. Category G – Shtan. Let G be a complex algebraic Lie group. by D_+, D_-, D_+^0, D_-^0, S^1 are denoted the following domains of the Riemann sphere: $|z| \le 1$, $|z| \ge 1$, $|z| < 1$, $|z| > 1$, $|z| = 1$.

An object of the G – Shtan category is a positive integer. A morphism $m \to n$ is the quadruple (R, F, r_i^+, r_j^-), $i \le m$, $j \le n$, where

1. R is a compact Riemannian surface (without boundary);
2. F is a principle bundle over R;
3. $r_k^\pm$ are morphisms of the principle G-bundles $D_\pm \times G \to F$. With this the corresponding mappings of bases $D_\pm \to R$ are onefold and holomorphic up to the border and their images do not intersect.

The morphisms (R, F, r_i^+, r_j^-), (Q, H, q_i^+, q_j^-): $m \to n$ are equivalent if there exists a morphism of priniciple G-bundles τ: $(R, F) \to (Q, H)$, such that $q_k^\pm = \tau\circ r_k^\pm$.

Let $(R, F, r_i^+, r_j^-) \in \mathrm{Mor}(m, n)$, $(P, H, p_j^+, p_\ell^-) \in \mathrm{Mor}(n, k)$. Then their product is the quadruple (Q, Z, r_i^+, p_ℓ^-), where the bundle is obtained from the nonconnected union of $(R, F)\setminus \bigcup_j r_j^-(D_-^0 \times G)$ and $(P, H)\setminus \bigcup_j p_j^+(D_+^0 \times G)$ by pasting together points $r_j^-(x)$ with $p_j^+(x)$, where $j = 1, \ldots, n$ and $x \in S^1 \times G$.

Example 1. If G consists of one unity, then G – Shtan coincides with Shtan. In this realization it is evident that there exists on $\mathrm{Mor}_{\mathrm{Shtan}}(m, n)$ a natural complex structure.

Example 2. Let $m = n = 1$, let R be the Riemann sphere, where $r^+(D_+^0 \times G)$ does not intersect with $r^-(D_-^0 \times G)$, and $r^+(D_+^0 \times G) \cup r^-(D_-^0 \times G) = F$. We will denote the set of such objects by ΓG. Formally ΓG does not enter the set of morphisms from 1 to 1, but it can be considered as a part of the border of this set. The multiplication by ΓG is introduced in an obvious way, and we obtain a group which is isomorphic to the semidirect product of Diff and the group of analytic mappings of the circle into the group G, that is one of the essential objects of the representation theory for infinite-dimensional groups (cf. [9, Sec. 7, 8.1]). (The author gives thanks to A. G. Reinman for discussion this point).

3.10. Embeddings of the Category SO(n)–Shtan into Category $\Gamma\Lambda$. The considerations of 3.4-3.8 can be carried over onto SO(n)–Shtan almost literally. One needs to take an n-dimensional vector bundle, associated with the principle bundle F and, instead of the word "form," we will always use "form with values in the bundle." To obtain another representations of the category G–Shtan, it is enough to embed G into $O(n)$.

3.11. Zigel–Krilov Domain K (see [6]) will be defined as the set of triples (R, r^+, z), where R is holomorphically equivalent to the circle, $(R, r^+) \in \mathrm{Mor}_{\mathrm{Shtan}}(1, 0)$, $z \in R$. The triples (R, r^+, z) and (Q, q^+, u) are equivalent if there exists a biholomorphic mapping τ: $R \to Q$, such that $q^+ = \tau\circ r^+$, $\tau(z) = u$. The semigroup Γ acts on K in an obvious way

Any nondegenerate (Sec. [9]) representation of the Virasoro algebra (= of semigroup Γ) with highest-order weight can be realized in cross sections of some holomorphic linear bundle over K.

Proposition. Any nondegenerate (see [9]) representation of the Virasoro algebra (= of semigroup Γ) with highest-order weight can be realized in cross sections of some holomorphic linear bundle over K.

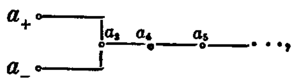
3.12. Krichever–Novikov Bases. Let us consider the morphism (R, r^+, r^-): $1 \to 1$ of category Shtan, being realized as in 3.9. Let μ_j be a form of the weight $\lambda \in Z$ on R, which has at the points $r^+(0)$ and $r^-(\infty)$ nulls of the order $\pm j + g/2 - \lambda(g + 1)$, where g is the sort of R. Let $e_j^{\pm}$ be the coimages of the form μ_j under the mappings $r^{\pm}$: $S^1 \to R$. Then e_j^+ (and, analogously, e_j^-) is a basis in the space A_λ of forms of the degree λ on S^1. Then the relation $L_\lambda(R, r^+, r^-)$ constructed in 3.4 maps e_j^+ to e_j^-. We will take now for every $e_j^{\pm}$ an odd variable $\xi_j^{\pm}$. Then the Berezin operator, constructed through $L_\lambda(R, r^+, r^-)$ as an operator from $\Lambda(\xi^+)$ to $\Lambda(\xi^-)$, maps $\prod_k \xi^+_{\alpha_k}$ to $\prod_k \xi^-_{\alpha_k}$.

In the counting system [7] this means the identification of the Fock spaces corresponding to the contours $r^+(e^{i\varphi})$ and $r^-(e^{i\varphi})$. In our case, these spaces have been identified with each other in another way by the choice of parametrization, and construction [7] in the counting system Shtan means the choice of bases in which the Berezin operator has the unit matrix.

3.13. Classification of Representations of Category Or. After having constructed a spinor representation of category Or, we come across a natural question: Which representations, in general, have the category Or ? We will restrict ourselves to subcategories Or_c of the category Or, whose objects are finite-dimensional, and their morphisms are the same as in Or (obviously, the representation theories for the category Or and Or_c coincide; in any case one can impose on the representations of Or continuity assumptions, so that this statement would be a theorem). It turns out that all representations of Or_c can be realized in contravariant tensors over the spinor representation. For purely aesthetic reasons, we will formulate a classification theorem not for Or_c, but for the following category D.

Its objects are the same as in Or_c. An orthogonal relation P: $V \rightrightarrows W$ is a morphism of the category D if the dimension of the space $P \cap (V_- \oplus W_+)$ is even. The product of morphisms and the null morphism are defined in the same way as in Or. Notice that the group of invertible morphisms of a 2n-dimensional object V_{2n} of the category D is isomorphic to SO(2n), and with each representation of the category D there is connected a representation of SO(2n) in the space $H(V_2n)$ (see Sec. 3.0).

Proposition. Irreducible holomorphic projective representations of the category D are enumerated by diagrams of the form



where a_j are nonnegative integers, among which only a finite number differs from null. Let a_α be the right ulmost nonnull index. If $n \geq \alpha - 1$, then the corresponding representation of SO(2n) is the irreducible representation of SO(2n) with the numerical indices $a_+, a_-, a_3, \ldots, a_n$ on the Dynkin diagram D_n. If $n < \alpha - 1$, then the space $H(V_{2n})$ is null-dimensional.

Analogous results are valid for categories connected also with other classical groups.

After this work had been already submitted to the editor, there appeared a preprint of Gr. Segal, in which independently from M. L. Kontsevich a definition of the Shtan category has been given. Moreover, there appeared [17], where operators have been constructed, the same as in our paper in Sec. 3.4 (the authors, however, are interested neither in the existence problem for the operators, nor in their multiplicative properties).

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In [2, 3] Sklyanin has constructed the family of the algebras $A(\mathscr{E}, \tau)$, parametrized by the set of pairs $(\mathscr{E}, \tau)$, where $\mathscr{E}$ is an elliptic curve and τ is a point on it. This family has the following properties:

1. The algebra $A(\mathscr{E}, \tau)$ is graded, $\dim A(\mathscr{E}, \tau)_i = 0$ for $i < 0$, and $\dim A(\mathscr{E}, \tau)_i = C_{i+3}^{i}$. The algebra $A(\mathscr{E}, \tau)$ is generated by the four-dimensional space $A(\mathscr{E}, \tau)_1$ and quadratic relations: the six-dimensional space

$$\operatorname{Ker}(A(\mathscr{E}, \tau)_1 \otimes A(\mathscr{E}, \tau)_1 \to A(\mathscr{E}, \tau)_2).$$

The algebra $A(\mathscr{E}, 0)$ is isomorphic to the algebra of polynomials in four variables.

2. Let the symbol Γ_n denote the finite Heisenberg group, i.e., the group generated by elements x, y, and ε and the relations $x^n = y^n = \varepsilon^n = 1$, $x\varepsilon = \varepsilon x$, $y\varepsilon = \varepsilon y$, $xy = \varepsilon yx$. The group Γ_4 acts by graduation-preserving automorphisms on the algebra $A(\mathscr{E}, \tau)$. The space $A(\mathscr{E}, \tau)_1$ is an irreducible representation Γ_4.

3. Let $\mathbf{C}[V]$ be the ring of polynomials generated by the space V and $a \in \operatorname{End} V$. Let us form the semidirect product of $\mathbf{C}[t]$ and $\mathbf{C}[V]$. This is the algebra generated by its subalgebra $\mathbf{C}[V]$ and the element t and the relations $tv = (av)t$, where v runs over V. Let $\mathbf{C}[V, a]$ denote the subalgebra of $\mathbf{C}[t] \ltimes \mathbf{C}[V]$ generated by the subspace $\mathbf{C}\cdot 1 \oplus tV$; $\mathbf{C}[V, a]$; it is called the algebra of skew polynomials.

Let τ be a point of fourth order on $\mathscr{E}$. Let us identify the group of points of fourth order on $\mathscr{E}$ with the quotient of Γ_4 modulo the center. Let $X(\tau)$ be a lifting of τ in Γ_4. The algebra $A(\mathscr{E}, \tau)$ is isomorphic to the algebra $\mathbf{C}[A(\mathscr{E}, \tau)_1, X(\tau)]$.

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