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Hausdorff Metric, the Construction of a Hausdorff Quotient Space, and Boundaries of Symmetric Spaces*

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As is known, in many interesting cases, the quotient space of a metric space (topological space, algebraic variety) turns out to be non-Hausdorff. In algebraic geometry, a number of methods are known for constructing a Hausdorff quotient space from a non-Hausdorff one (the closure in the Chow scheme, blow-up of singularities, and the spectrum of an algebra of invariant functions; see [11]). The objective of the present note is to suggest a very simple construction (in essence, of general topological nature) for a Hausdorff quotient space and to discuss some of its applications.

1. Non-Hausdorff quotient space. Let M be a compact metric space and let $M=\bigcup_{\alpha\in A} M_\alpha$ be its partition into pairwise disjoint sets. The topology of the quotient space A is defined as follows: $B\subset A$ is closed if and only if $\bigcup_{\alpha\in B} M_\alpha$ is closed in M. Accordingly, a sequence $\alpha_j\in A$ converges to $\alpha\in A$ provided that there exist $m_j\in M_{\alpha_j}$ and $m\in M_\alpha$ such that m_j converges to m in the topology of the space M. It is well known that this topology can be non-Hausdorff.

2. Admissible sets. Let S be the set of all subsets of M which are unions of elements of the partition M_α. Let the partition $\bigcup M_\alpha$ satisfy the following additional condition: if a set Q belongs to S, then its closure $\mathrm{Clos}(Q)$ also belongs to S.

In what follows, we choose an open dense set $\mathscr{M}\subset M$ such that $\mathscr{M}\in S$. Denote by $\mathscr{A}$ the set of all $\alpha\in A$ such that $M_\alpha\subset\mathscr{M}$.

We will now construct a Hausdorff metric space $\overline{\mathscr{A}}$ from these data.

Take a sequence $\alpha_j\in\mathscr{A}$. We say that α_j is *rigidly convergent* if any limit point of the sequence α_j in A is a limit of this sequence in A (we stress that the sequence α_j can have many limits, see Secs. 4 and 5 below).

A subset $T\subset A$ is said to be *admissible* if there exists a rigidly convergent sequence $\alpha_j\in\mathscr{A}$ such that T is the set of limits of the sequence α_j. We stress that the definition of an admissible set depends on the choice of the subset $\mathscr{M}\subset M$.

We define our space $\overline{\mathscr{A}}$ as the set of all admissible subsets of A. A natural question is how a topology can be defined in $\overline{\mathscr{A}}$.

3. Hausdorff metric. Denote by $O_\varepsilon(m)$ the ε-neighborhood of a point m and by $O_\varepsilon(Q)$ the ε-neighborhood of a subset $Q\subset M$. Let $[M]$ be the space of all closed nonempty subsets of M.

The *Hausdorff distance* (see [1, 2]) between the subsets $N_1, N_2\in[M]$ is defined as the infimum of the numbers $\varepsilon>0$ such that $N_1\subset O_\varepsilon(N_2)$ and $N_2\subset O_\varepsilon(N_1)$.

Recall a convergence test for $[M]$.

Lemma 1. *A sequence $N_j\in[M]$ converges to $N\in[M]$ if and only if, for any $\varepsilon>0$,*
(1) for any $m\in N$, beginning with some index j, the set $O_\varepsilon(m)\cap N_j$ is nonempty;
(2) for any $m\notin N$, beginning with some index j, the set $O_\varepsilon(m)\cap N_j$ is empty.

Consider the mapping $\psi\colon\mathscr{A}\to[M]$ that assigns to a point $\alpha\in\mathscr{A}$ the closure $\mathrm{Clos}(M_\alpha)$ of the set M_α. The space $\overline{\mathscr{A}}$ is defined as the closure of the image of the mapping ψ with respect to the Hausdorff metric.

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Lemma 2. *The following conditions are equivalent:*
(a) $N \in [M]$ *is an element of* $\bar{\mathscr{A}}$;
(b) *there exists an admissible set* $T \subset A$ *such that* $N = \bigcup_{\alpha \in T} M_\alpha$.

Thus, the above two definitions of the set $\bar{\mathscr{A}}$ are equivalent. Note that it follows from the second definition that the set $\bar{\mathscr{A}}$ is a compact metric space (since $[M]$ is compact; see [2]).

4. Example: "complete collineations" (see [4, 8, 9]). Denote by Gr_n the set of all n-dimensional subspaces of $\mathbb{C}^n \oplus \mathbb{C}^n$.

Example. Let A be an operator $\mathbb{C}^n \to \mathbb{C}^n$. Then its graph $\mathrm{graph}(A)$ is an element of Gr_n.

Let $V \in \mathrm{Gr}_n$ and let $\lambda \in \mathbb{C}^*$ (where $\mathbb{C}^*$ denotes the multiplicative group of the complex field). We define the subspace $\lambda V \in \mathrm{Gr}_n$ as the set of all $(x, \lambda y) \in \mathbb{C}^n \oplus \mathbb{C}^n$ such that $(x, y) \in V$.

Now let $M = \mathrm{Gr}_n$. Consider the partition of this space into orbits of the group $\mathbb{C}^*$ and denote the corresponding quotient space as $\mathrm{Gr}_n/\mathbb{C}^*$.

Remark. If $P \in \mathrm{Gr}_n$ has the form $P = (P \cap (\mathbb{C}^n \oplus 0)) \oplus (P \cap (0 \oplus \mathbb{C}^n))$, then P is a fixed point of the group $\mathbb{C}^*$. The other orbits are isomorphic to $\mathbb{C}^*$ as $\mathbb{C}^*$-homogeneous spaces. The orbits of the first type correspond to the closed points of $\mathrm{Gr}_n/\mathbb{C}^*$ and the orbits of the second type to the nonclosed points (i.e., to singletons that are not closed).

Remark. Any sequence $\mathrm{graph}(A_j)$, where A_j are invertible operators, is convergent in $\mathrm{Gr}_n/\mathbb{C}^*$ and has at least two limits, namely, $\mathbb{C}^n \oplus 0$ and $0 \oplus \mathbb{C}^n$, i.e., the usual definition of a convergent sequence turns out to be completely meaningless here.

Now we assume that $\mathscr{M} \subset M = \mathrm{Gr}_n$ is the set of the graphs of invertible operators, i.e., $\mathscr{M} = GL_n(\mathbb{C})$. Then $\mathscr{A} = \mathscr{M}/\mathbb{C}^* = PGL_n(\mathbb{C})$.

Example. Let the subspace $P_j \in \mathrm{Gr}_2$ consist of the vectors of the form $(x, y\,;\, x, jy) \in \mathbb{C}^2 \oplus \mathbb{C}^2$. Then the sequence $\mathbb{C}^* \cdot P_j$ is rigidly convergent in $\mathrm{Gr}_n/\mathbb{C}^*$, and the set of its limits is formed by $R_1 : (0, 0\,;\, x, y)$, $R_2 : (0, y\,;\, x, y)$, $R_3 : (x, 0\,;\, 0, y)$, $R_4 : (x, y\,;\, 0, y)$, and $R_5 : (x, y\,;\, 0, 0)$.

To describe the compactification $\bar{\mathscr{A}}$ of the space $\mathscr{A}$, we introduce a definition (see [9]).

Let $V := \mathbb{C}^n \oplus 0$ and $W := 0 \oplus \mathbb{C}^n$. Denote by π and π' the projections onto V and W in the space $\mathbb{C}^n \oplus \mathbb{C}^n$.

Definition. A *hinge* in $\mathbb{C}^n$ is the following collection of data:
(a) two flags of subspaces

$$V = L_0 \supset L_1 \supset \cdots \supset L_k = 0, \qquad 0 = M_0 \subset M_1 \subset \cdots \subset M_k = W$$

such that $\dim L_j + \dim M_j = n$ for any j;
(b) a family of subspaces $P_1, \ldots, P_k \in \mathrm{Gr}_n$ defined up to a factor $\lambda_j \in \mathbb{C}^*$ and such that $P_j \cap V = L_j$, $\pi(P_j) = L_{j-1}$, $\pi'(P_j) = M_j$, and $P_j \cap W = M_{j-1}$.

Theorem. (a) *For any hinge* $P_1, \ldots, P_k$, *the set* $P_1, \ldots, P_k$, $L_0 \oplus M_0, L_1 \oplus M_1, \ldots, L_k \oplus M_k$ *is admissible.*
(b) *Any admissible set has the form indicated in assertion* (a) (*for some hinge* $P_1, \ldots, P_k$).

Remark. The above compactification $\bar{\mathscr{A}}$ of the group $PGL_n(\mathbb{C})$ coincides with the variety of "complete collineations;" see [4, 8, 9]. It is known that $\bar{\mathscr{A}}$ is a smooth algebraic variety containing $PGL_n(\mathbb{C})$ as a Zariski open and dense set.

Remark. Natural compactifications of some other symmetric spaces, namely, the Satake–Furstenberg boundary [5, 7] and complete symmetric varieties [3, 4, 6, 8], can be described in terms of hinges in like manner (see [10, 12]).

5. Boundary of the Bruhat–Tits building. Let $\mathbf{Q}_p$ be the field of p-adic numbers, let $\mathbf{Z}_p$ be the set of integer p-adic numbers, and let $\mathbf{Q}_p^*$ be the multiplicative group of the field $\mathbf{Q}_p$. Let M be the space of all $\mathbf{Z}_p$-submodules of $\mathbf{Q}_p^n$ (it is natural to regard these modules as subsets of the projective space $\mathbf{QP}^n$). In this case M is a compact metric space with respect to the Hausdorff metric in $[\mathbf{QP}^n]$. Recall that by a *lattice* in $\mathbf{Q}_p^n$ we mean a $\mathbf{Z}_p$-submodule of the form $\mathbf{Z}_p v_1 \oplus \cdots \oplus \mathbf{Z}_p v_n$, where $v_1, \dots, v_n$ is an arbitrary basis in $\mathbf{Q}_p^n$. Denote by Ens_n the space of all lattices defined up to a dilation (i.e., up to the multiplication by an element of the group $\mathbf{Q}_p^*$). In other words, Ens_n is the set of vertices of the Bruhat–Tits building.

Consider now the space $A = M/\mathbf{Q}_p^*$ and take the set Ens_n as $\mathscr{A}$. Then the admissible subsets of $M/\mathbf{Q}_p^*$ have the form

$$0 = M_0, L_0, M_1, L_1, M_2, L_2, \dots, M_{k+1} = \mathbf{Q}_p^n,$$

where $M_0 \subset M_1 \subset \cdots \subset M_{k+1}$ are linear subspaces of $\mathbf{Q}_p^n$, $M_j \subset L_j \subset M_{j+1}$, and L_j/M_j are lattices in M_{j+1}/M_j.

Thus, the set Ens_n can be compactified in a natural way by the space of admissible subsets of $A = M/\mathbf{Q}_p^*$.

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Integration of Non-Abelian Langmuir Type Lattices by the Inverse Spectral Problem Method

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We consider the following Cauchy problem for a system of difference-differential equations whose coefficients are bounded operators in an arbitrary Banach space B:

$$\dot{C}_n = \sum_{i=1}^{q} C_{n+i} C_n - C_n \sum_{i=1}^{q} C_{n-i}, \qquad q \in \mathbb{N}, \tag{1}$$
$$C_n = C_n(t) \in \mathfrak{L}(B), \quad t \in [0, T), \quad 0 < T \le \infty, \quad \cdot = d/dt.$$

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