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Hinges and the Study–Semple–Satake–Furstenberg–De Concini–Procesi–Oshima Boundary

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To A. A. Kirillov on the occasion of his 60th anniversary

ABSTRACT. The aim of the present paper is twofold. One of the objectives is to present an elementary geometric description of the boundaries of symmetric spaces. These boundaries arise in mathematics independently and not simultaneously in quite different connections: enumerative algebraic geometry, harmonic analysis on symmetric spaces, the theory of automorphic forms. For spaces of rank one, e.g., for the Lobachevsky space $SO(n,1)/SO(n-1,1)$, the structure of the boundary is very simple. For symmetric spaces of rank > 1, there are many different boundaries and their structure is rather complicated. The second objective of the paper is to present the author's results announced in [**Ner4, Ner5, Ner7**]. We do not discuss applications of the constructions under consideration (except in the very simple §0) and also do not survey everything known about such objects. In Chapter I we describe in detail the variety of complete collineations defined by Semple in 1951 and show in Chapters II and III how to extract explicit constructions of other objects of the same kind from this description. Moreover, we give elementary constructions for the Satake-Furstenberg, Martin, and Karpelevich boundaries of symmetric spaces and construct some "new" boundaries.

Contents

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§0. The problem of five conics and complete conics

We recall an old problem of enumerative algebraic geometry, which was widely discussed in the second half of the nineteenth century (see remarks in §16). Now this problem is mainly of historical interest, but still is not trivial.

On the complex projective plane $\mathbb{C}P^2$, let five conics $L_1, \dots, L_5$ in general position be given. The question is: how many different conics are tangent to all five conics $L_1, \dots, L_5$?

0.1. **The space of conics.** A conic on the plane $\mathbb{C}P^2$ is given by the equation

$$\sum_{i,j=1}^{3} a_{ij} x_i x_j = 0, \tag{0.1}$$

where $a_{ij} \in \mathbb{C}$, $a_{ij} = a_{ji}$, and some a_{ij} are nonzero. Since the set of solutions of (0.1) is preserved under multiplication of all a_{ij} by a number, the space Q of all conics is identified with the 5-dimensional projective space $\mathbb{C}P^5$ (with coordinates $a_{11} : a_{12} : a_{13} : a_{22} : a_{23} : a_{33}$).

The group $PGL_3(\mathbb{C})$ acts on $\mathbb{C}P^2$ by projective transformations; therefore, this group acts also on the space $Q = \mathbb{C}P^5$.

As is known, the group $PGL_3(\mathbb{C})$ has the following three orbits in $Q = \mathbb{C}P^5$:

1° Nondegenerate conics. The stabilizer of the conic $x_1^2 + x_2^2 + x_3^2 = 0$ is the complex orthogonal group $O_3(\mathbb{C})$. Therefore, the space of all nondegenerate conics is the homogeneous (symmetric) space $PGL_3(\mathbb{C})/SO_3(\mathbb{C})$.

2° Pairs of intersecting lines.

3° Pairs of coinciding lines (double lines).

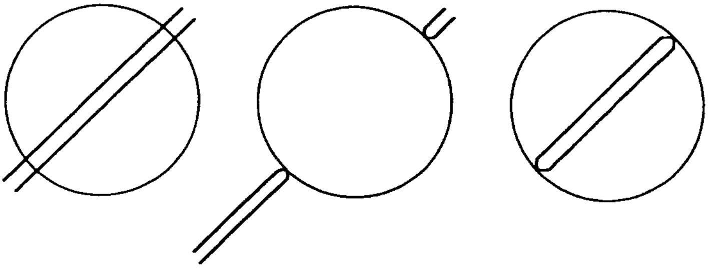

FIGURE 0.1

0.2. **An attempt to solve the problem.** Consider two conics given by the symmetric matrices $A = (a_{ij})$ and $P = (p_{ij})$.

The fact that the conics are tangent can be expressed by the following condition:

$(*)$ the equation $f(\lambda) = \det(A - \lambda P) = 0$ has a multiple root.

Therefore, the discriminant $D(A, P)$ of this equation must be zero. Now the problem of five conics can be restated in the following form: given five matrices $A_1, \ldots, A_5$, find the number of solutions of the system of equations

$$D(A_k, P) = 0, \qquad k = 1, \ldots, 5. \tag{0.2}$$

If the surfaces of sixth order $D(A_k, P) = 0$ were in general position, then (by the Bézout theorem) the answer would be equal to 6^5.

However, these surfaces are not in general position. The point is that each conic is tangent (in the sense of condition $(*)$) to every double line (see Figure 0.1). Therefore, the set of solutions to system (0.2) contains the two-dimensional surface of double lines, and we cannot apply the Bézout theorem.

0.3. **Complete conics.** We were interested in the set of solutions of system (0.2) in the space of nondegenerate quadrics $PGL_3(\mathbb{C})/SO_3(\mathbb{C})$. In fact we have set the problem in the completion $Q = \mathbb{C}P^5$ of the space $PGL_3(\mathbb{C})/SO_3(\mathbb{C})$.

An unpleasant phenomenon, namely, the appearance of a two-dimensional component in the space of solutions of system (0.2), occurred on the boundary of the space $PGL_3(\mathbb{C})/SO_3(\mathbb{C})$. It turns out that the situation will be modified if we take another boundary of the space $PGL_3(\mathbb{C})/SO_3(\mathbb{C})$.

Consider the space $\widehat{\mathbb{C}P^2}$ dual to $\mathbb{C}P^2$. Recall that, by definition, the points of $\widehat{\mathbb{C}P^2}$ are lines in $\mathbb{C}P^2$. Moreover, if $a \in \widehat{\mathbb{C}P^2}$, then the set $L(a)$ of all lines in $\mathbb{C}P^2$ (i.e., points of $\widehat{\mathbb{C}P^2}$) passing through a is a line in $\widehat{\mathbb{C}P^2}$.

To any *nondegenerate* conic $Q \subset \mathbb{C}P^2$ we assign the *dual conic* $\widehat{Q}$ that consists of all lines tangent to Q. If Q is defined by the matrix $A = (a_{ij})$ (see (0.1)), then $\widehat{Q}$ is defined by the matrix A^{-1}.

Consider the embedding

$$PGL_3(\mathbb{C})/SO_3(\mathbb{C}) \to \mathbb{C}P^5 \times \mathbb{C}P^5 \tag{0.3}$$

that assigns to each nondegenerate conic Q the pair of conics $(Q, \widehat{Q}) \in \mathbb{C}P^5 \times \mathbb{C}P^5$.

The variety C of "complete conics" is defined as the closure of the image of the embedding (0.3). This closure consists of four orbits of the group $PGL_3(\mathbb{C})$:

1. All pairs of the form $(Q, \widehat{Q})$, where Q is a nondegenerate quadric.

2. All pairs (R,T), where the quadric $R \subset \mathbb{C}P^2$ is a pair of lines ℓ_1 and ℓ_2 meeting at the point a and T is a double line $L(a)$.

3. All pairs (S,H), where $S \subset \mathbb{C}P^2$ is a double line and $H \subset \widehat{\mathbb{C}P^2}$ is a pair of lines $L(a)$ and $L(b)$, where $a, b \in S$.

4. All pairs (M,N), where $M \subset \mathbb{C}P^2$ is a double line and $N \subset \widehat{\mathbb{C}P^2}$ is a double line of the form $L(a)$, where $a \in M$.

The dimensions of these orbits are 5, 4, 4, and 3, respectively.

THEOREM 0.1. *C is a smooth algebraic variety.*

This assertion is rather simple (for details, see [**SR, DCP, GH**] and Theorem 4.6 below).

0.4. **Solution of the problem.** Now let X_Q be the set of all nondegenerate quadrics tangent to the given nondegenerate quadric Q. Let $\overline{X_Q}$ be the closure of X_Q in C. The following theorem is a pleasant exercise.

THEOREM 0.2. *"a)" The homology ring of C is generated by two cycles: the cycle λ that consists of all quadrics passing through a given point and the cycle μ that consists of all quadrics tangent to a given line.*

b) *The cycles λ and μ satisfy the relations*

$$\lambda^5 = 1, \quad \lambda^4\mu = 2, \quad \lambda^3\mu^2 = 4, \quad \lambda^2\mu^3 = 4, \quad \lambda\mu^4 = 2, \quad \mu^5 = 1\,. \tag{0.4}$$

c) *Let ν be the homology class of the cycle $\overline{X_Q}$. Then we have $\nu = 2(\lambda+\mu)$.*

REMARK 0.3. Equalities (0.4) have a clear geometric meaning. For example, the relation $\lambda^3\mu^2 = 4$ means that there exist four quadrics that pass through three points in general position and are tangent to two lines in general position.

Now it remains to compute

$$\nu^5 = 2^5(\lambda+\mu)^5 = 3264\,. \tag{0.5}$$

Moreover, we must verify that the trouble we wish to avoid is now really absent. Namely, verify that there is no subvariety $K \subset C$ such that $K \subset \overline{X_Q}$ for all Q. Clearly, if such a variety K exists, then K must be $PGL_3(\mathbb{C})$-invariant; therefore, it can only coincide with the orbit of the fourth type from the list in subsection 0.3. It remains to show that the variety $\overline{X_Q}$ does not contain the orbit of the fourth type.

Chapter I. Hinges and the canonical completion of the group $PGL_n(\mathbb{C})$

§1. Exterior algebras and the category GA

1.1. **Linear relations.** Let V and W be finite-dimensional linear spaces. By a *linear relation* $P\colon V \rightrightarrows W$ we mean an arbitrary subspace $P \subset V \oplus W$.

Denote by $\mathrm{Gr}^k(H)$ the Grassmannian of all k-dimensional subspaces of the space H. The set of all linear relations $V \rightrightarrows W$ is the union of Grassmann varieties:

$$\bigcup_{k=0}^{\dim V + \dim W} \mathrm{Gr}^k(V \oplus W)\,.$$

This set can be naturally endowed with the topology of disjoint union.

EXAMPLE 1.1. Let $A: V \to W$ be a linear operator. Consider its graph graph(A), that is, the set of pairs $(v, Av) \in V \oplus W$. Then graph(A) is a linear relation $V \rightrightarrows W$. Similarly, the graph of an operator $B: W \to V$ is also a linear relation $V \rightrightarrows W$. Below we do not distinguish between linear operators and their graphs.

Let $P: V \rightrightarrows W$ and $Q: W \rightrightarrows Y$ be linear relations. Then their product $QP: V \rightrightarrows Y$ is defined by the condition

$$(v, y) \in QP \iff \exists w \in W : (v, w) \in P,\ (w, y) \in Q\,.$$

Further, for every linear relation $P: V \rightrightarrows W$ we introduce the following objects:

a) by its *kernel* we mean $\operatorname{Ker} P = P \cap V$;
b) by its *image* $\operatorname{Im} P$ we mean the projection of P to W;
c) by its *domain* $\operatorname{Dom} P$ we mean the projection of P to V;
d) by its *indefiniteness* $\operatorname{Indef} P$ we mean the intersection of P with W.

REMARK 1.2. A linear relation P is the graph of an operator $A: V \to W$ if and only if we have $\operatorname{Dom} P = V$ and $\operatorname{Indef} P = \{0\}$. In this case we have $\operatorname{Ker} P = \operatorname{Ker} A$ and $\operatorname{Im} P = \operatorname{Im} A$.

For any linear relation $P: V \rightrightarrows W$ we introduce its *rank*

$$\begin{aligned}\operatorname{rk} P &= \dim \operatorname{Dom} P - \dim \operatorname{Ker} P = \dim \operatorname{Im} P - \dim \operatorname{Indef} P\\ &= \dim P - \dim \operatorname{Ker} P - \dim \operatorname{Indef} P\,.\end{aligned}$$

Further, for $P: V \rightrightarrows W$ we define the *pseudoinverse* linear relation $P^{\square}: W \rightrightarrows V$; namely, we consider the same linear subspace $P \subset V \oplus W$ as a subspace of $W \oplus V$.

Let $P: V \rightrightarrows W$ be a linear relation and $R \subset V$ a subspace. Define the subspace $PR \subset W$ as the set of all $w \in W$ for which there exists a $v \in R$ such that $(v, w) \in P$.

Finally, let us define the *multiplication of a linear relation by a number* $\lambda \neq 0$. Suppose $P: V \rightrightarrows W$ is a linear relation. Then the linear relation λP consists of vectors of the form $(v, \lambda w)$, where $(v, w) \in P$.

1.2. **Category** GA. The objects of the category GA are finite-dimensional complex linear spaces. The set $\operatorname{Mor}(V, W) = \operatorname{Mor}_{GA}(V, W)$ of morphisms $V \to W$ consists of

a) all linear relations $V \rightrightarrows W$ defined up to a scalar multiplier,
b) a formal morphism $null = null_{V,W}$ that is not identified with any linear relation.

Let $P \in \operatorname{Mor}(V, W)$ and $Q \in \operatorname{Mor}(W, Y)$. Then the product $QP \in \operatorname{Mor}(V, Y)$ is defined by the following rule:

a) if at least one of the factors is *null*, then the product is *null* as well;
b) if P and Q are linear relations and the following two conditions hold:

$$\operatorname{Im} P + \operatorname{Dom} Q = W, \tag{1.1}$$

$$\operatorname{Ker} Q \cap \operatorname{Indef} P = 0, \tag{1.2}$$

then the product QP can be calculated as the product of linear relations. For the case in which at least one of conditions (1.1) or (1.2) fails, the product is *null*.

We endow the set $\mathrm{Mor}_{GA}(V, W)$ with a non-Hausdorff topology. We assume that a set R is closed if and only if the following two conditions hold:

a) $null \in R$;

b) $R \setminus null$ is closed in the topology of the Grassmannian.

In particular, $null$ is the only closed point, and the closure of any other singleton defined by the point $P \neq null$ contains the point $null$.

THEOREM 1.3 (see [**Ner6**]). a) *The multiplication of morphisms of the category GA is associative.*

b) *If $P \in \mathrm{Mor}(V, W)$, $Q \in \mathrm{Mor}(W, Y)$, and if P, Q, and QP differ from null, then*

$$\dim QP = \dim Q + \dim P - \dim W. \tag{1.3}$$

c) *The multiplication is continuous.*

PROOF. We restrict ourselves to the proof of b) (the same arguments prove c)). Thus, let assumptions (1.1) and (1.2) hold. Denote by Z the space $V \oplus W \oplus W \oplus Y$, and denote by X its subspace that consists of vectors of the form (v, w, w, y). Denote by T the subspace of X that consists of vectors of the form $(0, w, w, 0)$. Then QP is the image of the subspace $(Q \oplus P) \cap X$ under the projection $X \to X/T = V \oplus Y$; by condition (1.1), we have $(Q \oplus P) + X = Z$, and therefore

$$\dim(Q \oplus P) \cap X = \dim(Q \oplus P) + \dim X - \dim Z = \dim Q + \dim P - \dim W.$$

By condition (1.2) we have $T \cap ((Q \oplus P) \cap X) = 0$; hence, the projection $X \to X/T$ is injective on T, and the assertion is proved. □

REMARK 1.4. This theorem is one of the reasons (possibly not the most important) to introduce the element $null$; otherwise, assertions b) and c) of Theorem 1.4 would fail for the category of linear relations.

1.3. **Exterior algebras.** Let us introduce some standard notation. Let V be a complex linear space of dimension n with basis $e_1, \dots, e_n$. Denote by $\Lambda^k V$ the kth exterior power of the space V, i.e., the set of all vectors of the form

$$\sum_{i_1 < \dots < i_k} \alpha_{i_1 \dots i_k} e_{i_1} \wedge \dots \wedge e_{i_k}.$$

Denote by $\Lambda(V) = \bigoplus_{k=0}^n \Lambda^k V$ the exterior algebra over V.

Let $A\colon V \to W$ be a linear operator. Denote by $\Lambda^k A\colon \Lambda^k V \to \Lambda^k W$ the kth exterior power of A and denote by ΛA the operator

$$\Lambda A = \bigoplus_{k=0}^{n} \Lambda^k A\colon \Lambda V \to \Lambda W.$$

Let $v \in V$. Denote by $a(v)$ the *operator of exterior multiplication* (creation operator) $a(v)h = v \wedge h$ in ΛV. This operator maps $\Lambda^k V$ into $\Lambda^{k+1} V$.

Let V' be the dual space of V. Let $f \in V'$. Denote by $a^+(f)$ the *operator of interior multiplication* (annihilation operator) in $\Lambda^k V$. This operator is defined by the relation

$$a(f) v_1 \wedge \dots \wedge v_k = \sum (-1)^{j+1} f(v_j) v_1 \wedge \dots \wedge v_{j-1} \wedge v_{j+1} \wedge \dots \wedge v_k .$$

The operators $a(v)$ and $a^+(f)$ satisfy the following identities (*canonical anti-commutation relations*):

$$a(v)a(w) + a(w)a(v) = 0, \qquad a^+(f)a^+(g) + a^+(g)a^+(f) = 0, \tag{1.4}$$

$$a^+(f)a(v) + a(v)a^+(f) = f(v) \cdot E. \tag{1.5}$$

1.4. **Plücker embeddings.** Let $S \subset V$ be a subspace of dimension k. In S we introduce a basis $s_1, \dots, s_k$ and consider the vector $s_1 \wedge \dots \wedge s_k \in \Lambda^k V$.

Let $t_j = \sum q_{ij} s_j$ be another basis in S. Let Q be the matrix with entries q_{ij}. Then we have

$$t_1 \wedge \dots \wedge t_k = \det(Q)\, s_1 \wedge \dots \wedge s_k.$$

Thus, the subspace S defines a vector in $\Lambda^k V$ which is determined up to a nonzero scalar factor.

Therefore, we have obtained the so-called *Plücker embedding*

$$\mathrm{Gr}^k(V) \to \mathbb{P}(\Lambda^k V)$$

(we denote by $\mathbb{P}W$ the projectivized space W).

1.5. **Fundamental representation of the category** GA. To any complex linear space V we assign the space ΛV. To any $P \in \mathrm{Mor}_{GA}(V, W)$ we assign an operator $\lambda(P)\colon \Lambda V \to \Lambda W$, which is defined up to a constant factor, so that for any V, W, Y and any $P \in \mathrm{Mor}(V, W)$ and $Q \in \mathrm{Mor}(W, Y)$ we have $\lambda(QP) = s(Q, P)\lambda(Q)\lambda(P)$, where $s(Q, P) \in \mathbb{C}$. In other words, λ is a *projective representation of the category* GA.

First we define the operator $\lambda(\,\cdot\,)$ for some special morphisms of the category GA.

1^+. We set $\lambda(\mathit{null}) = 0$ (this is another reason to introduce the morphism *null*).

2^+. Let X be a subspace of V. Let $T\colon V \rightrightarrows X$ be the graph of the embedding $X \to V$. Let $f_1, \dots, f_\alpha$ be a basis in the space of linear functionals that annihilate the subspace X. Then the operator $\lambda(T)\colon \Lambda V \to \Lambda X$ is defined by the formula

$$\lambda(T) = a^+(f_1) \cdots a^+(f_\alpha). \tag{1.6}$$

3^+. Let $Q\colon X \rightrightarrows Y$ be the graph of an operator $A\colon X \to Y$. Then we set $\lambda(Q) = \Lambda A = \bigoplus_j \Lambda^j A$.

4^+. Let Y be the quotient space $Y = W/L$. Let $R\colon Y \rightrightarrows W$ be the graph of the projection $W \to Y$. Choose a basis $e_1, \dots, e_\beta$ in L. Then

$$\lambda(R) = a(e_1)a(e_2) \cdots a(e_\beta). \tag{1.7}$$

It is important to note that the operator $\lambda(R)$ does not depend on the choice of a basis. Indeed, $\lambda(R)h = (e_1 \wedge \dots \wedge e_\beta) \wedge h$, and, as we have seen in the previous subsection the product $e_1 \wedge \dots \wedge e_\beta$ is defined by the subspace L uniquely up to a scalar factor. The same arguments show that the operator (1.6) is defined by the subspace X uniquely up to a scalar factor (and does not depend on the choice of the basis $f_1, \dots, f_\alpha$). Indeed, by identities (1.4), operator (1.6) is defined by the element $f_1 \wedge \dots \wedge f_\alpha \in \Lambda V^\circ$, where V° is the space dual to V.

Now let us take an arbitrary linear relation $P\colon V \rightrightarrows W$. We set $X = \mathrm{Dom}\,P$ and $Y = W/\,\mathrm{Indef}\,P$. Then P can be decomposed into the product $P = RQT$, where $T\colon V \rightrightarrows X$ has the form described in 2^+, $R\colon W/\,\mathrm{Indef}\,P \rightrightarrows W$ has the form

4^+, and Q is the graph of the operator $\operatorname{Dom} P \to W/\operatorname{Indef} P$, which is defined in an obvious way. Now we set $\lambda(P) = \lambda(R)\,\lambda(Q)\,\lambda(T)$.

THEOREM 1.5. *The functor $\lambda(\cdot)$ is a projective representation of the category GA, i.e., for any V, W, and Y and any $P \in \operatorname{Mor}(V, W)$ and $Q \in \operatorname{Mor}(W, Y)$ we have $\lambda(QP) = s(Q, P)\,\lambda(Q)\,\lambda(P)$, where $s(Q, P) \in \mathbb{C} \setminus \{0\}$.*

REMARK 1.6. Let $\dim V = n$, $\dim W = m$, and $\dim P = q$. Then it is clear that the operator $\lambda(P)\colon \Lambda V \to \Lambda W$ maps $\Lambda^j V$ into $\Lambda^{j+q-n} W$. In particular, if $V = W$ and $n = m = q$, then the operator $\lambda(P)$ leaves all exterior powers $\Lambda^j V$ invariant.

1.6. **Proof of Theorem 1.5.** Let V and W be linear spaces and let V° and W° be their dual spaces. Let $P\colon V \rightrightarrows W$ be a linear relation. We define the *dual linear relation* $P^\circ\colon V^\circ \rightrightarrows W^\circ$ as the set of all pairs $(f, g) \in V^\circ \oplus W^\circ$ such that for any $(v, w) \in P$ we have $f(v) = g(w)$.

We can readily verify that $(PQ)^\circ = P^\circ Q^\circ$.

REMARK 1.7. Let $P\colon V \rightrightarrows V$ be the graph of an invertible operator A. Then P° is the graph of the operator $(A^t)^{-1}$, where the operator $A^t\colon V^\circ \to V^\circ$ is adjoint to A.

THEOREM 1.8. *Let $P\colon V \rightrightarrows W$ be a linear relation.*

a) *The operator $\lambda(P)\colon \Lambda V \to \Lambda W$ satisfies the relation $a(w)\,\lambda(P) = \lambda(P)\,a(v)$ for all pairs $(v, w) \in P$.*

b) *The operator $\lambda(P)$ satisfies the relation $a^+(g)\,\lambda(P) = \lambda(P)\,a^+(f)$ for all $(f, g) \in P^\circ$.*

c) *If a nonzero operator $\Delta\colon \Lambda(V) \to \Lambda(W)$ satisfies the relations*

$$a(w)\,\Delta = \Delta a(v), \tag{1.8}$$

$$a^+(g)\,\Delta = \Delta a^+(f) \tag{1.9}$$

for all $(v, w) \in P$ and $(f, g) \in P^\circ$, then Δ coincides with $\lambda(P)$ up to a scalar factor.

PROOF. Choose bases $e_1, \dots, e_n$; $f_1, \dots, f_m$ such that the subspace $P \subset V \oplus W$ is a linear span of the vectors of the form

$$(e_1, f_1), \dots, (e_\alpha, f_\alpha), \qquad (e_{\alpha+1}, 0), \dots, (e_\beta, 0), \qquad (0, f_{\alpha+1}), \dots, (0, f_\gamma)\,.$$

Then all assertions become more or less evident.

Now let us pass directly to the proof of Theorem 1.5. Let $(v, w) \in P$ and $(w, y) \in Q$. Then

$$a(y)\,\lambda(Q)\,\lambda(P) = \lambda(Q)\,a(w)\,\lambda(P) = \lambda(Q)\,\lambda(P)\,a(v),$$

i.e., $\lambda(Q)\,\lambda(P)$ satisfies the same relations (1.8)–(1.9) as $\lambda(PQ)$. Therefore, by assertion c) of Theorem 1.8, $\lambda(QP)$ and $\lambda(Q)\lambda(P)$ coincide up to a scalar factor.

1.7. **Categories of linear relations.** The categories B, C, and GD defined below will be used in §8 only (for details, see [**Ner2, Ner6**]).

The objects of the category B are odd-dimensional complex linear spaces endowed with a nondegenerate *symmetric* bilinear form. Let V and W be objects of

B and let M_V and M_W be the corresponding bilinear forms. In $V \oplus W$ we introduce the symmetric bilinear form

$$M_{V\oplus W}((v,w),(v',w')) = M(v,v') - M(w,w').$$

The set $\mathrm{Mor}_B(V,W)$ of morphisms $V \to W$ consists of elements of two types:

a) linear relations $P\colon V \rightrightarrows W$ such that $P \subset V \oplus W$ is a maximal isotropic (with respect to the form $M_{V\oplus W}$) subspace of $V \oplus W$;

b) the element $null = null_{V,W}$.

The morphisms are multiplied in accordance with the rules of the category GA (note that conditions (1.1) and (1.2) are equivalent in this case).

REMARK 1.9. Recall that a subspace $H \subset Y$ is said to be *isotropic* with respect to the form M on Y if $M(h,h') = 0$ for all $h, h' \in H$. Note that in our case we have

$$\dim P = \frac{1}{2}(\dim V \oplus \dim W).$$

REMARK 1.10. Let $V = W$ and let A be an operator that preserves the form M_V, i.e.,

$$M(Av, Av') - M(v,v') = 0.$$

This is exactly equivalent to the condition that the graph P of the operator is isotropic in $V \oplus V$, that is, $P \in \mathrm{Mor}_B(V,V)$. We can readily show that the automorphism group of the object V is the orthogonal group of the space V.

The categories C and GD are defined in the same way; however, the objects of the category C are complex linear spaces endowed with a nondegenerate *skew-symmetric* bilinear form, and the objects of the category CD are complex *even-dimensional* spaces endowed with a nondegenerate *symmetric* bilinear form.

1.8. **Category** GA^*. Now we describe the category GA^*, which can be naturally regarded as a central extension of the category GA (we will first meet this category in §6).

The objects of the category GA^* are complex linear spaces. The morphisms from V to W are the operators $\Lambda(V) \to \Lambda(W)$ of the form $s \cdot \lambda(P)$, where $P \in \mathrm{Mor}_{GA}(V,W)$ and $s \in \mathbb{C}$.

The projection of GA^* onto GA is sufficiently evident (to the operator $s \cdot \lambda(P)$ we assign $P \in \mathrm{Mor}_{GA}(V,W)$ for $s \neq 0$ and $null$ for $s = 0$).

It would be of interest to obtain convenient explicit formulas for the product of operators $(s \cdot \lambda(P))(s' \cdot \lambda(P'))$. Clearly, this product has the form $s'' \cdot \lambda(PP')$, but the problem (possibly not very complicated) is to calculate s'' explicitly.

§2. Construction of a Hausdorff quotient space

2.1. Let M be a compact metric space and $M = \bigcup_{\alpha\in A} M_\alpha$ a partition of M into pairwise disjoint sets. Then the quotient set A is endowed with the *quotient topology*; namely, a subset $B \subset A$ is open if and only if the set $\bigcup_{\alpha\in B} M_\alpha$ is open in M.

Let $\alpha_j, \alpha \in A$. Then $\alpha = \lim_{j\to\infty} \alpha_j$ if and only if there exist $p \in M_\alpha$ and let $p_j \in M_{\alpha_j}$ be such that $p_j \to p$ in M.

In many interesting cases, the quotient topology on A is not Hausdorff (for instance, for the partition of $\mathbb{R}$ into positive numbers, negative numbers, and 0).

The following example is of main interest below.

2.2. **Example: the Grassmannian.** Let $M = \mathrm{Gr}_{2n}^n$ be the Grassmannian of all n-dimensional linear subspaces of $\mathbb{C}^n \oplus \mathbb{C}^n$. Let the multiplicative group of the field of complex numbers $\mathbb{C}^* = \mathbb{C} \setminus 0$ act on $M = \mathrm{Gr}_{2n}^n$ by multiplication of linear relations by numbers. Consider the partition of the space $M = \mathrm{Gr}_{2n}^n$ into the orbits of the group $\mathbb{C}^*$.

If a linear relation $L \subset \mathbb{C}^n \oplus \mathbb{C}^n$ has the form $L = P \oplus Q$, where $P \subset \mathbb{C}^n \oplus 0$ and $Q \subset 0 \oplus \mathbb{C}^n$, then the point L is fixed with respect to $\mathbb{C}^*$. All other orbits, regarded as homogeneous spaces, are isomorphic to $\mathbb{C}^*$. Their closures in the Grassmannian (regarded as complex varieties) are isomorphic to the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \infty = \mathbb{C}^* \cup 0 \cup \infty$.

The one-point orbits correspond to the closed points of the quotient space $M = \mathrm{Gr}_{2n}^n / \mathbb{C}^*$ (i.e., one-point set is closed). If $S \in \mathrm{Gr}_{2n}^n$ is not a fixed point, then the closure of the corresponding one-point set in M contains two points:

$$\operatorname{Dom} S \oplus \operatorname{Indef} S \subset \mathbb{C}^n \oplus \mathbb{C}^n, \qquad \operatorname{Ker} S \oplus \operatorname{Im} S \subset \mathbb{C}^n \oplus \mathbb{C}^n.$$

2.3. **Hausdorff metric: preliminaries.** Let M be a compact metric space with metric $\rho(\cdot\,,\cdot)$. Let $[M]$ be the set of all nonempty closed subsets of M. Recall the definition of the *Hausdorff metric* in M.

Let $m \in M$ and $N \in [M]$. Define the distance between m and N in the usual way by setting $\rho(m, N) = \min_{n \in N} \rho(m, n)$. Denote by N_ε the set of all points $m \in M$ at a distance $< \varepsilon$ from N, by $B_\varepsilon(m)$ the ball $\{x : \rho(x, m) < \varepsilon\}$, and by $\overline{Q}$ the closure of the set Q.

Let $N, L \in [M]$. Then the Hausdorff distance $h(N, L)$ is defined as the infimum of all $\varepsilon > 0$ such that $N \subset L_\varepsilon$ and $L \subset N_\varepsilon$.

Recall some simple facts about the Hausdorff metric.

LEMMA 2.1. *Let $N \in [M]$. Let $x_1, \dots, x_p \in M$ be a collection of points such that $\rho(x_j, N) < \varepsilon$ and the balls $B_\varepsilon(x_j)$ cover N. Then the distance from N to the set $\{x_1, \dots, x_p\}$ is at most ε.*

PROOF. The proof is obvious.

COROLLARY 2.2. *The space $[M]$ is compact.*

PROOF. Let $x_1, \dots, x_k$ be an ε-net in M. Then the subsets of the set $x_1, \dots, x_k$ form an ε-net in $[M]$.

Now let us give a constructive description of the convergence in $[M]$.

LEMMA 2.3. *Let $N_j, N \in [M]$. Then $N_j \to N$ provided the following two conditions hold:*

1° *for any $n \in N$ and any $\varepsilon > 0$, the set $B_\varepsilon(n) \cap N_j$ is not empty, starting from some index j;*

2° *for any $m \notin N$ there exists $\varepsilon > 0$ such that the intersection $B_\varepsilon(m) \cap N_j$ is empty, starting from some index j.*

Furthermore, let $N_j \in [M]$ be a sequence. Let Σ be the set of all its limit points. (The elements of the set Σ are closed subsets of M.) Let

$$Y = \bigcap_{K \in \Sigma} K, \qquad Z = \bigcup_{K \in \Sigma} K.$$

LEMMA 2.4. a) *We have $y \in Y$ if for any $\varepsilon > 0$, starting from some index j, the set $B_\varepsilon(y) \cap N_j$ is not empty.*

b) *We have $z \in Z$ if for any $\varepsilon > 0$ there exist arbitrarily large indices j such that the set $B_\varepsilon(z) \cap N_j$ is not empty.*

The proof is obvious.

Lemma 2.4 can also be rewritten in the following form.

LEMMA 2.5. a) $Y = \bigcap_{\Delta \subset \mathbb{N}} \overline{\bigcup_{j \in \Delta} N_j}$, *where Δ ranges over all infinite subsets of $\mathbb{N}$,*

b) $Z = \bigcap_{i \in \mathbb{N}} \overline{\bigcup_{j > i} N_j}$.

2.4. **Construction of a Hausdorff quotient space.** As above, let M be a compact metric space and $M = \bigcup M_\alpha$ $(\alpha \in A)$ a partition. Denote by S the set of all subsets of M that are the unions of elements of the partition. Let the partition satisfy the following property: if $N \in S$, then its closure $\overline{N}$ also satisfies $\overline{N} \in S$. This property certainly holds if we consider the partition of the space M into the orbits of some group G.

Furthermore, let $\mathcal{M} \subset M$ be an open dense subset and $\mathcal{M} \in S$. Consider the set $\mathcal{A} \subset A$ of all $\alpha \in A$ such that $M_\alpha \subset \mathcal{M}$. Consider the subset $\widetilde{\mathcal{A}} \subset [M]$ that consists of all sets of the form $\overline{M_\alpha}$, $\alpha \in \mathcal{A}$. Denote by $\overline{\mathcal{A}}$ the closure of $\widetilde{\mathcal{A}}$ in $[M]$. By construction, $\overline{\mathcal{A}}$ is a compact metric space that contains $\widetilde{\mathcal{A}}$ as a dense subset. We call the space $\overline{\mathcal{A}}$ the *Hausdorff quotient space* of the space M.

Certainly, the set $\overline{\mathcal{A}}$ depends not only on the metric space M, but also on the subset $\mathcal{M}$.

EXAMPLE 2.6. Let the multiplicative group $\mathbb{R}^*$ of positive numbers act on $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$ by multiplication of vectors by numbers. The orbits of $\mathbb{R}^*$ are the open rays starting from the origin 0 and the points 0 and ∞. If $\mathcal{M} = \overline{\mathbb{C}} \setminus \{0, \infty\}$, then the corresponding space $\overline{\mathcal{A}}$ is homeomorphic to the circle. The same assertion holds if $\mathcal{M}$ is obtained from $\overline{\mathbb{C}} \setminus \{0, \infty\}$ by deleting a finite number of rays. However, if $\mathcal{M} = \overline{\mathbb{C}}$, then the Hausdorff quotient is the disjoint union of two points and a circle.

Informally, we are interested in the situation where $\mathcal{M}$ is the union of the "generic" sets M_α.

2.5. **Description of the Hausdorff quotient $\overline{\mathcal{A}}$ in terms of the non-Hausdorff quotient $\mathcal{A}$.** Let $\alpha_j \in \mathcal{A}$ (we emphasize that α_j belongs not just to A, but to a certain subset of A). We say that the sequence α_j is *rigidly convergent* if all its limit points are limits. (Recall that a limit point of a sequence is a limit of a subsequence.)

We say that a subset $T \subset A$ is a *limit set* if there exists a rigidly convergent sequence $\alpha_j \in \mathcal{A}$ such that T is the set of limits of the sequence α_j.

EXAMPLE 2.7 (see subsection 2.2). Let Gr_4^2 be the set of all two-dimensional subspaces of $\mathbb{C}^2 \oplus \mathbb{C}^2$. Let $\mathrm{Gr}_4^2 / \mathbb{C}^*$ be the space of orbits of the group $\mathbb{C}^*$ (that acts by multiplication of a linear relation by numbers).

Consider the sequence V_j in Gr_4^2 whose elements are the subspaces that consist of points of the form $(x, y; x, jy) \in \mathbb{C}^2 \oplus \mathbb{C}^2$. In other words, V_j is the graph of the

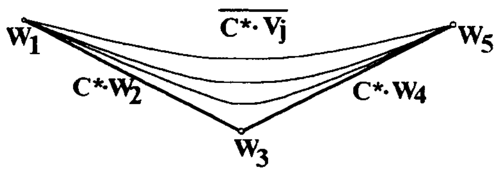

FIGURE 2.1

operator $\begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix}$. By abuse of language, we can say that the graphs V_j are points of the space $\mathrm{Gr}_4^2/\mathbb{C}^*$. Consider the following five sequences in Gr_4^2:

$$S_j^{(1)} = (1/j^2)V_j, \quad S_j^{(2)} = (1/j)V_j, \quad S_j^{(3)} = (1/\sqrt{j})V_j, \quad S_j^{(4)} = V_j, \quad S_j^{(5)} = jV_j.$$

Note that all $S_j^{(k)}$ (where $k \in \{1,2,3,4,5\}$) are representatives of the $\mathbb{C}^*$-orbit of the element V_j.

The limits of the sequences $S_j^{(1)}, \dots, S_j^{(5)}$ in the Grassmannian Gr_4^2 are the subspaces $W_1, \dots, W_5$, respectively, consisting of vectors of the form

$$W_1 : (x, y; 0, 0), \; W_2 : (x, y; 0, y), \; W_3 : (x, 0; 0, y), \; W_4 : (x, 0; x, y), \; W_5 : (0, 0; x, y).$$

Thus, the points $W_1, \dots, W_5$, regarded as elements of $\mathrm{Gr}_4^2/\mathbb{C}^*$, are limits of the sequence $V_j \in \mathrm{Gr}_4^2/\mathbb{C}^*$.

In $\mathrm{Gr}_4^2/\mathbb{C}^*$, there are no other limit points of the sequence V_j. Therefore, $\{W_1, \dots, W_5\}$ is a limit set.

PROPOSITION 2.8. *Let $N \subset M$ be a closed set. The following conditions are equivalent*:

a) *N is an element of the Hausdorff quotient $\overline{\mathcal{A}}$;*

b) *there is a limit set $T \subset A$ such that $N = \bigcup_{\beta \in T} M_\beta$.*

PROOF. Let $\overline{M_{\alpha_j}}$ converge to N in the Hausdorff metric. Then by Lemma 2.5 we have $N \in S$, i.e., N is composed of sets M_β. We can readily see that here the index β ranges over a limit set.

EXAMPLE 2.9. Let us return to the previous example. The sets $\overline{\mathbb{C}^* \cdot V_j}$ are complex curves isomorphic to the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C}P^1$. This sequence of curves converges, in the Hausdorff metric, to the union of two complex curves with a common point. In Figure 2.1 we show the arrangement of $\mathbb{C}^*$-orbits that correspond to the points V_j and W_1, W_2, W_3, W_4, and W_5.

2.6. **Example.** Let $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$ be the Riemann sphere. Let $M = (\overline{\mathbb{C}})^n$. Let the group $\mathbb{C}^*$ act on M by the formula $(x_1, \dots, x_n) \mapsto (\lambda x_1, \dots, \lambda x_n)$. The quotient $\overline{\mathbb{C}}^n/\mathbb{C}^*$ is not a Hausdorff space. We shall construct the Hausdorff quotient.

For $\mathcal{M} \subset M$ we take the set $(\mathbb{C}^*)^n$. Let $\mathcal{A}$ be the set of orbits of $\mathbb{C}^*$ in $(\mathbb{C}^*)^n$.

EXAMPLE 2.10. Consider the following sequence of orbits in $(\mathbb{C}^*)^4/\mathbb{C}^*$ (we indicate representatives of the orbits in $(\mathbb{C}^*)^4/\mathbb{C}^*$):

$$z_j = (1, j, 2j, j^2) \in (\mathbb{C}^*)^4.$$

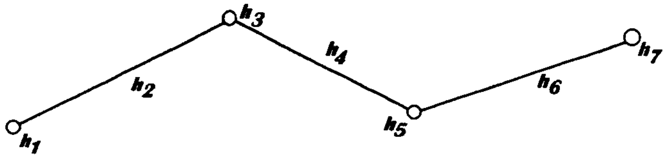

FIGURE 2.2

Then z_j is rigidly convergent, and the set of its limits (a limit set) is

$$h_1 = (0,0,0,0), \quad h_2 = (0,0,0,1), \quad h_3 = (0,0,0,\infty), \quad h_4 = (0,1,2,\infty),$$
$$h_5 = (0,\infty,\infty,\infty), \quad h_6 = (1,\infty,\infty,\infty), \quad h_7 = (\infty,\infty,\infty,\infty).$$

The points h_1, h_3, h_5, and h_7 are stable under the action of $\mathbb{C}^*$. The orbits of the points h_2, h_4, and h_6 are isomorphic to $\mathbb{C}^*$. Figure 2.2 illustrates the contiguity of the orbits.

THEOREM 2.11. a) *Every sequence*

$$z^{(j)} = (z_1^{(j)}, \dots, z_n^{(j)}) \in (\mathbb{C}^*)^n$$

(*we write down representatives of* $\mathbb{C}^*$*-orbits*) *that is rigidly convergent in* $(\mathbb{C}^*)^n/\mathbb{C}^*$ *must have the following form. There exists a family of subsets*

$$\varnothing = L_0 \subset L_1 \subset \cdots \subset L_k = \{1, \dots, n\}$$

of the n-tuple $\{1, \dots, n\}$ *such that* $L_{j+1} \neq L_j$ *and*

1^+ *for each* p *and any* $\alpha \in L_p$, $\beta \notin L_p$ *we have* $\lim_{j\to\infty} z_\alpha^{(j)}/z_\beta^{(j)} = \infty$.

2^+ *for any* $\alpha, \beta \in L_p \setminus L_{p+1}$ *there exists a limit* $\lim_{j\to\infty} z_\alpha^{(j)}/z_\beta^{(j)} \in \mathbb{C}^*$.

b) *Under the conditions above, the limit set for the sequence* $z^{(j)}$ *consists of elements of two types,* $u^0, \dots, u^k$ *and* $v^1, \dots, v^k$:

1° $u^p = (u_1^p, \dots, u_n^p)$, *where*

$$u_\alpha^p = \begin{cases} 0, & \alpha \notin L_p, \\ \infty, & \alpha \in L_p. \end{cases}$$

2° *Choose* $\gamma \in L_p \setminus L_{p-1}$. *Then* $v^p = (v_1^p, \dots, v_n^p)$, *where*

$$v_\alpha^p = \begin{cases} \infty, & \alpha \in L_{p-1}, \\ \lim_{j\to\infty} z_\alpha^{(j)}/z_\gamma^{(j)}, & \alpha \in L_p \setminus L_{p-1}, \\ 0, & \alpha \notin L_p. \end{cases}$$

REMARK 2.12. For the case considered in Example 2.10 we have a chain of subsets

$$\varnothing \subset \{4\} \subset \{2,3,4\} \subset \{1,2,3,4\},$$

and the corresponding limit set is

$$u^0 = h_1, \quad u^1 = h_3, \quad u^2 = h_5, \quad u^3 = h_7, \quad v^1 = h_2, \quad v^2 = h_4, \quad v^3 = h_6.$$

PROOF OF THEOREM 2.11. Since the Hausdorff quotient space is compact, any sequence $z^{(j)} \in (\mathbb{C}^*)^n/\mathbb{C}^*$ contains a rigidly convergent subsequence. Now we shall perform the promised construction, and the rigidly convergent sequence will have the form described in assertion a) of the theorem.

If our sequence is initially not of the form described in assertion a), then our procedure will make it possible to choose two different rigidly convergent subsequences with different limit sets.

Let us proceed with the description of the extracting procedure.

A point $z^{(j)} = (z_1^{(j)}, \dots, z_n^{(j)})$ can be regarded as a point of the projective space $\mathbb{C}P^{n-1}$. From the sequence $z^{(j)}$ we extract a subsequence $u^{(j)} = (u_1^{(j)}, \dots, u_n^{(j)})$ that is convergent in $\mathbb{C}P^{n-1}$, and let $r = (r_1, \dots, r_n)$ be its limit. Let $L_1 \subset \{1, \dots, n\}$ be the set of indices α such that $r_\alpha \neq 0$. For convenience of notation, assume that $L_1 = \{1, \dots, \beta\}$. Furthermore, consider the sequence of vectors $v^{(j)} = (u_{\beta+1}^{(j)}, \dots, u_n^{(j)})$, which we regard as a sequence in $\mathbb{C}P^{n-\beta-1}$ with homogeneous coordinates $u_{\beta+1} : u_{\beta+2} : \dots : u_n$.

Choose a convergent subsequence $w^{(j)} = (w_{\beta+1}^{(j)}, \dots, w_n^{(j)})$ of $v^{(j)}$. Let $q = (q_{\beta+1}, \dots, q_n)$ be its limit. Let K be the set of indices α such that $q_\alpha \neq 0$ and let $L_2 = L_1 \cup K$.

Now we can continue the arguments with the remaining coordinates, and so on. This completes the proof. □

Now we consider an arbitrary filtration

$$\mathfrak{A} \colon \varnothing \subset A_1 \subset \dots \subset A_k = \{1, \dots, n\}.$$

To any such filtration we assign the set $R_{\mathfrak{A}}$ whose elements are the k-tuples $\{P^{(1)}, \dots, P^{(k)}\}$ satisfying the following condition: each $P^{(\mu)} \in (\overline{\mathbb{C}})^n/\mathbb{C}^*$ has the form $\mathbb{C}^* \cdot (x_1^{(\mu)}, \dots, x_n^{(\mu)})$, where

$$\begin{aligned} x_\alpha^{(\mu)} &= 0 && \text{for } \alpha \notin A_k, \\ x_\alpha^{(\mu)} &= \infty && \text{for } \alpha \in A_{k-1}, \\ x_\alpha^{(\mu)} &\neq 0, \infty && \text{for } \alpha \in A_k \setminus A_{k-1}. \end{aligned}$$

The set $R_{\mathfrak{A}}$ is a complex variety isomorphic to $(\mathbb{C}^*)^{n-k}$.

Theorem 2.11 implies the following assertion.

THEOREM 2.13. *The Hausdorff quotient space coincides with the union of the sets $R_{\mathfrak{A}}$ over all filtrations $\mathfrak{A}$ of the set $\{1, \dots, n\}$.*

Denote by $\overline{\mathbb{T}^{n-1}}$ the space thus obtained. By construction, $\overline{\mathbb{T}^{n-1}}$ is a compact metric space. It turns out that, in fact, $\overline{\mathbb{T}^{n-1}}$ is a smooth complex algebraic variety.

Now we shall construct a complex analytic atlas on $\overline{\mathbb{T}^{n-1}}$. The charts of this atlas are indexed by the maximal filtrations (or the linear orderings, equivalently) of the set $\{1, \dots, n\}$:

$$\varnothing \subset A_1 \subset A_2 \subset \dots \subset A_n = \{1, \dots, n\},$$

where A_l consists of l elements. Let us describe the chart corresponding to the filtration

$$\mathfrak{h} \colon \varnothing \subset \{1\} \subset \{2\} \subset \dots \subset \{1, \dots, n\} \tag{2.1}$$

(the other charts can be obtained from this one by using the action of the symmetric group). Define the mapping $\Delta\colon \mathbb{C}^{n-1} \to \overline{\mathbb{T}^{n-1}}$ as follows. Let $z = (z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1}$ have the form

$$z = (z_1, \dots, z_{\alpha_1 - 1}, 0, z_{\alpha_1+1}, \dots, z_{\alpha_2-1}, 0, z_{\alpha_2+1}, \dots),$$

where all z_γ are nonzero, except for $z_{\alpha_1}, z_{\alpha_2}, \dots$. Then $\Delta(z)$ belongs to the set $R_{\mathfrak{A}}$ that corresponds to the filtration

$$\mathcal{W}\colon \varnothing \subset \{1, \dots, \alpha_1\} \subset \{1, \dots, \alpha_2\} \subset \cdots,$$

and we have $\Delta(z) = (P_1, P_2, \dots)$, where $P_s \in (\mathbb{C})^n$ has the form

$$z = \Bigl(\underbrace{\infty, \dots, \infty}_{\alpha_{s-1} \text{ times}}, 1, z_{\alpha_{s-1}+1}, z_{\alpha_{s-1}+1} z_{\alpha_{s-1}+2}, \dots, \prod_{\alpha_{s-1} < m < \alpha_s} z_{\alpha_m}, 0, 0, \dots\Bigr).$$

Now we consider the chart that corresponds to a maximal filtration

$$\mathfrak{b}\colon \varnothing \subset B_1 \subset B_2 \subset \cdots \subset B_n = \{1, \dots, n\}. \tag{2.2}$$

Denote by $i \prec j$ the ordering on the set $\{1, \dots, n\}$ given by the filtration (2.2), namely, we set $i \prec j$ if there exists B_α such that $i \in B_\alpha$ and $j \notin B_\alpha$.

Let

$$\varnothing \subset \{1, \dots, q_1\} \subset \{1, \dots, q_2\} \subset \cdots, \tag{2.3}$$

be the intersection of the filtrations (2.1) and (2.2).

As above, let $z_1, \dots, z_{n-1}$ be the coordinates in the chart corresponding to the filtration (2.1) and $u_1, \dots, u_{n-1}$ the coordinates in the chart corresponding to the filtration (2.2). The overlap functions

$$z_1 = z_1(u_1, \dots, u_{n-1}), \quad \dots, \quad z_{n-1} = z_{n-1}(u_1, \dots, u_{n-1})$$

are defined on the entire space $\mathbb{C}^{n-1}$, except for the hyperplanes $u_j = 0$, where j ranges over the entire set $\{1, \dots, n-1\}$ except for the points $q_1, q_2, \dots$ (see (2.3)). These functions can readily be calculated:

$$z_j = \prod_{\alpha \prec j} u_\alpha \Big/ \prod_{\alpha \prec j-1} u_\alpha. \tag{2.4}$$

Note that the variables u_{q_i} never occur in the denominator of formulas (2.4) (note that $z_{q_i} = u_{q_i}$); therefore, the overlap functions are indeed holomorphic on the domain under consideration.

2.7 **Remark: Universalization of separated quotient.** In subsection 2.4 for each $\mathcal{M} \in S$ we constructed separated quotient $\overline{\mathcal{A}} = \overline{\mathcal{A}}(\mathcal{M})$ of the space M. It is possible to construct separated quotient which does not depend on choice of subset $\mathcal{M}$. For this aim consider the space

$$\overline{\mathcal{A}_{\text{univ}}} = \Cap_{\mathcal{M} \in S} \overline{\mathcal{A}}(\mathcal{M}),$$

where intersection is given by all open dense sets $\mathcal{M} \in S$. All separated quotients which we consider in this paper coincide with $\overline{\mathcal{A}_{\text{univ}}}$.

§3. Hinges

3.1. **Definition.** A *hinge* in $\mathbb{C}^n$ is a sequence

$$\mathcal{P} = (P_1, \dots, P_k)$$

of n-dimensional linear relations $\mathbb{C}^n \rightrightarrows \mathbb{C}^n$ (*links of the hinge*) defined up to multiplication by scalar factors (which are different for different P_j) and satisfying the following conditions.

$$1^+ \quad \operatorname{Ker} P_j = \operatorname{Dom} P_{j+1}, \tag{3.1}$$

$$\operatorname{Im} P_j = \operatorname{Indef} P_{j+1}, \tag{3.2}$$

$$2^+ \quad P_j \neq \operatorname{Ker} P_j \oplus \operatorname{Indef} P_j, \tag{3.3}$$

$$3^+ \quad \operatorname{Indef} P_1 = 0, \tag{3.4}$$

$$\operatorname{Ker} P_k = 0. \tag{3.5}$$

Denote by Hinge_n the set of all hinges in $\mathbb{C}^n$.

REMARK 3.1. Certainly, here the main condition is stated in 1^+. Condition 3^+ is condition 1^+ interpreted for $j = 0$ and $j = k$. Condition 2^+ is not very essential and can be replaced by other conditions which are not worse by any means. For the sake of being definite, we chose one of the possibilities (which are formally nonequivalent but are essentially the same) in the statement; see Theorem 3.6 and the discussion in subsection 5.2. We would also like to note the following.

REMARK 3.2. By 2^+ we have the strict inclusions

$$\operatorname{Ker} P_j \supset \operatorname{Ker} P_{j+1}, \qquad \operatorname{Dom} P_j \supset \operatorname{Dom} P_{j+1},$$
$$\operatorname{Im} P_j \subset \operatorname{Im} P_{j+1}, \qquad \operatorname{Indef} P_j \subset \operatorname{Indef} P_{j+1}.$$

This means that the number k of links of the hinge is at most n.

EXAMPLE 3.3. The graph of an invertible operator is a hinge. The graph of a noninvertible operator does not satisfy condition 3^+.

EXAMPLE 3.4. Consider a two-link hinge $\mathcal{P} = (P_1, P_2)$. By condition 3^+, P_1 is the graph of an operator $A\colon \mathbb{C}^n \to \mathbb{C}^n$, and P_2 is the graph of an operator from the second copy of $\mathbb{C}^n$ into the first one. By condition 1^+, we have $AB = 0$ and $BA = 0$.

EXAMPLE 3.5. The subspaces W_2 and W_4 from Example 2.7 form a hinge.

3.2. **Space of hinges as a Hausdorff quotient of the Grassmannian.** Consider the action of the group $\mathbb{C}^*$ on Gr^n_{2n} described in subsection 2.2. For the set $\mathcal{M} \subset \mathrm{Gr}^n_{2n}$, we take the group $GL_n(\mathbb{C})$; to be more exact, consider the set of graphs of the invertible operators $\mathbb{C}^n \to \mathbb{C}^n$.

THEOREM 3.6. *For the subset $\mathcal{M} = GL_n(\mathbb{C})$, the limit sets in $\mathrm{Gr}^n_{2n}/\mathbb{C}^*$, are precisely the sets of the following form*:

$$Q_0, P_1, Q_1, P_2, \dots, P_k, Q_k, \tag{3.6}$$

where $\mathcal{P} = (P_1, \dots, P_k)$ is a hinge and

$$Q_j = \operatorname{Ker} P_j \oplus \operatorname{Im} P_j = \operatorname{Dom} P_{j+1} \oplus \operatorname{Indef} P_{j+1} \subset \mathbb{C}^n \oplus \mathbb{C}^n.$$

COROLLARY 3.7. *The metric space* Hinge_n *is compact. The group* $PGL_n(\mathbb{C})$ *is a dense open subset of* Hinge_n.

Below we shall return to the proof of the theorem and the corollary; now we continue discussing the definition of hinge.

3.3. **Another interpretation of hinges.** Let $\mathcal{P} = (P_1, \dots, P_n)$ be a hinge in $\mathbb{C}^n$. For the summands in the sum $\mathbb{C}^n \oplus \mathbb{C}^n$, we introduce the following notation:

$$V := \mathbb{C}^n \oplus 0, \qquad W := 0 \oplus \mathbb{C}^n.$$

Furthermore, define subspaces Y_s and Z_s for $s \in \{0, \dots, k\}$ as follows:

$$\begin{aligned} &Y_j = \operatorname{Ker} P_j = \operatorname{Dom} P_{j+1}, \qquad && j = 1, \dots, k-1, \\ &Y_0 = V, \qquad Y_k = 0, && \\ &Z_j = \operatorname{Im} P_j = \operatorname{Indef} P_{j+1}, \qquad && j = 1, \dots, k-1, \\ &Z_0 = 0, \qquad Z_k = W. && \end{aligned}$$

We obtained two flags:

$$V = Y_0 \supset Y_1 \supset \cdots \supset Y_k = 0, \tag{3.7}$$
$$0 = Z_0 \subset Z_1 \subset \cdots \subset Z_k = V. \tag{3.8}$$

The linear relation P_j defines an invertible linear operator

$$A_j \colon \operatorname{Dom} P_j / \operatorname{Ker} P_j \to \operatorname{Im} P_j / \operatorname{Indef} P_j .$$

In particular,

$$\dim Y_{j-1}/Y_j = \dim Z_j/Z_{j-1} . \tag{3.9}$$

Thus, every hinge defines the two flags (3.7) and (3.8) satisfying condition (3.9), and the collection of invertible operators

$$A_j \colon Y_{j-1}/Y_j \to Z_j/Z_{j-1} \tag{3.10}$$

defined up to a factor.

Conversely, let the flags (3.7) and (3.8) and a collection of invertible operators (3.10) defined up to a factor be given. Let us choose subspaces $R_j \subset Y_{j-1}$ and $Q_j \subset Z_j$ such that

$$Y_{j-1} = Y_j \oplus R_j, \qquad Z_j = Z_{j-1} \oplus Q_j .$$

Then the operator A_j defines an operator $A'_j \colon R_j \to Q_j$. Furthermore, consider a linear relation $P_j \colon V \rightrightarrows W$ defined as the sum of the three subspaces:

$$Y_j \subset V, \qquad Z_{j-1} \subset W, \qquad \operatorname{graph}(A'_j) \subset R_j \oplus Q_j \subset V \oplus W.$$

We can readily see that the collection $(P_1, \dots, P_k)$ is a hinge.

Thus, we have obtained a canonical one-to-one correspondence between the space of hinges and the space of collections (3.7), (3.8), and (3.10). These collections are said to be *framed pairs of flags.*

3.4. **Orbits of the group $PGL_n(\mathbb{C}) \times PGL_n(\mathbb{C})$.** As usual, denote by $PGL_n(\mathbb{C})$ the quotient group of the group $GL_n(\mathbb{C})$ by its center $\mathbb{C}^*$.

The group $G_n = PGL_n(\mathbb{C}) \times PGL_n(\mathbb{C})$ acts on Hinge_n in an obvious way:

$$(g_1, g_2)\colon \mathcal{P} \mapsto g_1^{-1}\mathcal{P}g_2 = (g_1^{-1}P_1g_2, \dots, g_1^{-1}P_kg_2).$$

PROPOSITION 3.8. *The hinges $\mathcal{P} = (P_1, \dots, P_k)$ and $\mathcal{R} = (R_1, \dots, R_k)$ belong to the same orbit of the group $PGL_n(\mathbb{C}) \times PGL_n(\mathbb{C})$ if and only if the sets of numbers*

$$\dim \operatorname{Ker} P_1, \ \dots, \ \dim \operatorname{Ker} P_{k-1} \tag{3.11}$$

and

$$\dim \operatorname{Ker} R_1, \ \dots, \ \dim \operatorname{Ker} R_{k-1}$$

coincide. The dimension of the orbit that contains the hinge $\mathcal{P} = (P_1, \dots, P_k)$ is equal to $n^2 - k$.

Let us give an equivalent formulation of this statement in terms given in subsection 3.3.

PROPOSITION 3.9. *The only $PGL_n(\mathbb{C}) \times PGL_n(\mathbb{C})$-invariant of a framed pair of flags is the set of numbers*

$$\dim Y_1, \ \dots, \ \dim Y_{k-1}. \tag{3.12}$$

The dimension of the corresponding orbit is equal to $n^2 - k$.

PROOF. Clearly, the set of invariants given in (3.11) (or in (3.12), which is the same) completely determines the orbit. Let us choose a canonical representative on any orbit. Namely, for $0 = j_0 < j_1 < \dots < j_k = n$ we define a *canonical hinge* $\mathcal{R}(j_1, \dots, j_{k-1}) = (R_1, \dots, R_k)$, where $R_s \subset \mathbb{C}^n \oplus \mathbb{C}^n$ consists of all vectors of the form $(x_1, \dots, x_n; y_1, \dots, y_n)$ such that

$$x_\alpha = 0, \quad \alpha > j_{s+1}; \qquad y_\beta = 0, \quad \beta \leqslant j_s; \qquad x_\gamma = y_\gamma, \quad j_s < \gamma \leqslant j_{s+1}. \tag{3.13}$$

The stabilizer of this hinge consists of the pairs of block matrices of order

$$(\lambda_1 + \dots + \lambda_k) \times (\lambda_1 + \dots + \lambda_k),$$

where $\lambda_s = j_s - j_{s-1}$, of the form

$$\begin{pmatrix} A_1 & * & * & \dots \\ 0 & A_2 & * & \dots \\ 0 & 0 & A_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \begin{pmatrix} t_1A_1^{-1} & 0 & 0 & \dots \\ * & t_2A_2^{-1} & 0 & \dots \\ * & * & t_3A_3^{-1} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $t_j \in \mathbb{C}^*$. The dimension of the stabilizer is equal to $n^2 + k$, and thus the assertion on the dimension of the orbit is proved.

3.5. **Proof of Theorem 3.6.** First we show that any set of the form (3.6) is a limit set in the sense of subsection 2.5. To this end, it suffices to restrict ourselves to the case of hinges of the form $\mathcal{R}(j_1, \dots, j_{k-1})$ (see the proof of Proposition 3.9).

Let $a_1 > \dots > a_k$. Consider the sequence

$$S_t = \begin{pmatrix} e^{a_1 t}E_{\lambda_1} & & \\ & e^{a_2 t}E_{\lambda_2} & \\ & & \ddots \end{pmatrix},$$

where $t = 1, 2, 3, \dots$ and $\lambda_s = j_s - j_{s-1}$ (we denote by E_λ the identity matrix of size $\lambda \times \lambda$). Our further arguments proceed as in Example 2.7. By considering the sequence of matrices $e^{-a_q t}S_t$ we obtain, as the limit, the linear relation (3.13). Furthermore, choose b_q so that $a_{q-1} < b_q < a_q$. By considering the sequence of matrices $e^{-b_q t}S_t$ we obtain, as the limit, a linear relation R_q that consists of vectors of the form $(x_1, \dots, x_n; y_1, \dots, y_n)$ such that

$$\begin{cases} x_\alpha = 0, & \alpha > \lambda_q, \\ y_\alpha = 0, & \alpha \leqslant \lambda_q. \end{cases}$$

The set of linear relations $(R_0, Q_1, R_1, \dots, R_{k-1}, Q_k, R_k)$ has the desired form.

Conversely, consider a sequence $A_j \in GL_n(\mathbb{C})$ regarded as a sequence in the (non-Hausdorff) quotient space $\mathrm{Gr}^n_{2n}/\mathbb{C}^*$. It suffices to show that A_j contains a rigidly convergent subsequence such that the set of its limits has the form (3.6).

Represent A_j in the form $A_j = B_j \Delta_j C_j$, where B_j and C_j are unitary matrices and Δ_j is a diagonal matrix:

$$\Delta_j = \begin{pmatrix} \delta_1^{(j)} & & \\ & \delta_2^{(j)} & \\ & & \ddots \end{pmatrix}, \qquad \delta_1^{(j)} \geqslant \delta_2^{(j)} \geqslant \dots > 0.$$

Without loss of generality we may assume that the sequences B_j and C_j converge in the unitary group (otherwise we can pass to a subsequence).

Now the convergence is completely determined by the middle factor, and we shall watch this factor only. To the vector $\delta^{(j)} = \{\delta_1^{(j)} \geqslant \dots \geqslant \delta_n^{(j)} > 0\}$ (composed of the eigenvalues of the matrix Δ_j) we can apply the procedure described in the proof of Theorem 2.11.

Namely, from the sequence $\delta^{(j)} = (\delta_1^{(j)} : \dots : \delta_n^{(j)}) \in \mathbb{R}P^{n-1}$ we extract a subsequence $\tau^{(j)} = (\tau_1^{(j)}, \dots, \tau_n^{(j)})$ that converges in $\mathbb{R}P^{n-1}$. Let $(p_1, \dots, p_\alpha, 0, \dots, 0)$ be its limit. Furthermore, from the sequence $(\tau_{\alpha+1}^{(j)} : \dots : \tau_n^{(j)}) \in \mathbb{R}P^{n-\alpha-1}$ we extract a subsequence that is convergent in $\mathbb{R}P^{n-\alpha-1}$, and so on.

Finally, from the sequence Δ_j we extract a subsequence $\Xi^{(\mu)}$ of the form

$$\Xi^{(\mu)} = \begin{pmatrix} a_1^{(\mu)} D_1^{(\mu)} & & \\ & a_2^{(\mu)} D_2^{(\mu)} & \\ & & \ddots \end{pmatrix},$$

where:

1. the matrices $D_m^{(\mu)}$ are diagonal; for chosen m and $\mu \to \infty$, the sequence $D_m^{(\mu)}$ has a limit, and this limit is an invertible matrix;
2. we have $a_m^{(\mu)} > 0$, and $\lim_{\mu \to \infty} a_m^{(\mu)} / a_{m+1}^{(\mu)} = \infty$ for any m.

The set of limits of the sequence $\Xi^{(\mu)} = \Delta_{j_r}$ clearly has the form (3.6) (see Example 2.7), and this completes the proof of the theorem.

3.6. **Contiguity of the orbits.** Thus, the metric space Hinge_n is compact. As was shown in subsection 3.4, this space is the union of 2^{n-1} orbits of the group $PGL_n(\mathbb{C}) \times PGL_n(\mathbb{C})$. Recall that the orbits are indexed by the collections of numbers

$$\Sigma(\mathcal{P}) = (\dim \operatorname{Ker} P_1, \dots, \dim \operatorname{Ker} P_{k-1}).$$

THEOREM 3.10. *The orbit of a hinge $\mathcal{R}$ is contained in the closure of the orbit of a hinge $\mathcal{R}'$ if and only if $\Sigma(\mathcal{R}') \subset \Sigma(\mathcal{R})$.*

COROLLARY 3.11. *The group $PGL_n(\mathbb{C})$ is an open dense set in* Hinge_n.

PROOF. The proof repeats the arguments of subsection 3.5. We only need to perform the selection procedure not for matrices but for points of a given orbit. By means of unitary matrices, the flags (3.7) and (3.8) can be transformed into canonical position, and then the problem is essentially reduced to the extraction of operators (3.10).

§4. Projective embedding

4.1. **Semigroup GL_n^*.** Let $V = \mathbb{C}^n$. Consider the semigroup GL_n^* formed by all linear relations in $\mathbb{C}^n$ of dimension n together with *null*.

Note that GL_n^* is a semigroup indeed (see relation (1.3)). We also note that the product of linear relations of dimension n itself can be not of dimension n. However, if the product has a "wrong" dimension, then this product in the category GA is equal to *null*.

Furthermore, we note that for $P \in GL_n^*$, the operator $\lambda(P)$ (see §1) preserves the subspaces $\Lambda^j\mathbb{C}^n \subset \Lambda\mathbb{C}^n$. Denote by $\lambda_j(P)$ the restriction of the operator $\lambda(P)$ to $\Lambda^j\mathbb{C}^n$ (certainly, if P is the graph of an operator A, then $\lambda_j(P) = \Lambda^j(A)$).

LEMMA 4.1. *Let $P \in GL_n^*$ be a linear relation.*

a) *The operator $\lambda_j(P)$ is nonzero if and only if j satisfies the condition*

$$\dim \operatorname{Indef} P \leqslant j \leqslant \dim \operatorname{Im} P. \tag{4.1}$$

b) *If j satisfies the condition* $\dim \operatorname{Indef} P < j < \dim \operatorname{Im} P$, *then the operator $\lambda_j(P)$ uniquely determines a linear relation P up to a factor.*

The assertion is quite clear from the explicit construction of the operators $\lambda_j(P)$.

We are mainly interested in the case of $j = \dim \operatorname{Indef} P$ and $j = \dim \operatorname{Im} P$.

Case A): $j = \dim \operatorname{Indef} P$.

Let $e_1, \dots, e_j$ be a basis in $\operatorname{Indef} P$ and let $f_1, \dots, f_j$ be a basis in the space of linear functionals that annihilate $\operatorname{Dom} P$ (note that the condition $\dim P = n$ implies $\dim \operatorname{Dom} P = n - \dim \operatorname{Indef} P$). Then, as can be readily verified, the operator $\lambda_j(P)$ coincides with the restriction of the operator

$$a(e_1) \cdots a(e_j) a^+(f_1) \cdots a^+(f_j) \tag{4.2}$$

to the subspace $\Lambda^j V \subset \Lambda V$.

We stress that in our case the operator $\lambda_j(P)$ is completely determined by the subspaces $\operatorname{Indef} P$ and $\operatorname{Dom} P$. We also stress that the rank of the operator (4.2)

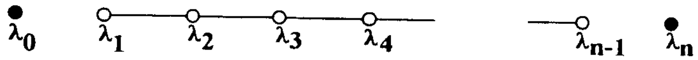

FIGURE 4.1

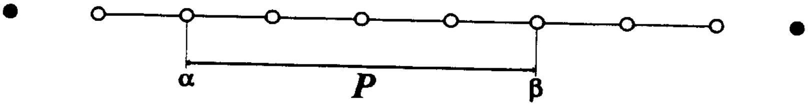

FIGURE 4.2

is equal to one. The image of this operator is the line spanned by the multivector $e_1 \wedge \cdots \wedge e_j \in \Lambda^j V$, and the kernel (of codimension one) coincides with the kernel of the linear functional $f_1 \wedge \cdots \wedge f_j \in \Lambda^j V^\circ = (\Lambda^j V)^\circ$.

Case B): $j = \dim \operatorname{Im} P$.

Let $e_1, \dots, e_j$ be a basis in $\operatorname{Im} P$ and let $f_1, \dots, f_j$ be a basis in the space of linear functionals that annihilate $\operatorname{Ker} P$. Then the operator $\lambda_j(P)$ coincides with the restriction of the operator

$$a(e_1) \cdots a(e_j) a^+(f_1) \cdots a^+(f_j) \tag{4.3}$$

to the subspace $\Lambda^j V$.

We stress that the operators (4.2) and (4.3) coincide.

Case C): Q=$\operatorname{Ker} Q \oplus \operatorname{Indef} Q$.

Consider a linear relation Q of the form

$$Q = \operatorname{Ker} Q \oplus \operatorname{Indef} Q \subset \mathbb{C}^n \oplus \mathbb{C}^n.$$

Let $\dim \operatorname{Indef} Q = j$, let $e_1, \dots, e_j$ be a basis in $\operatorname{Indef} Q$ and let $f_1, \dots, f_j$ be a basis in the space of linear functionals annihilating $\operatorname{Ker} Q$. Then we have

$$\lambda_j(Q) = a(e_1) \cdots a(e_j) a^+(f_1) \cdots a^+(f_j), \qquad \lambda_\alpha(Q) = 0, \quad \alpha \neq j\,. \tag{4.4}$$

Consider the Dynkin diagram of the group $A_{n-1} = GL_n(\mathbb{C})$ shown in Figure 4.1. Here the circles mark the fundamental representations λ_j of the group $GL_n(\mathbb{C})$, i.e., the representations in $\Lambda^j V$. It is convenient to add two black circles, from the left and from the right, that correspond to λ_0 and λ_n.

Let $P \in GL_n^*$ be a linear relation. By the *domain of action* of P we mean the set of all j that satisfy (4.1). We shall depict the domain of action in Figure 4.2, where $\alpha = \dim \operatorname{Indef} P$ and $\beta = \dim \operatorname{Im} P$. Outside of the domain of action of P, the operators $\lambda_j(P)$ are equal to 0 and on the boundary (i.e., for $j = \alpha$ and for $j = \beta$), the operators $\lambda_j(P)$ have rank 1.

4.2. **The operators** $\lambda_j(\mathcal{P})$. Let $\mathcal{P} = (P_1, \dots, P_k)$ be a hinge. Choose some $j \in \{1, \dots, n-1\}$. Consider the sequence of operators

$$\lambda_j(P_1), \dots, \lambda_j(P_k)\,. \tag{4.5}$$

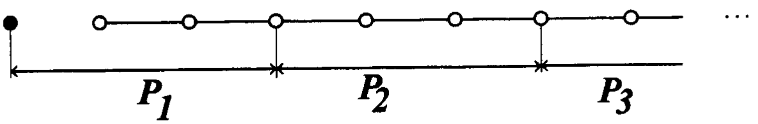

FIGURE 4.3

THEOREM 4.2. *There are exactly two possibilities concerning* (4.5).
1^+ *Exactly one term of the sequence* (4.5) *differs from* 0.
2^+ *There is a unique α such that $\lambda_j(P_\alpha) \neq 0$ and $\lambda_j(P_{\alpha+1}) \neq 0$. In this case*

$$\dim \operatorname{Dom}(P_\alpha) = j = \dim \operatorname{Indef}(P_{\alpha+1}),$$

the operators $\lambda_j(P_\alpha)$ and $\lambda_j(P_{\alpha+1})$ have rank 1 *and coincide, up to a scalar factor.*

PROOF. Indeed, by the definition of a hinge ($\operatorname{Im} P_\alpha = \operatorname{Dom} P_{\alpha+1}$), the domains of action of the linear relations $P_1, \dots, P_k$ (see (4.1)) border on each other as shown in Figure 4.3.

Consider the linear relations P_μ and $P_{\mu+1}$. By the definition of hinge, we have $\operatorname{Im} P_\mu = \operatorname{Indef} P_{\mu+1}$ and $\operatorname{Ker} P_\mu = \operatorname{Dom} P_{\mu+1}$. Let $\alpha = \dim P_\mu$. By the remarks in subsection 4.1 (see (4.2) and (4.3)) we have $\lambda_\alpha(P_\mu) = s \cdot \lambda_\alpha(P_{\mu+1})$, $s \in \mathbb{C}^*$, and this proves the theorem.

We define the operator $\lambda_j(\mathcal{P})\colon \Lambda^j\mathbb{C}^n \to \Lambda^j\mathbb{C}^n$ (which can be determined up to a scalar factor) as the nonzero element of the set

$$\lambda_j(P_1), \dots, \lambda_j(P_k).$$

REMARK 4.3. Let us supplement the hinge $\mathcal{P} = (P_1, \dots, P_k)$ to obtain the set $(Q_0, P_1, Q_1, \dots, P_k, Q_k)$; see Theorem 3.6. Choose some μ. Denote the number

$$\dim \operatorname{Im} P_\mu = \dim \operatorname{Im} Q_\mu = \dim \operatorname{Indef} Q_\mu = \dim \operatorname{Indef} P_{\mu+1}$$

by α. Then, by the remarks in subsection 4.1, we have

$$\lambda_\alpha(P_\mu) = s \cdot \lambda_\alpha(Q_\mu) = t \cdot \lambda_\alpha(P_{\mu+1}), \qquad s, t \in \mathbb{C}^*.$$

The domains of action of the linear relations $Q_0, P_1, Q_1, \dots$ are arranged as shown in Figure 4.4.

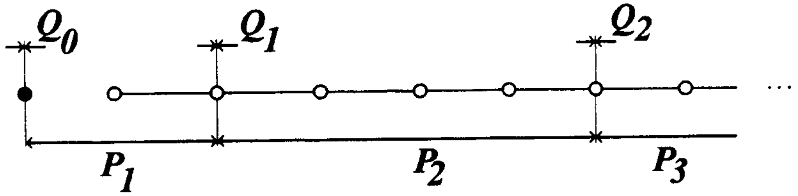

FIGURE 4.4

4.3. **Projective embedding.** Let W be a linear subspace and let $Op(W)$ be the set of linear operators on W. Denote by $\mathbb{P}(Op(W))$ the set of nonzero operators defined up to a factor. Consider the mapping

$$\text{Hinge}_n \to \prod_{j=1}^{n-1} \mathbb{P}(Op(\Lambda^j \mathbb{C}^n))$$

given by the formula

$$\mathcal{P} = (P_1, \dots, P_k) \mapsto \sigma(\mathcal{P}) = (\lambda_1(\mathcal{P}), \dots, \lambda_{n-1}(\mathcal{P}))\,.$$

THEOREM 4.4. *The mapping σ is continuous.*

PROOF. Let us prove that σ maps convergent sequences into convergent sequences. Let $\mathcal{P}_\alpha \in \text{Hinge}_n$ converge to $\mathcal{P}$. Without loss of generality we may assume that $\mathcal{P}$ is a canonical hinge of the form $\mathcal{R}(j_1, \dots, j_k)$ (see subsection 3.4). Furthermore, without loss of generality we may assume that all points $\mathcal{P}_\alpha$ belong to the same orbit of the group $GL_n \times GL_n$ (otherwise we can pass to a subsequence). Let $\mathcal{R}(h_1, \dots, h_s)$ be the canonical representative of this orbit. We stress that the set $h_1, \dots, h_s$ is a subset of the collection $j_1, \dots, j_k$. Furthermore, let the collection h_ν be empty (the general case differs from this one by complication of notation only), i.e., let the sequence $\mathcal{P}$ consist of invertible operators, which will be denoted by A_α.

The sequence A_α is representable in the form $A_\alpha = B_\alpha \Delta_\alpha C_\alpha$, where B_α and C_α are sequences of unitary matrices that tend to the identity matrix and Δ_α has the form

$$\Delta_\alpha = \begin{pmatrix} a_1^{(\alpha)} D_1^{(\alpha)} & & \\ & a_2^{(\alpha)} D_2^{(\alpha)} & \\ & & \ddots \end{pmatrix},$$

where

1°. $D_m^{(\alpha)}$ is a diagonal matrix of the size $(j_m - j_{m-1})$, and for any m, the sequence $D_m^{(\alpha)}$ tends to the identity matrix as $\alpha \to \infty$.

2°. $a_m^{(\alpha)} > 0$, and for any m we have $\lim_{\alpha\to\infty} a_m^{(\alpha)} / a_{m+1}^{(\alpha)} = \infty$.

The convergence $\Lambda^j(\Delta_\alpha) \to \lambda_j(\mathcal{R}(j_1, j_2, \dots))$ is more or less obvious.

We illustrate the assertion by the following example.

EXAMPLE 4.5. Consider the sequence of operators

$$A_j = \begin{pmatrix} j^2 & & & \\ & j & & \\ & & j & \\ & & & 1 \end{pmatrix} : \mathbb{C}^4 \to \mathbb{C}^4.$$

In Hinge_n, this sequence is convergent to a (three-link) hinge $\mathcal{R}(1,3)$. The sequence $j^{-2} A_j$ is convergent to the operator in $\mathbb{C}^4$ with the matrix

$$R = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}.$$

The sequence $j^{-3}\Lambda^2 A_j$ is convergent to the operator S in $\Lambda^2\mathbb{C}^4$ that has the form $S(e_1\wedge e_2)=e_1\wedge e_2$, $S(e_1\wedge e_3)=e_1\wedge e_3$, and $S(e_\alpha\wedge e_\beta)=0$ for all other pairs e_α, e_β. The sequence $j^{-4}\Lambda^3 A_j$ is convergent to the operator T given by the formula $T(e_1\wedge e_2\wedge e_3)=e_1\wedge e_2\wedge e_3$, and $T(e_\alpha\wedge e_\beta\wedge e_\gamma)=0$ for all other triples e_α, e_β, e_γ.

We can readily see that the collection of operators R, S, and T really corresponds to the hinge $\mathcal{R}(1,3)$.

4.4. **Smoothness.** It follows from our constructions that Hinge_n is a projective algebraic variety.

THEOREM 4.6. *The variety* Hinge_n *is smooth.*

Before passing to the proof of the theorem, we consider the following example.

EXAMPLE 4.7 (*diagonal hinges*). We return to the situation described in subsection 2.6. Construct the natural embedding $(\overline{\mathbb{C}})^n\to \mathrm{Gr}^n_{2n}$. Let us express a point of the jth copy of $\overline{\mathbb{C}}$ as the ratio a_j/b_j, where $a_j, b_j\in\mathbb{C}$ and at least one of these numbers is nonzero. Then to the point $a_1/b_1,\dots,a_n/b_n$ we assign the subspace of $\mathbb{C}^n\oplus\mathbb{C}^n$ that consists of the vectors of the form $(b_1x_1, b_2x_2,\dots,b_nx_n; a_1x_1, a_2x_2,\dots, a_nx_n)$. The constructed embedding commutes with the action of the group $\mathbb{C}^*$, and therefore it induces a mapping of the Hausdorff quotients

$$\overline{\mathbb{T}^{n-1}}\to \mathrm{Hinge}_n\,.$$

Furthermore, we note that the (smooth) variety $\overline{\mathbb{T}^{n-1}}$ is exactly the closure of the group of diagonal matrices in the space Hinge_n.

PROOF OF THEOREM 4.6. It suffices to introduce smooth coordinates in a neighborhood of an arbitrary point $\mathcal{R}(j_1,\dots,j_s)$. Let $\varkappa_\alpha=j_\alpha-j_{\alpha-1}$. Consider a subgroup G of $GL_n\times GL_n$, where G consists of pairs of block matrices that have the size $(\varkappa_1+\varkappa_2+\cdots)\times(\varkappa_1+\varkappa_2+\cdots)$ and are of the form

$$\begin{pmatrix} E & & & \cdots\\ * & E & & \cdots\\ * & * & E & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix};\qquad \begin{pmatrix} B_1 & * & * & \cdots\\ & B_2 & * & \cdots\\ & & B_3 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}.$$

This subgroup is supplementary to the stabilizer of the point $\mathcal{R}(j_1,\dots,j_s)$. Let us present a smooth transverse section to the orbits of this subgroup. To this end we consider the set D of operators of the form

$$\begin{pmatrix} x_1E_{\varkappa_1} & & \\ & x_2E_{\varkappa_2} & \\ & & \ddots\end{pmatrix}.$$

We can readily see that the closure $\overline{D}$ of the set D in the space of hinges is isomorphic to the variety $\overline{\mathbb{T}^s}$ from subsection 2.6. This is the desired section.

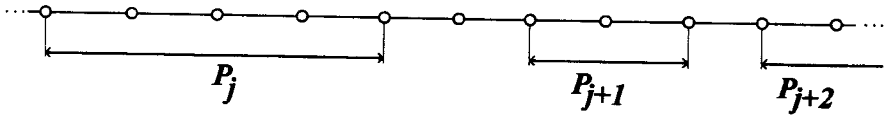

FIGURE 5.1

§5. Semigroup of hinges

5.1. **Weak hinges.** By a *weak hinge* in $\mathbb{C}^n$ we mean a family $\mathcal{P} = (P_1, \dots, P_k)$ of linear relations of dimension $\leqslant n$ that are determined up to a factor and satisfy the relations

$$\operatorname{Ker} P_j \supset \operatorname{Dom} P_{j+1} \quad \text{and} \quad \operatorname{Im} P_j \subset \operatorname{Indef} P_{j+1}.$$

EXAMPLE 5.1. A HINGE IS A WEAK HINGE. Sometimes we shall speak of an *exact hinge* instead of the term "hinge".

EXAMPLE 5.2. The empty set is a weak hinge.

EXAMPLE 5.3. Let $\mathcal{P} = (P_1, \dots, P_k)$ be a hinge. Then for any set of indices $0 < i_1 < \cdots < i_s \leqslant k$, the set $(P_{i_1}, \dots, P_{i_s})$ is a weak hinge.

We have the following chains of inclusions:

$$\operatorname{Dom} P_1 \supset \operatorname{Ker} P_1 \supset \operatorname{Dom} P_2 \supset \operatorname{Ker} P_2 \supset \cdots,$$
$$\operatorname{Indef} P_j \subset \operatorname{Im} P_1 \subset \operatorname{Indef} P_2 \subset \operatorname{Im} P_2 \subset \cdots.$$

Therefore, the domains of action of linear relations are roughly arranged as shown in Figure 5.1.

We stress that our definition (in contrast to those in subsection 3.1) does not forbid the case $P_\alpha = \operatorname{Ker} P_\alpha \oplus \operatorname{Indef} P_\alpha$.

5.2. **Equivalence of weak hinges.** Now we give a (technical and not very important) definition of the equivalence of weak hinges. Two weak hinges are equivalent if one of them can be obtained from the other by applying (possibly many times) the two operations, of addition and deletion of a linear relation, that are described below.

First operation. If the domains of action are arranged as shown in Figure 5.2 (for P_{j-1}, the domain of action is a singleton), then the linear relation P_{j-1} can be deleted (i.e., $(\dots, P_{j-2}, P_{j-1}, P_j, \dots) \sim (\dots, P_{j-2}, P_j, \dots)$). Note that in this case we have

$$P_{j-1} = \operatorname{Ker} P_{j-1} \oplus \operatorname{Indef} P_{j-1}, \qquad \operatorname{Ker} P_{j-1} = \operatorname{Dom} P_j,$$
$$\operatorname{Im} P_{j-1} = \operatorname{Indef} P_{j-1} = \operatorname{Indef} P_j\,.$$

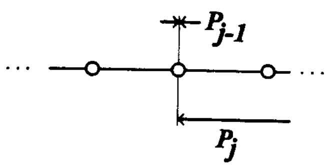

FIGURE 5.2

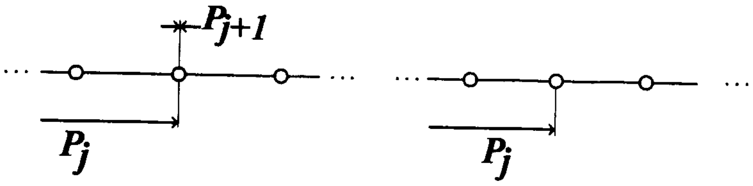

FIGURE 5.3

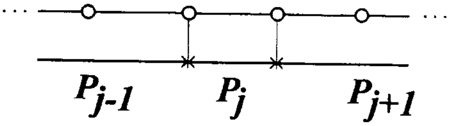

FIGURE 5.4

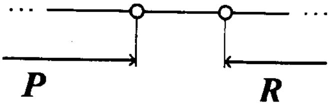

FIGURE 5.5

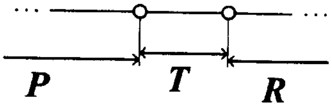

FIGURE 5.6

Conversely, if Q is a term of a weak hinge, then we can place the term $R = \operatorname{Dom} Q \oplus \operatorname{Indef} Q$ before Q. Similarly, the pictures shown in Figure 5.3 are equivalent.

Second operation. Let the domain of action of P_j consist of two points and the domains of action of P_{j-1} and P_{j+1} tightly border on the domain of action of P_j (see Figure 5.4). Then the link P_j can be deleted from the weak hinge. Conversely, let P and R be neighboring terms of a weak hinge and suppose that $\dim \operatorname{Ker} P - \dim \operatorname{Dom} R = 1$, i.e., the domains of the action are arranged as shown in Figure 5.5. Then $\dim \operatorname{Dom} R + \dim \operatorname{Im} P = n - 1$ and the linear relation T, of dimension n, such that $\operatorname{Indef} T = \operatorname{Im} P$ and $\operatorname{Ker} T = \operatorname{Dom} R$ is defined uniquely up to a factor. Now we can add T to the weak hinge (see Figure 5.6).

5.3. **Multiplication of hinges.** Denote by $\widetilde{\operatorname{Hinge}}_n$ the set of all weak hinges in $\mathbb{C}^n$ defined up to equivalence. Let

$$\mathcal{P} = (P_1, \dots, P_k), \mathcal{R} = (R_1, \dots, R_l) \in \widetilde{\operatorname{Hinge}}_n .$$

Consider the set of all possible products $A_{ij} = P_i R_j$ different from *null*.

THEOREM 5.4. *The set A_{ij} (being appropriately ordered) is a weak hinge.*

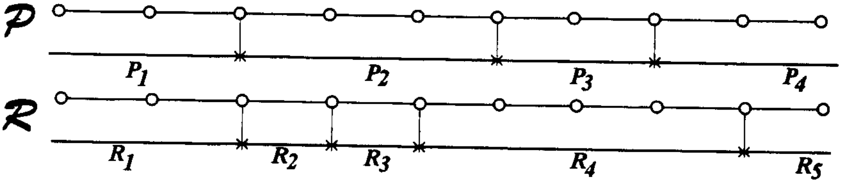

FIGURE 5.7

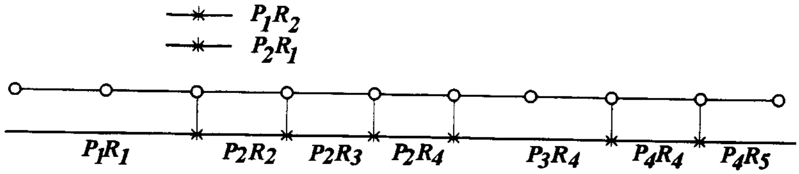

FIGURE 5.8

	R_1	R_2	R_3	R_4	R_5
P_1	×	×	•	•	•
P_2	×	×	×	×	•
P_3	•	•	•	×	•
P_4	•	•	•	×	×

FIGURE 5.9

EXAMPLE 5.5. In $\mathbb{C}^{11}$ consider two hinges $\mathcal{P} = (P_1, \ldots, P_4)$ and $\mathcal{R} = (R_1. \ldots, R_5)$ with the domains of action of the links shown in Figure 5.7. If these hinges are in general position, then the product has the form shown in Figure 5.8. If these hinges are not in general position, then any of the links can be skipped.

In Figure 5.9 the crosses denote the pairs $\{i, j\}$ for which (in the case of general position) we have $P_i R_j \neq null$.

Let us pass to the proof of the theorem.

LEMMA 5.6. *If $P_i R_j \neq null$, then the domains of action of P_i and R_j have a nonzero intersection.*

PROOF. Let the domains of action be disjoint, and, for the sake of being definite, let the domains of action be arranged as shown in Figure 5.10.

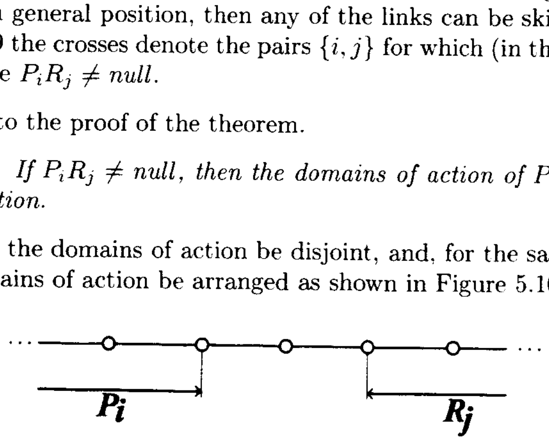

FIGURE 5.10

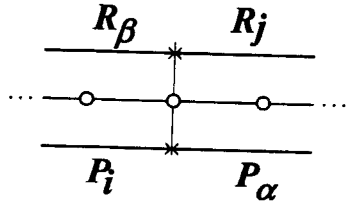

FIGURE 5.11

Then we have

$$\operatorname{Ker} P_i \supset \operatorname{Dom} R_j, \qquad \operatorname{Im} P_i \subset \operatorname{Indef} R_j,$$

both these inclusions being strict, and

$$\dim \operatorname{Ker} P_i + \dim \operatorname{Indef} R_j > \dim \operatorname{Ker} P_i + \dim \operatorname{Im} P_i = n\,.$$

Therefore, $\operatorname{Ker} P_i$ and $\operatorname{Indef} R_j$ have nonzero intersection, i.e., $P_i R_j = null$.

Lemma 5.6 implies the following assertion.

LEMMA 5.7. *Let $P_i R_j \neq null$ and $P_\alpha R_\beta \neq null$. Then at least one of the following three possibilities holds.*

a) $i \leqslant \alpha$, $j \leqslant \beta$.

b) $i \geqslant \alpha$, $j \geqslant \beta$.

c) *The domains of action of P_i, P_α, R_j, and R_β are arranged as shown in* Figure 5.11.

Note that the last case is of no interest because in this case $S = P_i R_j = P_\alpha R_\beta$ is $S = \operatorname{Ker} S \oplus \operatorname{Indef} S$. Moreover, since we have $P_i R_j \neq null$ and $P_\alpha R_\beta \neq null$, the linear relation S can be deleted by the first rule of subsection 5.2.

Thus, let $i \leqslant \alpha$ and $j \leqslant \beta$. Assume that $T = P_i R_j \neq null$ and $S = P_\alpha R_\beta \neq null$. We must show that either $T = S$ or

$$\operatorname{Ker} T \supset \operatorname{Dom} S, \qquad \operatorname{Im} T \subset \operatorname{Indef} S\,.$$

It suffices to consider three cases:

a) $i < \alpha$, $j < \beta$.

b) $i = \alpha$, $j < \beta$.

c) $i < \alpha$, $j = \beta$.

In the first case we have

$$\begin{aligned} \operatorname{Ker} T &\supset \operatorname{Ker} R_j \supset \operatorname{Dom} R_\beta \supset \operatorname{Dom} S, \\ \operatorname{Im} T &\subset \operatorname{Im} P_i \subset \operatorname{Indef} P_\alpha \subset \operatorname{Indef} S\,. \end{aligned} \tag{5.1}$$

In the second case we have the same chain (5.1) together with

$$\operatorname{Im} T = P_i(\operatorname{Im} R_j) \subset P_i(\operatorname{Indef} R_\alpha) \subset \operatorname{Indef} S\,.$$

The third case is similar to the second one. This completes the proof of the theorem.

Thus, we see that the set $\widetilde{\mathrm{Hinge}}_n$ is a semigroup.

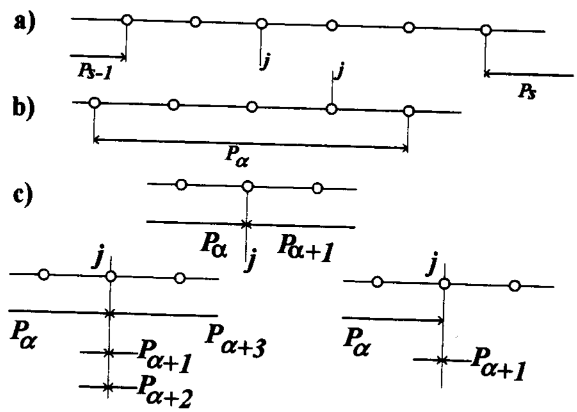

FIGURE 5.12

5.4. **Fundamental representations of the semigroup $\widetilde{\mathrm{Hinge}}_n$.** Let

$$j \in \{1, \dots, n-1\}.$$

Define the representation $\lambda_j(\cdot)$ of the semigroup $\widetilde{\mathrm{Hinge}}_n$ in the space $\Lambda^j\mathbb{C}^n$.

Let $\mathcal{P} = (P_1, \dots, P_k) \in \widetilde{\mathrm{Hinge}}_n$. Consider the set of operators

$$\lambda_j(P_1), \dots, \lambda_j(P_k). \tag{5.2}$$

The following assertion is obvious.

PROPOSITION 5.8. *For any $\mathcal{P}$, one of the following three properties holds.*
a) *All operators* (5.2) *are equal to* 0.
b) *There exists an α such that $\lambda_j(P_\alpha) \neq 0$, and this α is unique.*
c) *There exist indices $\alpha, \alpha+1, \dots, \beta$ such that $\lambda_j(P_\sigma) = 0$ for $\sigma < \alpha$ and for $\sigma > \beta$, and the operators $\lambda_j(P_\alpha), \dots, \lambda_j(P_\beta)$ are of rank one and coincide up to a factor.*

We illustrate cases a), b), and c) in Figure 5.12.

The last pictures show why many (> 2) links P_s such that $\lambda_j(P_s) \neq 0$ can occur. The reason for this occurrence is obvious: for any linear relation P entering the hinge, we can always add an arbitrary number of identical terms $Q = \operatorname{Ker} P \oplus \operatorname{Im} P$.

Now we assume that $\lambda_j(\mathcal{P})$ is an operator defined up to a factor and satisfying the following conditions.

1) If case a) of the theorem holds, then $\lambda_j(\mathcal{P}) = 0$.
2) If we have case b) or c), then $\lambda_j(\mathcal{P})$ is a nonzero term of the sequence $\lambda_j(P_\alpha)$.

We can readily see that $\mathcal{P} \mapsto \lambda_j(\mathcal{P})$ is a projective representation of the semigroup $\widetilde{\mathrm{Hinge}}_n$.

5.5. **Topology on $\widetilde{\mathrm{Hinge}}_n$.** Let V be a linear space. Denote by $\mathbb{P}^\circ V = \mathbb{P}V \cup 0$ the set of vectors $v \in V$ defined up to a factor, i.e., $\mathbb{P}^\circ V$ is the projective space $\mathbb{P}V$ to which we add 0.

The topology on $\mathbb{P}^\circ V$ is defined by the following condition: a set $S \subset \mathbb{P}^\circ V$ is closed if and only if $0 \in S$ and $S \cap \mathbb{P}V$ is closed in $\mathbb{P}V$.

REMARK 5.9. This is not a Hausdorff topology. The closure of any point $h \in \mathbb{P}V$ is the two-point set that consists of h and 0.

We have a mapping $\widetilde{\mathrm{Hinge}}_n \to \prod_{j=1}^{n-1} \mathbb{P}^\circ(\Lambda^j \mathbb{C}^n)$ given by the formula

$$\mathcal{P} \mapsto (\lambda_1(\mathcal{P}), \dots, \lambda_{n-1}(\mathcal{P})). \tag{5.3}$$

PROPOSITION 5.10. *The mapping* (5.3) *is an embedding.*

PROOF. In subsection 5.2, the equivalence of weak hinges was defined in such a way that (5.3) is an embedding.

Thus, $\widetilde{\mathrm{Hinge}}_n$ is embedded into $\prod_{j=1}^{n-1} \mathbb{P}^\circ(\Lambda^j \mathbb{C}^n)$, and this embedding induces a topology on $\widetilde{\mathrm{Hinge}}_n$. We can readily verify that it is not a Hausdorff topology.

EXAMPLE 5.11. Let us describe the closure of a point

$$\mathcal{P} = (P_1, \dots, P_k) \in \widetilde{\mathrm{Hinge}}_n. \tag{5.4}$$

Let us supplement the collection $\mathcal{P}$ to an equivalent hinge

$$\widehat{\mathcal{P}} = (S_1, P_1, T_1, S_2, P_2, T_2, \dots), \tag{5.5}$$

where $S_j = \mathrm{Dom}\, P_j \oplus \mathrm{Indef}\, P_j$ and $T_j = \mathrm{Ker}\, P_j \oplus \mathrm{Im}\, P_j$. Then the closure of the point $\mathcal{P}$ consists of all possible weak hinges obtained from $\widehat{\mathcal{P}}$ by deleting some set of links. In particular, the closure of any point contains an empty hinge.

5.6. Generators.

PROPOSITION 5.12. *The semigroup* $\widetilde{\mathrm{Hinge}}_n$ *is generated by the group* PGL_n *and by the canonical hinges* $\mathcal{R}(1), \dots, \mathcal{R}(n-1)$.

We omit the (more or less clear) proof. We only note that for $i_1 < \dots < i_k$ we have $\mathcal{R}(i_1) \cdots \mathcal{R}(i_k) = \mathcal{R}(i_1, \dots, i_k)$.

5.7. Central extension of the semigroup $\widetilde{\mathrm{Hinge}}_n$. An object with a non-Hausdorff topology can create the impression of something pathological. In fact, the semigroup $\widetilde{\mathrm{Hinge}}_n$ has something like a Hausdorff central extension, which we will now describe.

Let $Op(H)$ be the semigroup of operators in a linear space H. We introduce the semigroup $\widehat{\mathrm{Hinge}}_n$ as the subsemigroup of $\prod_{j=1}^{n-1} Op(\Lambda^j \mathbb{C}^n)$ that consists of the following collections of operators:

$$(s_1 \cdot \lambda_1(\mathcal{P}), \dots, s_{n-1} \cdot \lambda_{n-1}(\mathcal{P})), \tag{5.6}$$

where $\mathcal{P} \in \widetilde{\mathrm{Hinge}}_n$ and $s_j \in \mathbb{C}^*$.

The projection

$$\widehat{\mathrm{Hinge}}_n \to \widetilde{\mathrm{Hinge}}_n \tag{5.7}$$

is defined in an obvious way, namely, to the collection (5.6) we assign the weak hinge $\mathcal{P}$. The preimages of different points $\mathcal{P} \in \widetilde{\mathrm{Hinge}}_n$ in $\widehat{\mathrm{Hinge}}_n$ have different dimensions (namely, the fiber over $\mathcal{P}$ is $(\mathbb{C}^*)^m$, where m is the number of nonzero operators $\lambda_1(\mathcal{P}), \dots, \lambda_{n-1}(\mathcal{P})$), and this is the reason why the quotient space $\widetilde{\mathrm{Hinge}}_n = \widehat{\mathrm{Hinge}}_n / (\mathbb{C}^*)^{n-1}$ is a non-Hausdorff space.

It would be of interest to obtain explicit formulas for the product of collections of the form (5.6). This question can clearly be reduced to the problem of explicitly describing the category GA^* (see subsection 1.8).

§6. Representations

6.1. Representations of the group SL_n. Consider an irreducible representation of the group $SL_n(\mathbb{C})$ with numerical labels $a_1, \dots, a_{n-1}$ (see Figure 6.1).

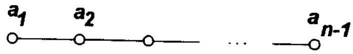

FIGURE 6.1

We recall the construction of this representation. Consider the space

$$W = \bigotimes_{j=1}^{n-1} (\Lambda^j(\mathbb{C}^n))^{\otimes a_j} \tag{6.1}$$

The group SL_n acts on this space by the formula $g \mapsto (\Lambda^j g)^{\otimes a_j}$. Let $h_j = e_1 \wedge \dots \wedge e_j \in \Lambda^j(\mathbb{C}^n)$, and let

$$v = v_a = \bigotimes_{j=1}^{n-1} (h_j)^{\otimes a_j} \in W.$$

Consider the cyclic span $V(a) = V(a_1, \dots, a_{n-1})$ of the vector v (i.e., the minimal SL_n-invariant subspace containing v). Then the irreducible representation $\rho_a = \rho_{a_1,\dots,a_{n-1}}$ with numerical labels $a = (a_1, \dots, a_{n-1})$ is the restriction of the representation (6.1) to the subspace $V(a_1, \dots, a_{n-1})$. The vector v_a is called the *highest weight vector.*

6.2. Projective embedding of Hinge_n**.** Now we construct a mapping

$$\pi_a = \pi_{a_1,\dots,a_{n-1}} : \mathrm{Hinge}_n \to \mathbb{P}(Op(V(a_1, \dots, a_{n-1}))) .$$

Let $\mathcal{P} = (P_1, \dots, P_l) \in \mathrm{Hinge}_n$. Consider the operator

$$\prod_{j=1}^{n-1} \lambda_j(\mathcal{P})^{\otimes a_j} \tag{6.2}$$

in W.

LEMMA 6.1. *The subspace $V(a)$ is invariant with respect to the operator* (6.2).

PROOF. $V(a)$ is invariant under SL_n, and SL_n is dense in Hinge_n.

Denote by $\pi_a(\mathcal{P})$ the restriction of the operator (6.2) to $V(a)$.

LEMMA 6.2. $\pi_a(\mathcal{P}) \neq 0$ *for all* $\mathcal{P} \in \mathrm{Hinge}_n$.

PROOF. It suffices to verify that $\pi_a(\mathcal{R}(i_1, \dots, i_k)) \neq 0$ for all canonical hinges $\mathcal{R}(i_1, \dots, i_k)$. However, we can readily see that $\mathcal{R}(i_1, \dots, i_k)v_a = s \cdot v_a$, $s \in \mathbb{C}^*$, which completes the proof.

EXAMPLE 6.3. The operator $\pi_a(\mathcal{R}(1,\dots,n-1))$ is the projection onto the highest weight vector v_a.

Denote by $\rho_a(SL_n)$ the set of all operators of the form $\rho_a(g)$, where $g \in SL_n$.

THEOREM 6.4. *The closure of the set $\rho_a(SL_n)$ in $\mathbb{P}Op(V(a))$ coincides with the image of the mapping*

$$\pi_a\colon \operatorname{Hinge}_n \to \mathbb{P}Op(V(a)). \tag{6.3}$$

PROOF. By Lemma 6.2, we have $\pi_a(\operatorname{Hinge}_n) \subset \mathbb{P}Op(V(a))$. The mapping π_a is clearly continuous and the set Hinge_n is compact.

PROPOSITION 6.5. *If all $a_1,\dots,a_{n-1}$ are nonzero, then the mapping* (6.3) *is an embedding.*

We shall prove this assertion later.

REMARK 6.6. If there are zeros among the numbers $a_1,\dots,a_{n-1}$, then the mapping π_a is not an embedding. Namely, in this case the mapping $\pi_a = \sigma\circ\tau$, i.e.,

$$\operatorname{Hinge}_n \xrightarrow{\tau} \prod_{j=1}^{n-1} \mathbb{P}(Op(\Lambda^j\mathbb{C}^n)) \xrightarrow{\sigma} \mathbb{P}(Op(V(a)))$$

can be factored through

$$\prod_{j:a_j\neq 0} \mathbb{P}(Op(\Lambda^j\mathbb{C}^n)). \tag{6.4}$$

In fact, in this case, the image of the mapping π_a is homeomorphic (and also equivalent as a variety) to the image of Hinge_n in (6.4).

6.3. **Representation $\widetilde{\pi_a}$ of the semigroup $\widetilde{\operatorname{Hinge}_n}$.** Let $\mathcal{P} \in \widetilde{\operatorname{Hinge}_n}$. Consider (see (6.1)) the operator $\bigotimes_{j=1}^{n-1} \lambda_j(\mathcal{P})^{\otimes a_j}$ in W. Denote by $\widetilde{\pi_a}(\mathcal{P})$ the restriction of this operator to the subspace $V(a)$. Clearly, $\mathcal{P} \mapsto \widetilde{\pi_a}(\mathcal{P})$ is a projective representation of the semigroup Hinge_n.

Let $\mathbb{P}^\circ H$ be the space of vectors from H defined up to a scalar factor.

THEOREM 6.7. *The closure of the image $\rho_a(SL_n)$ in $\mathbb{P}^\circ Op(V(a))$ coincides with $\pi_a(\widetilde{\operatorname{Hinge}_n})$.*

This assertion repeats Theorem 6.4.

REMARK 6.8. Let all $a_j \neq 0$. Let $\mathcal{P}$ be a weak hinge that is not equivalent to any exact hinge. Then $\pi_a(\mathcal{P}) = 0$. In other words, the closure of $\rho_a(SL_n)$ in $\mathbb{P}^\circ Op(V(a))$ is as follows. This is the semigroup that consists of exact hinges and 0. If $\mathcal{P}$ and $\mathcal{R}$ are exact hinges, then $\pi_a(\mathcal{R})\pi_a(\mathcal{P})$ is equal to 0 if and only if $\mathcal{RP}$ is a weak hinge that is not equivalent to any exact hinge.

REMARK 6.9. Let there be zeros among the numbers $a_1,\dots,a_{n-1}$. Assume that $\mathcal{P} = (P_1,\dots,P_n)$ is a hinge. Then $\pi_a(\mathcal{P}) \neq 0$ if and only if the union of the domains of action of the relations P_j contains all α such that $a_\alpha \neq 0$.

6.4. **Proof of Proposition 6.5.** Let Q be the orbit of the canonical hinge $\mathcal{R}(1,\dots,n-1)$ under the action of the group $PGL_n \times PGL_n$. Clearly, for any $\mathcal{P} \in \mathrm{Hinge}_n$ and any $S \in Q$ either we have $\mathcal{P}S \in Q$ or $\mathcal{P}S$ is not an exact hinge (and for the generic points we have the first case).

Furthermore, if $\mathcal{P}, \mathcal{P}' \in \mathrm{Hinge}_n$ and for any $S \in Q$ we have $\mathcal{P}S = \mathcal{P}'S$, then $\mathcal{P} = \mathcal{P}'$. In other words, a hinge $\mathcal{P}$ is completely defined by its products with the elements of the orbit Q (and it also suffices to consider the generic elements of Q).

Let us show that the mapping $\pi_a\colon \mathrm{Hinge}_n \to \mathbb{P}Op(V(a))$ defines an isomorphic embedding of the orbit Q in $\mathbb{P}Op(V(a))$. The group $PGL_n \times PGL_n$ acts on $Op(V(a))$ by multiplication by the operators $\rho_a(g)$ from the left and from the right. The operator $\pi_a(\mathcal{R}(1,\dots,n-1))$ is the projection onto the highest weight vector v_a. Therefore, the stabilizer of the operator $\pi_a(\mathcal{R}(1,\dots,n-1))$ in $PGL_n \times PGL_n$ is the product of two Borel subgroups $B \times B' \subset PGL_n \times PGL_n$, where B is the upper triangular subgroup and B' is the lower triangular subgroup. This stabilizer coincides with that of $\mathcal{R}(1,\dots,n-1)$ in $PGL_n \times PGL_n$ (see subsection 3.4). Thus, we have proved that the mapping π_a on Q is injective.

Now let $\pi_a(\mathcal{P}) = \pi_a(\mathcal{P}')$. Then $\pi_a(\mathcal{P}S) = \pi_a(\mathcal{P}'S)$ for all hinges S and, in particular, for $S \in Q$. However, this implies $\mathcal{P} = \mathcal{P}'$.

6.5. **Representations $\widehat{\pi_a}$ of the semigroup $\widehat{\mathrm{Hinge}_n}$.** Let the family of operators

$$\mathfrak{b} = (B_1,\dots,B_{n-1}) \in \prod_{j=1}^{n-1} Op(\Lambda^j\mathbb{C}^n)$$

belong to $\widehat{\mathrm{Hinge}_n}$. Define the operator $\bigotimes_{j=1}^{n-1} B_j^{\otimes a_j}$ in the space W. Denote by $\widehat{\pi_a}(\mathfrak{b})$ the restriction of this operator to $V(a)$. Clearly, $\mathfrak{b} \mapsto \widehat{\pi_a}(\mathfrak{b})$ is a *linear* representation of the semigroup $\widehat{\mathrm{Hinge}_n}$.

Now let ν be an arbitrary (in general, reducible) representation of the group SL_n in a linear space H. Let us decompose ν into irreducible representations. In any irreducible component, we have the action of the semigroup $\widehat{\mathrm{Hinge}_n}$, and hence $\widehat{\mathrm{Hinge}_n}$ acts on H.

REMARK 6.10. For the semigroup $\widetilde{\mathrm{Hinge}_n}$, this construction is impossible, because in the category of projective representations we have no direct sum operation.

Chapter II. Supplementary remarks

§7. Action of the semigroup $\widetilde{\mathrm{Hinge}_n}$ on the flag space

7.1. **Space $\mathcal{F}_n$.** Denote by $\mathcal{F}_n$ the space of all flags

$$0 \subset V_1 \subset V_2 \subset \dots \subset V_k \subset \mathbb{C}^n \tag{7.1}$$

(it is assumed that all inclusions are strict). Let $0 < i_1 < \dots < i_k < n$. Denote by $\mathcal{F}_n(i_1,\dots,i_k)$ the space of flags (7.1) such that $\dim V_1 = i_1$, $\dim V_2 = i_2, \dots$. The spaces $\mathcal{F}_n(i_1,\dots,i_k)$ are smooth varieties.

Let a number set $i_1, i_2, \dots$ contain a number set $j_1, j_2, \dots$. Then a projection

$$\pi^{i_1,i_2,\dots}_{j_1,j_2,\dots}\colon \mathcal{F}_n(i_1,\dots,i_k) \to \mathcal{F}_n(j_1,\dots,i_k)$$

is well defined: namely, from the flag $\mathcal{L} \in \mathcal{F}_n(i_1,\dots,i_k)$ we delete all subspaces except for the subspaces of the dimensions $j_1, j_2, \dots$.

Let us define a non-Hausdorff topology on the space

$$\mathcal{F}_n = \bigcup_{k,\, 0<i_1<\cdots<i_k<n} \mathcal{F}_n(i_1, \dots, i_k).$$

Let $\mathcal{L} \in \mathcal{F}(\alpha_1, \dots, \alpha_k)$ and $\mathcal{L}_j \in \mathcal{F}(\beta_1^j, \dots, \beta_{m_j}^j)$. Then the relation $\mathcal{L} = \lim_{j\to\infty} \mathcal{L}_j$ means that the following two conditions are satisfied.

1. Starting from some number j, the collections $(\beta_1^j, \dots, \beta_{m_j}^j)$ contain the collection $(\alpha_1, \dots, \alpha_k)$.
2. The projections of $\mathcal{L}_j$ to $\mathcal{F}(\alpha_1, \dots, \alpha_k)$ converge to $\mathcal{L}$ in the topology of $\mathcal{F}(\alpha_1, \dots, \alpha_k)$.

EXAMPLE 7.1. Consider a flag $\mathcal{L}\colon 0 \subset V_1 \subset \cdots \subset V_k \subset \mathbb{C}^n$. Then the closure of the point $\mathcal{L}$ consists of all points (flags) $V_{i_1} \subset \cdots \subset V_{i_\alpha}$ obtained from $\mathcal{L}$ by deleting some subspaces.

7.2. **Action of the semigroup $\widetilde{\mathrm{Hinge}}_n$ on the space $\mathcal{F}_n$.** Let

$$\mathcal{P} = (P_1, \dots, P_k) \in \widetilde{\mathrm{Hinge}}_n,$$

and let $\mathcal{L}\colon V_1 \subset V_2 \subset \cdots \subset V_s$ be an element of $\mathcal{F}$. Consider the set Σ of all pairs (i,j) such that

$$\operatorname{Ker} P_i \cap V_j = 0, \tag{7.2}$$

$$\operatorname{Dom} P_i + V_j = \mathbb{C}^n. \tag{7.3}$$

THEOREM 7.2. *The collection of subspaces $P_i V_j$, where the pair (i,j) ranges over Σ, form a flag.*

We omit the simple proof of this theorem, which is similar to the proof of Theorem 5.4.

Denote this flag by $\mathcal{P}\mathcal{L}$. We can readily see that $\mathcal{P}\colon \mathcal{L} \mapsto \mathcal{P}\mathcal{L}$ is an action of the semigroup $\widetilde{\mathrm{Hinge}}_n$ on $\mathcal{F}_n$.

7.3. **Hausdorff bundle over $\mathcal{F}_n$.** Denote by $\widehat{\mathcal{F}_n}$ the set of collections

$$(h_1, \dots, h_{n-1}) \in \prod_{j=1}^{n-1} \Lambda^j \mathbb{C}^n \tag{7.4}$$

such that h_j have the form

$$h_1 = t_1 \cdot v_1, \quad h_2 = t_2 \cdot v_1 \wedge v_2, \quad \dots, \quad h_{n-1} = t_{n-1} \cdot v_1 \wedge \cdots \wedge v_{n-1},$$

where $t_j \in \mathbb{C}$ and $v_1, \dots, v_{n-1}$ are linearly independent vectors in $\mathbb{C}^n$. We allow the possibility $t_j = 0$, i.e., some vectors h_j can be zero.

Let us define a projection $\widehat{\mathcal{F}_n} \to \mathcal{F}_n$.

Let $h_{i_1}, \dots, h_{i_k}$ be all nonzero vectors among $\{h_j\}$. Let V_{i_k} be a subspace spanned by the vectors $v_1, \dots, v_{i_k}$. Then to the element (7.4) there corresponds the flag

$$0 \subset V_{i_1} \subset \cdots \subset V_{i_k} \subset \mathbb{C}^n. \tag{7.5}$$

Note that the fiber over the point is of dimension k. As in subsection 5.7, the fibers over different points have different dimensions, and therefore the quotient $\mathcal{F}_n = \widehat{\mathcal{F}_n}/(\mathbb{C}^*)^{n-1}$ is a non-Hausdorff space.

7.4. **Action of the semigroup $\widehat{\mathrm{Hinge}_n}$ on $\widehat{\mathcal{F}_n}$.** Let $\mathfrak{b} = (B_1, \dots, B_{n-1}) \in \prod_{j=1}^{n-1} Op(\Lambda^j \mathbb{C}^n)$ be an element of $\widehat{\mathrm{Hinge}_n}$ and let $h = (h_1, \dots, h_{n-1}) \in \prod_{j=1}^{n-1} \Lambda^j V$ be an element of $\widehat{\mathcal{F}_n}$. Then the element $\mathfrak{b}h$ is defined by the formula $\mathfrak{b}h = (Bh_1, \dots, Bh_n)$.

§8. Complete symmetric varieties

Let G be a semisimple Lie group. Let σ be an involution on g (that is, σ is an automorphism and $\sigma^2 = \mathrm{id}$). Let H be the set of fixed points of σ. The homogeneous spaces G/H, where G and H are groups of the type described above, are called *symmetric spaces.*

In this section we are interested in the case of complex groups G and H. The completions of G/H constructed below are called *complex symmetric varieties.*

8.1. **Quasi-inverse hinge.** Recall that we denote by $P^\square$ the linear relation quasi-inverse to P (see subsection 1.1). Let $\mathcal{P} = (P_1, \dots, P_k)$ be a weak hinge. The *quasi-inverse hinge* $\mathcal{P}^\square$ is defined by

$$\mathcal{P}^\square = (P_k^\square, \dots, P_1^\square).$$

We can readily see that for any $\mathcal{P}, \mathcal{R} \in \widetilde{\mathrm{Hinge}_n}$ we have

$$(\mathcal{P}\mathcal{R})^\square = \mathcal{R}^\square \mathcal{P}^\square \tag{8.1}$$

8.2. **Transposed hinge.** Let $V \simeq \mathbb{C}^n$ be a linear space endowed with nondegenerate symmetric bilinear form

$$\langle x, y \rangle = \sum x_j y_j\,. \tag{8.2}$$

Introduce a skew-symmetric bilinear form on $V \oplus V$:

$$\{(x, u), (y, v)\} = \sum x_j v_j - \sum u_j y_j\,. \tag{8.3}$$

Let $P\colon V \rightrightarrows V$ be a linear relation. Denote by P^t the orthogonal complement to P.

EXAMPLE 8.1. Let P be the graph of a linear operator A. Then P^t is the graph of the transposed linear operator A^t. Indeed, A and A^t are related by the identity $\langle x, A^t y \rangle - \langle Ax, y \rangle = 0$, which exactly means that the vectors $(x, Ax) \in P$ and $(y, A^t y) \in P^t$ are orthogonal with respect to the form (8.3).

REMARK 8.2. For any linear relations P and Q, we have $(PQ)^t = Q^t P^t$.

Furthermore, let $\mathcal{P} = (P_1, \dots, P_k) \in \widetilde{\mathrm{Hinge}_n}$ be a weak hinge. Then the *transposed* hinge $\mathcal{P}^t$ is defined by the formula $\mathcal{P}^t = (P_1^t, \dots, P_k^t)$. We can readily see that

$$(\mathcal{P}\mathcal{R})^t = \mathcal{R}^t \mathcal{P}^t \tag{8.4}$$

for all $\mathcal{P}, \mathcal{R} \in \widetilde{\mathrm{Hinge}_n}$.

8.3. **Completion of the group $O_n(\mathbb{C})$.** Denote by $\overline{O_n}$ the space of exact hinges $\mathcal{P} \in \mathrm{Hinge}_n$ satisfying the condition $\mathcal{P}^\square = \mathcal{P}^t$. Denote by $\widetilde{O_n}$ the space of weak hinges satisfying the same condition.

It follows from (8.1) and (8.4) that the set $\widetilde{O_n}$ is closed with respect to the multiplication of hinges, that is, $\widetilde{O_n}$ is a semigroup.

The group $PO_n(\mathbb{C})$ (that is, the quotient group of the group $O_n(\mathbb{C})$ by its center) can be embedded in $\widetilde{O_n}$ in an obvious way. Namely, $O_n(\mathbb{C})$ consists of one-link hinges (Q), where Q ranges over the graphs of orthogonal operators. It is obvious that the group $O_n(\mathbb{C})$ is dense in $\overline{O_n}$ and in $\widetilde{O_n}$.

PROPOSITION 8.3. *The space $\overline{O_n}$ is a smooth algebraic variety.*

PROOF. Consider the mapping $\sigma\colon \mathrm{Hinge}_n \to \mathrm{Hinge}_n$ given by the formula $\sigma(\mathcal{P}) = (\mathcal{P}^\square)^t$. Then $\sigma^2 = \mathrm{id}$, and $\overline{O_n}$ is the set of fixed points of σ. Therefore, $\overline{O_n}$ is a smooth variety.

REMARK 8.4. Let us discuss in detail the conditions that must be satisfied by a hinge

$$\mathcal{P} = (P_1, \dots, P_k) \in \widetilde{O_n}\,. \tag{8.5}$$

For all j we have

$$P_j^t = P_{k-j}^\square\,. \tag{8.6}$$

Thus, the hinge $\mathcal{P}$ is completely determined by its terms:

$$(P_1, \dots, P_s), \tag{8.7}$$

where $s = k/2$ for k even and $s = (k+1)/2$ for k odd. Let us discuss the conditions satisfied by the linear relations $(P_1, \dots, P_s)$. There are two cases.

a) Let k be odd and $s = (k+1)/2$ (i.e., P_s is the middle term of the hinge (8.5)). In this case, relation (8.6) means that $P_s^t = P_s^\square$. The last relation is equivalent to the condition that P_s is a morphism of the category GD (see subsection 1.7).

b) Consider any other P_j with $j \leqslant s$. Then P_j and $(P_j^\square)^t$ occur in the same hinge, and hence

$$\operatorname{Ker} P_j \supset \operatorname{Dom}((P_j^\square)^t) \quad \text{and} \quad \operatorname{Im} P_j \subset \operatorname{Indef}((P_j^\square)^t).$$

These conditions are equivalent to the following three conditions:

1*. $\operatorname{Ker} P_j$ is co-isotropic, i.e., $\operatorname{Ker} P_j \supset (\operatorname{Ker} P_j)^\perp$, where the symbol $\perp$ stands for the orthogonal complement with respect to (8.2);

2*. $\operatorname{Im} P_j$ is isotropic, i.e., $\operatorname{Im} P_j \subset (\operatorname{Im} P_j)^\perp$;

3*. $\operatorname{Ker} P_j \supset \operatorname{Im} P_j$.

Conversely, if we have a weak hinge (8.7) satisfying conditions a) and b), then it is an element (8.5) of the semigroup $\widetilde{O_n}$.

8.4. **Complete quadrics.** Now we construct a completion of the space

$$PGL_n(\mathbb{C})/PO_n(\mathbb{C})\,.$$

Denote by $\widetilde{PGL_n/O_n}$ ($\overline{PGL_n/O_n}$, respectively) the space of weak hinges (exact hinges, respectively) $\mathcal{P} = (P_1, \dots, P_k)$ such that $\mathcal{P} = \mathcal{P}^t$ (where the transposition is the same as in subsection 8.2). This condition can be written in the form $P_j^t = P_j$ for all j.

REMARK 8.5. The condition $P = P^t$ is equivalent to the assumption that P is a maximal isomorphic subspace (*Lagrangian subspace*) of the space $\mathbb{C}^n \oplus \mathbb{C}^n$ endowed with skew-symmetric form (8.3).

The semigroup $\widetilde{O_n}$ acts on $\widetilde{PGL_n/O_n}$ by the formula $\mathcal{P} \mapsto \mathcal{R}\mathcal{P}\mathcal{R}^t$, where $\mathcal{P} \in \widetilde{PGL_n/O_n}$ and $\mathcal{R} \in \widetilde{O_n}$.

The *Study–Semple space of complete quadrics* is $\overline{PGL_n/O_n}$.

8.5. **Complete quadrics as a Hausdorff quotient.** Consider the Lagrangian Grassmannian $\mathcal{L}$ in the space $\mathbb{C}^n \oplus \mathbb{C}^n$. The group $\mathbb{C}^*$ acts on $\mathcal{L}$ by multiplication of a linear relation by a number (under this operation, a Lagrangian linear relation maps into a Lagrangian one).

The graphs of invertible operators $A\colon \mathbb{C}^n \to \mathbb{C}^n$ form an open dense set $\mathcal{L}^\circ$ in $\mathcal{L}$; here the condition $\operatorname{graph}(A) \in \mathcal{L}$ is equivalent to the condition $A = A^t$.

Note that the quadrics in $\mathbb{C}P^{n-1}$ are in a one-to-one correspondence with symmetric matrices A defined up to a scalar factor; i.e., $\mathcal{L}^\circ/\mathbb{C}^*$ can be regarded as the space of all quadrics.

Applying the construction of Hausdorff quotient space to $\mathcal{L}$ and its open set $\mathcal{L}^\circ$, we obtain exactly the space $\overline{PGL_n/O_n}$.

8.6. **Another description of complete quadrics.** Let $\mathcal{P} = (P_1, \dots, P_k)$, $P_j = P_j^t$, be an exact hinge. Then the following flag is defined:

$$\mathbb{C}^n \supset \operatorname{Ker} P_1 \supset \cdots \supset \operatorname{Ker} P_k \supset 0\,.$$

Furthermore, note that the form (8.3) defines a nondegenerate pairing between the two summands in $\mathbb{C}^n \oplus \mathbb{C}^n$. We can readily see that $\operatorname{Im} P_j \subset \{0\} \oplus \mathbb{C}^n$ is the annihilator of $\operatorname{Ker} P_j \subset \mathbb{C}^n \oplus \{0\}$, and therefore the spaces $\operatorname{Ker} P_j / \operatorname{Ker} P_{j+1}$ and $\operatorname{Im} P_{j+1} / \operatorname{Im} P_j$ are dual to each other. A linear relation P_{j+1} induces a nondegenerate linear operator $\operatorname{Ker} P_j / \operatorname{Ker} P_{j+1} = \operatorname{Dom} P_{j+1} / \operatorname{Ker} P_{j+1} \to \operatorname{Im} P_{j+1} / \operatorname{Im} P_j = \operatorname{Im} P_{j+1} / \operatorname{Indef} P_{j+1}$, and therefore we obtain a nondegenerate quadratic form on the space $\operatorname{Ker} P_j / \operatorname{Ker} P_{j+1}$.

Thus, the following collection of data can be regarded as a point of the space of complete quadrics:

1*. a flag $0 = V_0 \subset V_1 \subset \cdots \subset V_k \subset V_{k+1} = \mathbb{C}^n$;

2*. for any $j \in \{0, 1, \dots, k\}$, a nondegenerate quadratic form Q_j on the quotient space V_{j+1}/V_j defined up to a scalar factor.

8.7. **Completion of the group $Sp_{2n}(\mathbb{C})$.** Consider the space $V = \mathbb{C}^{2n}$ with nondegenerate skew-symmetric bilinear form

$$\{x, y\} = \sum_{i=1}^{n} x_i y_{i+n} - \sum_{j=n+1}^{2n} x_j y_{j-n}.$$

In the space $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ we introduce a symmetric bilinear form $M((x, y), (u, v)) = \{x, v\} - \{y, u\}$.

Let $P\colon \mathbb{C}^{2n} \rightrightarrows \mathbb{C}^{2n}$ be a linear relation. Denote by P^s the orthogonal complement to P with respect to the form M.

Now let $\mathcal{P} = (P_1, \dots, P_k) \in \widetilde{\mathrm{Hinge}}_n$. Then the hinge $\mathcal{P}^s$ is defined by $\mathcal{P}^s = (P_1^s, \dots, P_k^s)$.

Furthermore, the semigroup $\widetilde{Sp_{2n}}$ (the variety $\overline{Sp_{2n}}$) is defined as the set of weak hinges (exact hinges, respectively) that satisfies the condition $\mathcal{P}^s = \mathcal{P}^\square$.

Remark 8.4 (about the collections $\mathcal{P} \in \widetilde{O_n}$) remains valid for $\mathcal{P} \in \widetilde{Sp}$. The only difference is that, in our case, the question is in the category C and not in the categories B and GD.

8.8. **Space $\overline{PGL_n/Sp_{2n}}$.** We can naturally regard $PGL_n(\mathbb{C})/Sp_{2n}(\mathbb{C})$ as the space of nondegenerate skew-symmetric forms on $\mathbb{C}^{2n}$ defined up to a scalar factor. The space $\widetilde{PGL_n/Sp_{2n}}$ ($\overline{PGL_n/Sp_{2n}}$, respectively) is defined as the space of weak hinges (exact hinges, respectively) such that $\mathcal{P} = \mathcal{P}^s$.

All remarks of 8.4–8.6 on complete quadrics hold for $\overline{PGL_n/Sp_{2n}}$ as well.

8.9. **Completion of $GL_{p+q}(\mathbb{C})/GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$.** We can naturally regard the space $X_{p,q} = GL_{p+q}(\mathbb{C})/GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$ as the space of operators A on $\mathbb{C}^{p+q}$ such that $A^2 = E$ and q eigenvalues of A are equal to 1 and the other p are equal to (-1).

Let V_+ and V_- be the eigen subspaces of the operator A that correspond to the eigenvalues ± 1. Clearly, A is completely determined by the subspaces V_+ and V_-, and thus $X_{p,q}$ can be regarded as the set of pairs of subspaces $L, M \subset \mathbb{C}^{p+q}$ such that $\dim L = q$, $\dim M = p$, and $L \cap M = 0$.

Denote by $\overline{X_{p,q}} = \overline{GL_{p+q}/GL_p \times GL_q}$ the closure of $X_{p,q} \subset GL_{p+q}$ in Hinge. Assume for simplicity that $q \neq p$ and, for the sake of being definite, that $q > p$. Then $\overline{X_{p,q}} \setminus X_{p,q}$ consists of the hinges

$$\mathcal{P} = (P_1, \dots, P_k, \dots, P_{2k+1}) \tag{8.8}$$

such that

1. $P_j^\square = P_{2k+1-j}$;
2. the natural operator $A\colon \operatorname{Dom} P_k / \operatorname{Ker} P_k \to \operatorname{Im} P_k / \operatorname{Indef} P_k$ satisfies the condition $A^2 = E$; moreover, $\dim \operatorname{Ker}(A - E) - \dim \operatorname{Ker}(A + E) = q - p$.

8.10. The completions of the symmetric spaces $Sp_{2(n+k)}(\mathbb{C})/Sp_{2n}(\mathbb{C}) \times Sp_{2k}(\mathbb{C})$ and $O_{n+k}(\mathbb{C})/O_n(\mathbb{C}) \times O_k(\mathbb{C})$ can be constructed in just the same way as in the previous subsection; however, the hinge $\mathcal{P}$ must belong to $\overline{Sp_{2(n+k)}}$ and $\overline{O_{n+k}}$, respectively.

8.11. **Space $\overline{PO_{2n}/GL_n}$.** Let us describe the symmetric space

$$O_{2n}(\mathbb{C})/GL_n(\mathbb{C}). \tag{8.9}$$

Consider a $2n$-dimensional complex space V endowed with a nondegenerate bilinear form. It is natural to regard all possible decompositions of the space

$$V = V_+ \oplus V_- \tag{8.10}$$

into the direct sum of maximal isotropic subspaces as points of the space in (8.9). Indeed, the stabilizer of the pair $\{V_+, V_-\}$ is the group of matrices of the form $\begin{pmatrix} g & \\ & (g^t)^{-1} \end{pmatrix}$, $g \in GL_n(\mathbb{C})$, and the group $O(2n, \mathbb{C})$ transitively acts on the set of decompositions (8.10).

Furthermore, consider an operator $A \in O(2n, \mathbb{C})$ that is equal to iE on V_+ and to $(-iE)$ on V_-. Then we have $A^2 = -E$, and the set of these operators is in a one-to-one correspondence with the set of decompositions (8.10).

Let us take the closure of this set in $\overline{O_{2n}}$. We obtain the set of hinges $\mathcal{P} = (P_1, \dots, P_k)$ such that $\mathcal{P}^t = \mathcal{P}^\square$ (this is a condition on $\mathcal{P} \in \overline{O_{2n}}$) and $\mathcal{P}^\square = -\mathcal{P}$. In other words, $\mathcal{P}$ satisfies the conditions $P_j^t = -P_j$ and $P_j^\square = -P_{k-j}$.

8.12. The completion of the space $Sp_{2n}(\mathbb{C})/GL_n(\mathbb{C})$ is constructed in the same way.

8.13. **Smoothness.** The smoothness of all the above varieties is proved by the same arguments as in the proof of Proposition 8.3.

§9. Real forms of complete symmetric varieties. The Satake–Furstenberg boundary

9.1. **Real hinges.** Until now we discussed complex hinges only. However, we can also consider hinges over reals or over quaternions. For example, arguments just like those used in the complex case show that the set of all hinges in $\mathbb{R}^n$ is a smooth real analytic variety. Now we shall present a more involved example of a real form of a complex symmetric variety; namely, we consider a real form of complex quadrics.

9.2. **Boundaries of the spaces $SL(n, \mathbb{R})/SO(p, n-p)$.** In $\mathbb{R}^n$ we take the bilinear form $\langle x, y\rangle = \sum x_j y_j$. In the space $\mathbb{R}^{2n}$ we consider the skew-symmetric bilinear form $\{(x, y), (u, v)\} = \sum x_j v_j - \sum y_j u_j$.

Let $\mathcal{L}$ be a Lagrangian Grassmannian in $\mathbb{R}^n \oplus \mathbb{R}^n$. Let the multiplication group $\mathbb{R}^*$ of reals act on $\mathcal{L}$ by multiplication by scalars.

Consider an open dense subset $\mathcal{L}^\circ$ in $\mathcal{L}$ that consists of the graphs of the invertible operators $A\colon \mathbb{R}^n \to \mathbb{R}^n$; note that these operators satisfy the relation $A = A^t$.

Furthermore, apply the construction of Hausdorff quotient space to the $\mathcal{L}$ and to the open subset $\mathcal{L}^\circ$. We obtain the set C_n of exact real hinges $\mathcal{P} = (P_1, \dots, P_k)$ that satisfy the condition $P_j = P_j^t$ (where P_j^t was defined in subsection 8.2).

The set $\mathcal{L}^\circ$ consists of the one-link hinges.

The group $GL_n(\mathbb{R})$ acts on $\mathcal{L}^\circ$ by the transformations

$$A \mapsto g^t A g, \tag{9.1}$$

and on the space C_n by the transformations $\mathcal{P} \mapsto g^t \mathcal{P} g$.

An arbitrary real symmetric matrix A can be reduced, by transformations (9.1), to the form

$$J_s = \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline \dots & \dots & \dots & \dots & \dots & \dots \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\ \ddots\\ 1\end{array}}\right\} j \\ \\ \left.\vphantom{\begin{array}{c}-1\\ \ddots\\ -1\end{array}}\right\} n-s \end{array}$$

and the stabilizer of J_s is the group $O(s, n-s)$. The matrix A, in our case, is defined up to a factor, and thus the canonical forms J_s and J_{n-s} are equivalent.

Thus, the space $\mathcal{L}^\circ$ is the disjoint union of open orbits (symmetric spaces)

$$PGL(n,\mathbb{R})/SO(s, n-s), \tag{9.2}$$

where $s \in \{0, 1, \dots, [(n+1)/2]\}$.

The closure of the orbit

$$PGL(n,\mathbb{R})/PO(n) \tag{9.3}$$

is called the *Furstenberg–Satake compactification* of the Riemannian symmetric space (9.3), and the closures of the orbits (9.2) are compactifications of pseudo-Riemannian symmetric spaces $PGL(n,\mathbb{R})/SO(s, n-s)$.

9.3. **The Satake–Furstenberg compactification.** Now we shall give a more detailed description of the boundary of the Riemannian symmetric space $G/K = SL(n,\mathbb{R})/SO(n)$.

Consider a linear relation $\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ such that $P = P^t$. Then $\mathrm{Im}(P)$ is the orthogonal complement of $\mathrm{Ker}(P)$, and $\mathrm{Indef}(P)$ is the orthogonal complement to $\mathrm{Dom}(P)$ (with respect to the standard inner product in $\mathbb{R}^n$). Hence, a linear relation P defines a nondegenerate pairing

$$\mathrm{Dom}(P)/\mathrm{Ker}(P) \times \mathrm{Im}(P)/\mathrm{Indef}(P) \to \mathbb{R}. \tag{9.4}$$

Moreover, a linear relation P defines an operator

$$\mathrm{Dom}(P)/\mathrm{Ker}(P) \to \mathrm{Im}(P)/\mathrm{Indef}(P). \tag{9.5}$$

Hence, a symmetric linear relation P defines a nondegenerate symmetric bilinear form $\mathfrak{q}_P$ on the space $\mathrm{Dom}(P)/\mathrm{Ker}(P)$.

We say that a symmetric linear relation P is *nonnegative definite* if the form

$$[(v,w);(v',w')] := \langle v, w'\rangle + \langle v', w\rangle$$

is nonnegative definite on the subspace P. This condition holds if and only if the quadratic form defined by the bilinear form $\mathfrak{q}_P$ is positive.

REMARK 9.1. Let a linear relation P be the graph of an operator A. Then P is nonnegative definite if and only if A is nonnegative definite.

By our definition, a point of the *Satake–Furstenberg compactification* of the space $SL(n,\mathbb{R})/SO(n)$ is determined by the following data:

1* an integer $s \in \{1, \dots, n-1\}$;
2* a hinge $\mathcal{P} = (P_1, \dots, P_s)$ such that all linear relations P_j are nonnegative definite and satisfy the condition $P_j = P_j^t$ for all j.

Consider a point of the Satake–Furstenberg compactification (i.e., let the data of the form 1*–2* be given). Introduce the subspaces $V_j = \mathrm{Ker}(P_j) = \mathrm{Dom}(P_{j+1})$. Then the form related to $[\,\cdot\,,\,\cdot\,]$ defines a positive definite form on the quotient space $\mathrm{Dom}(P_j)/\mathrm{Ker}(P_j)$. We see that a point of the Satake–Furstenberg boundary can be defined by the following data:

1* an integer $s \in \{1, \dots, n-1\}$;
2* a flag $0 \subset V_1 \subset \cdots \subset V_s \subset \mathbb{R}^n$, where all subspaces $0, V_1, \dots, V_s$, $\mathbb{R}^n$, are distinct, and
3* for any $j \in \{1, \dots, s\}$, a positive definite quadratic form Q_j on the quotient space $\mathrm{Dom}(P_j)/\mathrm{Ker}(P_j)$.

§10. Category of hinges

10.1. **Spaces $\widetilde{\mathrm{Hinge}}_n(p,q)$.** By a *weak n-dimensional hinge* $\mathcal{P}\colon \mathbb{C}^p \rightrightarrows \mathbb{C}^q$ we mean the collection $\mathcal{P} = (P_1, \dots, P_s)$ of n-dimensional linear relations $P_j\colon \mathbb{C}^p \rightrightarrows \mathbb{C}^q$ such that $\operatorname{Ker} P_j \supset \operatorname{Dom} P_{j+1}$ and $\operatorname{Im} P_j \subset \operatorname{Indef} P_{j+1}$. Denote by $\widetilde{\mathrm{Hinge}}_n(p,q)$ the space of all n-dimensional weak hinges $\mathbb{C}^p \rightrightarrows \mathbb{C}^q$ and denote by $\widetilde{\mathrm{Hinge}}(p,q)$ the space

$$\widetilde{\mathrm{Hinge}}(p,q) = \bigcup_{n=0}^{p+q} \widetilde{\mathrm{Hinge}}_n(p,q).$$

10.2. **Multiplication.** The multiplication

$$\widetilde{\mathrm{Hinge}}(p,q) \times \widetilde{\mathrm{Hinge}}(q,r) \to \widetilde{\mathrm{Hinge}}(p,r)$$

is defined in just the same way as the multiplication of weak hinges. Let $\mathcal{P} = (P_1, \dots, P_k) \in \widetilde{\mathrm{Hinge}}(p,q)$ and $\mathcal{R} = (R_1, \dots, R_l) \in \widetilde{\mathrm{Hinge}}(q,r)$. Consider the set of all products $Q_{ij} = R_i P_j$ that differ from *null*. Then, after an appropriate ordering, the set Q_{ij} becomes a weak hinge $\mathbb{C}^p \rightrightarrows \mathbb{C}^r$.

Let us define the category *HINGE*. Its objects are linear spaces $\mathbb{C}^p$, and the set of morphisms of $\mathbb{C}^p$ into $\mathbb{C}^q$ is $\widetilde{\mathrm{Hinge}}(p,q)$.

10.3. **Action on the exterior powers.** Let $\mathcal{P} = (P_1, \dots, P_k) \in \widetilde{\mathrm{Hinge}}_n(p,q)$. Let us define the operator $\lambda_j(\mathcal{P})\colon \Lambda^j\mathbb{C}^p \to \Lambda^{j+n-p}\mathbb{C}^q$. To this end we consider the sequence of operators

$$\lambda_j(P_1), \dots, \lambda_j(P_k)\colon \Lambda^j\mathbb{C}^p \to \Lambda^{j+n-p}\mathbb{C}^q, \tag{10.1}$$

where $\lambda_j(P_s)$ is the restriction of the operator $\lambda(P_s)$ (see §1) to $\Lambda^j\mathbb{C}^p$. By definition, we take for $\lambda_j(\mathcal{P})$ the nonzero term of the sequence (10.1) if this nonzero term exists, and set $\lambda_j(\mathcal{P}) = 0$ otherwise.

If $\mathcal{P} \in \widetilde{\mathrm{Hinge}}_n(p,q)$ and $\mathcal{R} \in \widetilde{\mathrm{Hinge}}_m(q,r)$, then $\lambda_j(\mathcal{RP}) = t \cdot \lambda_{j+n-p}(\mathcal{R})\lambda_j(\mathcal{P})$, where $t \in \mathbb{C}^*$.

10.4. **Action on flags.** Let $0 \subset V_1 \subset \cdots \subset V_k \subset \mathbb{C}^p$ be a flag in the space $\mathbb{C}^p$, i.e., an element of the space $\mathcal{F}_p$, in the notation of §7. Let $\mathcal{P} = (P_1, \dots, P_k) \in \widetilde{\mathrm{Hinge}}_n(p,q)$. Furthermore, consider the set Σ of all pairs (i,j) such that $\operatorname{Dom} P_i + V_j = \mathbb{C}^p$ and $\operatorname{Ker} P_i \cap V_j = 0$. Then the set of subspaces P_iV_j, where (i,j) ranges over Σ, forms a flag in $\mathbb{C}^q$; that is, a hinge $\mathcal{P}$ defines a mapping $\mathcal{F}_p \to \mathcal{F}_q$.

Chapter III. Other boundaries

§11. Velocity compactifications of symmetric spaces

In this section we consider the symmetric spaces $SL(n,\mathbb{R})/SO(n)$ only.

We recall that a point of the space $SL(n,\mathbb{R})/SO(n)$ can be identified with a positive definite real matrix that is defined up to a scalar factor.

11.1. **The simplest velocity compactification.** Consider a positive definite matrix $A \in Q = SL(n,\mathbb{R})/SO(n)$. Let $a_1 \geqslant \cdots \geqslant a_n$ be the eigenvalues of A. Let $\lambda_j = \ln a_j$. Denote by $\Lambda(A)$ the set

$$\Lambda(A) = (\lambda_1, \dots, \lambda_n), \qquad \lambda_1 \geqslant \lambda_2 \geqslant \cdots. \tag{11.1}$$

The matrix A is defined up to a factor, and hence $\Lambda(A)$ is defined up to an additive constant:

$$(11.2) \qquad (\lambda_1, \dots \lambda_n) \sim (\lambda_1 + \sigma, \dots, \lambda_n + \sigma).$$

We denote by Σ_n the space of all sets $\Lambda(A)$ (see (11.2)). We can readily see that $\Lambda(A)$ is an $(n-1)$-dimensional simplicial cone. We can assume that $\lambda_n = 0$, and hence the cone Σ_n can be regarded as the space of sets $\{\lambda_1 \geqslant \dots \geqslant \lambda_{n-1} \geqslant 0\}$. We denote by $\Delta_n = \partial\Sigma_n$ the $(n-2)$-dimensional simplex $1 \geqslant \mu_2 \geqslant \mu_3 \geqslant \dots \geqslant \mu_{n-1} \geqslant 0$ and set $\mu_1 = 1$ and $\mu_n = 0$. Then Δ_n is called the *velocity simplex.* Consider the natural projection $\pi\colon (\Sigma_n \setminus 0) \to \Delta_n$ defined by the rule

$$\pi(\lambda_1, \dots \lambda_{n-1}, 0) = (\lambda_2/\lambda_1, \lambda_3/\lambda_1, \dots, \lambda_{n-1}/\lambda_1).$$

Now we define the compactification $\overline{\Sigma}_n = \Sigma_n \cup \Delta_n$ of Σ_n as follows. We say that a sequence $L_j = (\lambda_1^{(j)}, \dots, \lambda_n^{(j)}) \in \Sigma_n$ converges to an element $M \in \Delta_n$ if the following two conditions hold:

1) $\lambda_1^{(j)} - \lambda_n^{(j)} \to \infty$ as $j \to \infty$, and
2) the sequence $\pi(L_j) \in \Delta_n$ converges to M.

Moreover, we introduce the *velocity compactification* of the symmetric space $SL(n,\mathbb{R})/SO(n)$ by the relation $\overline{Q}^{\mathrm{vel}} = (SL(n,\mathbb{R})/SO(n)) \cup \Delta_n$, where a sequence A_j in Q is convergent to $M \in \Delta_n$ whenever $\Lambda(A_j)$ converges to M with respect to the topology of $\overline{\Sigma}_n$.

11.2. **The polyhedron of Karpelevich velocities.** Now we describe a more delicate compactification of the simplicial cone Σ_n (namely, the compactification by Karpelevich velocities). Consider a sequence $\lambda^{(j)} = \{\lambda_1^{(j)} \geqslant \dots \geqslant \lambda_n^{(j)}\} \in \Sigma_n$. Let this sequence have the limit $1 \geqslant \mu_2 \geqslant \dots \geqslant \mu_{n-1} \geqslant 0$ in Δ_n. It may happen that some of the numbers μ_i are equal, i.e., $\mu_k = \mu_{k+1} = \dots = \mu_l$. In this case we can separate velocities that correspond to the subset $\{\lambda_k^{(j)} \geqslant \dots \geqslant \lambda_l^{(j)}\} \in \Sigma_{l-k+1}$ by the same rule as above.

Definition of the polyhedron. Denote by $I_{\alpha,\beta}$ the set $\{\alpha, \alpha+1, \dots, \beta\} \subset \mathbb{N}$.

Consider an interval $I_{\alpha,\beta} = \{\alpha, \alpha+1, \dots, \beta\}$ and denote by $\Sigma(I_{\alpha,\beta})$ the simplicial cone $\lambda_\alpha \geqslant \lambda_{\alpha+1} \geqslant \dots \geqslant \lambda_\beta$, where the elements of the cone $\Sigma(I_{\alpha,\beta})$ are defined up to an additive constant (see (11.2)). Moreover, introduce the simplex $\Delta(I_{\alpha,\beta})$ defined by the inequalities $1 = \mu_\alpha \geqslant \mu_{\alpha+1} \geqslant \dots \geqslant \mu_{\beta-1} \geqslant \mu_\beta = 0$ and consider the compactification $\overline{\Sigma}(I_{\alpha,\beta}) = \Sigma(I_{\alpha,\beta}) \cup \Delta(I_{\alpha,\beta})$.

REMARK 11.1. For the case $\alpha = \beta$, the set $\Sigma(I_{\alpha,\alpha}) = \overline{\Sigma}(I_{\alpha,\alpha})$ is a singleton (a real defined up to an additive constant).

For $k \leqslant \alpha \leqslant \beta \leqslant l$ we define the mapping $\pi^{k,l}_{\alpha,\beta}\colon \Sigma(I_{k,l}) \to \Sigma(I_{\alpha,\beta})$ by the formula $\pi^{k,l}_{\alpha,\beta}(\lambda_k, \dots, \lambda_l) = (\lambda_\alpha, \dots, \lambda_\beta)$. We introduce two polyhedra:

$$\Xi(k,l) := \prod_{\alpha,\beta\,:\,k\leqslant\alpha\leqslant\beta\leqslant l} \Sigma(I_{\alpha,\beta}), \qquad \overline{\Xi}(k,l) := \prod_{\alpha,\beta\,:\,k\leqslant\alpha\leqslant\beta\leqslant l} \overline{\Sigma}(I_{\alpha,\beta}).$$

We clearly have $\Xi(k,l) \subset \overline{\Xi}(k,l)$. Consider the (diagonal) embedding $i : \Sigma(I_{k,l}) \to \Xi(k,l)$ (which is the product of the mappings $\pi^{k,l}_{\alpha,\beta}$).

We define the *polyhedron of Karpelevich velocities* $\mathcal{K}(k,l)$ as the closure of the subset $i(\Sigma(I_{k,l}))$ in $\overline{\Xi}(k,l)$.

A CRITERION FOR THE CONVERGENCE OF A SEQUENCE OF INTERIOR POINTS TO A BOUNDARY POINT. Consider a sequence $\Lambda^{(j)} = \{\lambda_k^{(j)}, \lambda_{k+1}^{(j)}, \dots, \lambda_l^{(j)}\}$. Then the sequence $\Lambda^{(j)}$ is convergent in $\mathcal{K}(k,l)$ if and only if all sequences of the form $\pi_{\alpha,\beta}^{k,l}(\Lambda^{(j)}) = (\lambda_\alpha^{(j)}, \dots, \lambda_\beta^{(j)})$ are convergent in $\overline{\Sigma}(I_{\alpha.\beta})$.

The polyhedron of Karpelevich velocities is thus defined. Now we give an explicit (but cumbersome) description of its combinatorial structure.

11.3. **Combinatorial description of the polyhedron of Karpelevich velocities.**

TREE-PARTITIONS. Consider the set $I_{k.l} := \{k, k+1, \dots, l\}$. By a *partition* of $I_{k.l}$ we mean a representation of $I_{k.l}$ in the form $I_{k.m_1} \cup I_{m_1+1.m_2} \cup \dots \cup I_{m_{s-1}+1.l}$, where $s > 1$. A system $\mathfrak{a}$ formed by subsets of $I_{k.l}$ is called a *tree partition* if the following conditions hold:

a) $I_{k.l} \in \mathfrak{a}$;
b) any element $J \in \mathfrak{a}$ has the form $I_{\alpha.\beta} = \{\alpha, \alpha+1, \dots, \beta\}$;
c) If $J_1, J_2 \in \mathfrak{a}$, then we have either $J_1 \cap J_2 = \varnothing$ or one of the conditions $J_1 \supset J_2$ and $J_2 \subset J_1$;
d) for any $J = I_{\alpha.\beta} \in \mathfrak{a}$ we have one of the following two possibilities:
 (1) 1* there is no $K \in \mathfrak{a}$ such that $K \subset J$ (in this case, $I_{\alpha.\beta}$ is said to be *irreducible*);
 (2) 2* $J = I_{\alpha.\beta}$ is the

$$I_{\alpha,\beta} = I_{\alpha.\gamma_1} \cup I_{\gamma_1+1.\gamma_2} \cup I_{\gamma_2+1.\gamma_3} \cup \dots \cup I_{\gamma_{s-1}+1.\beta}, \tag{11.3}$$

where $I_{\alpha,\gamma_1}, I_{\gamma_1+1.\gamma_2}, \dots, I_{\gamma_{s-1}+1.\beta} \in \mathfrak{a}$ and there is no $K \in \mathfrak{a}$ such that $I_{\alpha,\beta} \supset K \supset I_{\gamma_{i-1}+1.\gamma_i}$, $K \neq I_{\alpha.\beta}, I_{\gamma_{i-1}.\gamma_i}$. In this case, J is said to be *reducible* and (11.3) is called the *canonical decomposition* of J.

REMARK 11.2. Let $I_{\alpha,\beta} \in \mathfrak{a}$. Let $\mathfrak{b}$ be the set of all $J \subset I_{\alpha.\beta}$ such that $J \in \mathfrak{a}$. Then $\mathfrak{b}$ is a tree partition of $I_{\alpha.\beta}$.

REMARK 11.3. In other words, a tree partition can be defined by the following data: a partition of the segment $I_{k.l} \subset \mathbb{N}$ into subsegments, partitions of some subsegments, etc.

We denote by $TP(k,l)$ the set of all tree partitions of $I_{k.l}$. Introduce the canonical partial ordering on $TP(k,l)$. Let $\mathfrak{a}, \mathfrak{b} \in TP(k,l)$. We say that $\mathfrak{a} > \mathfrak{b}$ whenever $J \in \mathfrak{a}$ implies $J \in \mathfrak{b}$ (i.e., $\mathfrak{b} \supset \mathfrak{a}$).

The partially ordered set $TP(k,l)$ contains a unique maximal element $\mathfrak{a}_0$. This is the tree partition that consists of a single element $I_{k.l}$.

An element $\mathfrak{b} \in TP(k,l)$ is minimal if the following two conditions hold:

a) any irreducible element of $\mathfrak{b}$ is a singleton;
b) if $J \in \mathfrak{b}$ is reducible, then the canonical decomposition of J contains exactly two elements ($s = 2$ in (11.3)).

Consider a partition $\mathfrak{t}$ of $I_{\alpha.\beta}$:

$$I_{\alpha.\beta} = I_{\alpha.\gamma_1} \cup I_{\gamma_1+1.\gamma_2} \cup I_{\gamma_2+1.\gamma_3} \cup \dots \cup I_{\gamma_{s-1}+1.\beta}. \tag{11.4}$$

Denote by $\widetilde{\Delta}(I_{\alpha.\beta} \,|\, \mathfrak{t})$ the open simplex

$$\begin{aligned} 1 = \mu_\alpha = \dots = \mu_{\gamma_1} > \mu_{\gamma_1+1} \\ = \mu_{\gamma_1+2} = \dots = \mu_{\gamma_2} > \dots > \mu_{\gamma_{s-1}+1} = \dots = \mu_\beta = 0 \end{aligned} \tag{11.5}$$

and by $\Delta(I_{\alpha,\beta}\,|\,\mathfrak{t})$ the compact simplex

$$1 = \mu_\alpha = \dots = \mu_{\gamma_1} \geqslant \mu_{\gamma_1+1} = \mu_{\gamma_1+2} = \dots = \mu_{\gamma_2} \geqslant \dots \geqslant \mu_{\gamma_{s-1}+1} = \dots = \mu_\beta = 0. \tag{11.6}$$

In $\Delta(I_{\alpha,\beta}\,|\,\mathfrak{t})$ and $\widetilde{\Delta}(I_{\alpha,\beta}\,|\,\mathfrak{t})$ we introduce the natural coordinates

$$\tau_2 := \mu_{\gamma_1+1} = \dots = \mu_{\gamma_2}, \quad \dots, \quad \tau_{s-1} := \mu_{\gamma_{s-2}+1} = \dots = \mu_{\gamma_{s-1}}.$$

REMARK 11.4. If s=2, then $\Delta(J\,|\,\mathfrak{t}) = \widetilde{\Delta}(J\,|\,\mathfrak{t})$ is a singleton $\{1 > 0\}$.

REMARK 11.5. We have $\Delta(I_{\alpha,\beta}) = \bigcup_{\mathfrak{t}} \widetilde{\Delta}(I_{\alpha,\beta}\,|\,\mathfrak{t})$, where the union is taken over all partitions of $I_{\alpha,\beta}$.

Let us choose a tree partition $\mathfrak{a} \in TP(k,l)$. For any element $J \in \mathfrak{a}$ we consider its canonical decomposition $\mathfrak{t}$ and denote the simplex $\widetilde{\Delta}(J\,|\,\mathfrak{t})$ by $\widetilde{\Delta}(\mathfrak{a}, J)$. For any $\mathfrak{a} \in TP(k,l)$ we define the *face* $F(\mathfrak{a})$ as follows:

$$F(\mathfrak{a}) = \Big(\prod_{J=I_{\alpha,\beta}\in\mathfrak{a} \text{ is irreducible}} \Sigma(I_{\alpha,\beta}) \Big) \prod_{J\in\mathfrak{a} \text{ is redicible}} \widetilde{\Delta}(\mathfrak{a}, J)\,. \tag{11.7}$$

REMARK 11.6. For the trivial tree partition $\mathfrak{a}_0$, we have $F(\mathfrak{a}_0) = \Sigma(I_{k,l})$. If $\mathfrak{b}$ is a minimal tree partition, then $F(\mathfrak{b})$ is a singleton.

Now we can represent the *polyhedron of Karpelevich velocities* $\mathcal{K}(k,l)$ as the union $\mathcal{K}(k,l) = \bigcup_{\mathfrak{a}\in TP(k,l)} F(\mathfrak{a})$. Let us define a certain topology of a compact metric space on $\mathcal{K}(k,l)$. The face $F(\mathfrak{a}_0) = \Sigma(I_{k,l})$ will be an open dense subset of $\mathcal{K}(k,l)$.

REMARK 11.7. Let $l = k$. Then $\mathcal{K}(k,k)$ is a singleton. Let $l = k+1$. Then there are two tree partitions of the set $\{k, k+1\}$: the trivial tree partition $\mathfrak{a}_0$ and the minimal tree partition $\mathfrak{a}_1$ with the elements $(k, k+1)$, (k), and $(k+1)$. The face $F(\mathfrak{a}_0)$ is the closed semi-axis $\lambda_1 > 0$. The face $F(\mathfrak{a}_1)$ is a singleton. Hence, $\mathcal{K}(k, k+1)$ is the segment $[0, \infty]$.

THE CONVERGENCE OF INTERIOR POINTS TO A BOUNDARY POINT. For the definition of convergence we proceed by induction. We assume that the convergence is defined for all Karpelevich polyhedra $\mathcal{K}(\alpha,\beta)$ such that $\beta - \alpha < l - k$. We define the convergence of a sequence

$$x^{(j)} = \{x_k^{(j)} \geqslant \dots \geqslant x_l^{(j)}\} \in \Sigma(I_{k,l}) = F(\mathfrak{a}_0)$$

in two stages.

Step 1. The convergence of $x^{(j)}$ in $\overline{\Sigma}(I_{k,l})$ is necessary for the convergence of this sequence in $\mathcal{K}(k,l)$. Let y be the limit of $x^{(j)}$ in $\overline{\Sigma}(I_{k,l})$.

If $y \in \Sigma(k,l)$, then the sequence is said to be convergent in $\mathcal{K}(k,l)$, and y is called the limit of $x^{(j)}$ in $\mathcal{K}(k,l)$.

Step 2. Let $y \notin \Sigma(I_{k,l})$. Then y belongs to an open simplex $\widetilde{\Delta}(I_{k,l}\,|\,\mathfrak{t})$, that is, y is of the form

$$\{1 = y_k = \dots = y_{\gamma_1} > y_{\gamma_1+1} = \dots = y_{\gamma_2} > \dots > y_{\gamma_{s-1}+1} = \dots = \gamma_l = 0\}\,.$$

In this case, we say that the sequence $x^{(j)}$ is convergent in $\mathcal{K}(k,l)$ if and only if all sequences $x^{(j)}_{[\psi]} := (x^{(j)}_{\gamma_\psi+1}, \dots, x^{(j)}_{\gamma_{\psi+1}}) \in \Sigma(I_{\gamma_\psi+1,\gamma_{\psi+1}})$ converge in the corresponding polyhedra of Karpelevich velocities $\mathcal{K}(\gamma_\psi+1, \gamma_{\psi+1})$ (their convergence being defined by the induction assumption).

This concludes the definition.

EXAMPLE 11.8. Let $k=1$ and $l=8$. Consider a sequence $x^j = (x^j_1, \dots, x^j_8)$, where

$$x^{(j)}_1 = 2j^3, \quad x^{(j)}_2 = j^3, \quad x^{(j)}_3 = j^2+j+2, \quad x^{(j)}_4 = j^2+j+1,$$
$$x^{(j)}_5 = j^2+j, \quad x^{(j)}_6 = 2j, \quad x^{(j)}_7 = j, \quad x^{(j)}_8 = 0.$$

Then the associated tree partition has the form

$$\begin{array}{c} (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8) \\ (1)\ (2)\ (3\ 4\ 5\ 6\ 7\ 8) \\ (3\ 4\ 5\ )\ (6\ 7\ 8) \\ (6)\ (7)\ (8) \end{array}$$

and the limit of $x^{(j)}$ in $\overline{\Sigma}(I_{1,8})$ is the set

$$\{1 > 1/2 > 0 = 0 = 0 = 0 = 0 = 0\} \in \Delta(I_{1,8})\,. \tag{11.8}$$

The sequence $x^{(j)}$ defines the sequence $y^{(j)} = (x^{(j)}_3, \dots, x^{(j)}_8) \in \Sigma(I_{3,8})$. The limit of $y^{(j)}$ in $\overline{\Sigma}(I_{3,8})$ is the set

$$\{1 \geqslant 1 \geqslant 1 \geqslant 0 \geqslant 0 \geqslant 0\} \in \Delta(I_{3,8})\,. \tag{11.9}$$

Moreover, we have the sequences $z^{(j)} = (x^{(j)}_3, x^{(j)}_4, x^{(j)}_5) \in \Sigma(I_{3,5})$ and $u^{(j)} = (x^{(j)}_6, x^{(j)}_7, x^{(j)}_8) \in \Sigma(I_{6,8})$. We have $z^{(j)} = (j^2+j+2, j^2+j+1, j^2+j) = (2,1,0)$ (recall that the collection $z^{(j)}$ is defined up to an additive constant), and $\lim z^{(j)}$ is the set

$$\{2 > 1 > 0\} \in \Sigma(I_{3,5})\,. \tag{11.10}$$

Finally, $u^{(j)} = (2j, j, 0)$, and the limit of $u^{(j)}$ in $\overline{\Sigma}(I_{6,8})$ is the point

$$\{1 > 1/2 > 0\} \in \widetilde{\Delta}(I_{6,8})\,. \tag{11.11}$$

The limit of the sequence $x^{(j)}$ is the family of sets (11.8)–(11.11).

THE TOPOLOGY ON THE BOUNDARY OF $I_{k,l}$. This topology must satisfy the following property: the closure of $F(\mathfrak{a})$ consists of all faces $F(\mathfrak{b})$ such that $\mathfrak{b} < \mathfrak{a}$.

We assume that the topology is defined for all polyhedra $\mathcal{K}(\alpha,\beta)$ such that $\beta - \alpha < l - k$.

Let us define the convergence of a sequence $Z^{(j)} \in F(\mathfrak{a})$ in two stages.

Step 1. Let $h^{(j)} = \{1 = h^{(j)}_k \geqslant h^{(j)}_{k+1} \geqslant \cdots \geqslant h^{(j)}_l = 0\}$ be the component of $Z^{(j)}$ related to the factor $\widetilde{\Delta}(\mathfrak{a}, I_{k,l})$ in the product (11.7). Then the convergence of $h^{(j)}$ in $\Delta(\mathfrak{a}, I_{k,l})$ is necessary for the convergence of this sequence in $\mathcal{K}(k,l)$. Let u be the limit of $h^{(j)}$ in $\Delta(\mathfrak{a}, I_{k,l})$.

Step 2. Consider the partition of $I_{k,l}$ related to $\mathfrak{a}$, that is, $I_{k,l} = I_{k,\gamma_1} \cup I_{\gamma_1+1,\gamma_2} \cup \dots \cup I_{\gamma_{s-1}+1,l}$. Then the set u has the form

$$u = \{1 = u_k = \dots = u_{\gamma_1} \geqslant u_{\gamma_1+1} = \dots = u_{\gamma_2} \geqslant \dots\}.$$

Introduce the indices $\tau_1, \tau_2, \dots$ such that

$$\{1 = u_k = \dots = u_{\tau_1} > u_{\tau_1+1} = \dots = u_{\tau_2} > \dots\}.$$

Then $\{\tau_1, \tau_2, \dots\}$ is a subset of $\{\gamma_1, \gamma_2, \dots\}$), and hence any segment of the form $I_{\tau_\alpha+1}, \tau_{\alpha+1}$ is the union of segments $I_{\gamma_m+1,\gamma_{m+1}}$.

On any set of the form $\{\tau_\alpha+1, \tau_\alpha+2, \dots, \tau_{\alpha+1}\}$, we introduce the tree partition $\mathfrak{b}_\alpha$ induced by the tree partition $\mathfrak{a}$. The sequence $Z^{(j)}$ induces a sequence $Z^{(j)}_{[\alpha]}$ on any face $F(\mathfrak{b}_\alpha) \subset \mathcal{K}(\tau_\alpha + 1, \tau_{\alpha+1})$.

The sequence $Z^{(j)}$ is said to be convergent if and only if any sequence $Z^{(j)}_{[\alpha]}$ in the Karpelevich polyhedron $\mathcal{K}(\tau_\alpha + 1, \tau_{\alpha+1})$ is convergent.

11.4. **The compactification of the symmetric space by the Karpelevich velocities.** Consider the boundary $\partial\mathcal{K}(1,n) := \mathcal{K}(1,n) \setminus \Sigma(I_{1,n})$ of the polyhedron $\mathcal{K}(1,n)$ and define the compactification $(SL(n,\mathbb{R})/SO(n)) \cup (\partial\mathcal{K}(1,n))$ of the symmetric space $SL(n,\mathbb{R})/SO(n)$. Let $x^{(j)}$ be a sequence in $SL(n,\mathbb{R})/SO(n)$ and let $y \in \partial\mathcal{K}(I_{1,n})$. Then we have $x^{(j)} \to y$ if and only if the following conditions hold:

1) the distance $d(x^{(j)}, 0)$ satisfies the relation $d(x^{(j)}, 0) \to \infty$, and
2) we have $\Lambda(x^{(j)}) \to y$ in the topology of $\mathcal{K}(I_{1,n})$ (where $\Lambda(\cdot)$ is defined by formula (11.1)).

§12. Tits building on the matrix sky

In this section we consider the symmetric spaces $SL(n,\mathbb{R})/SO(n)$ only. Recall that the points of the space $SL(n,\mathbb{R})/SO(n)$ can be identified with real positive definite matrices defined up to a scalar factor.

We recall that any geodesic curve in the space $SL(n,\mathbb{R})/SO(n)$ has the form

$$\gamma(s) = A \begin{pmatrix} \exp(\lambda_1 s) & & & \\ & \exp(\lambda_2 s) & & \\ & & \ddots & \\ & & & \exp(\lambda_n s) \end{pmatrix} A^t, \tag{12.1}$$

where

$$A \in SL(n,\mathbb{R}), \qquad \lambda_1 \geqslant \dots \geqslant \lambda_n. \tag{12.2}$$

In this section, by a *geodesic* we mean an oriented geodesic, without specifying any parametrization.

12.1. **The matrix sky (the visibility boundary).** Consider a noncompact Riemannian symmetric space G/K. Choose a point $x_0 \in G/K$ (for the case under consideration, $G/K = SL(n,\mathbb{R})/SO(n)$, it is natural to take $x_0 = E$). Let T_{x_0} be the tangent space at the point x_0 (in our case, the tangent space is the space of symmetric matrices defined up to a scalar summand, that is, $Q \simeq Q + \lambda E$). Let S be the space of rays in T_{x_0} that start from the origin (that is, $S = (T_{x_0} \setminus 0)/\mathbb{R}^*_+$, where $\mathbb{R}^*_+$ is the multiplicative group of positive reals). Let $v \in S$ and let $\widetilde{v} \in T_{x_0}$ be a tangent vector on the ray v. Let $\gamma_v = \gamma_v(t)$ be the geodesic such that $\gamma_v(0) = x_0$ and $\gamma_v'(0) = \widetilde{v}$. We are not interested in the parametrization of the geodesic γ, but its direction is essential for us.

Let Sk be another copy of the sphere S. The points of the sphere Sk are regarded as points at infinity of G/K. The sphere Sk is called the *matrix sky* or the *visibility boundary*. Let us describe the topology on the space

$$\overline{(G/K)}^{\text{vis}} := G/K \cup Sk.$$

We equip the spaces G/K and Sk with the standard topology. Let y_j be a sequence in G/K. Let $v \in Sk$. Let $\gamma^{(j)}$ be a geodesic joining the points x_0 and y_j. Consider the vectors $v_j \in S$ such that $\gamma^{(j)} = \gamma_{v_j}$. The sequence $y_j \in G/K$ is said to be convergent to a point $v \in Sk$ whenever the following conditions hold:

1) $\rho(x_0, y_j) \to \infty$, and
2) $v_j \to v$ in the natural topology of the sphere Sk.

12.2. **The projection of the matrix sky to the velocity simplex.** Let $G/K = SL(n,\mathbb{R})/SO(n)$. Consider a geodesic γ that starts from $x_0 = E$. Then γ has the form

$$\gamma(s) = A \begin{pmatrix} \exp(\lambda_1 s) & & & \\ & \exp(\lambda_2 s) & & \\ & & \ddots & \\ & & & \exp(\lambda_n s) \end{pmatrix} A^t, \qquad s \in \mathbb{R}, \tag{12.3}$$

where $A \in SO(n)$ and $\lambda_1 \geqslant \dots \geqslant \lambda_n = 0$. Let $\Delta = \Delta_n$ be the simplex $1 \geqslant \mu_2 \geqslant \dots \geqslant \mu_{n-1} \geqslant 0$ (see subsection 11.1). To any geodesic γ we assign the point

$$D(\gamma) : 1 \geqslant \frac{\lambda_2}{\lambda_1} \geqslant \frac{\lambda_3}{\lambda_1} \geqslant \dots \geqslant \frac{\lambda_{n-1}}{\lambda_1} \geqslant 0$$

of the simplex Δ. We obtain a map D from Sk to velocity simplex Δ.

It is clear that $D(\gamma)$ is the limit of the geodesic γ in the simplest velocity compactification of $SL(n,\mathbb{R})/SO(n)$. We say that $D(\gamma) \in \Delta$ is the *velocity of the geodesic* γ.

12.3. **The projection of the matrix sky to the space of flags.** Let $\mathcal{F}$ be the space of all flags

$$0 = V_0 \subset V_1 \subset \dots \subset V_s = \mathbb{R}^n$$

in $\mathbb{R}^n$ ($s = 0, 1, \dots, n$), see §7. Denote by $\mathcal{F}_{\text{complete}}$ the space of complete flags in $\mathbb{R}^n$ (i.e., $s = n$).

Let a geodesic γ be given by the expression (12.3). Assume that the collection $\lambda_1, \dots, \lambda_n$ has the form

$$\lambda_1 = \lambda_2 = \dots = \lambda_{s_1} > \lambda_{s_1+1} = \lambda_{s_1+2} = \dots = \lambda_{s_2} > \dots . \tag{12.4}$$

Let T_α be the subspace in $\mathbb{R}^n$ formed by the vectors $(x_1, \dots, x_{s_\alpha}, 0, 0, \dots)$. Let $V_\alpha = AT_\alpha$ (see (12.3)). By $F(\gamma)$ denote the flag

$$V_1 \subset V_2 \subset V_3 \subset \dots . \tag{12.5}$$

We obtain the mapping $F\colon Sk \to \mathcal{F}$. We can readily see that the geodesic γ is defined by the pair

$$(D(\gamma); F(\gamma)) \in \Delta \times \mathcal{F}.$$

A pair (velocity (12.4), flag (12.5)) is not arbitrary and must satisfy the condition $\dim V_j = s_j$.

12.4. **Limits of geodesics on the matrix sky.** Consider an arbitrary geodesic γ given by (12.1)–(12.2) (generally speaking, $E \notin \gamma$). Introduce a geodesic κ_s joining the points $x_0 = E$ and $\gamma(s)$, where s is a parameter on γ. Let us calculate the limit $\lim_{s\to\infty} \kappa(s)$.

To this end we represent the matrix $A \in GL(n, \mathbb{R})$ in the form $A = UB$, where $U \in O(n)$ and B is an upper triangular matrix. It is easy to prove that the limit of the family of geodesics γ_s is the geodesic $\sigma(t)$ given by the formula

$$\sigma(t) = U \begin{pmatrix} \exp(\lambda_1 t) & & \\ & \ddots & \\ & & \exp(\lambda_n t) \end{pmatrix} U^{-1}.$$

This remark has several simple consequences.

A. *The construction of the matrix sky does not depend on the point* x_0.

Indeed, consider two points x_0 and x_1 and denote the corresponding matrix skies by $Sk(x_0)$ and $Sk(x_1)$. Consider a geodesic γ starting at x_1. Then γ has a limit on $Sk(x_0)$. This defines the canonical mapping $\psi_{10}\colon Sk(x_1) \to Sk(x_0)$. We also have the canonical mapping $\psi_{01}\colon Sk(x_0) \to Sk(x_1)$. We can easily show that $\psi_{01} \circ \psi_{10} = \mathrm{id}$ and $\psi_{10} \circ \psi_{01} = \mathrm{id}$, and we obtain the canonical bijection $Sk(x_0) \leftrightarrow Sk(x_1)$.

B. *In particular, for any point* $x \in G/K$ *and any point* $y \in Sk$, *there exists a unique geodesic joining* x *and* y.

C. *The group* G/K *naturally acts on the space* $(G/K)^{\mathrm{vis}}$.

Indeed, the group G acts on the space of geodesics. □

Note that for any $g \in G$ and any $\gamma \in Sk$ we have

$$D(g \cdot \gamma) = D(\gamma), \qquad F(g \cdot \gamma) = g \cdot F(\gamma).$$

12.5. **A simplicial structure on the matrix sky.** Consider a complete flag $L \in \mathcal{F}_{\mathrm{complete}}$ of the form $L : 0 \subset W_1 \subset \cdots \subset W_{n-1} \subset \mathbb{R}^n$, where $\dim W_j = j$.

Now for each $L \in \mathcal{F}_{complete}$ we will construct a canonical embedding $\sigma_L\colon \Delta \to Sk$. Consider an orthonormal basis $e_1, \dots, e_n \in \mathbb{R}^n$ such that $e_j \in W_j$ and e_j is orthogonal to W_{j-1}. Let $M = \{1 = \mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{n-1} \geqslant \mu_n = 0\}$ be a point of Δ. Consider a family of operators $R(s)$ defined by

$$R(s)e_k = \exp(\mu_k s)e_k$$

Then $R(s)$ is a geodesic and we define $\sigma_L(M)$ as the limit of this geodesic.

The map $\sigma_L\colon \Delta \to Sk$ satisfies the conditions:

1) $D \circ \sigma_L$ is the identity mapping $\Delta \to \Delta$; and
2) the image of the mapping $F \circ \sigma_L\colon \Delta \to \mathcal{F}$ consists of the subflags of the flag L.

We obtain the tiling of the sphere Sk by the simplexes $\sigma_L(\Delta)$. These simplexes are indexed by the points L of the space of *complete* flags. We can readily show that this tiling satisfies the following conditions.

a) Let $g \in SL(n, \mathbb{R})$. Then $\sigma_{gL}(\Delta) = g \cdot \sigma_L(\Delta)$.
b) If $L \neq L'$, then the interiors of the simplexes $\sigma_L(\Delta)$ and $\sigma_{L'}(\Delta)$ are disjoint.

c) Let $L\colon V_1 \subset V_2 \subset \cdots \subset V_{n-1}$ and $L'\colon V_1' \subset V_2' \subset \cdots \subset V_{n-1}'$ be complete flags. If $V_j \neq V_j'$ for all j, then $\sigma_L(\Delta) \cap \sigma_{L'}(\Delta) = \varnothing$. Otherwise the intersection $\Lambda = \sigma_L(\Delta) \cap \sigma_{L'}(\Delta)$ is a common face of the simplexes $\sigma_L(\Delta)$ and $\sigma_{L'}(\Delta)$. Now let us describe the face Λ. Let $\alpha_1, \dots, \alpha_s$ be the indices j such that $V_j = V_j'$ (i.e., $V_{\alpha_i} = V_{\alpha_i}'$ and $V_j \neq V_j'$ for all $j \neq \alpha_i$). Consider the face

$$1 = \lambda_1 = \cdots = \lambda_{\alpha_1} \geqslant \lambda_{\alpha_1+1} = \lambda_{\alpha_1+2} = \cdots = \lambda_{\alpha_2} \geqslant \cdots$$

of the simplex Δ. Then $\Lambda = \Sigma_L(N) = \Sigma_{L'}(N)$.

Now the sphere Sk is endowed with the structure of a Tits building (see [**Tit**]).

12.6. **The Tits metric on the matrix sky.** Let $y_1, y_2 \in \sigma_L(\Delta)$. We introduce the distance $d(y_1, y_2)$ between y_1 and y_2 as the angle between the geodesics x_0y_1 and x_0y_2. Let $z, u \in Sk$. Consider a chain $z = z_1, z_2, \dots, z_\beta = u$ ($z_j \in Sk$) such that for any j, the points z_j, z_{j+1} belong to the same element of our tiling.

Let us define the *Tits metric* $D(\cdot\,,\cdot)$ on Sk by the formula

$$D(z,u) = \inf\left(\sum_j d(z_j, z_{j+1})\right),$$

where the infimum is taken over all chains $z_1, \dots, z_\beta$.

REMARK 12.1. The topology on the sphere Sk introduced by the Tits metric is not equivalent to the standard topology of the sphere.

EXAMPLE 12.2. Let $n = 3$ and $G/K = SL(3,\mathbb{R})/SO(3)$. Then Sk is a 4-dimensional sphere S^4 and $\dim \Delta = 1$, and thus the simplexes $\sigma_L(\Delta)$ are segments. Let us describe the simplicial structure on $Sk = S^4$. Let P be the space of all 1-dimensional linear subspaces of $\mathbb{R}^3$ and Q the space of 2-dimensional subspaces of $\mathbb{R}^3$ (certainly, $P \simeq Q$ are projective planes). We intend to construct a graph, say, Γ. The set of vertices of Γ is $P \cup Q$. Assume that $p \in P$, $q \in Q$, and $p \subset q$. Then p and q are joined by an edge, and all edges have this form. Assume that the length of any edge is $\pi/3$. Then the graph Γ is isometric to the sphere $Sk = S^4$ endowed with the Tits metric.

12.7. **Abel subspaces.** Let A be an orthogonal matrix. Consider the subvariety $R[A] \subset SL(n,\mathbb{R})/SO(n)$ that consists of all matrices of the form

$$\psi_A(s_1, \dots, s_{n-1}) = A \begin{pmatrix} \exp(s_1) & & & & \\ & \exp(s_2) & & & \\ & & \ddots & & \\ & & & \exp(s_{n-1}) & \\ & & & & 1 \end{pmatrix} A^{-1},$$

where $s_1, \dots, s_{n-1} \in \mathbb{R}$.

The mapping $(s_1, \dots, s_{n-1}) \mapsto \psi(s_1, \dots, s_{n-1})$ is an isometric embedding

$$\mathbb{R}^{n-1} \to SL(n,\mathbb{R})/SO(n)$$

(with respect to the standard metrics in $\mathbb{R}^{n-1}$ and in $SL(n,\mathbb{R})/SO(n)$).

Consider the trace $S[A]$ of the space $R[A]$ on the surface Sk. It is obvious that $S[A]$ is the union of $(n-1)!$ simplexes $\sigma_L(\Delta)$. These simplexes are separated by the hyperplanes $s_i = s_j$.

§13. Hybridization: The Dynkin–Olshanetsky and Karpelevich boundaries

13.1. **Hybridization.** Let $i_1: G/K \to X$ and $i_2: G/K \to Y$ be embeddings of a symmetric space G/K into compact metric spaces X and Y. Let the images of these embeddings in X and Y be dense.

Consider the embedding $i_1 \times i_2: G/K \to X \times Y$ defined by the formula $h \mapsto (i_1(h), i_2(h))$, where $h \in G/K$. Let Z be the closure of the image of G/K in $X \times Y$. Then Z is a new compactification of G/K. We say that Z is the *hybrid* of X and Y.

Now we apply this construction for the case in which X is a velocity compactification and Y is the Satake–Furstenberg compactification.

13.2. **The Dynkin–Olshanetsky boundary.** Consider the hybrid Z of the simplest velocity compactification (see subsection 11.1) and the Satake–Furstenberg compactification for some noncompact Riemannian symmetric space. We again restrict ourselves to the case $G/K = SL(n, \mathbb{R})/O(n)$.

A point of the space Z is given by the following data:

0^* $s \in \{1, \dots, n-1\}$;

1^* a hinge $\mathcal{P} = (P_1, \dots, P_s)$ such that $P_j = P_j^t$ and P_j are nonnegative definite for all $j \in \{1, \dots, s\}$ (see §9); and

2^* a point of the simplex Δ_s of the form $1 \geqslant \mu_2 \geqslant \dots \geqslant \mu_{s-1} \geqslant 0$.

Let $x^{(j)} \in SL(n, \mathbb{R})/O(n)$ be an unbounded sequence and $a_1^{(j)} \geqslant \dots \geqslant a_n^{(j)}$ the set of eigenvalues of $x^{(j)}$. Let $\lambda_\alpha^{(j)} = \ln a_\alpha^{(j)}$. Then the point $\Lambda(x^{(j)}) := (\lambda_1^{(j)}, \lambda_2^{(j)}, \dots)$ is a point of the simplicial cone Σ_n (see subsection 12.1), and the sequence $x^{(j)} \in SL(n, \mathbb{R})/O(n)$ converges in Z if and only if $x^{(j)}$ converges in the Furstenberg–Satake compactification and $\Lambda(x^{(j)})$ converges in the velocity simplex $\overline{\Sigma}_n = \Sigma_n \cup \Delta$.

Now let us calculate the data 0^*–2^* corresponding to the limit $\lim x^{(j)}$. Let $\mathcal{P} = (P_1, \dots, P_s)$ be the limit of $x^{(j)}$ in the Satake–Furstenberg compactification. Let $\gamma_j = \dim \operatorname{Im}(P_j)$. Let $(\tau_2, \dots, \tau_{s-1})$ be the limit of $\Lambda(x^{(j)})$ in the simplex Δ. Then the set $\tau_2 \geqslant \tau_3 \geqslant \dots$ has the form

$$1 = \tau_1 = \dots = \tau_{\gamma_1} \geqslant \tau_{\gamma_1+1} = \dots = \tau_{\gamma_2} \geqslant 4.1 \cdots . \tag{13.1}$$

We assume that

$$\mu_j := \tau_{\gamma_{j-1}+1} = \dots = \tau_{\gamma_j} \tag{13.2}$$

and obtain data of the form 0^*–2^*.

13.3. **The projection of the Dynkin–Olshanetsky boundary to the matrix sky.** Let the data 0^*–2^* be given. Consider the new data:

1^+ the flag $\operatorname{Ker}(P_1) \supset \operatorname{Ker}(P_2) \supset \operatorname{Ker}(P_3) \supset \cdots$; and

2^+ the set of numbers $\tau_2, \dots, \tau_{s-1}$ defined by formula (13.2).

These data define a point of the matrix sky (see subsections 12.2–12.3).

13.4. **Limits of geodesics.** Let us consider a geodesic given by

$$\gamma(s) = A \begin{pmatrix} \exp(\lambda_1 s) & & & \\ & \exp(\lambda_2 s) & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} A^t,$$

where $A \in SL(n,\mathbb{R})$ and $\lambda_1 \geqslant \cdots \geqslant \lambda_n = 0$. Let $\mathcal{P} = (P_1, \ldots, P_s)$ be the limit of γ in the space of hinges, and let the limit of γ in the velocity simplex Δ_n be $\tau_2, \ldots, \tau_{n-1}$, where $\tau_j = \lambda_j/\lambda_1$.

Let $\gamma_\alpha = \dim \operatorname{Im}(P_\alpha)$. We introduce the numbers $\mu_\alpha := \tau_{\gamma_{\alpha-1}+1} = \cdots = \tau_{\gamma_\alpha}$ Now we obtain data of the form 13.2 0^*–2^*.

REMARK 13.1. Some points of the Dynkin–Olshanetsky boundary are not limits of geodesics. For example, the point defined by the data 13.2 0^*–2^* is the limit of a geodesic if and only if $1 > \mu_2 > \cdots > \mu_{n-1} > 0$.

13.5. **The Karpelevich compactification.** The Karpelevich compactification is the hybrid of the compactification by the Karpelevich velocities and of the Satake–Furstenberg compactification. Namely, a boundary point of the Karpelevich compactification is given by following data:

0^* $s \in \{2, \ldots, n-1\}$;

1^* a hinge $\mathcal{P} = (P_1, \ldots, P_s)$ such that $P_j = P_j^t$ are positive definite for all $j \in \{1, \ldots, s\}$ (see §10); and

2^* a point of the boundary of the polyhedron $\mathcal{K}(1,s)$ of Karpelevich velocities (see subsection 11.2).

The topology on the Karpelevich compactification can be defined in an obvious way. The natural projection $\partial\mathcal{K}(1,s) \to \Delta(I_{1,s})$ defines a projection of the Karpelevich boundary to the Dynkin–Olshanetsky boundary.

§14. The space of geodesics and sea urchins

14.1. **The space of geodesics.** Consider the noncompact Riemannian symmetric space $G/K = SL(n,\mathbb{R})/SO(n)$. Denote by $\mathfrak{G}$ the space of all oriented geodesics in G/K.

Here the definition of the topology on the space $\mathfrak{G}$ is rather delicate. We shall describe the topology that seems to be the most natural. Consider a collection of integers $A = (\alpha_0, \ldots, \alpha_\sigma)$ such that $1 = \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_\sigma = n$. Denote by $\Delta(A)$ the open simplex

$$1 = \lambda_1 = \cdots = \lambda_{\alpha_1} > \lambda_{\alpha_1+1} = \cdots = \lambda_{\alpha_2} > \cdots > \lambda_{\alpha_{\sigma-1}+1} = \cdots = \lambda_n = 0\,.$$

For different A, the simplexes $\Delta(A)$ are disjoint, and the union $\cup_A \Delta(A)$ coincides with the simplex Δ_n. Consider a geodesic $\gamma \in \mathfrak{G}$. The velocity of this geodesic belongs to some simplex $\Delta(A)$. The space of all geodesics with given velocity $\Lambda \in \Delta(A)$ is an $SL(n,\mathbb{R})$-homogeneous space. The stabilizer $G(A)$ of the geodesic γ (considered up to conjugacy) depends only on the set A (and depends neither on Λ nor on the geodesic itself): $G(A) = \mathbb{R}_+^* \times \prod O(\alpha_{j+1} - \alpha_j)$. Denote by $\mathfrak{G}(A)$ the space of all geodesics whose velocities are elements of $\Delta(A)$. Then

$$\mathfrak{G}(A) \simeq \Delta(A) \times SL(n,\mathbb{R})/G(A)\,.$$

We endow this space with the standard direct product topology and the space

$$\mathfrak{G} = \bigcup_A \mathfrak{G}(A) \simeq \bigcup_A \Delta(A) \times SL(n,\mathbb{R})/G(A)$$

with the topology of disjoint union.

REMARK 14.1. Thus, the space of geodesics is disconnected. This is natural. Indeed, let $A_0 = \{0, 1, \dots, n\}$. Consider the set of limits of the geodesic $\gamma \in \mathfrak{G}(A_0)$ on the matrix sky. Then this set is open and dense. The set of limits of $\gamma \in \mathfrak{G}(A)_0$ on the Satake–Furstenberg boundary is compact. Namely, it is the minimal compact $SL(n, \mathbb{R}^n)$-orbit on the boundary.

14.2. **The space of geodesics as a boundary of the symmetric space.** Let us define a natural topology on the space $\mathfrak{R} = G/K \cup \mathfrak{G}$. We equip the space G/K with the standard topology. The space $\mathfrak{G}$ is equipped with the above topology, and thus the space $\mathfrak{G}$ is closed in $\mathfrak{R}$. Choose a point $b_0 \in G/K$. Let $x_j \in G/K$ be an unbounded sequence. The sequence x_j converges in $\mathfrak{R}$ if and only if it satisfies the following conditions:

1) the sequence of geodesics $b_0 x_j$ converges, and we denote by y the limit of this sequence on the matrix sky; and
2) there exists a limit z of the sequence of geodesics $x_j y$.

By the limit of the sequence x_j we mean the geodesic z.

REMARK 14.2. In our case, the dimension $\dim \mathfrak{G}$ of the boundary is given by the formula $\dim \mathfrak{G} = 2 \dim G/K - 2$, which is greater than $\dim G/K$ (even in the case $G/K = SL(2, \mathbb{R})/SO(2)$, i.e., the Lobachevsky plane).

REMARK 14.3. The space $\mathfrak{R}$ is not compact (since $\mathfrak{G}$ is not compact)

14.3. **Sea urchins.** Recall that for any geodesic $\gamma \in \mathfrak{G}$ we can define a velocity $\{\mu_2, \mu_3, \dots\}$, which is a point of the simplex Δ (see subsection 11.1). We denote by $\mathfrak{G}^{\mathrm{rat}}$ the space of geodesics with rational velocities (i.e., the numbers μ_j are rational). Consider the so-called *sea urchin*, i.e., the set $\mathfrak{R}^{\mathrm{rat}} := G/K \cup \mathfrak{G}^{\mathrm{rat}} \subset \mathfrak{R}$. We are not interested in the topology on the sea urchin (it seems natural to endow the set of velocities with the discrete topology, consider the ordinary topology on the space of geodesics with a given velocity, and introduce the natural (see subsection 14.2) convergence of sequences in G/K to geodesics).

14.4. **Projective universality.** Let ρ_j be a finite family of linear irreducible representations of the group G in the spaces V_j. We assume that for any j there exists a nonzero K-fixed vector $v_j \in V_j$. Consider the direct sum $\rho = \bigoplus \rho_j$ of representations ρ_j and take the vector $w = \bigoplus v_j \in \bigoplus V_j$. Let $\mathcal{O} \simeq G/K$ be the G-orbit of the vector $w \in \mathbb{P}(\mathcal{O})$ and $\overline{\mathcal{O}}$ the closure of $\mathcal{O}$ in the projective space $\mathbb{P}(\bigoplus V_j)$. The G-spaces $\overline{\mathcal{O}}$ are called the *projective compactifications* of G/K.

Now we construct a mapping $\pi \colon \mathfrak{R} = G/K \cup \mathfrak{G} \to \overline{\mathcal{O}}$. The mapping $G/K \to \mathcal{O}$ is natural. Let us consider a geodesic $\rho(s) \in \mathfrak{G}$. We can readily prove that the limit $\lim_{s \to \infty} \pi(\gamma(s))$ in $\mathbb{P}(\bigoplus V_j)$ exists. By definition, $\pi(\gamma)$ is this limit.

PROPOSITION 14.4. a) *The mapping* $\pi \colon \mathfrak{R} \to \overline{\mathcal{O}}$ *is surjective.*
b) *Moreover, the* π*-image of the sea urchin* $\mathfrak{R}^{\mathrm{rat}}$ *is the entire space* $\overline{\mathcal{O}}$.

§15. The boundary of a Bruhat–Tits building

An analog of a Riemannian symmetric space in the p-adic case is the so-called Bruhat–Tits building. We discuss an example.

15.1. **An analog of the Satake–Furstenberg boundary.** Let $\mathbb{Q}_p$ be p-adic field, $\mathbb{Z}_p$ be the ring of p-adic integers, and let $\mathbb{Q}_p^* = \mathbb{Q}_p \setminus 0$ be the multiplicative group of $\mathbb{Q}_p$.

By a *lattice* in the linear space $\mathbb{Q}_p^n$ we mean an open compact $\mathbb{Z}_p$-submodule of $\mathbb{Q}_p^n$ (any lattice has the form $\mathbb{Z}_p v_1 \oplus \cdots \oplus \mathbb{Z}_p v_n$, where $v_1, \ldots, v_n$ are linearly independent vectors). The group $\mathbb{Q}_p^*$ acts by dilations on the set Lat_n of all lattices in $\mathbb{Q}_p$. The *Bruhat–Tits building* Ens_n is the quotient space $Ens_n = Lat_n/\mathbb{Q}_p^*$.

The space Ens_n possesses a beautiful geometric structure, which is not essential for our purposes. We only note that the group $PGL_n(\mathbb{Q}_p)$ acts on Ens_n in an obvious way and the stabilizer of the lattice $(\mathbb{Z}_p)^n \subset \mathbb{Q}_p^n$ is the group $PGL_n(\mathbb{Z}_p)$, that is, $Ens_n = PGL_n(\mathbb{Q}_p)/PGL_n(\mathbb{Z}_p)$.

We shall describe the analog of the Satake boundary for Ens_n.

To this end we embed Lat_n into the space Sub_n of all $\mathbb{Z}_p$-submodules in $\mathbb{Q}_p^n$. Introduce some natural metric on Sub_n. For example, any submodule under consideration can be regarded as a closed subset of $\mathbb{P}\mathbb{Q}_p^n$, and thus the Hausdorff metric on $[\mathbb{P}\mathbb{Q}_p^n]$ induces a metric on Sub_n.

Furthermore, we apply the construction of the Hausdorff quotient space to the compact space Sub_n, its open subset Lat_n, and the orbits of the group $\mathbb{Q}_p^*$. Then the limit sets in $Sub_n/\mathbb{Q}_p^*$ are as follows: they are the collections

$$0 = L_0 \subset M_1 \subset L_1 \subset \cdots \subset L_{k-1} \subset M_k \subset L_k = \mathbb{Q}_p^n,$$

where for each j, the L_j is a linear subspace and M_j/L_{j-1} is a lattice in L_j/L_{j-1} (defined up to a scalar factor). We call these collections *filtered flags*.

15.2. **Velocity compactification.** A pair of lattices $R, T \subset \mathbb{Q}_p^n$ can be written in the form of the direct sums

$$R = \mathbb{Z}_p v_1 \oplus \mathbb{Z}_p v_2 \oplus \cdots \oplus \mathbb{Z}_p v_n, \qquad T = p^{k_1}\mathbb{Z}_p v_1 \oplus p^{k_2}\mathbb{Z}_p v_2 \oplus \cdots \oplus p^{k_n}\mathbb{Z}_p v_n,$$

where $k_1 \geqslant \cdots \geqslant k_n$. The set $D(R,T) = (k_1, \ldots, k_n)$ is called the *complex distance* between R and T. The complex distance is an invariant of a pair of lattices with respect to the action of the group $GL(n, \mathbb{Q}_p)$ on the space of lattices, that is, $D(g \cdot R, g \cdot T) = D(R,T)$ for $R, T \subset \mathbb{Q}_p^n$ and $g \in GL(n, \mathbb{Q}_p)$. The vertices of the buildings are lattices defined up to a factor, and hence for a Bruhat–Tits building, the complex distance is defined up to an additive constant. Now we can repeat both constructions of §11.

15.3. **Hybrids.** We can repeat both constructions of §13 as well. A point of the *analog of the Dynkin–Olshanetsky boundary* of a building Ens_n is given by the following data:

- 0^* $s \in \{1, \ldots, n-1\}$;
- 1^* a filtered flag $0 = L_0 \subset M_1 \subset L_1 \subset M_2 \subset \cdots \subset L_s \subset M_s \subset L_{s+1} = \mathbb{Q}_p^n$, where L_j are subspaces and M_j are $\mathbb{Z}_p$-submodules (defined up to a factor) such that M_j/L_j are lattices in L_{j+1}/L_j; and
- 2^* a point of the simplex Δ_s (see subsection 11.1).

A point of the *analog of the Karpelevich boundary* is given by similar data. namely, by data of the form 0^* and 1^* together with data of the form

- 2^{**} a point of the polyhedron $\mathcal{K}(1, s)$ of Karpelevich velocities.

15.4. **The lattice sky.** A point of the *lattice sky* is given by the following data:

- $0^\dagger$ $s \in \{1, \ldots, n-1\}$;
- $1^\dagger$ a flag $0 = L_0 \subset L_1 \subset \cdots \subset L_{s+1} = \mathbb{Q}_p^n$; and
- $2^\dagger$ a point of the simplex Δ_s.

There exist a natural tiling of the lattice sky into simplexes. The simplexes $\Delta_{\mathcal{L}}$ are enumerated by the complete flags

$$\mathcal{L}:\ 0 = L_0 \subset L_1 \subset \cdots \subset L_n = \mathbb{Q}_p^n .$$

Let $\mathcal{A}(\mathcal{L})$ be the set of all nontrivial subflags of the flag $\mathcal{L}$. Then the simplex $\Delta_{\mathcal{L}}$ consists of all elements of the lattice sky such that the data 1† are elements of $\mathcal{A}(\mathcal{L})$ (the faces of the simplex $\Delta_{\mathcal{L}}$ are enumerated by the subflags of the flag $\mathcal{L}$).

§16. Remarks

16.0. **Remarks on §0.** α) The five conics problem was formulated and (incorrectly) solved by Steiner in 1848; in 0.2 we followed this solution. The correct answer and a correct solution were obtained by Chasles in 1864 [**Chas**]. A similar problem on the number of quadrics in $\mathbb{C}P^3$ that are tangent to nine generic quadrics was solved by Schubert [**Schu**]; this number is equal to

$$666841088 . \tag{16.1}$$

For the history of the problem, see [**Kle**].

β) The variety of "complete conics" was introduced by Study [**Stu**]. For the discussion on this variety, see [**Sev**]. A similar completion of $PGL_n(\mathbb{C})/PO_n(\mathbb{C})$ (i.e., of the space of quadrics in $\mathbb{C}P^{n-1}$) was constructed by Semple [**Sem1, Sem3**]. This variety is called that of "complete quadrics", see our §8.

For a discussion of the five conics problem from another point of view, see [**GH,** VI.1].

For the calculation of the number (16.1) and for other problems of the same type, see [**SR,** Chapter XI] and also [**DCP1**].

16.1. **Remarks on §1.** The category GA and its fundamental representation are defined in [**Ner1**]; see also [**Ner2, Ner3**]. For the categories B, C, and GD and for the theory of representations for the categories GA, B, C, and GD, see [**Ner2, Ner3, Ner6**].

16.2. **Remarks on §2.** α) This section contains an exposition of the preprint [**Ner5**].

β) The Hausdorff metric was introduced in [**Pom, Hau**].

γ) The standard compactifications of $\mathbb{R}$ are those by means of the points $+\infty$ and $-\infty$ and by means of the singleton ∞ (or $\pm\infty$). Some other compactifications of $\mathbb{R}$ can be obtained as follows: we can embed $\mathbb{R}$ into a compact metric space and take the closure of the image (for instance, we can take a winding of a torus or the curve $r = (\pi/2 + \arctan t)$, $\varphi = t$ on $\mathbb{R}^2$).

An example of a nontrivial and interesting compactification of $\mathbb{R}^n$ is given by the closure of the set of diagonal matrices in the model of §13 for the Karpelevich compactification.

δ) To 2.6. Certainly, the space $(\mathbb{C}^*)^k$ has many nontrivial compactifications. By a *toric variety* we mean a complex algebraic variety L, possibly singular, on which the group $(\mathbb{C}^*)^k$ (the "complex torus") acts in such a way that one of the orbits of $(\mathbb{C}^*)^k$ on L is open.

Let the torus $(\mathbb{C}^*)^k$ act by linear transformations on the space $\mathbb{C}^N$. Then the closures of orbits of $(\mathbb{C}^*)^k$ in $\mathbb{C}P^{N-1}$ are toric varieties, and all toric varieties have this form.

Our space $\overline{\mathbb{T}^{n-1}}$ is an example of such toric variety. For details on toric varieties, see the review [**Dan**].

Certainly, in the construction of 2.6, we can replace the group $\mathbb{C}^*$ by $\mathbb{R}^* = \mathbb{R} \setminus 0$ (in this case we obtain a smooth analytic real variety) or by the multiplicative group of positive reals.

ε) Instead of the words "the closure in the Hausdorff metric", it is customary to speak of "the closure in the Hilbert scheme or Chow scheme". The latter operation is more refined but harder to visualize [**BiS, Kap**]; see [**Kap**] for the construction of a separated quotient of the Grassmannian in $\mathbb{C}^n$ by the action of the torus $\mathbb{C}^*n$.

16.3. **Remarks on §3.** α) The author believes that the language of 3.3 was first used in [**Alg**]. Hinges (under the title "resolving sequences") were introduced in [**Ner4**].

β) The description of the closure of a conjugacy class in $PGL_n(\mathbb{C})$ in the variety Hinge_n seems to be of interest.

16.4. **Remarks on §4.** α) The smooth algebraic variety Hinge_n was first constructed by Semple [**Sem2**] in 1951 as a natural generalization of "complete quadrics". The Semple construction is as follows.

To any operator $A \in PGL_n(\mathbb{C})$ one assigns the collection of operators

$$A = \Lambda^1 A, \Lambda^2 A, \ldots, \Lambda^{n-1} A, \tag{16.2}$$

that are defined up to a factor, in the exterior powers $\Lambda^j \mathbb{C}^n$ of the space $\mathbb{C}^n$. This collection can be regarded as a point of the product of projective spaces

$$\prod_{j=1}^{n-1} \mathbb{P}(\Lambda^j V) = \prod_{j=1}^{n-1} \mathbb{C}\mathbb{P}^{C_n^j - 1}. \tag{16.3}$$

Semple defined the variety S_n of "complete collineations" as the closure of the image of the group $PGL_n(\mathbb{C})$ in (16.3). Points of this closure are called *Semple complexes.* In our language, Semple complexes are collections of operators of the form $(\lambda_1(\mathcal{P}), \ldots, \lambda_{n-1}(\mathcal{P}))$, where $\mathcal{P}$ is a hinge. In this connection, the varieties S_n and Hinge_n coincide. For the variety S_n, see also [**Tyr, Lak, DCP1**].

β) The papers of Semple [**Sem1, Sem2, Sem3**] are among the first applications of the construction of the *closure of an equivariant embedding.* Let a group G act on a (noncompact) space M and on a compact space N. Let an embedding $\tau \colon M \to N$ be given such that $\tau(g \cdot m) = g \cdot (\tau m)$ (such embeddings are said to be *equivariant*). Furthermore, consider the closure $\overline{\tau(M)}$ of the image of the mapping τ in N. It turns out that a number of meaningful and nontrivial objects can be obtained in this way.

In the case of "complete collineations," the group G is $PGL_n(\mathbb{C}) \times PGL_n(\mathbb{C})$, the space M is the group $PGL_n(\mathbb{C})$ (the group $G = PGL_n(\mathbb{C}) \times PGL_n(\mathbb{C})$ actually acts on $PGL_n(\mathbb{C})$ by left and right multiplications), and N is given by (16.3). The equivariant embedding is defined by formula (16.2).

Let us present some examples of this construction that differ from the examples above and from the examples below.

1*. *Bohr compacta* (see [**Dix, Rup**]). Consider the product N of a continual set of circles $S^1 = \{z : |z| = 1\}$ endowed with the Tychonoff topology. It is natural to regard the elements of this space as functions $f \colon \mathbb{R} \to S^1$ (without any conditions of continuity and measurability on the functions f). Clearly, N is a group with respect to the pointwise multiplication. Furthermore, the group $\mathbb{R}$ is embedded into N by the rule $s \mapsto F_s(t) = e^{ist}$, $t \in \mathbb{R}$. The *Bohr compactum* is the closure of the image of the group $\mathbb{R}$ in N. It seems that this object can hardly be described in reasonable language, and it is pathological in this sense. On the other hand, sometimes it plays an auxiliary role in very useful and reasonable discussions; see [**Rup**].

2^*. *Thurston boundary* (see [**Thu, MS**]). Let Γ_g be the fundamental group of a sphere C_g with g handles. Let Δ_g be the set of homomorphisms from Γ_g into $PSL_2(\mathbb{R})$ whose images are lattices in $PSL_2(\mathbb{R})$ (defined up to conjugation in $PSL_2(\mathbb{R})$) (this set is called the Teichmüller space). Let G be the group of automorphisms of the group Γ_g. Let N be the space of real functions on Γ_g defined up to a factor. We endow this space with the topology of pointwise convergence. Further, to any homomorphism $\Pi\colon \Gamma_g \to PSL_2(\mathbb{R})$ we assign the function $\alpha_\pi(\gamma) = \log \operatorname{tr} \pi(\gamma)$, $\gamma \in \Gamma_g$, regarded as an element of N.

The Thurston compactification of the Teichmüller space is the closure of the image of Δ_g under this mapping in the projective space N.

3^*. *Olshanski mantles* [**Ols, Ner3, Ner6**]. Let $M = G$ be an infinite-dimensional group. Let $N = \mathcal{B}(H)$ be the set of linear operators in a Hilbert space H with norm $\leqslant 1$. The set $\mathcal{B}(H)$ is compact with respect to the weak operator topology.

Let ρ be a unitary representation of the group G in the space H. Then ρ can be regarded as a mapping of G into $\mathcal{B}(H)$. Let $\Gamma(G,\rho)$ (a mantle of the group G) be the closure of the image of this mapping. We can readily see that $\Gamma(G,\rho)$ is a semigroup. It turns out that in many cases this semigroup can be described explicitly, and the answer is often nontrivial.

γ) The semigroup $\widetilde{\mathrm{Hinge}}_n$ is defined in [**Ner4**]. Apparently, the semigroup $\widetilde{\mathrm{Hinge}}_n$ coincides with that constructed in [**Vin**]; see also [**Pop**].

16.5. **Remarks on §5.** None.

16.6. **Remarks on §6.** α) The construction of 6.2 is a special case of that introduced by de Concini–Procesi [**DCP1**]; see 16.8. It is an example of the closure of an equivariant embedding. Namely, the $PGL_n \times PGL_n$-homogeneous space PGL_n can be embedded in the projectivized space of operators in $V(a)$, and we take the closure of the image.

The construction of 6.3 was obtained in [**Ner4**].

β) Putcha and Renner (see [**Put, Ren**]) studied the following problem. Let G be a complex semisimple Lie group and let ρ be its finite-dimensional holomorphic representation in a space V (in general, reducible). Consider the set $\mathbb{C}\cdot\rho(G)$ of operators of the form $\lambda\cdot\rho(g)$, where $g \in G$ and $\lambda \in \mathbb{C}$. Let $\Gamma(\rho)$ be the closure of the set $\mathbb{C}\cdot\rho(G)$ in the space of all operators on the space V. Clearly, $\Gamma(\rho)$ is a semigroup. It turns out that the semigroups $\Gamma(\rho)$ can be nonisomorphic for different ρ. To be more precise, $\Gamma(\rho) \simeq \Gamma(\rho')$ if the convex hull $Conv(\rho)$ of the weights of the representation ρ can be obtained by a dilation from the convex hull $Conv(\rho')$ of the weights of the representation ρ'.

For the case $G = GL_n$, the semigroup $\widetilde{\mathrm{Hinge}}_n$ is a universal object for almost the same problem. Namely, for all *irreducible* representations ρ, the quotient semigroup $\Gamma(\rho)/\mathbb{C}^*$ is a homomorphic image of the semigroup $\widetilde{\mathrm{Hinge}}_n$.

On the other hand, it seems probable (I do not know the proof) that for any representation ρ (not necessarily irreducible), the semigroup $\Gamma(\rho)$ coincides with the image of the semigroup $\widetilde{\text{Hinge}}_n$ (in all events, a semigroup $\Gamma \supset GL_n$ with this property is constructed in [**Vin**]).

16.7. **Remarks on §7.** We follow [**Ner4**]. The described construction is apparently a special case of some general construction that holds for all homogeneous spaces; see [**Vin, Pop**].

16.8. **Remarks on §8.** α) *De Concini–Procesi construction.* Complete symmetric varieties were introduced by de Concini and Procesi [**DCP1**] in 1983. Their construction (which is the same for all complex symmetric spaces) is as follows.

Let G/H be a symmetric space, where G and H are complex groups. Let π be a spherical representation of G, that is, an irreducible representation of G that has a nonzero H-stable vector, say, v. The description of all these representations is given by the well-known Helgason theorem ([**Hel1**]; see also [**Hel2,** V.4.1]).

Let $\mathbb{P}_\pi$ be the projectivized representation space of π. Let $\overline{(G/H)_\pi}$ be the closure of the G-orbit of the vector v in $\mathbb{P}_\pi$. Then $\mathbb{P}_\pi$ is a compactification of the symmetric space G/H.

This construction *a priori* depends on π. We say that π is nondegenerate if all numerical labels of π on the Dynkin diagram that can be nonzero (for spherical representations) are nonzero. It turns out that for all nondegenerate irreducible representations π, the varieties $\overline{(G/H)_\pi}$ are isomorphic. Moreover, for all G/H these are smooth complex varieties.

The complete collineations (i.e., elements of the completion of the space $PGL_n \times PGL_n/PGL_n$) fit within the framework of the construction above as follows. Let ρ be a holomorphic representation of GL_n and let ρ' be the dual representation. Consider the representation $\rho \otimes \rho'$ of the group $GL_n \times GL_n$. This is just the representation of $GL_n \times GL_n$ in the space of linear operators in the representation space of ρ (see §6).

For the properties of complete symmetric varieties, see [**DCP1, DCP2, DCS**].

β) Hinge constructions react rather painfully to modifications of G/H that seem to be unessential from any other point of view. For example, the completion of the group SO_n [**Ner4**] can be described in the language of hinges in a much more cumbersome way than the completion $\overline{O_n}$ described in §8.

γ) *Projective compactifications.* Let $\rho_1, \dots, \rho_k$ be irreducible H-spherical representations of the group G in some spaces $V_1, \dots, V_k$, and let $v_j \in V_j$ be their spherical vectors. Let $\rho = \bigoplus \rho_j$, $V = \bigoplus V_j$, and $v = \bigoplus v_j \in V$. Consider the G-orbit of the vector v in the projective space $\mathbb{P}V$. Denote by $\mathbb{P}_\rho$ the closure of the orbit Gv in $\mathbb{P}V$.

It turns out that for different representations ρ, the varieties $\mathcal{P}_\rho$ are, in general, different; as a rule, they are not smooth.

Such objects were studied quite intensively, for example, see [**Vus, CX, Kus3**]. We also note that the constructions of this type are possible for any (in general, not symmetric) subgroup H, see [**LV**].

δ) *Closure in the Grassmannian in the adjoint representation.* Consider an irreducible representation ρ of a simple complex linear group in a space V. For the sake of simplicity we assume that on the space V, a nondegenerate symmetric bilinear form $B(\,\cdot\,,\,\cdot\,)$ is given. Consider the symmetric bilinear form $\widetilde{B}((v,w);(v',w')) = B(v,v') - B(w,w')$, $v, v', w, w' \in V$, on $V \oplus V$. Denote by $\mathrm{Gr} = \mathrm{Gr}(V \oplus V)$ the

Grassmannian of maximal isotropic subspaces of $V \oplus V$. To any element $g \in G$ we assign the graph of the operator $\rho(g)$. Thus, we obtain an embedding of G in Gr. The problem of describing the closure of G in Gr naturally arises.

Now we show that this problem can be reduced to that of item γ. By [**Ner6**, Chap. 2], the spinor representation Spin of the group $O(V)$ can be extended to a continuous embedding of $\mathrm{Gr}(V \oplus V)$ into the projectivized space of operators. Therefore, our problem is reduced to the problem of describing the closure of the group G in the representation $\mathrm{Spin} \circ \rho$.

Now we consider the case where ρ is the adjoint representation Ad of the group G. According to [**Kos1, Kos2**], for any simple Lie group G, the representation $\mathrm{Spin} \circ \mathrm{Ad}$ is the sum of equal representations of the group G whose highest weight is the half-sum of the positive roots. Therefore, the closure of G in the representation $\mathrm{Spin} \circ \mathrm{Ad}$ and in the Grassmannian $\mathrm{Gr}(\mathfrak{g} \oplus \mathfrak{g})$ coincides with the De Concini–Procesi compactification (with the complete symmetric variety) of the group G.

16.9. **Remarks on §9.** α) The Satake construction [**Sat**] for the compactification of a Riemannian symmetric space G/K (where G is a simple Lie group and K is a compact subgroup) given in 1960 is precisely the real version of the construction of complete symmetric varieties. We consider a finite-dimensional K-spherical irreducible representation ρ of the group G, and then take the closure of the orbit of a spherical vector in the projective space. Let ρ be nondegenerate in the same sense as in 16.8, α). It turns out that, in this case, the obtained compactification does not depend on ρ.

If ρ can be degenerate, then in this way we can obtain $2^{\mathrm{rk}(G/K)} - 1$ different compactifications, which are also called Satake compactifications (as in our Remark 6.6). The ordinary compactifications, for example, the matrix ball [**Pya**, Chapter II; **Ner2**] belong to the family of Satake compactifications.

For pseudo-Riemannian symmetric spaces (see the list in [**Ber**]), we can repeat the Satake construction.

Oshima and Sekiguchi [**Osh, OS, Schl**] also studied the gluing of compactifications of this type into smooth analytic varieties. In [**Osh**] (see also [**Schl**]) for a Riemannian symmetric space G/K, $2^{\mathrm{rk}(G/K)}$ counterparts of Satake compactifications are glued together. In [**OS**] different real forms G/K_α of the same symmetric space are glued together. The construction [**OS**] is much more involved than that of §9, and the relationship between these constructions remains unclear.

β) The same compactifications of Riemannian symmetric spaces, by a substantially different method, were obtained by Furstenberg [**Fur1, Fur2**].

16.10. **Remarks on §10.** A more general "hinge superstructure" over the category GA was discussed in [**Ner4**]. Similar "hinge superstructures" exist for the categories B, C, and GD from 1.7.

16.11. **Remarks on §11.** α) I have not seen these constructions in the literature. For the discussion of the closures of Weyl chambers, see [**Tay**]. There exist many other compactifications of the simplicial cone Σ_n, and hence there exist many other velocity compactifications of symmetric spaces. Some examples are given below in β) and γ). An analog of the set $(\ln a_1, \dots, \ln a_n)$ for an arbitrary noncompact Riemannian symmetric space is the complex distance (for example, see [**Ner6**, 6.3]).

β) *Satake velocities.* Here we will discuss the closure of a Weyl chamber in the Satake compactification (see also §2). Consider a partition $\mathfrak{t}$ of $I_{1,n}$, where $I_{1,n} =$

$I_{1,\alpha_1} \cup I_{\alpha_1+1,\alpha_2} \cup \cdots \cup I_{\alpha_{s-1},n}$, and assign to t the *face* $G(\mathfrak{t}) = \prod_m \Sigma(I_{\alpha_m+1,\alpha_{m+1}})$. We define the polyhedron of Satake velocities by the relation $\mathcal{S}(n) = \cup_{\mathfrak{t}} G(\mathfrak{t})$. Consider a sequence $h^j = \{h_1^{(j)} \geqslant \cdots \geqslant h_n^{(j)} = 0\} \in \Sigma(I_{1,n})$. The sequence $h^{(j)}$ is convergent to a point

$$u = (u_1, \dots, u_s) \in \prod_m \Sigma(I_{\alpha_m+1,\alpha_{m+1}}) = G(\mathfrak{t})$$

if there exist sequences $p_1^{(j)}, p_2^{(j)}, \dots$ such that

1) for all σ we have $\lim_{j\to\infty}(p_\sigma^{(j)} - p_{\sigma+1}^{(j)}) = +\infty$, and
2) for all σ, the sequences

$$h_{[\alpha]}^{(j)} := (h_{\alpha_{\sigma-1}+1}^{(j)} - p_\sigma^{(j)}, \dots, h_{\alpha_\sigma}^{(j)} - p_\sigma^{(j)}) \in \Sigma(I_{\alpha_{\sigma-1}+1,\alpha_{\sigma+1}})$$

are convergent, with the limit $u_\sigma \in \Sigma(I_{\alpha_{\sigma-1}+1}, \alpha_{\sigma+1})$.

The polyhedron $\mathcal{S}(n)$ is a compactification of Σ_n.

The projection of the polyhedron of Karpelevich velocities to the polyhedron of Satake velocities. For any tree partition we can delete all its reducible elements, and thus obtain an ordinary partition. Then in the product (11.7) we can omit the second factor.

γ) *Toric velocities.* Consider the space L formed by the sets $(t_1, \dots, t_m) \in \mathbb{R}^n$ that are defined up to an additive constant. Let $Q \subset L$ be a convex polyhedron with rational vertices contained in the hyperplane $t_1 + \cdots + t_n = 0$. Let Q be invariant with respect to the action of the symmetric group S_n on L. Starting from this polyhedron, we shall construct a compactification of Σ_n. Let $M_j = (m_1^j, \dots, m_n^j)$, $j = 1, \dots, N$, be the vertices of Q. For any vertex we introduce the expression

$$\chi_j(t_1, \dots, t_n) = \exp\left(\sum_{k=1}^n m_k^j t_k\right).$$

Consider the embedding of L in the projective space $\mathbb{RP}^N$ given by the formula

$$\pi\colon (t_1, \dots, t_n) \to (\chi_1(t_1, \dots, t_n), \dots, \chi_N(t_1, \dots, t_n)).$$

Let $\overline{\pi(L)}$ be the closure of $\pi(L)$ in $\mathbb{RP}^N$. The structure of these closures is well known [**Dan**], and they have a nice and simple description in terms of the geometry of the polyhedron Q. The quotient space $\overline{\pi(L)}/S_n$ is a compactification of Σ_n.

REMARK. The polyhedron of Satake velocities has the form above. Apparently, this is not the case for the polyhedron of Karpelevich velocities.

REMARK. Apparently, projective compactifications (see 14.4) are hybrids of Satake compactifications and toric velocity compactifications.

16.12. **Remarks on §12.** The heroes of our paper are mainly very exotic objects from the point of view of "ordinary" differential geometry. The construction of 12.1 is an exception. The corresponding object of differential geometry is more or less standard, see [**EO, BGS, VEL**]. The Tits metrics on the boundary can also be defined for a general Cartan–Hadamard manifold, see [**BGS**] ([**BGS**] also contains the reference [**ImH**]). Nevertheless, it seems that now it is known only for a few interesting examples.

16.13. **Remarks on §13.** α) The Dynkin–Olshanetsky boundary [**Dyn, Olse1, Olse2**] is the Martin boundary for the ordinary diffusion on a symmetric space (see 16.16,β).

β) The Karpelevich boundary was constructed in [**Kar**].

16.14. **Remarks on §14.** α) I have not seen this construction in the literature. Kushner [**Kus1, Kus2**] constructed a compactification for a noncompact Riemannian symmetric space that is universal in the sense of 14.3. Our space $\mathfrak{R}$ is not compact.

β) *Examples of sea-urchin-type constructions*

EXAMPLE (blowing up of cusps). Consider the subgroup $PSL(2,\mathbb{Z}) \subset PSL(2,\mathbb{R})$ and its action on the Lobachevsky plane $\mathcal{L} = PSL(2,\mathbb{R})/SO(n)$ as an upper half-plane. Denote by R the space of oriented geodesics on $\mathcal{L}$ whose limits are rational points on $\overline{\mathbb{R}} = \mathbb{R}\cup\infty$ (it is essential for our purposes that any rational point of the absolute of the Lobachevsky plane is a fixed point for some cusp element of $PSL(2,\mathbb{Z})$). The group $PSL(2,\mathbb{Z})$ has a natural discrete action on the spaces $\mathcal{L}$ and R, and hence we can construct the quotient space $(\mathcal{L}\cup R)/PSL(2,\mathbb{Z})$. This space is a compactification of the space $\mathcal{L}/PSL(2,\mathbb{Z})$. For more general constructions of this type, see [**BS1**].

EXAMPLE (universal toric variety). Consider the torus $(\mathbb{C}^*)^n$. For any

$$L = (l_1,\dots,l_n) \in \mathbb{Z}^n$$

we consider the one-parameter subgroup $\gamma_L \subset \mathbb{C}^n$ that consists of the elements $(z^{l_1},\dots,z^{l_n}) \in (\mathbb{C}^*)^n$ (where $z \in (\mathbb{C}^*)^n$). A point of the boundary $\mathfrak{Q}$ of $(\mathbb{C}^*)^n$ can be defined by the following data:

1* an element $L = (l_1,\dots,l_n) \in \mathbb{Z}^n$, and

2* an element q of the quotient group $(\mathbb{C}^*)^n/\gamma_L$.

A sequence $u^{(j)} = (u_1^{(j)},\dots,u_n^{(j)}) \in (\mathbb{C}^*)^n$ converges to an element $(L,q) \in \mathfrak{Q}$ whenever it satisfies the following conditions:

(i) there exists a real sequence $b^{(j)} \to +\infty$ such that for any $m = 1,\dots,n$, there exists a finite limit $\lim_{j\to\infty} |u_m^{(j)}/(b^{(j)})^{l_m}|$;

(ii) let $\widetilde{u^{(j)}}$ be the image of $u^{(j)}$ in the quotient group $(\mathbb{C}^*)^n/\gamma_L$. Then $\widetilde{u^{(j)}}$ converges to $q \in (\mathbb{C}^*)^n/\gamma_L$.

EXAMPLE (the complex sea urchin). Consider the space $PSL(n,\mathbb{C})/SO(n,\mathbb{C})$. Choose $s \in \{2,3,\dots,n\}$. Choose a set of positive integers $K = (k_1,\dots,k_s)$ such that $\sum k_\sigma = n$. Choose a set of integers $L : \{l_1 \geqslant \cdots \geqslant l_s\}$ that is defined up to an additive constant. Let $\mathcal{B}(K)$ be the group of $(k_1+k_2+\dots)\times(k_1+k_2+\dots)$ block upper triangular matrices B. Consider the complex curve $\gamma[K,L]$ given by

$$\gamma[K,L](t) = \begin{pmatrix} t^{l_1}E_{k_1} & & \\ & \ddots & \\ & & t^{l_s}E_{k_s} \end{pmatrix},$$

where $t \in \mathbb{C}^*$ and E_ψ is the $\psi\times\psi$ identity matrix. The curves $C\gamma[K,L](t)C^t$, where $C \in GL(n,\mathbb{C})$, are called *geodesic curves*. Choose an element $A \in PSL(n,\mathbb{C})$. Define the family (a *pencil*) $P(K,L\,|\,A)$ of all curves $\mathbb{C}^* \to PSL_n/SO_n$ that can be represented in the form $A^tB^t\gamma[K,L]BA$, where $B \in \mathcal{B}(K)$. We can readily see that the curves of a pencil either coincide or are disjoint. The set

$$S(K,A) = \bigcup_{B\in\mathcal{B}(K)} A^tB^t\gamma[K,L]BA$$

is open and dense in $PSL(n,\mathbb{C})/SO(n,\mathbb{C})$.

Now we consider the space

$$PSL(n,\mathbb{C})/SO(n,\mathbb{C}) \cup \bigcup_{K,L}\Big(\bigcup_{A\in GL(n,\mathbb{C})/\mathcal{B}(K)} P(K,L\,|\,A)\Big)$$

endowed with the topology for which a sequence $x_j \in PSL(n,\mathbb{C})/SO(n,\mathbb{C})$ converges to $\lambda \in P(K,L\,|\,A)$ whenever it satisfies the following conditions.

1. $x_j \in S(K,A)$ for large j.
2. Let $\gamma_j \in P(K,L\,|\,A)$ be a geodesic curve that contains x_j. Then γ_j converges to γ in the natural topology of the pencil $P(K,L\,|\,A)$.
3. Let $B_j \in \mathcal{B}(K)$ be a sequence such that $B_j \to E$ and $B_j\gamma_j = \gamma$. Then the sequence $B_jx_j \in \gamma$ converges to $+\infty$ in γ.

This topology is natural on sets of the form $\bigcup_{A\in GL(n,\mathbb{C})/\mathcal{B}(K)} P(K,L\,|\,A)$ and coincides with the topology of disjoint union on the set

$$\bigcup_{K,L}\Big(\bigcup_{A\in GL(n,\mathbb{C})/\mathcal{B}(K)} P(K,L\,|\,A)\Big).$$

16.15. **Remarks on §15.** For the boundaries of the Bruhat–Tits buildings, see [**BS2, Ger1, Ger2**]. Consider the space of lattices in $\mathbb{R}^n$ defined up to dilatations. This space can be compactified by the same way as in subsection 15.1 (see also [**How**]).

16.16. **Some general constructions.** Here we briefly discuss several general constructions of compactifications for metric (and topological) spaces.

α) *Maximal ideals.* Let X be a noncompact space. Let $\mathcal{A}$ be an algebra of (continuous, holomorphic, algebraic, almost periodic, etc.) functions on X. Let $\mathcal{A}$ satisfy the following conditions:

a) $\mathcal{A}$ contains the constants; and

b) $\mathcal{A}$ separates the points of X (for any $x_1, x_2 \in X$, there exists $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$).

Denote by Spec($\mathcal{A}$) the spectrum (the space of maximal ideals) of $\mathcal{A}$. There is a natural embedding $\lambda\colon X \to$ Spec($\mathcal{A}$). If $x \in X$, then the set of functions $f \in \mathcal{A}$ that vanish at the point x is a maximal ideal of $\mathcal{A}$. As a rule, Spec($\mathcal{A}$) is a compact space and X is dense in $\leq Spec(\mathcal{A})$. Let $\mathcal{A} \subset \mathcal{A}'$ be two algebras satisfying conditions a) and b). Then there is a natural mapping Spec($\mathcal{A}'$) $\to$ Spec($\mathcal{A}$) given by the formula $J \mapsto J \cap \mathcal{A}$.

Example: Stone-Čech compactification. For a topological space X we introduce the algebra $C(X)$ of all continuous functions and the algebra $\mathcal{B}(X)$ of all bounded continuous functions. The space Spec($\mathcal{B}(X)$) is the universal compactification of X in the following sense. For any dense embedding i of X into a compact topological space K, there exists a mapping $\pi\colon$ Spec($\mathcal{B}(X)$) $\to K$ such that $i = \pi \circ \lambda$. Indeed, $C(K) \subset \mathcal{B}(X)$, and this embedding induces the mapping

$\pi\colon \mathrm{Spec}(\mathcal{B}(X)) \to K = \mathrm{Spec}(C(K))$. This construction seems to be pathological because no point of $\mathrm{Spec}(\mathcal{B}(X)) \setminus X$ can be described explicitly.

β) *Martin boundary* [**Mar, Doo, Shu, KSK, Kai**] Consider a domain $\Omega \subset \mathbb{R}^n$ and a certain boundary problem for the Laplace operator Δ. Let $G(x,y)$ be its Green function; assume that this function is positive. Consider a family of functions $g_x(y) = G(x,y)$, $x,y \in \Omega$, defined up to a positive factor. We thus obtain an embedding of Ω to the projectivized space of functions on Y. By the *Martin compactification* of Ω we mean the pointwise closure of the image of Ω in the projective space. This definition is not constructive, and to describe the Martin boundary, it is necessary to find the asymptotic behavior of Green's function. The Dynkin–Olshanetsky boundary is the Martin boundary for the equation $\Delta u = \mu u$ on a symmetric space [**Dyn, Olse1, Olse2**]. The Martin boundary for the Laplace equation is only a particular case of the very general construction known under the name of Martin boundary. For simplicity, we consider a discrete countable space Ω and a *Markov transition function* $p(x,y)$ on $\Omega \times \Omega$. Recall that for all x we have $\sum_{y\in\Omega} p(x,y) = 1$. A function $f\colon \Omega \to \mathbb{R}$ is said to be λ-harmonic if

$$\lambda f(x) = \sum_{y\in\Omega} p_n(x,y) f(y).$$

Now introduce the functions $p_1(x,y) = p(x,y)$, $p_2(x,y)$, $p_3(x,y)$, ... such that

$$p_n(x,y) = \sum_{y\in\Omega} p(x,z) p_{n-1}(z,y)$$

(i.e., consider the powers of the Markov transition matrix) and define the λ-Green function $G_\lambda(x,y) = \sum_{n=1}^{\infty} \lambda^{-n} p_n(x,y)$. If $G_\lambda(x,y)$ is finite, then we can repeat the arguments above.

γ) *Distance functions* [**Gro, BGS**]. Consider a noncompact metric space X with distance function $d(x,y)$. Let $C(X)$ be the space of continuous functions on X. For any $x \in X$ we introduce the function $a_x(y) = d(x,y)$, $y \in X$. The relation $x \mapsto a_x$, $x \in X$, defines an embedding $X \to C(X)$. Consider the quotient space $C(X)/\mathbb{R}$, i.e., the space of continuous functions on X defined up to an additive constant. Now we take the closure of the image of X with respect to the uniform-on-compacta topology. For symmetric spaces, this construction yields the visibility boundary.

16.17. **Formal geometric constructions.** α) *Gluing of the quotient space.* Consider a noncompact metric space X and a compact set $K \subset X$. Let Y be a compact space and $\sigma\colon X \setminus K \to Y$ a surjective map. Introduce the space $X \cup Y$ endowed with a topology that satisfies the following condition: let $x_j \in X$ be a sequence that has no limit points in X; then $\lim x_j = y \in Y$ whenever the sequence $\sigma(x_j)$ converges to $y \in Y$.

EXAMPLE. In 11.1, $X = SL(n,\mathbb{R})/SO(n)$, K is the point E, and Y is Δ. The mapping σ is the composition of the mappings $X\setminus E \to \Sigma\setminus 0 \to \Delta$.

EXAMPLE. In 12.1, X is a symmetric space, K is the point x_0, and $Y = Sk$.

EXAMPLE. See the construction of the universal toric variety in 16.14.

β) *Gluing of the boundary of the quotient space (velocity compactifications).* Consider a noncompact metric space X and a compact set $K \subset X$. Let Z be a compact space and let $Y \subset Z$ be a compact subset. Let a mapping $\tau\colon X\backslash K \to Z\backslash Y$ be given. Define a topology on $X \cup Y$ by the following condition: a sequence $x_j \in X$ without limit points in X converges to $y \in Y$ whenever $\tau(x_j)$ converges to Y with respect to the topology of Z.

EXAMPLE. Let X_1 and X_2 be two metric spaces. We construct another compactification of $X_1 \times X_2$. Choose points $a_1 \in X_1$ and $a_2 \in X_2$. Let $A = [0,\infty) \times [0,\infty)$. Consider the mapping $X_1 \times X_2 \to A$ given by the rule $(x_1, x_2) \mapsto (d(x_1, a_1), d(x_2, a_2))$. Take a certain boundary of A. For instance, let us consider the mapping $A \to [0,\infty]$ given by the formula $(d_1, d_2) \mapsto d_1/d_2$. Now we can glue the quotient space $[0,\infty]$ to A, and then we can glue $[0,\infty]$ to $X_1 \times X_2$.

γ) *The projective limit of hypersurfaces.* Let us consider a noncompact metric space X and a compact set $K \subset X$. Assume that we have a family S_t, $t > 0$, of closed subsets of X such that

1) $S_t \cup S_\tau = \varnothing$ for $t \neq \tau$,
2) $X \setminus K = \cup_{t>0} S_t$, and
3) for any $t > \tau$, there exists a continuous mapping $\pi_{t\tau}\colon S_t \to S_\tau$ such that for $t > \tau > \sigma$ we have $\pi_{t\sigma} = \pi_{\tau\sigma}\pi_{t\tau}$.

Let S_∞ be the projective limit of the family S_t and let $\pi_{\infty,t}$ be the natural projection of S_∞ onto S_t. We shall define a topology on $X \cup S_\infty$ by the following condition: for a sequence $x_j \in X$ we define $t = t(x_j) \in (0,\infty)$ by the condition $x_j \in S_{t(x_j)}$, and assume that, as $t(x_j) \to \infty$, x_j converges to $y \in S_\infty$ whenever for any $\sigma > 0$, the sequence $\pi_{t(x_j)\sigma}(x_j) \in S_\sigma$ converges to $\pi_{\infty\sigma}(y)$ with respect to the topology of S_σ.

EXAMPLE (visibility boundary). Let $a \in X$. We assume that for any $x \in X$, there exists a unique shortest curve γ_x joining a and x. Let $l = l(x)$ be the length of γ_x. Let $s > 0$. Consider the set S_t given by the equation $l(x) = t$. Let $t > \tau$ and $x \in S_t$. Define the point $\pi_{t\tau}(x) \in S_\tau$ as $S_\tau \cap \gamma_x$. Now we can apply our construction.

EXAMPLE. *Tits-type metrics.* In the situation of the preceding example we denote by ρ the metric on X. Let $p, q \in S_\infty$. Then

$$d(p,q) = \limsup_{t\to\infty} \frac{1}{t}\, \rho(\pi_{\infty t}(p), \pi_{\infty t}(q))$$

is a metric on S_∞.

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