Universal Completions of Complex Classical Groups

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Let ρ be a representation of a group G in a space H. Consider two sets: the closure $\overline{\rho(G)}$ of the set $\rho(G)$ of operators $\rho(g)$, $g \in G$, and the closure $\overline{\mathbb{C}\cdot\rho(G)}$ of the set $\mathbb{C}\cdot\rho(G)$ of all operators of the form $\lambda\rho(g)$, $g \in G$, $\lambda \in \mathbb{C}$. The problem of description of the semigroups $\overline{\rho(G)}$ and $\overline{\mathbb{C}\cdot\rho(G)}$ have arisen many times in a number of contexts (in particular, the semigroup $\overline{\mathbb{C}\cdot \mathrm{Spin}(O(2n, \mathbb{C}))}$ related to the spinor representation Spin was considered in detail in [7], the case of unitary representations of semisimple groups was investigated in [9], and the case of finite dimensional representations was discussed in [13–15]).

The crucial role of this problem in representation theory of infinite dimensional groups was perceived by G. I. Ol'shanskii (see, e.g., [12]). In the course of investigation of these semigroups some new interesting algebraical structures were discovered (see [2–6, 10–11]). In particular, it turned out that the semigroup $\overline{\mathbb{C}\cdot \mathrm{Spin}(O(2n, \mathbb{C}))}$ has a nice and clear description in terms of linear relations.

In a series of articles (see [13–15]), arbitrary semigroups of the form $\Gamma = \overline{\mathbb{C}\cdot\rho(G)}$ were considered, where G is a complex semisimple Lie group and ρ is a finite dimensional representation. These semigroups were described in terms of canonical forms of elements of Γ under the action of the group $G\times G$ ($\gamma \mapsto g_1\gamma g_2$, where $\gamma \in \Gamma$ and g_1, $g_2 \in G$). It was also discovered that the semigroups related to two representations ρ_1, ρ_2 (of the same group G) are isomorphic if and only if the weight sets of ρ_1 and ρ_2 have similar convex hulls.

So it may seem that different representations of the group G give rise to different semigroups. However, it is not quite so. The aim of our paper is to construct, for each classical complex group G, an universal (not separated) semigroup $\overline{G}$ such that

1) G is dense in $\overline{G}$ and any irreducible holomorphic representation of the group G can be canonically extended to a projective representation $\overline{\rho}$ of the semigroup $\overline{G}$;

2) the sets $\mathbb{C}\cdot\overline{\rho}(\overline{G})$ and $\overline{\mathbb{C}\cdot\rho(G)}$ coincide for each irreducible ρ.

The explicit construction of $\overline{G}$ can be described in much the same terms as for the classical categories GA, B, C, D of [6]; however, the construction of $\overline{G}$ itself is less clear.

The semigroups $\overline{G}$ are described for $G = \mathrm{SL}_n(\mathbb{C})$ in § 2 and for $G = \mathrm{SO}(n, \mathbb{C})$, $\mathrm{Sp}(2n, \mathbb{C})$ in § 3. In any case, an element of the semigroup is some sequence of linear relations.

Moreover, it turns out that the semigroups thus constructed have certain fairly interesting nonlinear actions. First, they act on flag spaces (see 2.6), where the term "flag space" denotes the space of all flags (complete and noncomplete as well) endowed with the natural (not separated) topology.

Further, let σ be a holomorphic involutive automorphism of the group G. Let $H \subset G$ be the subgroup of all fixed points of σ. It turns out that symmetric spaces of the form G/H have some natural (not separated) completions $\overline{G/H}$ similar to those of classical groups, therewith the semigroup $\overline{G}$ acts on $\overline{G/H}$; in 2.7 we discuss an example of this action.

After this paper had been submitted to the editors, C. de Concini informed the author that the completions for classical groups, constructed via exact resolving sequences (see § 2), are known in algebraic geometry (see [16–19]). However, the existence of a natural multiplication in the space of resolving sequences was not noticed; accordingly, the representations of corresponding semigroups were not considered. In order to clarify connections of our paper with [16–19], we add Subsection 2.7.

Recently it has been discovered (see [3–6]) that semigroups usually appear in representation theory because we actually deal not with representations of groups or semigroups, but with representations of

Moscow Institute for Electronic Engineering. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 26, No. 4, pp. 30–44, October–December, 1992. Original article submitted May 21, 1991.

0016–2663/92/2604–0001 $12.50

categories. The situation is maybe the same in our case, too. Keeping this in mind, we consider th category LA related to the semigroups $\overline{\mathrm{SL}}_n$ (see § 3).

I am grateful to G. I. Ol'shanskii, C. de Concini, S. L. Tregub for discussions of these topics.

Throughout this paper, the notion of convergence and continuity can be realized in the sense of usu topology and Zariski topology as well.

§1. Preliminaries

1.1. Representations of categories. Suppose that $\mathcal{K}$ is a category and $\mathrm{Mor}_{\mathcal{K}}(V, W)$ is th set of morphisms from V to W. By definition, a *projective representation* of $\mathcal{K}$ is a functor $(T,$ which assigns a linear space $T(V)$ to each object V of $\mathcal{K}$ and a linear operator $\tau(P)$ to each morphis $P\colon V \to W$; moreover, we have

$$\tau(QP) = c(P, Q)\tau(Q)\tau(P),$$

for each $P \in \mathrm{Mor}(V, W)$ and $Q \in \mathrm{Mor}(W, Y)$, where $c(P, Q)$ is a complex number.

1.2. Linear relations. Let V, W be linear spaces. A *linear relation* is a subspace $P \subset V \oplus W$ Informally speaking, these subspaces can be regarded as graphs of linear "operators" $V \to W$ which ca be multivalued and not everywhere defined.

For any linear relation $P\colon V \rightrightarrows W$ we define

a) the kernel $\mathrm{Ker}\, P = \{v \in V : (v, 0) \in P\}$,

b) the image $\mathrm{Im}\, P =$ the projection of P onto W,

c) the domain $\mathcal{D}(P) =$ the projection of P onto V,

d) the indefiniteness $\mathrm{Indef}(P) = \{w \in W : (0, w) \in P\}$.

If $P\colon V \rightrightarrows W$, $Q\colon W \rightrightarrows Y$ are linear relations, then we define their product to be a linear relatio $QP\colon V \rightrightarrows Y$. An element $(v, y) \in V \oplus Y$ is contained in QP whenever there exists an element $w \in W$ such that $(v, w) \in P$, $(w, y) \in Q$.

1.3. Category GA (see [4, 6]). Objects of the category GA are finite dimensional complex linea spaces. Let V, W be objects of GA; then the set $\mathrm{Mor}_{\mathrm{GA}}(V, W)$ of morphisms consists of elements c two kinds:

a) linear relations $P : V \rightrightarrows W$;

b) a formal element $\mathrm{null} = \mathrm{null}_{V,W}$ (which is not to be identified with any linear relation).

The topology on $\mathrm{Mor}(V, W)$ is defined as follows: it is the topology of disjoint union of Grassmannian on $\mathrm{Mor}(V, W) \setminus \mathrm{null}_{V,W}$ and the point $\mathrm{null}_{V,W}$ is contained in the closure of any other point.

Define a multiplication of morphisms as follows:

a) the product of the null with any other morphism is the null;

b) if $P\colon V \rightrightarrows W$, $Q\colon W \rightrightarrows Y$ are linear relations and the conditions

$$\mathrm{Ker}\, Q \cap \mathrm{Indef}(P) = 0, \qquad \mathrm{Im}\, P + \mathcal{D}(Q) = W, \tag{1.1}$$

hold, then QP is the usual product of linear relations; otherwise QP is the null.

Remark. The reasons for introducing this strange element "null" are discussed in [6]; one of them is that the multiplication of linear relations is discontinuous at the points

$$(P, Q) \in (\mathrm{Mor}(V, W) \setminus \mathrm{null}) \times (\mathrm{Mor}(W, Y) \setminus \mathrm{null}),$$

where $QP = \mathrm{null}$.

1.4. The fundamental representation of the category GA. The representations of the category GA were classified in [6]. The only (projective) representation of GA we need is the fundamenta representation (Λ, λ).

To each object V of GA we assign the exterior algebra $\Lambda(V)$.

Any nonzero morphism $P\colon V \rightrightarrows W$ of the category GA is decomposed into the product of three morphisms $P = QRT$, where the factors are defined as follows:

a) $T\colon V \rightrightarrows \mathcal{D}(P)$ is the graph of the embedding $\mathcal{D}(P) \to V$;

2.1. Fundamental representations of SL_n and their semigroup extensions. Let $0 \le j \le n$. Denote by λ_j the natural representation of SL_n in the jth exterior power $\Lambda^j(\mathbb{C}^n)$ of $\mathbb{C}^n$. We have observed that λ_j can be canonically extended to a projective representation of the semigroup GL_n^*; denote this representation also by λ_j.

Lemma 2.1. $\overline{\mathbb{C}\cdot\lambda_j(\mathrm{SL}_n)} = \mathbb{C}\cdot\lambda_j(\mathrm{GL}_n^*)$.

(This follows easily from assertion of Subsections 1.5–1.6.)

Let $P \in \mathrm{GL}_n^* \setminus \mathrm{null}$. We define the *domain of action* of a linear relation P to be the set of all λ_j such that $\lambda_j(P) \neq 0$.

Lemma 2.2. *The domain of action of the linear relation P consists of all λ_j satisfying the condition*

$$\dim \mathrm{Indef}(P) \le j \le \dim \mathrm{Im}\, P. \tag{2.1}$$

We are especially interested in the behavior of $\lambda_j(P)$ on the boundary of the domain of action, i.e., at those j for which some of inequalities (2.1) become equalities.

Lemma 2.3. a) *Let* $\dim \mathrm{Im}\, P = j$. *Then $\lambda_j(P)$ is a rank one operator and, up to a scalar multiple, it is determined uniquely by the subspaces* $\mathrm{Im}\, P$ *and* $\mathrm{Ker}\, P$, *i.e., if* P, $P' \in \mathrm{GL}_n^* \setminus \mathrm{null}$ *and* $\mathrm{Im}\, P = \mathrm{Im}\, P'$, $\mathrm{Ker}\, P = \mathrm{Ker}\, P'$, *then* $\widetilde{\lambda}_j(P) = \widetilde{\lambda}_j(P')$.

b) *Let* $\dim \mathrm{Indef}(Q) = j$. *Then $\lambda_j(Q)$ is a rank one operator and, up to a scalar multiple, it is determined uniquely by the subspaces* $\mathrm{Indef}\, Q$ *and* $\mathcal{D}(Q)$.

c) *Let* $\dim \mathrm{Indef}(P) = j = \dim \mathrm{Im}\, Q$ *and* $\mathrm{Indef}(P) = \mathrm{Im}\, Q$, $\mathrm{Ker}\, Q = \mathcal{D}(P)$. *Then* $\widetilde{\lambda}_j(P) = \widetilde{\lambda}_j(Q)$.

Proof. In the case a) the subspace $\mathrm{Im}\,\lambda_j(P)$ is generated by the vector $e_1 \wedge \ldots \wedge e_j$, where $\{e_\alpha\}$ is a basis in $\mathrm{Im}\, P$, and the subspace $\mathrm{Ker}\,\lambda_j(P)$ is generated be vectors $f \wedge g_2 \wedge \ldots \wedge g_{n-j}$, where f runs over $\mathrm{Ker}\, P$.

In the case b) the subspace $\mathrm{Im}\,\lambda_j(P)$ is generated by the vector $e_1 \wedge \ldots \wedge e_j$, where $\{e_\alpha\}$ is a basis in $\mathrm{Im}\, P$, and the subspace $\mathrm{Ker}\,\lambda_j(P)$ is generated by the vector $f \wedge g_2 \wedge \ldots \wedge g_{n-j}$, where f belongs to $\mathcal{D}(P)$.

The case c) also follows from what was said above.

2.2. Exact resolving sequences. By definition, an exact resolving sequence is a sequence of linear relations $P_1, P_2, \ldots, P_s \in \mathrm{GL}_n^*$ such that

$$\mathrm{Indef}\, P_1 = 0, \quad \mathrm{Ker}\, P_s = 0, \quad \mathcal{D}(P_{j+1}) = \mathrm{Ker}\, P_j, \quad \mathrm{Indef}(P_{j+1}) = \mathrm{Im}\, P_j \quad \text{for all } j.$$

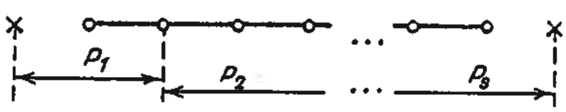
Let $\pi = (P_1, \ldots, P_s)$ be an exact resolving sequence. Fix a number j and consider the sequence of operators $\lambda_j(P_1), \lambda_j(P_2), \ldots, \lambda_j(P_s)$. Lemma 2.2 implies that this sequence contains only one or two nonzero elements. If there are two nonzero elements, then their indices are consecutive integers. Lemma 2.3 also implies that $\widetilde{\lambda}_j(P_\alpha) = \widetilde{\lambda}_j(P_{\alpha+1})$ for $\lambda_j(P_\alpha) \neq 0$, $\lambda_j(P_{\alpha+1}) \neq 0$.

Define an operator $\widetilde{\lambda}_j(\pi) \in \mathrm{Mt}(\Lambda^j \mathbb{C}^n)$ by

$$\widetilde{\lambda}_j(\pi) = \widetilde{\lambda}_j(P_\alpha),$$

where $\widetilde{\lambda}_j(P_\alpha) \neq 0$.

It is convenient to represent the action of exact resolving sequences by diagrams of the form



where the circles denote the representations $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ of the group SL_n, therewith the fundamental representations $\lambda_1, \ldots, \lambda_{n-1}$ are linked by edges, and so we obtain the Dynkin diagram of type A_{n-1}. The crosses denote the representations λ_0 and λ_n. The arrows denote the domains of action of linear relations $P_1, P_2, \ldots$; they contact but do not overlap.

$\mathcal{D}(P) \oplus W \to \mathcal{D}(P) \oplus (W/\operatorname{Indef}(P))$; obviously, R is simply the graph of a linear operator;

c) $Q\colon W/\operatorname{Indef}(P) \rightrightarrows W$ is the graph of the projection $W \to W/\operatorname{Indef}(P)$.

Define an operator $\lambda(P)$ to be $\lambda(Q)\lambda(R)\lambda(T)$, where $\lambda(Q)$, $\lambda(R)$, $\lambda(T)$ are defined as follows:

a) $\lambda(T): \Lambda(V) \to \Lambda(\mathcal{D}(P))$ is the interior multiplication by $f_1 \wedge \ldots \wedge f_s$, where $\{f_j\}$ is a basis in th space of functionals on V which are zero on $\mathcal{D}(P)$;

b) $\lambda(R)\colon \Lambda(\mathcal{D}(P)) \to \Lambda(W/\operatorname{Indef}(P))$ is the natural functorial mapping of exterior algebras corre sponding to the operator with graph R;

c) $\lambda(Q)\colon \Lambda(W/\operatorname{Indef}(P)) \to \Lambda(W)$ is the exterior multiplication by $e_1 \wedge \ldots \wedge e_t$, where $\{e_i\}$ is a bas in $\operatorname{Indef}(P)$.

Denote by $\mathrm{GL}^*(V)$ the semigroup whose elements are the null and all the linear relations of dimensio $\dim V$ (it is not quite obvious that $\mathrm{GL}^*(V)$ is closed under multiplication; the conditions (1.1) are heavil exploited in proving this fact). If $\dim V = n$, then we also denote the semigroup $\mathrm{GL}^*(V)$ by GL^*_n.

Remark. Suppose that $P\colon V \rightrightarrows W$ is a linear relation and $\dim P - \dim V = s$. Then the operatc $\lambda(P)\colon\ \Lambda(V) \to \Lambda(W)$ maps $\Lambda^k(V)$ into $\Lambda^{k+s}(V)$.

1.5. Toric manifolds. Suppose that $\mathbb{T} = (\mathbb{C}^*)^k$ is the complex torus. Let $g \mapsto \rho(g)$ be a holomor phic representation of $\mathbb{T}$, defined up to a coordinate transform, $g = (z_1, \ldots, z_k) \in \mathbb{T}$. The representatio $\rho(g)$ has the following form:

$$\rho(g) = \operatorname{diag}(\chi_1(g), \ldots, \chi_s(g)) = \begin{pmatrix} \chi_1(g) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \chi_s(g) \end{pmatrix},$$

where $\chi_j(g) = z_1^{a_{j1}} \cdots z_n^{a_{jn}}$ and a_{jp} are (real) integers. It is known that the sequences $a_j = (a_{j1}, \ldots, a_{jn})$ are said to be weights of the representation ρ.

It is not difficult to describe the closure $\overline{\rho(\mathbb{T})}$ of the set $\rho(\mathbb{T})$ in the space of all operators. We stat the answer [1]. Let $\mathcal{P}$ be the convex cone in $\mathbb{R}^n$ generated by the weights a_j of the representation ρ Consider some face A of the cone $\mathcal{P}$. Put $\chi_j^A(g) = \chi_j(g)$ if $a_j \in A$ and $\chi_j^A(g) = 0$ if $a_j \notin A$. Le $X(A)$ be the set of all the matrices of the form $\operatorname{diag}(\chi_1^A(g), \ldots, \chi_s^A(g))$, where g runs over $\mathbb{T}$. Then

$$\overline{\rho(\mathbb{T})} \setminus \rho(\mathbb{T}) = \bigcup_A X(A),$$

the union being taken over all the faces of the cone $\mathcal{P}$.

1.6. The set $\overline{\rho}(G)$. Suppose that G is a complex reductive Lie group and ρ is a representatior of G. Let $\overline{\rho(G)}$ be the closure of the set of operators $\rho(g)$, $g \in G$, in the space of all operators. Let $\mathbb{T} \subset G$ be a maximal torus.

Theorem (see [14, 15]).

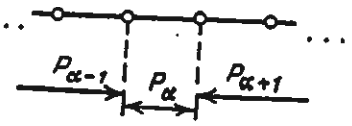
$$\overline{\rho(G)} = G \cdot \overline{\rho(\mathbb{T})} \cdot G.$$

Thus, the problem of describing $\overline{\rho(G)}$ is reduced to the same problem for $\rho(\mathbb{T})$.

Now let G be a semisimple group and ρ be its irreducible representation. We are interested in describing the set $\overline{\mathbb{C} \cdot \rho(G)}$. This problem can be reduced to the previous one, because the reduction group $G' = \mathbb{C}^* \cdot G$ acts in the representation space of ρ (the group $\mathbb{C}^*$ acts by scalar multiplication and G acts by the operators $\rho(g)$).

§2. Completion of the Group SL_n

Let V be a linear space and $\operatorname{Mat}(V)$ be the semigroup of all operators in V; we define $\operatorname{Mt}(V) = \operatorname{Mat} V/\mathbb{C}^*$ to be the quotient of $\operatorname{Mat}(V)$ with respect to the action of the complex multiplicative group $\mathbb{C}^*$. Introduce the natural quotient topology in the semigroup $\operatorname{Mt}(V)$; this topology is not separated: the point 0 is contained in the closure of any point. If $\rho\colon \Gamma \to \operatorname{Mat}(V)$ is a representation of a group or of a semigroup Γ, then we denote by $\widetilde{\rho}$ the natural mapping of Γ into $\operatorname{Mt}(V) = \operatorname{Mat}(V)/\mathbb{C}^*$.



(2.2)

Then $\widetilde{\lambda}_j(\pi) = \widetilde{\lambda}_j(\pi')$ for any j.

Remark 2. Suppose that $\pi = (P_1, \dots, P_s)$ is a resolving sequence and the domain of action for P_α is the single point which is contained in the domain of action for $P_{\alpha-1}$ or $P_{\alpha+1}$.

(2.3)

Consider the sequence $\pi' = (P_1, \dots, P_{\alpha-1}, P_{\alpha+1}, \dots, P_s)$. Then $\widetilde{\lambda}_j(\pi) = \widetilde{\lambda}_j(\pi')$ for all j.

So we see that in some cases different resolving sequences can give rise to the same set of operators $\widetilde{\lambda}_j(\pi)$.

A resolving sequence $\pi = (P_1, \dots, P_s)$ is said to be *correct* if the following conditions hold:

a) for any j the condition $\mathcal{D}(P_{j+1}) \neq \operatorname{Ker} P_j$ implies $\dim \operatorname{Ker} P_j - \dim \mathcal{D}(P_{j+1}) \geq 2$, i.e., either the domains of action for neighboring relations P_j, P_{j+1} have contacts or they are at a distance not less than 2;

b) if the domain of action of P_α is the single point, then it is not contained in the domains of action of $P_{\alpha-1}$ and $P_{\alpha+1}$; then, the pictures of the form (2.3) are forbidden.

The following assertion is more or less obvious.

Lemma 2.6. *For each resolving sequence π there exists a unique correct resolving sequence π' such that $\widetilde{\lambda}_j(\pi) = \widetilde{\lambda}_j(\pi')$ for all j.*

Define $\overline{\mathrm{SL}}_n$ to be the semigroup of all correct resolving sequences. Note that the multiplication of correct resolving sequences does not break the condition a); if the condition b) is broken then we simply drop unnecessary relations.

Remark. The group $\mathrm{GL}_n \subset \overline{\mathrm{SL}}_n$ consists of sequences $\pi = \{P_1\}$ which have the single element, P_1, being an invertible operator ($\operatorname{Ker} P_1 = 0$, $\operatorname{Indef}(P_1) = 0$).

Now we define the topology in $\overline{\mathrm{SL}}_n$ as follows. The sequence $\pi_j \in \overline{\mathrm{SL}}_n$ is said to converge to $\pi \in \overline{\mathrm{SL}}_n$ if the sequence $\widetilde{\lambda}_k(\pi_j)$ converges to $\widetilde{\lambda}_k(\pi)$ in the topology of $\mathrm{Mt}(\Lambda^k \mathbb{C}^n)$ for any k. The topology thus obtained on $\overline{\mathrm{SL}}_n$ is not separated. For example, the closure of the point $\pi = (P_1, \dots, P_k) \in \overline{\mathrm{SL}}_n$ contains all the resolving sequences of the form $(P_{i_1}, \dots, P_{i_s})$.

By Lemma 2.5, the group SL_n is dense in the semigroup $\overline{\mathrm{SL}}_n$.

2.5. Extension of irreducible representations of SL_n to $\overline{\mathrm{SL}}_n$. A fixed irreducible representation of the group SL_n can have several extensions that are continuous projective representations of the semigroup $\overline{\mathrm{SL}}_n$. This does not contradict the fact that SL_n is dense in $\overline{\mathrm{SL}}_n$; the reason is that the topology on $\overline{\mathrm{SL}}_n$ is not separable. However, among these extensions there exists a certain canonical one $\overline{\rho}$. Now we are going to construct it.

Let ρ be an irreducible representation of SL_n, and let $a_1, \dots, a_{n-1}$ be its labels at the Dynkin diagram of type A_n. It is known (see, e.g., [8]) that this representation has the following realization. Consider the tensor product of the form $S = \lambda_1^{\otimes a_1} \otimes \lambda_2^{\otimes a_2} \otimes \dots \otimes \lambda_{n-1}^{\otimes a_{n-1}}$; then the representation ρ acts on the subspace $\mathcal{L}$ in S generated by the highest weight vector. The representation S can be extended to the semigroup $\overline{\mathrm{SL}}_n$ by the same formula. Since the group SL_n is dense in the semigroup $\overline{\mathrm{SL}}_n$, the subspace $\mathcal{L}$ is invariant under the action of $\overline{\mathrm{SL}}_n$. The canonical extension $\overline{\rho}$ of ρ is constructed.

Lemma 2.4. *Let π be an exact resolving sequence. Then there exists a curve $g_\varepsilon \subset \mathrm{SL}_n$ such that $\widetilde{\lambda}_j(g_\varepsilon) \to \widetilde{\lambda}_j(\pi)$ as $\varepsilon \to 0$, for all j.*

Proof. The group $\mathrm{GL}_n \times \mathrm{GL}_n$ acts on the set of all exact resolving sequences by left and right multiplications. Under this action, each exact resolving sequence can be transformed to the canonical form $Q_1, \dots, Q_t$ as follows.

Let $\mathbb{C}^n = V_1 \oplus V_2 \oplus \dots \oplus V_t$ and let $V_1' \oplus \dots \oplus V_t'$ be another copy of the space $\mathbb{C}^n$. Then the desired linear relation $Q_\alpha \subset \mathbb{C}^n \oplus \mathbb{C}^n$ is generated by $(V_1 \oplus \dots \oplus V_{\alpha-1}) \oplus (V_{\alpha+1}' \oplus \dots \oplus V_t')$ and by the graph of the identical map $V_\alpha \to V_\alpha$.

Now we define g_ε to be an operator in $\mathbb{C}^n$ whose restriction to V_α is the multiplication by $\varepsilon^{\alpha-1}$. This proves the Lemma.

The converse is also true.

Lemma 2.5. *Let $0 < i_1 < i_2 < \dots < i_s < n$, and let $g_k \in \mathrm{SL}_n$ be a sequence such that $\widetilde{\lambda}_{i_\alpha}(g_k) \to A_i \in \mathrm{Mt}(\Lambda^{i_\alpha}\mathbb{C}^n) \setminus 0$ for each i_α. Then there exists an exact resolving sequence π such that*

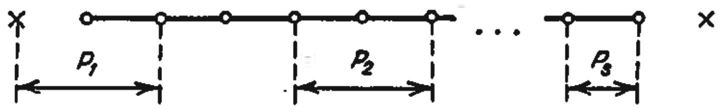
$$A_{i_\alpha} = \widetilde{\lambda}_{i_\alpha}(\pi).$$

Proof. Consider the group G of operators in $\bigoplus_\alpha \Lambda^{i_\alpha}(\mathbb{C}^n)$ generated by operators $\bigoplus \lambda_{i_\alpha}(g)$, $g \in \mathrm{SL}_n$, and by operators which act as a scalar multiplication on each $\Lambda^{i_\alpha}(\mathbb{C}^n)$. The group G is the quotient of the group $\mathrm{SL}_n \times (\mathbb{C}^*)^s$ by some finite subgroup. Thus, the technique of Subsections 1.5 and 1.6 can be applied.

2.3. Resolving sequences. By definition, a resolving sequence is a set (possibly void) of linear relations such that

$$\mathcal{D}(P_{j+1}) \subset \mathrm{Ker}\, P_j\,, \qquad \mathrm{Indef}\,(P_{j+1}) \supset \mathrm{Im}\, P_j.$$

It is convenient to represent resolving sequences by diagrams of the form



As before, the arrows indicate the domain of action of P_α; however, these domains may have no contact points.

For each resolving sequence $\pi = (P_1, \dots, P_s)$ and for each j such that $0 \le j \le n$, we now define an operator $\widetilde{\lambda}_j(\pi) \in \mathrm{Mt}(\Lambda^j\mathbb{C}^n)$. As before, consider the sequence of operators $\lambda_j(P_1), \dots, \lambda_j(P_s)$. Let $\widetilde{\lambda}_j(\pi) = 0$ if $\lambda_j(P_\alpha) = 0$ for all P_α and $\widetilde{\lambda}_j(\pi) = \widetilde{\lambda}_j(P_\mu)$ if $\lambda_j(P_\mu) \neq 0$. Using the same reasons as above, we prove that the operator $\widetilde{\lambda}_j(\pi) \in \mathrm{Mt}(\Lambda^j\mathbb{C}^n)$ is well defined.

Define a multiplication of resolving sequences $\pi = (P_1, \dots, P_s)$ and $\varkappa = (Q_1, \dots, Q_t)$. In order to do this, we consider all the products of the form $Q_\alpha P_\beta$ different from the null. Note that if the domains of action for Q_α and P_β do not intersect, then their product is null. Moreover, the domain of action for $Q_\alpha P_\beta$ is contained in the intersection of the domains of action for Q_α and P_β. It is easy to prove that the set $R_{\alpha\beta} = Q_\alpha P_\beta$ becomes a resolving sequence when properly ordered; the corresponding ordering is defined by the condition $R_{\alpha\beta} \le R_{\gamma\delta}$ if $\alpha \le \gamma$, $\beta \le \delta$. It follows from what was said above that this ordering is total.

The universal completion of SL_n could be defined as a semigroup of resolving sequences in $\mathbb{C}^n$; however, we have preferred the definition which is a bit more complicated.

2.4. Correct resolving sequences.

Remark 1. Suppose that $\pi = (P_1, \dots, P_s)$ and $\pi' = (P_1, \dots, P_{\alpha-1}, P_{\alpha+1}, \dots, P_s)$ are resolving sequences and the domains of action have the following form near P_α:

The circles denote the genuine points of the object A, the crosses denote the adjoined points; other elements of $\mathbb{Z}$ are denoted by dots. Consecutive genuine points are linked with edges.

To each interval I of A consisting of k points we assign a complex linear space $V(I)$ of dimension $(k+1)$. Thus, to any object of A we have assigned a collection of linear spaces $V(I_1), \dots, V(I_\alpha)$, where $I_1, \dots, I_\alpha$ is the collection of all the intervals of our object.

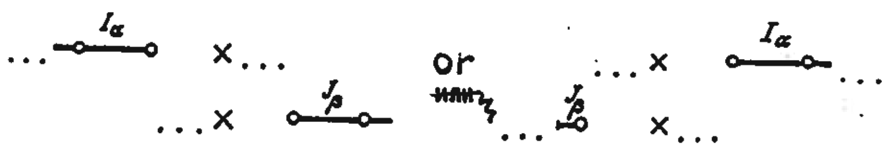
3.2. Morphisms of category LA. Suppose that A, B are objects of the category LA. Let $I_1, \dots, I_s$ be the intervals of the object A, and let $J_1, \dots, J_t$ be the intervals of the object B. In both cases the intervals are indexed from left to right.

By definition, a *morphism* is a (possibly empty) collection π of linear relations $P_j \colon V(I_\alpha) \rightrightarrows V(J_\beta)$. The exact description of admissible collections is rather complicated and occupies the rest of this subsection.

A. Dimensions. Let $I_\alpha = [s, s+k-1]$, $J_\beta = [t, t+l-1]$. Then the dimension of the relation P_j acting from $V(I_\alpha)$ to $V(J_\beta)$ equals to $(k+1)+(s-t) = \dim V(I_\alpha) + s - t$.

In particular, a linear relation $P_j \in \pi$ can act from $V(I_\alpha)$ to $V(J_\beta)$ only provided the extended intervals $\widetilde{I}_\alpha$ and $\widetilde{J}_\beta$ have a common point.

Example. Suppose that I_α and J_β are disposed as follows:



and P_j acts from $V(I_\alpha)$ to $V(J_\beta)$. Then P_j is zero-dimensional in the first case; $P_j = V(I_\alpha) \oplus V(J_\beta)$ in the second case.

B. The domain of action of P_j. By definition, the domain of action of P_j is the set of integers $k \in \mathbb{Z}$ such that

$$t + \dim \operatorname{Indef}(P_j) \le k \le t + \dim \operatorname{Im}(P_j).$$

Note that the domain of action of P_j is contained in $\widetilde{I}_\alpha \cap \widetilde{J}_\beta$.

It is convenient to suppose at once that the domains of action do not overlap for different relations $P_j \in \pi$ (but they may have contacts, i.e., common endpoints); however, this requirement is a consequence of further conditions.

C. Concordance. Let $P, Q \in \pi$ be linear relations, P acts from $V(I_\alpha)$ to $V(J_\beta)$ and Q acts from $V(I_\alpha)$ to $V(J_\gamma)$, therewith $\gamma \ge \beta$ and the domain of action of P is from the left of the domain of action of Q (i.e., neither point of the domain of action of P is from the right of any point of the domain of action of Q). Then

$$\mathcal{D}(Q) \subset \operatorname{Ker} P.$$

Now, if P acts from $V(I_\alpha)$ to $V(J_\beta)$, Q acts from $V(I_\mu)$ to $V(J_\beta)$, $\mu \ge \alpha$, and the domain of action for P is from the left of the domain of action for Q, then

$$\operatorname{Indef}(Q) \supset \operatorname{Im} P.$$

D. Exactness in pendent points. Suppose that the extended intervals $\widetilde{I}_\alpha$ and $\widetilde{I}_{\alpha+1}$ have a common endpoint k.

Theorem 2.1. $C \cdot p(\overline{SL}_n) = C \cdot p(SL_n)$.

We omit the proof of this theorem based on the arguments of Subsections 1.5–1.6.

Remark. This Theorem is valid only for irreducible representations.

2.6. The action of $\overline{SL}_n$ on flag spaces. Let S be a set of integers $s_1, \dots, s_\alpha$ such that $0 < s_1 < s_2 < \dots < s_\alpha < n$. Suppose $\mathcal{F}_S$ is the flag space of type S in $\mathbb{C}_n$. A point of this space is the set of subspaces $V_1, \dots, V_\alpha$ such that $V_1 \subset V_2 \subset \dots \subset V_\alpha$ with $\dim V_j = s_j$. We define the flag space $\mathcal{F}$ to be $\bigcup \mathcal{F}_S$, the union being taken over all sets S (including the empty one).

If $S' \supset S$, then the natural projection $P_S^{S'} : \mathcal{F}_{S'} \to \mathcal{F}_S$ is defined (we simply drop the unnecessary subspaces). Define a (not separated) topology on $\mathcal{F}$. The sequence $v^{(1)}, v^{(2)}, \dots$ is said to converge to the point $v \in \mathcal{F}_S$ if

a) if $v^{(i)} \in \mathcal{F}_{S^{(i)}}$, then $S^{(i)} \supset S$ for sufficiently large i;

b) $P_S^{S^{(i)}} v^{(i)} \to v$ in the natural topology of $\mathcal{F}_S$.

Thus, the closure of the point $v = (V_1, \dots, V_s) \in \mathcal{F}$ consists of all sequences obtained by dropping some subspaces from v.

Now let $\pi = (P_1, \dots, P_s) \in \overline{SL}_n$, $v = (V_1, \dots, V_t) \in \mathcal{F}$. Define an element $\pi v \in S$. Toward this end, consider the set of all pairs (P_α, V_β) such that

$$\operatorname{Ker} P_\alpha \cap V_\beta = \varnothing, \qquad \mathcal{D}(P_\alpha) + V_\beta = \mathbb{C}^n.$$

For each pair of this type we consider the subspace $P_\alpha V_\beta \subset \mathbb{C}^n$ given by $P_\alpha V_\beta = \{x \in \mathbb{C}^n \mid \exists y\ (x, y) \in P_\alpha, y \in V_\beta\}$. It is easy to verify that the set of subspaces $P_\alpha V_\beta$ is a flag. Thus, we have constructed the action of $\overline{SL}_n$ on $\mathcal{F}$.

2.7. The action of $\overline{SL}_n$ on the "completed quadric space" $\overline{GL_n/O_n}$. Suppose that a space $V \simeq \mathbb{C}^n$ is equipped with a nondegenerate symmetric bilinear form $B(\cdot, \cdot)$. Introduce a nondegenerate skew-symmetric bilinear form on $V \oplus V$ by the formula

$$C((v, w), (v', w')) = B(v, w') - B(v', w).$$

By definition, the space $\overline{GL_n/O_n}$ consists of resolving sequences $\gamma = (Q_1, \dots, Q_k)$ in the space V such that any Q_j is a maximal isotropic subspace in $V \oplus V$ with respect to the form $C(\cdot, \cdot)$.

Introduce an antiautomorphism $\pi \mapsto \pi^t$ in $\overline{SL}_n$ as follows. If $\pi = (P_1, \dots, P_k)$, then we set $\pi^t = (P_1^t, \dots, P_k^t)$, where P_j^t is the orthogonal complement to P_j with respect to the form $C(\cdot, \cdot)$.

The semigroup $\overline{SL}_n$ acts on $\overline{GL_n/O_n}$ by the formula

$$\pi : \gamma \mapsto \pi^t \gamma \pi.$$

Remark. Consider the subspace S in $\overline{GL_n/O_n}$ that consists of all exact resolving sequences. Then S is the completed quadric space defined by Semple (see [16–17]).

§3. The Big Category LA

In § 2 we have constructed natural operators which act on irreducible representations of SL_n. Similarly, we can construct some operators relating representations of different groups SL_n and SL_m and, moreover, those of different groups of the form $\bigoplus_i SL_{n_i}$.

3.1. Objects of the category LA. We define an object of the category LA to be a finite subset A of $\mathbb{Z}$. The points of A are said to be genuine points of the object A. An interval $I = [\alpha, \alpha + k]$ of the object A is a maximal subset of the form $\alpha, \alpha + 1, \dots, \alpha + k$ contained in A. An extended interval $\widetilde{I}$ is defined as the set $\alpha - 1, \alpha, \dots, \alpha + k + 1$. The points $\alpha - 1$ and $\alpha + k + 1$ are called adjoined points of the interval I and of the object as well. We denote by $\widetilde{A}$ the set of all the genuine and adjoined points of the object A. If α is an adjoined point and the points $\alpha \pm 1$ are genuine, then α is called a pendent point.

It is convenient to represent the objects by diagrams

Note that this definition is a bit ambiguous for pendent points; namely, if $k \in \tilde{I}_\alpha$ and $k \in \tilde{I}_{\alpha+1}$ simultaneously, then our definition implies that $M_k(A)$ is simultaneously equal to $\Lambda^0(V(I_{\alpha+1}))$ and to $\Lambda^{\dim V(I_\alpha)}(V(I_\alpha))$. However, these spaces are one-dimensional and so they can be identified arbitrarily (because our representations are projective).

Define an operator $\mu_k(\pi)\colon M_k(A) \to M_k(B)$ for each morphism $\pi = (P_1, \dots, P_s)\colon A \to B$. We set $\mu_k(\pi) = 0$ if k is not contained in the domain of action for any relation P_j. Suppose, that k belongs to the domain of action for some $P_j\colon V(I_\alpha) \rightrightarrows V(J_\beta)$. Then $\mu_k(\pi)$ is the natural mapping of exterior powers related to the linear relation P_j.

Remark. The last assertion may seem ambiguous for a pendent point k of A or B. However, the mapping is well defined by virtue of exactness in pendent points.

It is easy to check that (M_k, μ_k) is a projective representation of the category LA.

3.5. Representations of the category LA. Suppose that h_k $(k \in \mathbb{Z})$ is a sequence of nonnegative integers with a finite number of nonzero elements. We use this sequence to construct an irreducible representation (T_h, τ_h) of the category LA. Toward this end, we consider the tensor product $S_h = \bigotimes_k M_k^{\otimes h_k}$ and take a highest weight vector $v(A)$ in any $\bigotimes_k M_k(A)^{\otimes h_k}$; then we form the cyclic subspace $T_h(A)$ generated by the vector $v(A)$ under the action of the group $\mathrm{Aut}_{\mathrm{LA}}(A)$. The collection of subspaces $T_h(A) \subset S_h(A)$ determines an irreducible subrepresentation in $S_h(A)$.

Seemingly, these are all the irreducible representations of the category LA.

§4. Completions of Groups Sp_{2n} and SO_k

We restrict ourselves only to describing the universal semigroups.

4.1. The case of groups Sp_{2n}. Let V be a $2n$-dimensional complex space equipped with a nondegenerate skew-symmetric bilinear form $\{\cdot, \cdot\}$. The group Sp_{2n} consists of operators which preserve $\{\cdot, \cdot\}$. The group Sp_{2n} is included in a larger semigroup $\mathrm{End}_C(V)$ which consists of endomorphisms of objects of the category C (the category C is defined in [6]). This semigroup is a subsemigroup in $\mathrm{GL}^*(V)$ and consists of null and of the maximal isotropic (lagrangian) subspaces in $V \oplus V$ with respect to the form

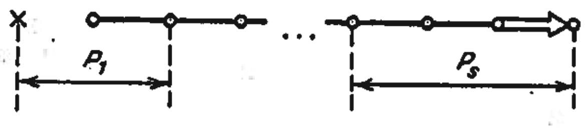
$$\{(v, w), (v', w')\}_{V\oplus V} = \{v, v'\} - \{w, w'\}. \tag{4.1}$$

The semigroup Sp_{2n} consists of correct resolving sequences $\pi = (P_1, \dots, P_s)$, where $P_j \in \mathrm{GL}^*_{2n}$ satisfy some additional conditions. Namely, the elements P_j are of two kinds:

1) $P_j \in \mathrm{End}_C(V)$;
2) $\mathrm{Ker}\, P_j$ is coisotropic and $\mathrm{Im}\, P_j$ is isotropic.

Note that the elements of the first kind may be absent; otherwise, the only element of this kind, containing in the sequence $\pi = (P_1, \dots, P_s)$, is P_s.

It is convenient to represent the elements of $\overline{\mathrm{Sp}}_{2n}$ by diagrams of the form



The domains of action of linear relations P_j are the same as for $\overline{\mathrm{SL}}_{2n}$ (although the action of P_s on the rightest fundamental representation requires a more detailed discussion).

4.2. The groups $\overline{\mathrm{SO}}_{2n+1}$. These groups can be treated similarly verbatim.

4.3. The case of groups $\overline{\mathrm{SO}}_{2n}$. Let V be a complex linear space of even dimension endowed with a nondegenerate symmetric bilinear form $\{\cdot, \cdot\}$. Let us introduce a bilinear form on $V \oplus V$ by means of (4.1). Consider the Grassmannian Gr of maximal isotropic subspaces in $V \oplus V$. The group $\mathrm{SO}(V)$ is naturally embedded into Gr. It is easy to check that Gr has two connected components and that the group $\mathrm{SO}(V)$ is dense in one of them; denote this component by Gr_+.

Then we require the following conditions to be equivalent:

a) k is contained in the domain of action of a relation $P\colon V(I_\alpha) \rightrightarrows V(J_\beta)$, $P \in \pi$;

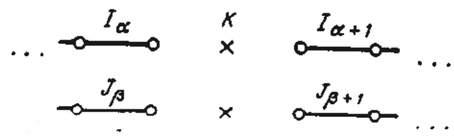
b) k is contained in the domain of action of a relation $Q\colon V(I_{\alpha+1}) \rightrightarrows V(J_\beta)$, $Q \in \pi$.

Similarly, suppose that extended intervals $\widetilde{J}_\beta$ and $\widetilde{J}_{\beta+1}$ have a common pendent point k. Then we require equivalence of the following conditions:

a) k is contained in the domain of action of a relation $P\colon V(I_\alpha) \rightrightarrows V(J_\beta)$, $P \in \pi$;

b) k is contained in the domain of action of a relation $Q\colon V(I_\alpha) \rightrightarrows V(J_{\beta+1})$, $Q \in \pi$.

Example. In the case



the collection π either does not contain any relation whose domain of action includes k or contains four relations of this type:

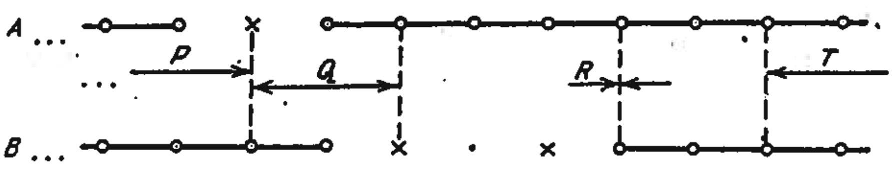
$$V(I_\alpha) \rightrightarrows V(J_\beta), \quad V(I_\alpha) \rightrightarrows V(J_{\beta+1}), \quad V(I_{\alpha+1}) \rightrightarrows V(J_\beta), \quad V(I_{\alpha+1}) \rightrightarrows V(J_{\beta+1}).$$

E. Correctness (as in § 2, this condition is reasonable from the aesthetic view-point). Suppose that the relations P, $Q \in \pi$ act from $V(I_\alpha)$ to $V(J_\beta)$. We require the following conditions:

a) the domain of action for P does not contain that of Q;

b) the neighboring domains of action for relations $V(I_\alpha) \rightrightarrows V(J_\beta)$ either have a contact point or they are at a distance not less then 2.

It is convenient to represent morphisms of the category LA by diagrams



arrows denoting the domains of action of linear relations from the collection π.

3.3. Multiplication of morphisms. Suppose that A, B, C are the objects of LA and I_α, J_β, K_γ are intervals of A, B, C. Let $\pi\colon A \to B$, $\varkappa\colon B \to C$ be morphisms of the category LA, $\pi = (P_1, \dots, P_s)$, $\varkappa = (Q_1, \dots, Q_t)$. Consider all the products of the form $Q_i P_j$, where P_j acts from $V(I_\alpha)$ to $V(J_\beta)$ and Q acts from $V(J_\beta)$ to $V(K_\gamma)$. Dropping all the elements equal to the null from the collection $Q_i P_j$ we obtain the collection of linear relations which satisfies all the conditions that are to be satisfied by a morphism $A \to C$, except the condition a) of correctness. As in 2.5, we obtain a morphism of the category LA from A to C by dropping unnecessary relations from $Q_i P_j$.

Remark. The group $\mathrm{Aut}_{\mathrm{LA}}(A)$ of automorphisms of an object A is isomorphic to $\bigoplus_\alpha \mathrm{GL}(V(I_\alpha))$, where I_α runs over all intervals of the object A.

3.4. Fundamental representations. Let $k \in \mathbb{Z}$. Assign a linear space $M_k(A)$ to each object A of the category LA.

If a point k is neither genuine nor adjoined for the object A, then $M_k(A)$ is zero-dimensional. Now we define $M_k(A)$ at other points.

Let I_α be an interval in A, $I_\alpha = [a, a+m-1]$. Then for all k such that $a-1 \le k \le a+m$ we set

$$M_k(A) := \Lambda^{k-a+1}(V(I_\alpha)).$$

Define a semigroup $\mathrm{End}_{\mathcal{D}}(V)$ to be the subsemigroup in $\mathrm{End}_{\mathrm{GA}}(V)$ that consists of null and of the elements of Gr_+ (on the category $\mathcal{D}$ see [6]).

The semigroup $\overline{\mathrm{SO}_{2n}}$ consists of collections of linear relations $\pi = (P_1, \dots, P_s)$ from V satisfying some additional conditions given below.

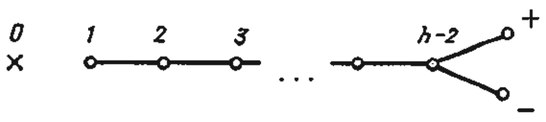
There are elements P_j of two kinds:

1) $P_j \in \mathrm{End}_{\mathcal{D}}(V)$;

2) $\mathrm{Ker}\, P_j$ is coisotropic, $\mathrm{Im}\, P_j$ is isotropic, and in the case $\dim \mathrm{Ker}\, P_j = \dim \mathrm{Im}\, P_j = n$ the dimension of the subspace $\mathrm{Ker}\, P_j \cap \mathrm{Im}\, P_j$ is even (this complication appears, because we deal with SO_{2n} instead of O_{2n}).

There are domains of action of two kinds for linear relations P_j: i.e., natural and prescribed domains.

Enumerate the fundamental representations σ_α of the group SO_{2n} as follows ($\alpha = 1, 2, \dots, n-2, +, -$):



σ_0 is a representation of SO_{2n} in $\Lambda^0(V)$.

Linear relations of the type 1) have natural domains of action if $\dim \mathrm{Ker}\, P_j \le n-3$ or, equivalently, if $\dim \mathrm{Indef}(P_j) \le n-3$. In this case the domain of action consists of $\sigma_\pm$ and all the σ_k $(k \in \mathbb{Z})$ satisfying the condition $k \ge \dim \mathrm{Ker}\, P_j$. Linear relations P_j of type 2) have natural domains of action if $\dim \mathrm{Indef}(P_j) \le n-3$ and also if $\dim \mathrm{Indef}(P_j) = \dim \mathrm{Im}\, P_j = n-2$. In this case the domain of action contains all the σ_k $(k \in \mathbb{Z})$ such that $0 \le k \le n-2$, $\dim \mathrm{Indef}(P_j) \le k \le \dim \mathrm{Im}\, P_j$. If $\dim \mathrm{Im}\, P_j > n-2$, then the domain of action also contains σ_+ and σ_-.

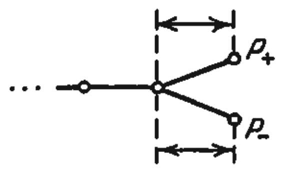
Prescribed domains of action occur only if $\dim \mathrm{Indef}(P_j) \ge n-2$. If $\dim \mathrm{Indef}(P_j) = n-2$, then the domain of action contains σ_{n-2}. The domain of action also contains σ_+ or σ_-. We stress that the domain of action is not determined completely by P_j only; we must also keep in mind whether it acts in σ_+ or in σ_-. Recall once more that in the case $\dim \mathrm{Indef}(P_j) = \dim \mathrm{Im}\, P_j = n-2$ the domain of action is natural.

Now we pass to the list of conditions on $\pi = (P_1, \dots, P_s)$.

a) The collection π is a resolving one, i.e., if the domain of action for P_j is from the left of the domain of action for P_k, then

$$\mathrm{Ker}\, P_j \supset \mathcal{D}(P_k), \qquad \mathrm{Im}\, P_j \subset \mathrm{Indef}(P_k).$$

Besides, if the domains of action for P_+ and P_- are disposed as follows:



then $\mathrm{Indef}(P_+) = \mathrm{Indef}(P_-)$, $\mathcal{D}(P_+) = \mathcal{D}(P_-)$.

b) The collection $\{P_j\}$ is correct, i.e., the domain of action for P_α is not contained in the domain of action for P_β and either two neighboring domains of action have a contact point or they are at a distance not less than 2.

4.4. Representations of the semigroups $\overline{\mathrm{Sp}_{2n}}$, $\overline{\mathrm{SO}_{2n+1}}$, $\overline{\mathrm{SO}_{2n}}$. In this case an analog of Theorem 2.1 is valid. The main step in the construction of representation of this semigroups is the construction of fundamental representations. The principal difficulty lies in the construction of the action of linear relations in spinor representations (partly this was done in [4, 6]; however, in our case this construction is inadequate, because it can occur that the operators $\pi_\pm(P_s)$ are not Berezin operators in the sense of [4]).

The construction of the big category LA and of the action of SL_n on flag spaces can be also extende to the three other series.

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Translated by R. S. Ismagilov