Amer. Math. Soc. Transl.
(2) Vol. **137**, 1987

The Variety A_n of n-Dimensional Lie Algebra Structures

A. A. KIRILLOV AND YU. A. NERETIN

1. In many applications of Lie groups and Lie algebras, a situation arises where the group or algebra involved may depend on the parameters of the problem. In such cases it is useful to know the structure of the set of all Lie algebras of a given dimension (at least in a neighborhood of a given point). Of particular interest would be a description of the possible limit passages between Lie algebra structures.

One way of making a precise formulation of this problem is the following. We suppose an n-dimensional space V over a field K, with fixed basis $X_1, \dots, X_n$. To define a Lie algebra structure on V, we must specify the commutators of the basis vectors, i.e., the constants c_{ij}^k in the equations([1])

$$[X_i, X_j] = c_{ij}^k X_k. \tag{1}$$

These constants are not arbitrary. They are subject to two sets of conditions:

$$c_{ij}^k = -c_{ji}^k \quad \text{(skew-symmetry)} \tag{2}$$

and

$$c_{ij}^l c_{lk}^m + c_{jk}^l c_{li}^m + c_{ki}^l c_{lj}^m = 0 \quad \text{(Jacobi identity)}. \tag{3}$$

Equations (2) and (3) determine a certain algebraic variety in n^3-dimensional affine space with coordinates c_{ij}^k, $1 \le i, j, k \le n$. We denote this variety by A_n and call it the variety of structure constants of n-dimensional Lie algebras.

By definition, A_n lies in the tensor space of type $(1,2)$ over V, i.e., in $V \otimes V^* \otimes V^*$. The group $\mathrm{GL}(V) = \mathrm{GL}(n, K)$ acts on this space and takes the variety A_n into itself. The orbits of $\mathrm{GL}(V)$ in A_n correspond to the isomorphism classes of n-dimensional Lie algebras: two sets of structure constants generate isomorphic

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 17B05; Secondary 14M99.
Translation of Some Questions in Modern Analysis (V. M. Tikhomirov, editor), Mekh.-Mat. Fak. Moskov. Gos. Univ., Moscow, 1984, pp. 42–56.

([1])We use the standard technique for writing tensor expressions: a repeated index indicates summation.

Lie algebras if and only if they go into each other under action of the group $\mathrm{GL}(V)$. The orbit space will be denoted by L_n.

We are concerned with the structure of the variety A_n; in particular, with answers to the following questions:

(1) What irreducible components does A_n fall into?

(2) What are the dimensions and degrees of these components?

(3) What are their generic points?

The answers to these questions would allow us to describe the structures of the "typical" Lie algebras of dimension n.

Many important characteristics of Lie algebras (e.g., dimension of the center and of the commutator ideal, solvability or nilpotency class, index, etc.) are locally constant on A_n and therefore almost constant on each component (i.e., constant on the complement of a variety of lower dimension). An explicit computation of these characteristics would also be of considerable interest.

Unfortunately, answers to all these questions are in general unknown. In this paper we shall give a description of the components for small dimensions ($n \leq 6$) and present certain arguments and estimates for the general case. The ground field is taken everywhere to be the field $\mathbf{C}$ of complex numbers.([2])

2. Complete answers to the questions in §1 are known only in the two simplest cases $n = 2$ and $n = 3$.

For $n = 2$, condition (3) is satisfied automatically, and there remains only the linear condition (2). Therefore the variety A_2 is the affine plane $\mathbf{C}^2$. Relative to the action of the group $\mathrm{GL}(2)$ this plane splits into two orbits: the origin $\{0\}$ and its complement $\mathbf{C}^2 \setminus \{0\}$. Thus, the space L_2 consists of two points: a "large" open point, for which a representative is the Lie algebra $\mathrm{aff}(1)$ of the group of affine transformations of a one-dimensional space; and a "small" closed point, for which a representative is the commutative algebra $\mathbf{C}^2$.

For $n = 3$, it is convenient to make use of the decomposition of the space of tensors of the form c_{ij}^k satisfying condition (2) into a sum of two irreducible subspaces relative to the action of $\mathrm{GL}(3)$.

Namely, consider the space of vectors $a_m = c_{mk}^k$ and the space of symmetric tensors (more precisely, tensor densities)

$$s^{kl} = \tfrac{1}{2}(\varepsilon^{ijk} c_{ij}^l + \varepsilon^{ijl} c_{ij}^k),$$

where ε^{ijk} is the standard antisymmetric tensor of rank 3. The original tensor c_{ij}^k can be reconstructed from a_m and s^{kl}:

$$c_{ij}^k = \tfrac{1}{2}(\delta_j^k a_i - \delta_i^k a_j + \varepsilon_{ijl} s^{kl}).$$

In the "coordinates" a_m and s^{kl}, condition (3) takes the simple form

$$s^{km} a_m = 0. \tag{4}$$

([2])As regards the classification of real Lie algebras, see [1] and [2].

The variety A_3 defined by the system of equations (4) splits into two six-dimensional components:

$$A_3^{(1)}: a_m = 0, \ s^{kl} \text{ arbitrary.}$$

$$A_3^{(2)}: \det(s^{kl}) = 0, \ s^{km} a_m = 0.$$

The first component is given by the linear equations $a_m = 0$ and is isomorphic to the affine space $\mathbf{C}^6$ (and consequently has degree 1). Relative to the action of GL(3) the points of this component behave like symmetric tensor densities and split into four orbits Ω_i, $0 \le i \le 3$, according to the rank of the matrix s^{kl}.

The dimensions of these orbits are respectively 0, 3, 5, 6. For representatives we can take the following Lie algebras:

(1) In Ω_3: the Lie algebra sl(2) of 2×2 matrices with zero trace.

(2) In Ω_2: the Lie algebra m(2) of the group of motions of the euclidean plane.

(3) In Ω_1: the Heisenberg algebra Γ_3 with generators X, Y, Z and commutator $[X, Y] = Z$.

(4) In Ω_0: the commutative Lie algebra $\mathbf{C}^3$.

The second component has degree 7, since the whole variety A_3 is given in the 9-dimensional space with coordinates a_m, s^{kl} by three independent quadratic equations and is therefore of degree 8.

Thus, the component $A_3^{(2)}$ is seven times more "massive" than the component $A_3^{(1)}$. This can be given a precise meaning in two ways (see [3]). First, almost every three-dimensional linear subvariety in the 9-dimensional space with coordinates a_m, s^{kl} intersects the first component in one point, but intersects the second in seven. Second, relative to the natural volume form in 8-dimensional complex projective space, the image of the first component has volume 1; the image of the second, volume 7.

Let us observe also that the component $A_3^{(2)}$ is not a complete intersection, i.e., cannot be given by three independent equations.

The intersection $A_3^{(1)} \cap A_3^{(2)}$ is the union of the orbits Ω_0, Ω_1, and Ω_2 of the group GL(3).

On the complement of this intersection in $A_3^{(2)}$ the group GL(3) acts so as to preserve the invariant

$$J = \frac{c_{il}^k c_{kj}^l}{c_{ik}^k c_{jl}^l} = \frac{\operatorname{tr}(\operatorname{ad} X_i \operatorname{ad} X_j)}{\operatorname{tr} \operatorname{ad} X_i \operatorname{tr} \operatorname{ad} X_j}. \tag{5}$$

(It can be verified that this expression is independent of the choice of the indices i and j.) A level surface $J = c$ for $c \neq \frac{1}{2}$ is a GL(3)-orbit, for which a representative is the Lie algebra

$$[X, Y] = \lambda Y, \quad [X, Z] = \mu Z, \quad [Y, Z] = 0,$$

where $(\lambda^2 + \mu^2)/(\lambda + \mu)^2 = c$. The surface $J = \frac{1}{2}$ splits into two GL(3)-orbits.

3. For the varieties A_n with $4 \leq n \leq 6$, all that is known is a description of the irreducible components and their dimensions.(3) We exhibit this result here in the form of a table. The five columns of this table list respectively:*

(1) The notation for the component $A_n^{(k)}$.

(2) The dimension of the component.

(3) The codimension of the generic $\mathrm{GL}(V)$-orbit (i.e., the dimension of the corresponding component in L_n).

(4) The structure of the nilpotent radical, coinciding with the commutator ideal $[\mathfrak{g}, \mathfrak{g}]$, of the generic Lie algebra g in the given component (when this algebra is solvable).

(5) A representative of the generic $\mathrm{GL}(V)$-orbit. Note that the number of parameters on which this representative depends is not always chosen smallest possible; the smallest number is given in column 3.

The notation for the nilpotent Lie algebras in column 4 is explained in the next section.

Let us point out also how to find, for a given Lie algebra $\mathfrak{g}$, at least one component $A_n^{(k)}$ that contains it. If $\mathfrak{g}$ is not solvable, then it either appears in column 5 or is isomorphic to $\mathrm{sl}(2) \ltimes \mathbf{C}^3$ and then lies in $A_6^{(1)}$. If $\mathfrak{g}$ is solvable, then its maximal proper nilpotent ideal appears in column 4.

4. The description of the components of A_n for small n is based on certain properties of solvable and nilpotent Lie algebras of small dimension. We present the relevant facts in this section.

THEOREM 1. (a) *There exist only finitely many nonisomorphic nilpotent Lie algebras of dimension* ≤ 6.

(b) *Every such algebra admits a nondegenerate derivation.*

A proof of (a) is given in [4]. It was found there that for $n \leq 6$ all n-dimensional nilpotent Lie algebras can be obtained by simple limit passages from a single Lie algebra Γ_n.

To describe Γ_n, consider a graded n-dimensional Lie algebra with homogeneous basis $\{X_i\}$, where the index i runs from 1 to n for $n \leq 5$ and takes the values 1, 2, 3, 4, 5, 7 for $n = 6$. The commutation law is

$$[X_i, X_j] = a_{ij} X_{i+j}. \tag{6}$$

The Jacobi identity is satisfied automatically for $n \leq 5$, and for $n = 6$ reduces to the single relation

$$a_{12}a_{34} + a_{41}a_{52} = 0. \tag{7}$$

(3)For $n = 6$ this result is due to Yu. A. Neretin. The case $n = 5$ was worked out in a course paper of S. E. Belkin (1976, unpublished).

* *Translator's note.* The numbering of the columns here differs from their order in the table.

	k	$[\mathfrak{g},\mathfrak{g}]$	$\dim A_n^{(k)}$	Generic algebra (representation of the generic GL(V)-orbit)	Number of parameters in $L_n^{(k)}$
A_2	1	$\mathbf{C}^2$	2	$\mathrm{aff}(1)$	0
A_3	1	–	6	$\mathrm{sl}(2,\mathbf{C})$	0
	2	$\mathbf{C}^2$	6	$[X,Y_i]=\lambda_i Y_i,\ i=1,2$	1
A_4	1	–	12	$\mathrm{sl}(2,\mathbf{C})\oplus\mathbf{C}$	0
	2	$\mathbf{C}^3$	12	$[X,Y_i]=\lambda_i Y_i,\ i=1,2,3$	2
	3	Γ_3	12	$[X,Y_i]=\lambda_i Y_i,\ i=1,2,\ [Y_1,Y_2]=Z$	1
	4	$\mathbf{C}^2$	12	$[X,Z]=(\lambda_1+\lambda_2)Z$ $\mathrm{aff}(1)\oplus\mathrm{aff}(1)$	0
A_5	1	–	19	$\mathrm{sl}(2,\mathbf{C})\ltimes\mathbf{C}^2$	0
	2	–	20	$\mathrm{sl}(2,\mathbf{C})\oplus\mathrm{aff}(1)$	0
	3	$\mathbf{C}^4$	20	$[X,Y_i]=\lambda_i Y_i,\ 1\le i\le 4$	3
	4	$\Gamma_3+\mathbf{C}$	20	$[X,Y_i]=\lambda_i Y_i,\ 1\le i\le 4,\ \lambda_1+\lambda_2=\lambda_3$ $[Y_1,Y_2]=Y_3$	2
	5	Γ_4	20	$[X,Y_i]=\lambda_i Y_i,$ $\lambda_3=\lambda_1+\lambda_2,\ \lambda_4=2\lambda_1+\lambda_2$ $[Y_1,Y_2]=Y_3,\ [Y_1,Y_3]=Y_4$	1
	6	Γ_3	20	$[X_1,Y_1]=Y_1,\ [X_2,Y_2]=Y_2$ $[Y_1,Y_2]=[X_1,Y_3]=[X_2,Y_3]=Y_3$	0
	7	$\mathbf{C}^3$	21	$[X_i,Y_j]=\lambda_{ij}Y_j,\ i=1,2,\ j=1,2,3$	2
A_6	1	–	30	$\mathrm{sl}(2,\mathbf{C})\oplus\mathrm{sl}(2,\mathbf{C})$	0
	2	–	30	$\mathrm{sl}(2,\mathbf{C})\oplus\mathfrak{g},\ \mathfrak{g}\in A_3^{(2)}$	1
	3	–	30	$\mathrm{aff}(2)=\mathrm{gl}(2,\mathbf{C})\ltimes\mathbf{C}^2$	0
	4	–	30	$\mathrm{sl}(2,\mathbf{C})\ltimes\Gamma(3)$	0
	5	$\mathbf{C}^5$	30	$[X,Y_i]=\lambda_i Y_i,\ 1\le i\le 5$	4
	6	$\Gamma_3+\mathbf{C}^2$	30	$[X,Y_i]=\lambda_i Y_i,\ 1\le i\le 5,\ \lambda_3=\lambda_1+\lambda_2$ $[Y_1,Y_2]=Y_3$	3
	7	$\Gamma_4+\mathbf{C}$	30	$[X,Y_i]=\lambda_i Y_i,\ 1\le i\le 5,\ \lambda_3=\lambda_1+\lambda_2,$ $\lambda_4=2\lambda_1+\lambda_2,\ [Y_1,Y_2]=Y_3,\ [Y_1,Y_3]=Y_4$	2
	8	$\Gamma_{5,1}$	30	$[X,Y_i]=\lambda_i Y_i,\ 1\le i\le 5,$ $gl_1+\lambda_4=\lambda_2+\lambda_3=\lambda_5$ $[Y_1,Y_4]=[Y_2,Y_3]=Y_5$	2
	9	$\Gamma_{5,2}$	30	$[X,Y_i]=\lambda_i Y_i,\ 1\le i\le 5,\ \lambda_1+\lambda_2=\lambda_3,$ $\lambda_1+\lambda_4=\lambda_5,\ [Y_1,Y_2]=Y_3,\ [Y_1,Y_4]=Y_5$	2
	10	$\Gamma_{5,3}$	30	$[X,Y_i]=\lambda_i Y_i,\ 1\le i\le 5,\ \lambda_1+\lambda_3=\lambda_4,$ $2\lambda_1+\lambda_3=\lambda_5,\ \lambda_2=2\lambda_1,\ [Y_1,Y_3]=Y_4,$ $[Y_1,Y_4]=Y_5,\ [Y_2,Y_3]=Y_5$	
	11	$\Gamma_{5,4}$	30	$[X,Y_i]=\lambda_i Y_i,\ 1\le i\le 5,\ \lambda_3=\lambda_1+\lambda_2,$ $\lambda_4=2\lambda_1+\lambda_2,\ \lambda_5=2\lambda_2+\lambda_1$ $[Y_1,Y_2]=Y_3,\ [Y_1,Y_3]=Y_4,\ [Y_2,Y_3]=Y_5$	1
	12	$\Gamma_{5,5}$	30	$[X,Y_i]=\lambda_i Y_i,\ 1\le i\le 5,$ $\lambda_k=\lambda_2+(k-2)\lambda$ $[Y_1,Y_k]=Y_{k+1},\ k=2,3,4$	1
	13	$\Gamma_{5,6}$	30	$[X,Y_k]=kY_k,\ [Y_i,Y_j]=Y_{i+j}$ $i,j,k=1,2,3,4,5,\ i<j$	0
	14	$\mathbf{C}^4$	32	$[X_i,Y_j]=\lambda_{ij}Y_j,\ i=1,2,\ j=1,2,3,4$	4
	15	$\Gamma_3\oplus\mathbf{C}$	31	$[X_i,Y_j]=\lambda_{ij}Y_j,\ [Y_1,Y_2]=Y_3$ $\lambda_{i1}+\lambda_{i2}=\lambda_{i3},\ i=1,2,\ j=1,2,3,4$	2
	16	Γ_4	30	$[X_1,Y_1]=Y_1,\ [X_2,Y_2]=Y_2,$ $[Y_1,Y_2]=Y_3,\ [Y_1,Y_3]=Y_4,$ $[X_1,Y_3]=[X_2,Y_3]=Y_3,\ [X_1,Y_4]=2Y_4,$ $[X_2,Y_4]=Y_4$	0
	17	$\mathbf{C}^3$	30	$\mathrm{aff}(1)\oplus\mathrm{aff}(1)\oplus\mathrm{aff}(1)$	0

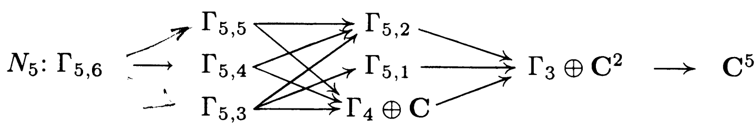
The homogeneous basis transformation $X_i \to \lambda_i X_i$ takes the set $\{a_{ij}\}$ into the set $\{(\lambda_i\lambda_j/\lambda_{i+j})a_{ij}\}$. It is easily verified that the group $(\mathbf{C}^*)^{n-1}$ acts transitively on the collection of all sets $\{a_{ij}\}$ for which $a_{ij} \neq 0$ for $i \neq j$. This means that all such sets define the same Lie algebra (up to isomorphism). This algebra is Γ_n. All other n-dimensional nilpotent Lie algebras, for $n \leq 5$, are obtained by setting certain a_{ij} equal to zero for $i \neq j$.

Part (b) of the theorem follows for $n \leq 5$ from the fact that the mapping $X_k \to kX_k$ is a nondegenerate derivation of Γ_n and of all other Lie algebras of the type (6). For $n = 6$ this part is not needed here, and we omit its proof.

We can describe now the sets N_n of isomorphism classes of nilpotent Lie algebras, of dimension $n \leq 5$, by indicating the limit passages between them:

$$N_3\colon \Gamma_3 \longrightarrow \mathbf{C}^3$$

$$N_4\colon \Gamma_4 \longrightarrow \Gamma_3 \oplus \mathbf{C} \longrightarrow \mathbf{C}^4$$



Here the $\Gamma_{5,k}$, $1 \leq k \leq 6$, are the indecomposable five-dimensional nilpotent Lie algebras in Dixmier's list [5]. They are all obtained from $\Gamma_{5,6} \cong \Gamma_5$ by a so-called contraction, i.e., by setting equal to zero certain coefficients a_{ij}:

k	The a_{ij}, $i < j$, to be set equal to zero
1	a_{12} and a_{13}
2	a_{13} and a_{23}
3	a_{12} or a_{13}
4	a_{14}
5	a_{23}
6	All a_{ij} are nonzero for $i \neq j$.

A similar analysis can be made of N_6 (see [4]). One obtains twenty types of Lie algebras that are contractions of Γ_6. For $n = 7$, the variety of isomorphism classes of nilpotent Lie algebras has positive dimension and consists of several components (see [6] and [7]). Furthermore, there exist seven-dimensional nilpotent Lie algebras all of whose derivations are nilpotent [8].

Now let $\mathfrak{n}$ be a nilpotent Lie algebra of dimension $n - k$, and $\mathbf{C}^k$ the k-dimensional commutative Lie algebra. A Lie algebra $\mathfrak{g}$ is called an extension of $\mathfrak{n}$ by $\mathbf{C}^k$ if it can be included in an exact sequence

$$0 \to \mathfrak{n} \to \mathfrak{g} \to \mathbf{C}^k \to 0.$$

The subset of A_n that corresponds to all such extensions will be denoted by $L(k, \mathfrak{n})$, and its closure by $R(k, \mathfrak{n})$.

THEOREM 2. *For $n \leq 6$, the sets $R(1, \mathfrak{n})$ are $(n^2 - n)$-dimensional components of A_n. Two components $R(1, \mathfrak{n})$ and $R(1, \mathfrak{n}')$ coincide only if $\mathfrak{n}$ and $\mathfrak{n}'$ are isomorphic.*

PROOF. Since almost all derivations of the algebra $\mathfrak{n}$ are nondegenerate, in $L(1, \mathfrak{n})$ an open set consists of those algebras $\mathfrak{g}$ for which the derived algebra $[\mathfrak{g}, \mathfrak{g}]$ has dimension $n - 1$ and is isomorphic to $\mathfrak{n}$. From this follows the second assertion. To prove the first, observe that the dimension of the space $\operatorname{Der} \mathfrak{n}$ of derivations of $\mathfrak{n}$ is equal to the dimension of the group $\operatorname{Aut} \mathfrak{n}$ of automorphisms of $\mathfrak{n}$ (since $\operatorname{Der} \mathfrak{n}$ is the Lie algebra of $\operatorname{Aut} \mathfrak{n}$). Denote this dimension by $\delta(\mathfrak{n})$. The dimension of $R(1, \mathfrak{n})$ is the sum of (1) the dimension of the variety of $(n-1)$-planes in $\mathbf{C}^n$, (2) the dimension of the variety of structure constants defining a Lie algebra isomorphic to $\mathfrak{n}$ on a given $(n-1)$-plane, and (3) the dimension of all extensions of the Lie algebra structure from the $(n-1)$-plane to the n-dimensional space. This gives in totality the number

$$n - 1 + (n-1)^2 - \delta(\mathfrak{n}) + \delta(\mathfrak{n}) = n^2 - n.$$

It remains to verify that $R(1, \mathfrak{n})$ cannot lie in $R(k, \mathfrak{n}')$ for $k > 1$. But this follows from the fact that, for $\mathfrak{g}$ in $R(k, \mathfrak{n}')$, $\dim[\mathfrak{g}, \mathfrak{g}] \leq n - k$.

THEOREM 3. *Any solvable n-dimensional Lie algebra contains a nilpotent ideal of dimension $\geq n/2$.*

PROOF. Let $\mathfrak{n}$ be the maximal nilpotent ideal. Reduce the adjoint representation of $\mathfrak{g}$ in $\mathfrak{n}$ to triangular form. If $\dim \mathfrak{n} < \operatorname{codim}_{\mathfrak{g}} \mathfrak{n}$, then there exists an $x \in \mathfrak{g} \setminus \mathfrak{n}$ for which the principal diagonal consists of zeros. This contradicts the maximality of $\mathfrak{n}$.

Thus, every component of A_n consisting of solvable Lie algebras is contained in the union of the $R(k, \mathfrak{n})$, where $k \leq n/2$.

Analysis of the irreducibility of $R(k, \mathfrak{n})$ for $k > 1$ leads to an interesting problem concerning the structure of the variety of sets of k commuting elements in a given Lie algebra. This problem is discussed in the next section.

The results presented in the table can be derived from the theorems of this section. A detailed exposition will be published elsewhere.

5. Let $\mathfrak{g}$ be a Lie algebra. Consider the space $M_k(\mathfrak{g})$ whose points are the sets $x_1, \ldots, x_k$ of k pairwise commuting elements of $\mathfrak{g}$. Clearly, $M_k(\mathfrak{g})$ is an algebraic variety. Problem: describe the irreducible components of this variety. An important special case is for $\mathfrak{g}$ to be the algebra of $n \times n$ matrices with the ordinary definition of commutator. The corresponding variety will be denoted by $M_{k,n}$. It is proved in [9] that $M_{2,n}$ is irreducible for every n. The generic point of this variety is the pair $(A, p(A))$, where A is an arbitrary matrix and p an arbitrary polynomial of degree $n - 1$.

For our purposes we need to know the irreducibility of the variety $M_{3,3}$. This follows from a more general assertion:

THEOREM 4. *The variety $M_{k,3}$ is irreducible for all k.*

The proof is based on the following fact.

LEMMA. *In the space of third-order matrices, the maximal commutative subalgebras have dimension* 3.

Indeed, suppose $(x_1, \dots, x_k) \in M_{k,3}$. Among the matrices $x_1, \dots, x_k$, and 1, no more than three are linearly independent. Therefore we can suppose that $x_i = \lambda_i x_1 + \mu_i x_2 + \nu_i \cdot 1$ for $i > 2$.

In view of Gerstenhaber's result [9], the pair (x_1, x_2) can be represented as the limit of pairs of the form $(y, p(y))$. Then our chosen set is the limit of sets of the form

$$(y, p(y), p_3(y), \dots, p_k(y)),$$

where $p_i(y) = \lambda_i y + \mu_i p(y) + \nu_i \cdot 1$. This implies that $M_{k,3}$ consists of a single component.

A similar proof shows that $M_{k,2}$ is irreducible for every k.

The variety $M_{4,4}$ contains at least two components: in one, a dense set consists of the sets of matrices reducible simultaneously to diagonal form; in the other, to the form

$$\left(\begin{array}{c|c} \lambda \cdot 1 & A \\ \hline 0 & \lambda \cdot 1 \end{array}\right)$$

with cells of order two.

6. With increasing n, the number $s(n)$ of components of the variety A_n increases rather quickly.

THEOREM 6. $e^{\sqrt{n}} < s(n) < 2^{n^4/6}$.

The upper bound is obtained from trivial considerations: the number of components cannot exceed the degree of the whole variety, which in turn cannot exceed the product of the degrees of the equations defining it.

To obtain the lower bound, observe that if the n-dimensional Lie algebra $\mathfrak{g}$ has second cohomology group $H^2(\mathfrak{g}, \mathfrak{g})$ equal to zero (i.e., if $\mathfrak{g}$ has no nontrivial deformations), then the the set $R(0, \mathfrak{g})$ (see §4) is a component of A_n. An example of such an algebra is the semidirect product $\mathrm{sl}(2, \mathbf{C}) \ltimes \mathbf{C}^{2N}$, where $\mathbf{C}^{2N}$, regarded as an sl(2)-module, is the sum of even-dimensional irreducible submodules. (This is connected with the fact that the tensor product of two even-dimensional irreducible representations of sl(2) splits into a sum of odd-dimensional irreducible components.) The number of such algebras is equal to the number $p(N)$ of partitions of N into unordered summands. By the Hardy-Ramanujan formula,

$$p(N) \sim \frac{1}{4N\sqrt{3}} e^{\pi\sqrt{2N/3}},$$

which provides the required lower bound if we take $n = 2N + 3$.

The dimensions of the irreducible components become more diversified with increasing n than might appear from our table.

The example of the algebra $\mathfrak{g} = \mathrm{sl}(2) \ltimes (\mathbf{C}^2)^N$ shows that A_n has a component with dimension of order $\frac{3}{4}n^2$. It can be shown that there is no component whose dimension increases more slowly than $\frac{3}{4}n^2$.

On the other hand, there exist components of A_n whose dimensions are of order $\frac{2}{27}n^3$. These are components of the form $R(1, \mathfrak{n})$, where $\mathfrak{n}$ is an extension of $\mathbf{C}^{2N}$ by $\mathbf{C}^N$. A simple calculation shows that $\dim R(1, \mathfrak{n})$ is of order $2N^3$, while $\mathfrak{g}$ has dimension $3N + 1$.

We conjecture that the whole variety A_n has dimension of order $\frac{2}{27}n^3$ for large n; but up to now no better bound has been obtained for the order of $\dim A_n$ than the trivial bound $n^3/2$, which follows from the skew-symmetry of the structure constants.

7. We comment here on some problems closely related to the description of the variety A_n.

First, there is the problem of describing the associative commutative algebra structures on a given n-dimensional space V.

This problem has been studied in [10] for $n \leq 5$.

From the point of view of the theory of representations of a symmetric group, a Lie algebra structure is dual to an associative commutative algebra structure.

A natural generalization of these two structures is the Lie superalgebra structure, defined on a $\mathbf{Z}_2$-graded space V of dimension (p, q).

A description of the variety of these structures even for small dimensions would be of great interest, since the Lie superalgebras are more and more frequently finding applications in current research, both in mathematics and in mathematical physics. Only the very first steps have been made in this direction.([4])

The problem remains unsolved of computing the degrees of the irreducible components $A_n^{(k)}$ for $n \geq 4$. For example, for $n = 4$ all four components $A_n^{(k)}$ have the same dimension. Therefore the answer to the question of what is the "typical" 4-dimensional Lie algebra depends on the degrees of these components.

Attempts to compute the Hilbert polynomial for these varieties by the methods of representation theory have so far not produced any result.

BIBLIOGRAPHY

1. G. M. Mubarakzyanov, *Classification of real structures of Lie algebras of fifth order*, Izv. Vyssh. Uchebn. Zaved. Mat. **1963**, no. 3 (34), 99–106. (Russian)

2. J. Patera, R. T. Sharp, P. Winternitz, and H. J. Zassenhaus, *Casimir operators of subalgebras of the Poincaré Lie algebra and of real Lie algebras of low dimension*, Group Theoretical Methods in Physics (Fourth Internat. Colloq., Nijmegen, 1975), Lecture Notes in Phys., Vol. 50, Springer-Verlag, 1976, pp. 500–515.

3. David Mumford, *Algebraic geometry*. I: *Complex projective varieties*, Springer-Verlag, 1976.

4. V. V. Morozov, *Classification of nilpotent Lie algebras of sixth order*, Izv. Vyssh. Uchebn. Zaved. Mat. **1958**, no. 4 (5), 161–171. (Russian)

([4])The diploma paper [Russian equivalent of a Master's thesis] of A. S. Sergeev (1979) studied the case $p = 3$, $q = 2$.

5. J. Dixmier, *Sur les représentations unitaries des groupes de Lie nilpotents.* III, Canad. J. Math. **10** (1958), 321–348.

6. Michèle Vergne, *Réductibilité de la variété des algèbres de Lie nilpotentes*, C. R. Acad. Sci. Paris Sér. A-B **263** (1966), A4-A6.

7. ——, *Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes*, Bull. Soc. Math. France **98** (1970), 81–116.

8. Gabriel Favre, *Une algèbre de Lie caractéristiquement nilpotente de dimension* 7, C. R. Acad. Sci. Paris Sér. A-B **274** (1972), A1338–A1339.

9. Murray Gerstenhaber, *On dominance and varieties of commuting matrices*, Ann. of Math. (2) **73** (1961), 324–348.

10. Guerino Mazzola, *The algebraic and geometric classification of associative algebras of dimension five*, Manuscripta Math. **27** (1979), 81–101.

Translated by J. A. ZILBER