Addendum

A. Classical groups. Notation

Let $\mathbb{R}$ be real numbers, $\mathbb{C}$ complex numbers, $\mathbb{H}$ the quaternions. We denote by $\mathbb{R}^*$, $\mathbb{C}^*$ the multiplicative groups of real and complex numbers, by $\mathbb{T} \subset \mathbb{C}^*$ the group of complex numbers with absolute value $= 1$.

A.1. Forms. We shall regard the following 7 types of forms on linear spaces over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$.

1) Symmetric bilinear forms over $\mathbb{R}$
2) Skew-symmetric bilinear forms over $\mathbb{R}$
3) Symmetric bilinear forms over $\mathbb{C}$
4) Skew-symmetric bilinear forms over $\mathbb{C}$
5) Hermitian forms over $\mathbb{C}$
6) Hermitian forms over $\mathbb{H}$
7) Anti-Hermitian forms over $\mathbb{H}$.

We recall that a *sesquilinear* form on a left $\mathbb{H}$-module is an $\mathbb{H}$-valued function $\langle\cdot,\cdot\rangle$ satisfying the conditions

$$\langle v, w_1 + w_2\rangle = \langle v, w_1\rangle + \langle v, w_2\rangle, \qquad \langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle \tag{0.1}$$

$$\langle \lambda v, \mu w\rangle = \lambda\langle v, w\rangle\overline{\mu} \tag{0.2}$$

In coordinates, we can write out

$$\langle x, y\rangle = \sum_{k,l} x_k a_{k,l} \overline{y}_l, \qquad x_k, y_l, a_{k,l} \in \mathbb{H}$$

A form is *Hermitian* if $\langle v, w\rangle = \overline{\langle w, v\rangle}$. A form is *anti-Hermitian* if $\langle v, w\rangle = -\overline{\langle w, v\rangle}$. Examples of Hermitian and anti-Hermitian forms are respectively

$$\langle v, w\rangle = \sum v_k \overline{w}_k, \qquad \langle v, w\rangle = \sum v_k i \overline{w}_k$$

Below the term 'form' means a nondegenerate form.

A.2. Classical groups. We define the groups of the following 10 series.

1) The (real) *general linear group* $\mathrm{GL}(n,\mathbb{R})$ is the group of all invertible real matrices.

2) $\mathrm{GL}(n,\mathbb{C})$ is the group of all invertible complex matrices.

3) $\mathrm{GL}(n,\mathbb{H})$ is the group of all invertible quaternionic matrices.

4) The *pseudo-orthogonal group* $\mathrm{O}(p,q)$ is the group of real matrices preserving a symmetric bilinear form with the inertia indices (p,q)

5) The *real symplectic group* $\mathrm{Sp}(2n,\mathbb{R})$ is the group of matrices preserving a skew-symmetric bilinear form on $\mathbb{R}^{2n}$.

6) The *complex orthogonal group* $\mathrm{O}(n,\mathbb{C})$ is the group preserving a symmetric bilinear form on $\mathbb{C}^n$.

7) The *complex symplectic group* $\mathrm{Sp}(2n,\mathbb{C})$ is the group of matrices preserving a skew-symmetric bilinear form on $\mathbb{C}^{2n}$.

8) The *pseudo-unitary group* $\mathrm{U}(p,q)$ is the group of matrices preserving a Hermitian form over $\mathbb{C}$ with inertia indices (p,q)

9) The *pseudo-symplectic group* or *pseudo-unitary quaternionic group* $\mathrm{Sp}(p,q)$ is the group of matrices preserving Hermitian form over $\mathbb{H}$ with inertia indices (p,q)

10) The *nameless classical group* $\mathrm{SO}^*(2n)$ is the group of matrices preserving an anti-Hermitian form on n-dimensional space over $\mathbb{H}$.

REMARK. The standard notation for $\mathrm{U}(n,0)$, $\mathrm{O}(n,0)$, $\mathrm{Sp}(n,0)$ is $\mathrm{U}(n)$, $\mathrm{O}(n)$, $\mathrm{Sp}(n)$. $\square$

A.3. Standard realizations. Below we choose the forms in canonical ways and write equations on elements g of a group. A field is the same as above.

$$\begin{aligned}
&\mathrm{O}(p,q): && g\begin{pmatrix}-1_p & 0\\ 0 & 1_q\end{pmatrix} g^t = \begin{pmatrix}-1_p & 0\\ 0 & 1_q\end{pmatrix},\\
&\mathrm{Sp}(2n,\mathbb{R}): && g\begin{pmatrix}0 & 1_n\\ -1_n & 0\end{pmatrix} g^t = \begin{pmatrix}0 & 1_n\\ -1_n & 0\end{pmatrix},\\
&\mathrm{O}(n,\mathbb{C}): && gg^t = 1,\\
&\mathrm{Sp}(2n,\mathbb{C}): && g\begin{pmatrix}0 & 1_n\\ -1_n & 0\end{pmatrix} g^t = \begin{pmatrix}0 & 1_n\\ -1_n & 0\end{pmatrix},\\
&\mathrm{U}(p,q): && g\begin{pmatrix}-1_p & 0\\ 0 & 1_q\end{pmatrix} g^* = \begin{pmatrix}-1_p & 0\\ 0 & 1_q\end{pmatrix},\\
&\mathrm{Sp}(p,q): && g\begin{pmatrix}-1_p & 0\\ 0 & 1_q\end{pmatrix} g^* = \begin{pmatrix}-1_p & 0\\ 0 & 1_q\end{pmatrix},\\
&\mathrm{SO}^*(2n): && gig^* = 1
\end{aligned}$$

A.4. Lie algebras. In all the cases enumerated above, we can easily describe the Lie algebra as it was explained in Subsection 2.2.4. We obtain the following conditions for a matrix $X \in \mathfrak{g}$

$$\begin{aligned}
&\mathfrak{o}(p,q): && X\begin{pmatrix}-1_p & 0\\ 0 & 1_q\end{pmatrix} + \begin{pmatrix}-1_p & 0\\ 0 & 1_q\end{pmatrix}X^t = 0,\\
&\mathfrak{sp}(2n,\mathbb{R}): && X\begin{pmatrix}0 & 1_n\\ -1_n & 0\end{pmatrix} + \begin{pmatrix}0 & 1_n\\ -1_n & 0\end{pmatrix}X^t = 0,\\
&\mathfrak{o}(n,\mathbb{C}): && X + X^t = 0,\\
&\mathfrak{sp}(2n,\mathbb{C}): && X\begin{pmatrix}0 & 1_n\\ -1_n & 0\end{pmatrix} + \begin{pmatrix}0 & 1_n\\ -1_n & 0\end{pmatrix}X^t = 0,\\
&\mathfrak{u}(p,q): && X\begin{pmatrix}-1_p & 0\\ 0 & 1_q\end{pmatrix} + \begin{pmatrix}-1_p & 0\\ 0 & 1_q\end{pmatrix}X^* = 0,\\
&\mathfrak{sp}(p,q): && X\begin{pmatrix}-1_p & 0\\ 0 & 1_q\end{pmatrix} + \begin{pmatrix}-1_p & 0\\ 0 & 1_q\end{pmatrix}X^* = 0,\\
&\mathfrak{so}^*(2n): && Xi + iX^* = 0
\end{aligned}$$

A.5. Homomorphisms to Abelian groups and connected components.

1) $\mathrm{GL}(n,\mathbb{R})$ has a homomorphism $g \mapsto \det(g)$ to $\mathbb{R}^*$. The kernel is the (real) *special linear group* $\mathrm{SL}(n,\mathbb{R})$. The group $\mathrm{GL}(n,\mathbb{R})$ consists of two connected components separated by the conditions $\det g > 0$ and $\det g < 0$.

2) $\mathrm{GL}(n,\mathbb{C})$ has a homomorphism $g \mapsto \det(g)$ to $\mathbb{C}^*$. The kernel is $\mathrm{SL}(n,\mathbb{C})$.

3) $\mathrm{GL}(n,\mathbb{H})$ has a homomorphism $g \mapsto \det(g)$ to the multiplicative group of positive real numbers. The kernel is $\mathrm{SL}(n,\mathbb{H})$.

Let us regard a quaternionic operator g in $\mathbb{H}^n$ as a real operator $g_{\mathbb{R}}$ in $\mathbb{R}^{4n}$. By definition its *determinant* is

$$\det g := \sqrt[4]{\det(g_{\mathbb{R}})} \in \mathbb{R}$$

The kernel is $\mathrm{SL}(n,\mathbb{H})$.

4) $\mathrm{O}(n,\mathbb{C})$ has a homomorphism $g \mapsto \det(g) = \pm 1$ to $\mathbb{Z}_2$. The kernels is the *special orthogonal* group $\mathrm{SO}(n,\mathbb{C})$. The group $\mathrm{O}(n,\mathbb{C})$ consists of two connected components.

5) $\mathrm{O}(p,q)$ has a homomorphism ψ to $\mathbb{Z}_2 \times \mathbb{Z}_2$. For $g = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ we define $\psi(g) := \big(\mathrm{sgn}(\det a), \mathrm{sgn}(\det d)\big)$. Also, $\det(g) = \mathrm{sgn}\det(a)\,\mathrm{sgn}\det(d)$. The kernel of det is $\mathrm{SO}(p,q)$, the kernel of ψ is $\mathrm{SO}_0(p,q)$. The last group is connected.

6) $\mathrm{U}(p,q)$ has a homomorphism $\det(g)$ to $\mathbb{T}$. The kernel is the *special pseudo-unitary group* $\mathrm{SU}(p,q)$.

To obtain the Lie algebras $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{su}(p,q)$ of the groups $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{SU}(p,q)$ we add the condition $\mathrm{tr}\,X = 0$. For $\mathfrak{sl}(n,\mathbb{H})$ we require $\mathrm{Re}\,\mathrm{tr}\,X = 0$.

A.6. Letter P. If G is a classical group then PG is the quotient of G with respect to the center. For instance,

$$\mathrm{PGL}(n,\mathbb{C}) := \mathrm{GL}(n,\mathbb{C})/\mathbb{C}^*,\ \mathrm{PO}(n,\mathbb{C}) := \mathrm{O}(n,\mathbb{C})/\{\pm 1\},\ \mathrm{PU}(p,q) := \mathrm{SU}(p,q)/\mathbb{T}$$

They are called *projective linear group*, *projective orthogonal group projective pseudo-unitary group* etc..

A.7. Alternative notation.

$$\mathrm{GL}(n,\mathbb{K}) =: \mathrm{GL}_n(\mathbb{K}),\ \ \mathrm{O}(n,\mathbb{C}) =: \mathrm{O}_n(\mathbb{C}),\dots$$

and also, personally

$$\mathrm{SL}(n,\mathbb{H}) =: \mathrm{SU}^*(n),\ \ \mathrm{Sp}(p,q) =: \mathrm{U}(p,q;\mathbb{H}),\ \ \mathrm{Sp}(n) =: \mathrm{SpU}(n)$$

Also $\mathrm{Spin}(n)$ is a two-sheet cover of $\mathrm{SO}(n)$. Other covers have no personal notation.

A relatively common notation for the universal covering group of a connected group G is $G^\sim$.

A.8. Tables of real classical groups (matrix balls, categories linear relations, symmetric spaces) are contained in [120], [121], [118], Addendum A.

A.9. Definition of classical groups over arbitrary fields. Let K be a commutative or non-commutative field of characteristic $\neq 2$. Let $a \mapsto a^\circ$ be its antiinvolution[5] i.e., $a^{\circ\circ} = a$ and $(ab)^\circ = b^\circ a^\circ$. A sesquilinear form on a left K-module V is a map $V \times V \to V$ (notation: $\langle\cdot,\cdot\rangle$), such that all the maps

$$v \mapsto \langle v, a\rangle, \qquad w \mapsto \langle b, w\rangle^\circ$$

are linear. A sesquilinear form is *Hermitian* if $\langle v, w\rangle = \langle w, v\rangle^\circ$ and *anti-Hermitian* if $\langle v, w\rangle = -\langle w, v\rangle^\circ$.

A *classical group* in a narrow sense is a general linear group over K or the group of all the isometries of a Hermitian or anti-Hermitian form over K.

B. Representations. Several definitions

B.1. Subrepresentations. Let ρ be a finite-dimensional representation of a group G in a space V. A *subrepresentation* is a subspace invariant with respect to G.

A representation is *reducible* if it contains a nontrivial subrepresentation. Otherwise it is called *reducible.*

A *subrepresentation of a unitary representation* is a closed invariant subspace.

If $W \subset V$ is a subrepresentation of a unitary representation, then its orthocomplement is a subrepresentation.

[5]For $K = \mathbb{C}$ one has two variants, $a^\circ := a$ and $a^\circ := \overline{a}$; for the quaternions, $a^\circ = \overline{a}$; for real numbes, $a^\circ = a$.

A *character*[6] of a group G is an one-dimensional unitary representation. In other words, a character is a map $\chi : G \to \mathbb{T}$ such that

$$\chi(g_1 g_2) = \chi(g_1)\chi(g_2)$$

B.2. Equivalence of representations. Let $[\rho_1, V_1]$, $[\rho_2, V_2]$ be two representations of a group G. They are equivalent if there exists an invertible operator $A : V_1 \to V_2$ such that

$$A\rho_1(g)A^{-1} = \rho_2(g)$$

B.3. Morphisms and the Schur Lemma. Let $[\rho_1, V_1]$, $[\rho_2, V_2]$ be two representations of a group G. An operator $A : V_1 \to V_2$ is *intertwining* if

$$A\rho_1(g) = \rho_2(g)A$$

In fact, intertwining operators are morphisms in the category of representations. The following 4 statements are called the *Schur lemma*

Theorem B.1 *If ρ_1, ρ_2 are irreducible finite-dimensional and non-equivalent, then $A = 0$*

PROOF. Let $v \in \ker A$. Then

$$A\rho_1(g)v = \rho_2(g)A\rho_1(g)v = 0 \qquad \Longrightarrow \qquad v \in \ker A$$

Next, let $w \in \operatorname{im} A$, $w = Ah$. Then

$$\rho_2(g)Ah = A(\rho_1(g)h) \Longrightarrow \quad \rho_2(g)w \in \operatorname{im} A$$

Thus, $\ker A$ and $\operatorname{im} A$ are subrepresentations. Since ρ_1, ρ_2 are irreducible, $A = 0$. □

Theorem B.2 *Let (ρ, V) be an irreducible finite-dimensional representation. Any intertwining operator $A : V \to V$ has a form* const $\cdot 1$.

PROOF. Let λ be an eigenvalue of A. The subspace $\ker(A - \lambda)$ is a subrepresentation, and hence $\ker(A - \lambda) = V$. □

Theorem B.3 *Let (ρ, V) be an irreducible unitary representation. Any intertwining operator $A : V \to V$ has a form* const $\cdot 1$.

PROOF. The previous proof breaks down. Note first

$$A\rho(g) = \rho(g)A \quad \Longrightarrow \quad (A\rho(g))^* = (\rho(g)A)^* \quad \Longrightarrow \quad A^*\rho(g^{-1}) = \rho(g^{-1})A^*$$

Therefore A^* also is intertwining operator. Hence, the same is valid for $A + A^*$, $i(A - A^*)$. The latter operators are self-adjoint. By the Spectral Theorem (see [145]), an operator, commuting with a self-adjoint operator commutes with all the spectral projectors. Hence the spectral projectors are intertwining operators. Therefore the images of the spectral projectors are subrepresentations. Since ρ is irreducible, all the spectral projectors are 0 or 1. Hence both $A - A^*$ and $i(A - A^*)$ are scalar operators. □

[6] This term also has other meanings

Theorem B.4 *If $[\rho_1, V_1]$, $[\rho_2, V_2]$ are unitary irreducible and non-equivalent, then any intertwining operator $A : V_1 \to V_2$ is zero.*

PROOF. The operator $A^*A : V_1 \to V_1$ is an intertwining operator. By the previous theorem $A^*A = 1$. For the same reason $AA^* = 1$. Hence A is unitary and, in particular, it is invertible. Thus representations are equivalent. □

Proposition B.5 *An operator intertwining two irreducible unitary representations is unitary within to a scalar factor.*

PROOF. Indeed, A^*A and AA^* must be scalar operators. □

B.4. Some remarks.

Proposition B.6 *Let ρ be a unitary irreducible representation of a group G. Let γ be an element of the center of G. Then $\rho(\gamma)$ is scalar.*

PROOF. Indeed $\rho(\gamma)$ is an intertwining operator. □

Proposition B.7 *Each irreducible unitary representation of an Abelian group is one-dimensional.*

PROOF. By the previous proposition, it follows that all the operators $\rho(g)$ are scalar. □

In particular, a unitary representation of G determines a character of its center.

B.5. Matrix elements.

Proposition B.8 *Let ρ, ρ' be irreducible unitary representations of a group G in Hilbert spaces V_1, V_2 respectively. Assume that for some $v \in V$, $v' in V'$,*

$$\langle \rho(g)v, v\rangle_V = \langle \rho'(g)v', v'\rangle_{V'}$$

Then ρ and ρ' are equivalent.

PROOF. Since

$$\langle \rho(g)v, \rho(h)v\rangle = \langle \rho(h)^*\rho(g)v, v\rangle = \langle \rho(h)^{-1}\rho(g)v, v\rangle = \langle \rho(h^{-1}g)v, v\rangle$$

it follows that

$$\langle \rho(g)v, \rho(h)v\rangle = \langle \rho'(g)v', \rho(h)v'\rangle$$

Hence there exists a unique unitary operator $U : V \to V'$ such that $U\rho(g)v = \rho'(g)v'$ for all g. □

B.6. Stone-von Neumann theorem.

Consider the Heisenberg group Heis_n as it was described in Subsection 1.2.6 and the representation U given by the formula (1.2.8). For nonzero $s \in \mathbb{R}$, we define a representation of Heis_n in $L^2(\mathbb{R})$ given by

$$U_s\big[v_+ \oplus v_-, \xi\big] = f(x + v_-)\exp\big\{is(xv_+^t + \xi + \frac{1}{2}v_+v_-^t)\big\}$$

In other words, we consider the automorphism $\tau_s : [v_+ \oplus v_-; \xi] \mapsto [sv_+ \oplus v_-; s\xi]$ of Heis_n and assume

$$U_s = U \circ \tau_s$$

All the representations U_s are twins, but officially they are not equivalent.

Theorem 1.1 (Stone–von Neumann) *Each irreducible unitary represen-taion of the Heisenberg group of dimension > 1 has the form U_s.*

The most natural way of proof is a reference to Mackey's theorems on induced representations (see [80], 13.2). Here we outline the original proof of von Neumann, [175].

Sketch of proof. Let U be a unitary irreducible representation of Heis_n. Without loss of generality, we assume $U[0;\xi] = e^{i\xi}$. Thus,

$$U[v;0]U[w;0] = \exp(\frac{i}{2}\{v,w\})U[v+w;0]$$

where $\{\cdot,\cdot\}$ is the bilinear form from Subsection 1.2.6.

Let us consider the operator

$$A := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} e^{-|v|^2/2}U[v;0]\,dv$$

Lemma B.9 *a) $A \neq 0$*

b) $AU[c;0]A = e^{-|c|^2/2}A$.

c) A is self-adjoint and $A^2 = A$.

Proof of the lemma. The claim a) is proved by reductio ad absurdum. If $A = 0$, then

$$0 = U[-c;0]A[c;0] = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} e^{i\{c,v\}/2}e^{-|v|^2/2}U[v;0]\,dv$$

Considering linear combinations of such operators and their limits, one can easily show that

$$\int A(v)U[v;0]\,dv = 0$$

for all the functions v.

b) follows from a straightforward calculation. Substituting $c = 0$, we obtain $A^2 = A$. □

Proof of the theorem. Let h be a unit vector satisfying $Ah = h$. Let us evaluate its matrix element,

$$\langle U[c;0]h,h\rangle = \langle U[c;0]Ah,Ah\rangle = \langle AU[c;0]Ah,h\rangle = e^{-|c|/2}\langle Ah,h\rangle = e^{-|c|/2}$$

But a matrix element determines an irreducible representation. □

Problem B.1 a) Restore details.

b) Invent how it was possible to write the operator A. Apparently, this is an imitation of the Weil symbol calculus (see Section 4.5). The function $e^{-(x^2+y^2)/2}$ is the Weil symbol of the orthogonal projection to the Gaussian $e^{-x^2/2}$. Formal verifivation of $A^2 = A$ coincide (as a calculation) with evaluation of the Weil symbol of the product.

C. Lie groups. Some logical arrows

Here we present sketches of proof of several basic theorems of Lie group theory. This section is not a regular exposition.

C.1. Lie algebras. A *Lie algebra* $\mathfrak{g}$ is a linear space endowed with an operation $(X, Y) \mapsto [X, Y]$ (*commutator*) such that

$$[X,Y] = -[Y,X], \qquad [X_1 + X_2, Y] = [X_1, Y] + [X_2, Y], \qquad [\lambda X, Y] = \lambda[X,Y]$$
$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0 \qquad \text{(Jacobi identity)}$$

The basic example is the algebra $\mathfrak{gl}(n)$ of all $n \times n$ matrices with the *commutator*

$$[X,Y] = XY - YX$$

Theorem C.1 (Ado) *Each finite-dimensional Lie algebra is a subalgebra in* $\mathfrak{gl}(n)$.

A natural proof is in [**?**].

C.2. Representations of Lie algebras. A finite-dimensional representation of a Lie algebra $\mathfrak{g}$ is a map $\psi : \mathfrak{g} \to \mathfrak{gl}(n)$ satisfying

$$[\psi(X), \psi(Y)] = \psi([X,Y])$$

C.3. Lie algebra of a matrix group.

Theorem C.2 *Each closed subgroup G in* $\mathrm{GL}(n,\mathbb{R})$ *is a smooth manifold.*

Let $\Omega \subset \mathbb{R}^n$ be an arbitrary set. We say that a vector v at point $a \in \Omega$ is tangent if one can find a sequence of points $x_j \in \Omega$ converging to a such that directions of segments $[a, x-j]$ converge to v.

Lemma C.3 *The tangent cone $\mathfrak{g}$ to G at 1 is a linear subspace. For each $X \in \mathfrak{g}$, all the points* $\exp(tX)$*, where $X \in \mathbb{R}$, are contained in G. Moreover, all the elements of G, lying in a small neigborhood of 1, have this form.*

Proof of the lemma. Consider a small neigborhood U of 1 in the group $\mathrm{GL}(n,\mathbb{R})$. We represent $g \in U$ in the form $g = \exp(X)$, where X in a small neigborhood of 0 (also in the space of all matrices). We regard $G \cap U$ as the set Ω in the definition of a tangent vector given above.

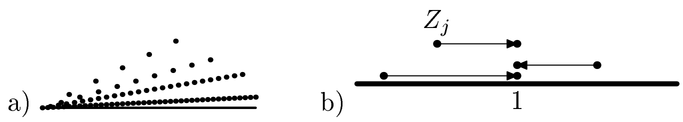

Fig. 1: To proof of Lemma C.3

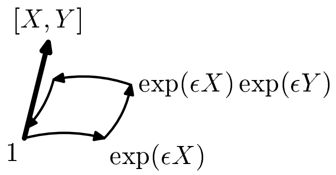

Fig. 2: To Theorem C.4.

It is almost obvious, that for each tangent vector $Y \in \mathfrak{g}$, we have $\exp(tY) \in G$; see Fig.1, but formally some proof is necessary.

Further let X, $Y \in \mathfrak{g}$ be tangent vectors. Then the group G contains

$$\exp\{\varepsilon tX\}\exp\{\varepsilon sY\} = \exp\{\varepsilon tX + \varepsilon tY\} + O(\varepsilon^2)$$

and hence $tX + sy \in \mathfrak{g}$. Thus $\mathfrak{g}$ is a linear space.

It remains to prove the last statement of the lemma. Let $\exp(Z_j) \in G$ be a sequence convergent to 1, let $Z_j \notin \mathfrak{g}$. Then we can choose $X_j \in \mathfrak{g}$ such that directions of the vectors $X_j + Z_j$ are convergent to a line $\notin \mathfrak{g}$, see Fig.1b). This contradiction completes our quasi-proof. $\square$

Proof of the theorem. Thus, a neighborhood of the unit in G is a submanifold in $\mathrm{GL}(n,\mathbb{C})$. Further the map $h \mapsto hg$ identifies a neighborhood of unit and a neighborhood of g. Therefore all the points of G have equal rights, and G is a smooth manifold. $\square$

Theorem C.4 *If* X, $Y \in \mathfrak{g}$, *then* $[X,Y] \in \mathfrak{g}$

Proof. Consider the curve

$$\gamma(\varepsilon) = \exp\{\varepsilon X\}\exp\{\varepsilon Y\}\exp\{-\varepsilon X\}\exp\{-\varepsilon Y\} \in G$$

Expanding $\gamma(\varepsilon)$ in a Taylor series, after simple calculations, we obtain

$$\gamma(\varepsilon) = 1 + \frac{1}{2}\varepsilon^2(XY - YX)$$

and hence $[X,Y] \in \mathfrak{g}$. $\square$

Thus, the tangent space $\mathfrak{g}$ has a canonical structure of a Lie algebra, $\mathfrak{g}$ is said to be *the Lie algebra of the Lie group* $\mathfrak{g}$.

C.4. Lie groups. A Lie group G is a manifold endowed with a smooth multiplication $G \times G \to G$ satisfying the axioms of a group.

Now let describe briefly (without auxiliary proofs) how to define the Lie algebra of G in this case. Denote by $\mathfrak{g}$ the tangent space to G at 1.

An *one-parameter subgroup* in G is a smooth homomorphism τ of the additive group $\mathbb{R}$ to G,

$$\tau(\alpha)\tau(\beta) = \tau(\alpha+\beta), \qquad \alpha, \beta \in \mathbb{R}$$

Then $X := \tau'(0)$ is an element of $\mathfrak{g}$. Moreover *this is one-to-one correspondence between the set of all the one-parameter subgroups and* $\mathfrak{g}$. We say that X is the *generator* of the one-parameter subgroup τ, and denote τ by $\tau(\alpha) =: \exp(\alpha X)$.

Now we define a structure of Lie algebra on $\mathfrak{g}$ by

$$[X,Y] := \frac{d^2}{d\varepsilon^2} \exp\{\varepsilon X\} \exp\{\varepsilon Y\} \exp\{-\varepsilon X\} \exp\{-\varepsilon Y\}$$

C.5. Inverse construction.

Theorem C.5 *Each finite-dimensional Lie algebra* $\mathfrak{g}$ *is a Lie algebra of some Lie group.*

PROOF. By the Ado theorem, $\mathfrak{g} \subset \mathfrak{gl}(N, \mathbb{R})$. We regard $\mathfrak{g}$ as a subspace in the tangent space to $\mathrm{GL}(N, \mathbb{R})$ at 1. For each $g \in \mathrm{GL}(n, \mathbb{R})$, we construct a subspace A_g in the tangents space T_g as follows. We consider right shifts $\mathfrak{R}_g : h \mapsto hg$ on $\mathrm{GL}(n, \mathbb{R})$ and $A_g \subset T_g$ is the push-forward of $\mathfrak{g}$ with respect to $\mathfrak{R}_g$.

It can be easily checked that A_g is an integrable distribution in the sense of Frobenius. Consider the integral manifold containing 1. It is not hard to prove that this manifold is an (incidentally, non-closed) subgroup in $\mathrm{GL}(N, \mathbb{R})$. It is a desired Lie group. $\square$

C.6. Functoriality.

Theorem C.6 *Let* G, G' *be Lie groups. Let* $\psi : G \to G'$ *be a smooth homomorphisms. Then its differential at unit is a homomorphism* $\mathfrak{g} \to \mathfrak{g}'$ *of corresponding Lie algebras*

This immediately follows from the definition of commutator.

Theorem C.7 *Let* G, G' *be connected Lie groups. We denote by* $\mathfrak{g}$ *the corresponding Lie algebras. Suppose* G *is simply connected. Then each homomorphism* $\mathfrak{g} \to \mathfrak{g}'$ *is a differential of a unique homomorphism* $G \to G'$.

PROOF. Let $\psi : \mathfrak{g} \to \mathfrak{g}'$ be a homomorphism. The graph $\mathfrak{q}$ of ψ is a subalgebra in $\mathfrak{g} \oplus \mathfrak{g}'$.

Next we consider the group $G \times G'$ and the subspace $\mathfrak{q}$ in the tangent space at 1. Further, construct a distribution of subspaces as in proof of Theorem C.5. Again, this distribution is integrable in the Frobenius sense. Denote by Q the integral manifold containing the unit. It can be readily checked that $Q \subset G \times G'$ is a subgroup. Evidentely, the projections $\pi : Q \to G$ and $\pi' : Q \to G'$ are homomorphisms. Moreover, the map $\pi : Q \to G$ is covering; since G is simply connected, the map π is a bijection. Thus, $\pi' \circ \pi^{-1} : G \to G'$ is the desired homomorphism. $\square$

C.7. Universal covering group. Let G be a connected Lie group. Consider the universal covering manifold $\widetilde{G}$ of G. Let us define a structure of a Lie group on $\widetilde{G}$.

Denote by $\pi : \widetilde{G} \to G$ the natural projection map. For the unit $e \in G$, fix an element $\widetilde{e} \in \widetilde{G}$ such that $\pi(\widetilde{e}) = e$ and declare that $\widetilde{e}$ is a unit.

Define the product of two elements u, v. For this, consider smooth paths $\mu(t), \nu(t) : [0,1] \to \widetilde{G}$, connecting $\widetilde{e} =: \mu(0) = \nu(0)$ with $u =: \mu(1)$ and $v =: \nu(1)$ respectively.

We consider the path $\pi(\mu(t)\nu(t)) \in G$ and its lift $\varkappa(t) \in \widetilde{G}$ such that $\varkappa(0) = \widetilde{e}$. We assume $uv := \varkappa(1)$.

C.8. Groups with a given Lie algebra.

Theorem C.8 *a) For a given finite-dimensional Lie algebra $\mathfrak{g}$, there exists a unique connected simply connected Lie group G, whose Lie algebra is $\mathfrak{g}$.*

b) Moreover, each connected Lie group G°, whose Lie algebra is $\mathfrak{g}$, is a quotient G/Γ, where Γ is a discrete subgroup in the center of G.

PROOF. As we seen above, a group G° with a given Lie algebra exists; we define G as the universal covering group of G°.

Further, we refer to the proof of Theorem C.7. There is a homomorphism $\theta : G \to G^\circ$ which is bijective in a neibourhood of unit. Evidently, this map is covering. The preimage Γ of 1 must be a discrete subgroup. Also Γ must be normal; these easily imlies the last claim of the theorem.

C.9. Adjoint representation. Denote by $\mathrm{ad}(X)$ the linear operator in a Lie algebra $\mathfrak{g}$ given by

$$\mathrm{ad}(X) = [X, Y]$$

By the Jacobi identity

$$[\mathrm{ad}X, \mathrm{ad}Y] = \mathrm{ad}[X, Y]$$

Hence ad is a representation of $\mathfrak{g}$ on itself, it is named the *adjoint representation*

The *adjoint representation* $\mathrm{Ad}(g)$ of a Lie group G is defined in the following way:

$\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ *is the differential of the map $h \mapsto ghg^{-1}$ at the unit of the group G.*

For matrix groups it is given by

$$\mathrm{Ad}(g)X = gXg^{-1}$$

C.10. Haar measure. By a measure on a manifold we mean a positive Borel measure, i.e., a measure defined on all the Borel subsets.

Let M be a manifold, $M = \cup V_n$ where V_n are open set diffeomorphic to open balls in $\mathbb{R}^n$. We say that a measure μ on M is *smooth* if for each function supported by some V_j we have $\int_M f \, d\mu = \int_{V_j} f(x)\tau_j(x) \, dx$, where dx is the Lebesgue measure on the ball and $\tau_j(x)$ is a smooth non-negative function. In fact $\tau_j(x)\,dx$ can be considered as a differential form of a maximal order. If M is orientable, then the collection $\tau_j(x)dx$ determines a globally existing differential form on M, otherwise $\tau_j(x)\,dx = \pm\tau_k(y)\,dy$ on the intersection $V_k \cap V_l$.

Let G be a Lie group.

Theorem C.9 *a) There is a unique to within proportionality ('Haar') measure on G invariant for all the left shifts $g \mapsto hg$.*

b) Haar measure is smooth.

c) If $\det \mathrm{Ad}(h) = 1$ for all $h \in G$, then this measure also is invariant for all the right shifts $g \mapsto gh$.

PROOF. a)–b). Existence. We consider an exterior form of the maximal degree on $\mathfrak{g}$ (it is unique to within a scalar factor) and consider its left shifts to tangent spaces at all other points of the group. We obtain a nonzero invariant differential form (denote it by dg) of a maximal degree on G. The same argument also proves the uniqueness of the invariant measure in the class of smooth measures.

Next, we should show that an invariant measure is smooth. Let μ be a left-invariant Borel measure. Let φ be a positive compactly supported smooth function on G such that $\int f\,dg = 1$. Define the measure[7] μ_φ

$$\int_G f(g)\,d\nu_\varphi(g) := \int_G \int_G f(hg)\varphi(h)\,d\mu(g)\,dh$$

It is easy to show that this measure is smooth, therefore, it is const $\cdot dg$. Consider a sequence φ_j, whose supports tend to e. Then μ_{φ_j} tends to μ weakly, and hence μ itself is const $\cdot dg$.

c) Let $\Omega(g)$ be a left-invariant volume form on G. Then its right shift $\Omega(gh)$ also is a left-invariant volume form, and hence $\Omega(gh) = \lambda(g)\Omega(g)$ with some scalar factor λ. Since $\Omega(g) = \Omega(h^{-1}g)$, we have $\Omega(h^{-1}gh) = \lambda(g)\Omega(g)$. We compare these forms at $g = 1$. Obviously, the factor is $\det \mathrm{Ad}(g)$. ☐

There are numerous explicit expressions for Haar measures on given Lie groups, for instance, in this book in (2.10.9), (2.11.16).

C.11. Invariant measures on homogeneous spaces. Let G be a (real) Lie group and H be its closed subgroup. Let $\mathfrak{g} \supset \mathfrak{h}$ be their Lie algebras. Consider a homogeneous space G/H. Recall that such a space admits a canonical structure of a smooth manifold (indeed, the set of manifolds gH, where g ranges in G, is a smooth foliation with closed fibres, and we have a natural smooth structure on the quotient space). The tangent space to G/H at the initial point is $\mathfrak{g}/\mathfrak{h}$.

For $h \in H$, the adjoint transformation $\mathrm{Ad}(h) : \mathfrak{g} \to \mathfrak{g}$ preserves the subspace $\mathfrak{h} \subset \mathfrak{g}$ and hence it induces an operator $\mathrm{Ad}^\circ(h) : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$. Equivalently, this transformation can be regarded as an operator in the tangent space to G/H.

[7] It is the convolution of μ and φ

Theorem C.10 *a) A G-invariant measure on a homogeneous space G/H exists iff* $|\det \mathrm{Ad}^\circ(h)| = 1$ *for all* $h \in H$.

b) An invariant measure is smooth.

c) G-invariant measure on G/H is unique to within a constant factor.

PROOF is the same as for the previous theorem. We only must remember that an invariant differential form on G/H is determined modulo a sign. $\square$