Chapter 10

Classical p-adic groups. Introduction

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In Section 10.1 we present a brief informal introduction to p-adic numbers. I am not sure that this introduction is sufficient; anyhow this simple and nice topic requires some acclimatization. This is the aim of numerous problems in this section. See also systematic books [85], [29], [20], [162] and a nonsystematic book [50].

Sections 10.2–10.7 contain a (not too advanced) discussion of p-adic classical groups. Our main purpose here are phenomena that are new in comparison the real case.

10.1 p-adic numbers

1.1. 10-adic numbers. We know rules of addition and multiplication of finite decimal fractions. For instance,

$$
\begin{array}{rr}
 & 8765.32 \\
+ & 3526.67 \\
\hline
 & 12291.99
\end{array}
\qquad\qquad
\begin{array}{rr}
 & 3.45 \\
\times & 2.13 \\
\hline
 & .1035 \\
 & .345 \\
 & 6.90 \\
\hline
 & 7.3485
\end{array}
$$

A usual decimal fraction is a sequence of digits infinite to the right-hand side (and finite to the left-hand side)

The 10-*adic numbers* are decimal fractions that are infinite to the left-hand side and finite to the right-hand side. For instance,

$$...12121212121212121.3456789$$

There is no sign in the front of 10-adic numbers.

A 10-*adic integer* is a sequence of digits infinite to the left-hand-side and finished by a comma[1]. For instance, $\dots 1111111111111000000.$

We apply the usual rules of addition and multiplication, for instance

$$\begin{array}{r} \dots 8765.32 \\ + \quad \dots 3526.67 \\ \hline \dots 2291.99 \end{array} \qquad\qquad \begin{array}{r} \dots 3.45 \\ \times \quad \dots 2.13 \\ \hline \dots 7.3485 \end{array}$$

Problem 1.1 a) Find -1.

b) Convince yourself that 10-adic numbers form a commutative ring. Integer p-adic numbers form a subring.

c) Evaluate $1/2$, $1/1024$, $1/3$

d) Show that for positive p, $q \in \mathbb{Z}$, the equation $qx = p$ has a unique solution.

Problem 1.2 Show that the equation $x^2 = x$ has four solutions, namely, $x_1 = 0$, $x_2 = 1$, $x_3 = ...890625$, $x_4 = 1 - x_3 = ...109376$. Since a quadratic equation has > 2 solutions, the ring of 10-adic numbers is not a field. Also, $x_3 x_4 = 0$.

Define the norm $|r|$ of a 10-adic number r as 10^{j-1}, there j is the number of the last nonzero figure, for instance,

$$|...11.| = 1, \qquad |..111000.| = 10^{-3}, \qquad |..1111.1111| = 10^4, \qquad |0| = 10^{-\infty} = 0$$

Problem 1.3 a) Show that $|a + b| \leqslant \max(|a|, |b|)$.

b) Show that $\rho(a, b) := |a - b|$ is a metric on 10-adic numbers

c) Show that the set of 10-adic numbers is a complete metric space

d) Denote by D_{10} the set $\{0, 1, 2, \dots, 9\}$ endowed with the discrete topology. The space of integer 10-adic numbers is homeomorphic to the infinite product $D_{10} \times D_{10} \times \dots$. In particular, a sequence of 10-adic numbers converges iff the sequence of figures in each digit stops startings some moment. For instance, the sequence

$$\begin{aligned} a_1 &= \dots 00000001. \\ a_2 &= \dots 00000011. \\ a_3 &= \dots 00000111. \\ a_4 &= \dots 00001111. \\ &\dots\dots\dots\dots \end{aligned}$$

converges to $\dots 11111111.$. Also, 10^n tends to 0 as $n \to \infty$

e) Show that the set of 10-adic integers is the closure of the set of the usual integers.

[1] Following traditions of programming, we write commas as points

In particular, we can consider 10-adic numbers as sums of convergent series of the form

$$\sum_{j=k}^{\infty} \alpha_j \cdot 10^j, \qquad \alpha_j \text{ ranges in } \{0, 1, \dots, 9\}$$

1.2. p-adic numbers. Let p be a prime integer. We define *p-adic numbers* in the same way, only consider p number system instead of decimal one.

Theorem 1.1 *The ring of p-adic numbers is a field (in other words, a division is well defined)*

All other properties of 10-adic numbers survive literally.

Denote by $\mathbb{Q}_p$ the field of p-adic numbers, by $\mathbb{O}_p$ the ring of $p-adic$ integers. By $\mathbb{Q}_p^*$, we denote the multiplicative group of the field $\mathbb{Q}_p$.

We also use the notation

$$r_1 - r_2 \equiv 0(\mod p^n) \Longleftrightarrow |r_1 - r_2| \leqslant p^{-n-1}$$

Proof of the theorem. First, we can represent a p-adic number in the form $p^k \cdot r$, where $|r| = 1$. It suffices to show, that for $\alpha_0 \neq 0$, the number $r = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \dots$ is invertible. In other words, we must solve the equation

$$(\alpha_0 + \alpha_1 p + \alpha_2 p^2 + \dots) \cdot (x_0 + x_1 p + x_2 p^2 + \dots) = 1, \qquad x_j = 0, 1, \dots, p-1$$

Assume that we found a solution $\mod (p^n)$, i.e.,

$$(\alpha_0 + \alpha_1 p + \dots + \alpha_n p^n + \dots) \cdot (x_0 + x_1 p + \dots + x_n p^n + \dots) = 1 + c \cdot p^{n+1} + \dots$$

Then

$$(\alpha_0 + \alpha_1 p + \dots + \alpha_n p^n + \alpha_{n+1} p^{n+1} + \dots) \cdot (x_0 + x_1 p + \dots + x_n p^n + x_{n+1} p^{n+1}) =$$
$$= 1 + \Big[\alpha_0 + \alpha_1 x_n + \dots + \alpha_n x_n + x_{n+1}\Big] \cdot p^{n+1} + \dots$$

Evidently, we can obtain $[\dots] = 0$ for an appropriate choice of x_{n+1}. □

Problem 1.4 Why this proof is not correct for 10-adic numbers?

Problem 1.5 a) Let $p \neq 2$. Show, that for $r = 1 + \alpha_1 p + \alpha_2 p^2 + \dots$, the equation $y^2 = r$ has a solution. Look for it in the form $y = 1 + x_1 p + x_2 p^2 + \dots$.

b) Let $p \neq 2$. Let $r = p^k(\alpha_0 + \alpha_1 p + \alpha_2 p^2 + \dots)$, and $\alpha_0 \neq 0$. The equation $y^2 = p$ has a solution iff k is even and $\left(\frac{\alpha_0}{p}\right) = 1$

c) Show, that the equation $x^{p-1} = 1$ has $(p-1)$ solutions $x_1, \dots, x_{p-1}$ satisfying $x_j = j \mod p$ respectively

d)* Investigate the equation $y^n = r$

Thus, we have

$$\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \qquad \text{for } p \neq 2$$

Problem 1.6 a) The usual integers $\mathbb{Z}$ are dense in $\mathbb{O}_p$.

b) All the closed subgroups of the additive group of $\mathbb{Q}_p$ have the form $p^j\mathbb{O}_p$

c) The additive group $\mathbb{Q}_p/p^j\mathbb{O}_p$ is isomorphic to the inductive limit

$$\mathbb{Z}_{p^\infty} := \lim_{k\to\infty} \mathbb{Z}_{p^k} \tag{1.1}$$

where $\mathbb{Z}_{p^k}$ is regarded as a subgroup of $\mathbb{Z}_{p^{k+1}}$. Equivalently, we can say that $\mathbb{Q}_p/p^j\mathbb{O}_p$ is isomorphic to the multiplicative group of all the complex numbers λ, such that $\lambda^{p^N} = 1$ for sufficiently large N.

Problem 1.7 a) Show that $p\mathbb{O}_p$ is an ideal in $\mathbb{O}_p$. Show that the quotient ring is the field $\mathbb{F}_p$.

b) For $k > 0$, the quotient ring $\mathbb{O}_p/\mathbb{O}_{p^k}$ is isomorphic to the residue ring $\mathbb{Z}/p^k\mathbb{Z}$.

Problem 1.8 Since $\mathbb{Q}_p$ is a field, it contains all the rational numbers. Show that numbers of the form $a\cdot p^k$, where $a \in \mathbb{Z}$, have the form $\dots 000000000\alpha_m\alpha_{m-1}\alpha_{m-2}\dots$ in our notation.

1.3. More formal definition. Represent $t \in \mathbb{Q}$ as

$$t = \frac{u}{v}\cdot p^k, \qquad \text{where } u,\, v \in \mathbb{Z};\, p \text{ is not a divisor of } u,\, v$$

We define the *norm* of t as $|t|_p = p^{-k}$. We have

$$|t+s|_p \leqslant \max(|t|_p, |s|_p), \qquad |ts|_p = |t|_p \cdot |s|_p$$

Also $\rho(t,s) := |t-s|_p$ is a metric on $\mathbb{Q}$.

The field $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to this metric.

Problem 1.9 Show that the field defined in the previous subsection is complete and contains $\mathbb{Q}$ as a dense set. This implies the equivalence of the two definitions.

1.4. Still more formal definition. Consider the natural chain of homorphisms of rings

$$\mathbb{Z}/p\mathbb{Z} \leftarrow \mathbb{Z}/p^2\mathbb{Z} \leftarrow \mathbb{Z}/p^3/\mathbb{Z} \leftarrow \dots$$

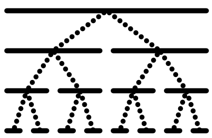
In each case, we take the unit to the unit.

We define the ring $\mathbb{O}_p$ of p-adic integers as the inverse limit of this chain. We define $\mathbb{Q}_p$ as the quotient field of this ring.

Problem 1.10 Convince yourself that we obtain the same object.

Problem 1.11 Show that $\mathbb{O}_{10} = \mathbb{O}_5 \times \mathbb{O}_2$, $\mathbb{Q}_{10} = \mathbb{Q}_5 \times \mathbb{Q}_2$.

1.5. Balls. We define a closed *ball* $B(a,p^k) \subset \mathbb{Q}_p$ in the usual way, namely $z \in B(a,p^k)$ iff $|z-a| \leqslant p^k$.



$B(0,1)$
$B(0,1/2)$, $B(1,1/2)$
$B(0,1/4)$, $B(1,1/4)$, $B(2,1/4)$, $B(3,1/4)$,

Fig. 10.1: To problem 1.12 Partitions of balls into smaller balls in $\mathbb{Q}_2$. The complete picture is an infinite 3-valent tree

Problem 1.12 a) If $c \in B(a,p^k)$, then $B(c,p^k) = B(a,p^k)$. Thus a ball has a well-defined radius, but each point of a ball is its center.

b) Two balls are disjoint or one ball contains the another.

c) Closed balls are open sets.

d) Each ball consists of points having the form

$$\sum_{j=-\infty}^{k} x_j p^j + \sum_{i=k+1}^{l} \alpha_i p^i$$

where α_i are fixed and x_j are arbitrary.

e) Each ball of radius $r = p^k$ is a disjoint union of p balls of radius p^{k-1}.

1.6. Additive characters. A character χ of the additive group $\mathbb{Q}_p$ is a continuous map $\mathbb{Q}_p \to \mathbb{C}^*$ satisfying

$$\chi(u+v) = \chi(u)\chi(v)$$

We define a standard character χ_{st} by

$$\chi_{st}(z) := \exp\{2\pi i z\}$$

More precisely,

$$\chi_{st} := \Big(\sum_{j=\infty}^{n} \alpha_j p^j\Big) = \exp\Big\{2\pi i \sum_{j<0} \alpha_j p^j\Big\}$$

In fact all the values of this character have form $\sqrt[p^m]{1}$

Problem 1.13 Each character of the additive group $\mathbb{Q}_p$ has the form

$$\chi_a(z) = \exp\{2\pi i a z\}$$

A possible way of a proof follows.

The subgroups $p^j\mathbb{O}_p$ form a base of neighborhoods of zero. Hence, for some j the image $\chi(p^j\mathbb{O}_p)$ is contained in a small neighborhood of unit $\in \mathbb{C}^*$. Since $\chi(p^j\mathbb{O}_p)$ is a subgroup, we get $\chi(p^j\mathbb{O}_p) = 1$. Hence, it suffices to describe characters of $\mathbb{Q}_p/p^j\mathbb{O}_p$. The last group is the direct limit of the groups $\mathbb{Z}_{p^n}$. Therefore, the group of its characters is the inverse limit of the groups $\mathbb{Z}_{p^n}$.

1.7. Multiplicative group $\mathbb{Q}_p^*$. Define p-adic exponent and logarithm by

$$\exp z := \sum_{j\geqslant 0} \frac{z^n}{n!}, \qquad \ln(1+z) := \sum_{j>0} \frac{(-1)^n z^n}{n}$$

here z, $\exp z$, $\ln(1+z) \in \mathbb{Q}_p$.

Problem 1.14 Let $p \neq 2$.

a) Both series converge in the ball $|z| \leqslant p^{-1}$.

b) The maps $\ln(1+z)$ and $\exp(z) - 1$ transfer this ball to itself bijectively.

c) These functions have the usual properties.

d) The set $1 + p\mathbb{O}_p$ is a multiplicative subgroup in $\mathbb{Q}_p^*$. This group is isomorphic to the additive group $p\mathbb{O}_p$.

As a corollary of this problem, we obtain the following theorem

Theorem 1.2 *Let $p \neq 2$. Then $\mathbb{Q}_p^* \simeq \mathbb{Z} \times \mathbb{F}_p^* \times \mathbb{O}_p$ where the subgroup $\mathbb{Z}$ consists of numbers p^k, the subgroup $\mathbb{F}_p^*$ consists of solutions of the equation $x^{p-1} = 1$, and the last factor consists of elements $1 + pz$, where z ranges in $\mathbb{O}_p$.*

In other words, each element $z \in \mathbb{Q}_p^*$ admits a unique representation

$$z = p^k \cdot \theta \cdot \exp\{pu\}, \qquad \text{where } \theta^{p-1} = 1,\ u \in \mathbb{O}_p$$

Corollary 1.3 *Let $p \neq 2$. Then $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The generators of the quotient group are p and arbitrary θ such that $\theta^{p-1} = 1$, $\theta^{(p-1)/2} \neq 1$.*

1.8. Multiplicative group of $\mathbb{Q}_2$. The case $p = 2$ rather differs from the general case and below we are (unsuccessfully) trying to exclude it from our considerations.

Theorem 1.4 *Each element of $\mathbb{Q}_2^*$ admits a unique representation in the form*

$$z = p^k \cdot (-1)^\sigma \exp(4u), \qquad u \in \mathbb{O}_2$$

We omit a proof. In this case, the functions $\exp z - 1$ and $\ln(1+z)$ provide a bijection of the set $4\mathbb{O}_2$ to itself.

Corollary 1.5 *The quotient group $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

Problem 1.15 An element $2^m(1 + a_1 \cdot 2 + a_2 \cdot 2^2 + a_2 \cdot 2^3 + \dots)$ is a square iff m is even and $a_1 = a_2 = 0$.

1.9. Invariant measure. We define the standard measure $\mathrm{vol}(\cdot)$ on $\mathbb{Q}_p$ by the conditions:

— $\mathrm{vol}(a + U) = \mathrm{vol}(U)$ for each $a \in \mathbb{Q}_p$ and each ball U

— $\mathrm{vol}(\mathbb{O}_p) = 1$

Then the measure of an arbitrary ball of radius p^k is p^{-k}.

Problem 1.16 a) Prove the last statement.
b) For each measurable $U \subset \mathbb{Q}_p$ and each nonzero $\lambda \in \mathbb{Q}_p$, $\mathrm{vol}(\lambda \cdot U) = |\lambda| \cdot \mathrm{vol}(U)$

We also define the measure $\mathrm{vol}(\cdot)$ on the linear space $\mathbb{Q}_p^n$ by the conditions:
— $\mathrm{vol}(\cdot)$ is shift-invariant
— $\mathrm{vol}(\mathbb{O}_p^n) = 1$.

Problem 1.17 For $U \subset \mathbb{Q}_p^n$ and $g \in \mathrm{GL}(n, \mathbb{Q}_p)$,

$$\mathrm{vol}(g \cdot U) = |\det g| \, \mathrm{vol}(U)$$

10.2 Classification of quadratic forms

2.1. Symmetric forms.

Lemma 2.1 *Any symmetric bilinear form $B(v,w)$ over $\mathbb{Q}_p$ is uniquely determined by the quadratic form $B(v,v)$*

This statement is valid for arbitrary field of characteristic $\neq 2$, see (1.5). The following statement also is abstract and general (see Subsection 2.2.1).

Proposition 2.2 *Each nondegenerate quadratic form can be reduced to*

$$\sum a_j z_j^2, \qquad a_j \neq 0 \tag{2.1}$$

There arises a natural question:

$$\textit{When forms } \sum_j a_j x_j^2 \textit{ and } \sum_j b_j x_j^2 \textit{ are equivalent?}$$

We can multiply x_j by constants, $x_j \mapsto \lambda_j x_j$; then a_j goes to $\lambda_j^2 a_j$. Hence the answer is 'definitely yes' if $a_j = b_j$ as elements of the group $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$.

On the other hand, for a non-degenerate quadratic form zAz^t, we defined its *discriminant* $\mathrm{Dis}(A)$ being the determinant A considered as an element of the group $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$. As we seen above (Observation 9.1.2) it is an invariant of a form.

However there is an additional (non-obvious) invariant.

Problem 2.1 Prove the Witt Theorem 2.2.6 for quadratic forms over $\mathbb{Q}_p$. One of the proofs given above survives for an arbitrary field of characteristic $\neq 2$.

2.2. Notation. In what follows, $p \neq 2$ and *quadratic forms under a discussion are non-degenerate.*

Define two homomorphisms $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2 \to \mathbb{Z}_2$. Let $z = p^k(\alpha_0 + \alpha_1 p + \alpha_2 p^2 + \dots)$, where $\alpha_0 \neq 0$. We put

$$\eta(z) := \eta(\alpha_0) = \left(\frac{\alpha_0}{p}\right) \tag{2.2}$$

$$\upsilon(z) := k \mod 2 \tag{2.3}$$

(the Legendre symbol $\left(\cdot\right)$ was defined in Chapter 9).

Define the function ξ on $\mathbb{Q}^p/(\mathbb{Q}_p^*)^2$ given by

$$\xi(z) := \begin{cases} 1 & \text{if } \upsilon(z) = 0 \\ \eta(z) & \text{if } \upsilon(z) = 1 \text{ and } p = 4k+1 \\ \eta(z)i & \text{if } \upsilon(z) = 1 \text{ and } p = 4k+3 \end{cases} \tag{2.4}$$

2.3. The main statement. The invariant Ξ of a quadratic form. For a quadratic form $\Phi(x) = \sum a_j x_j^2$, define the number

$$\Xi := \prod_j \xi(a_j) \tag{2.5}$$

Theorem 2.3 *a) $\Xi = \Xi(\Phi)$ is an invariant of a nondegenerate quadratic form.*

b) Moreover, $\mathrm{Dis}(\Phi)$ *and* $\Xi(\Phi)$ *is a complete system of invariants of a non-degenerate quadratic form.*

A proof is done below in Subsections 2.4–2.11; the invariance of Ξ is proved again below in Section 11.1. It seems that the second proof is more simple and more natural); in fact, Ξ is a phase of a Gaussian integral.

Problem 2.2 For $p = 4k+3$ our invariant takes values in the set ± 1, $\pm i$. Show that $\Xi = \pm 1$ or $\pm i$ according $\upsilon(\mathrm{Dis}(\Phi)) = 0$ or 1. Thus, only the sign of $\Xi(\Phi)$ is informative.

2.4. Ranges of quadratic forms. Non-degenerate quadratic forms over $\mathbb{R}$ are separated according their signs (positive, negative, indefinite). We are going to carry out a similar analysis in the p-adic case.

Denote by $\mathrm{Ran}(\Phi)$ the range of a non-degenerate quadratic form Φ on $\mathbb{Q}_p^n \setminus 0$. Evidently, if $z \in \mathrm{Ran}(\Phi)$, then $\lambda^2 z \in \mathrm{Ran}(\Phi)$ for all $\lambda \in \mathbb{Q}_p^*$. Hence it is natural to regard $\mathrm{Ran}(\Phi)$ as a subset of the 5-element set

$$\mathbb{Q}_p/(\mathbb{Q}_p^*)^2 = 0 \cup \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$$

Denote by r an arbitrary element of $\mathbb{Q}_p$ such that $|r| = 1$, $\eta(r) = -1$. Denote by

$$\langle 1 \rangle, \qquad \langle r \rangle, \qquad \langle p \rangle, \qquad \langle pr \rangle,$$

the $(\mathbb{Q}_p^*)^2$-cosets in $\mathbb{Q}_p^*$ with given representatives, i.e., $\langle r \rangle := r(\mathbb{Q}_p^*)^2$, etc.

We are going to present a constructive description of ranges of arbitrary quadratic forms.

Problem 2.3 Let $p = 4k + 1$. Find ranges of the forms $x_1^2 + x_2^2$, $x_1^2 + rx_2^2$, $x_1^2 + px_2^2$, $x_1^2 + prx_2^2$. It is not too interesting to read a written solution; however the exercise is pleasant.

2.5. Preliminary remarks.

Observation 2.4 *If* $\mathrm{Ran}(\Phi)$ *contains 0 (i.e., Φ has an isotropic vector), then* $\mathrm{Ran}(\Phi)$ *is the whole* $\mathbb{Q}_p$.

PROOF. Indeed, in this case, a form can be reduced to

$$2x_1x_2 + \sum_{k,l\geqslant 3} c_{kl}x_kx_l \tag{2.6}$$

(see Lemma 2.1.3). We assume $x_2 = 1$, $x_3 = x_4 = \cdots = 0$. $\square$

Observation 2.5 *Let* $\Phi(x) = \sum a_jx_j^2$. *If* $a_i = -a_j$ *in* $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$, *then* $\Phi(x)$ *has an isotropic vector.*

PROOF. Say, $a_1 = -a_2$. Then $(1, 1, 0, 0, \dots)$ is isotropic. $\square$

Observation 2.6 *If* $\mathrm{Ran}(a_1x_1^2 + a_2x_2^2 + a_3x_3^2) = \mathbb{Q}_p$, *then* $(-a_3) \in \mathrm{Ran}(a_1x_1^2 + a_2x_2^2)$

PROOF. Choose a nonzero (z_1, z_2, z_3) such that $a_1z_1^2 + a_2z_2^2 + a_3z_3^2 = 0$. If $z_3 = 0$, then $a_1z_1^2 + a_2z_2^2 = 0$, and the range of this form $\mathbb{Q}_p$. Otherwise, $a_1(z_1/z_3)^2 + a_2(z_2/z_3)^2 = -a_3$. $\square$

Observation 2.7 *There exist a form on* $\mathbb{Q}_p^3$, *whose range is not* $\mathbb{Q}_p$

PROOF. Put $\Phi(x) = x_1^2 + rx_2^2 + px_3^2$. Show that $\Phi(x) \neq 0$ for nonzero x. Indeed, the norms of the first two summands are even and the norm of the last summand is odd. The only chance to obtain zero is $|x_1^2| = |rx_2^2| > |px_3^2|$. But the range of form $x_1 + rx_2^2$ over $\mathbb{F}_p$ does not contain 0. $\square$

2.6. Addition of cosets in $\mathbb{Q}/(\mathbb{Q}_p^*)^2$.

Theorem 2.8

$$\textit{For } p = 4k+1, \qquad \langle u\rangle + \langle v\rangle = \begin{cases} \mathbb{Q}_p & \textit{if } u = v \\ \{\langle u\rangle, \langle v\rangle\} & \textit{otherwise} \end{cases}$$

$$\textit{For } p = 4k+3, \qquad \langle u\rangle + \langle v\rangle = \begin{cases} \mathbb{Q}_p & \textit{if } u = rv \\ \{\langle u\rangle, \langle ru\rangle\} & \textit{if } u = v \\ \{\langle u\rangle, \langle v\rangle\} & \textit{otherwise} \end{cases}$$

It is easy to carry out a case-by-case analysis (see Problem 2.3. We prefer another approach.

PROOF. For definiteness, assume $p = 4k+1$, therefore (-1) is a square. Only the second row requires a proof.

By Observation 2.6,

$$\langle y\rangle \in \langle v\rangle + \langle w\rangle \quad \Longleftrightarrow \quad \langle y\rangle + \langle v\rangle + \langle w\rangle = \mathbb{Q}_p$$

We emphasis, that the last relation is symmetric. Next,

$$\langle y\rangle + \langle v\rangle + \langle w\rangle = \mathbb{Q}_p \quad \Longleftrightarrow \quad \langle yvw\rangle + \langle v\rangle + \langle w\rangle = \mathbb{Q}_p$$

we merely multiply both sides by vw and observe $\langle vw\cdot v\rangle = \langle w\rangle$, $\langle vw \cdot w\rangle = \langle v\rangle$.

We get the following observation:

— *If $\langle y\rangle + \langle v\rangle + \langle w\rangle = \mathbb{Q}_p$ for some pairwise distinct y, v, w, then the same relation holds for all triples of pairwise distinct y, v, w.*

Indeed, we can change an arbitrary term in our relation.

By Observation 2.7, the conclusion of the last deduction is false. This proves the theorem. □

Now one can easily find the range of an arbitrary form.

Corollary 2.9 *Each form of $n \geqslant 5$ variables has an isotropic vector.*

PROOF. Let $\sum_{j=1}^{n} a_j x_j^2$ has no isotropic vectors.

If $p = 4k+1$, then all $\langle a_j\rangle$ must be pairwise distinct and hence $n \leqslant 4$.

If $p = 4k+3$, then coincidences $\langle a_i\rangle = \langle ra_j\rangle$ and $\langle a_i\rangle = \langle a_j\rangle = \langle a_k\rangle$ are forbidden. Hence $n \leqslant 4$. □

Problem 2.4 Find a (unique) form on $\mathbb{Q}_p^4$ without isotropic vectors

2.7. Classification of binary forms.

Observation 2.10 *a) Forms $u(x_1^2 - x_2^2)$ are pairwise equivalent for all u.*

b) Forms $ux_1^2 + vx_2^2$ and $vx_1^2 + ux_2^2$ are equivalent.

c) There are no other equivalences between binary forms.

PROOF. Let $\Phi = a_1x_1^2 + a_2x_2^2$, $\Psi = b_1x_1^2 + b_2x_2^2$ be equivalent forms. Then $\operatorname{Dis}\Phi = \operatorname{Dis}\Psi$, i.e., $a_1a_2 = b_1b_2$ in $\mathbb{Q}_p/(\mathbb{Q}_p)^2$. Equivalently $a_1/b_1 = a_2/b_2$. Thus $\Psi = \lambda\Phi$.

It remains to watch a possibility of equivalence $\lambda\Phi \sim \Phi$.

If $\Phi = x_1^2 - x_2^2$, we refer to Observation 2.4 (this is variant a).

If $\lambda = a_2/a_1$, then we come to variant b).

Evidently, in all the remaining situations $\operatorname{Ran}\lambda\Phi \neq \operatorname{Ran}\Phi$ because Ran remembers coefficients of a form. □

2.8. Invariants of binary forms.

Observation 2.11 *Ξ is an invariant of binary forms.*

PROOF. If $\Phi(x) = a(x_1^2 - x_2^2)$, then $\Xi = 1$. □

Lemma 2.12 *Discriminant and Ξ constitute a complete system of invariants of binary forms.*

PROOF. We refer to the proof of Observation 2.10. We must show that Ξ separates nonequivalent Φ and $\lambda\Phi$.

For definiteness, consider $p = 4k + 1$. We must watch that Ξ changes under the transformations

$$x_1^2 + rx_2^2 \longrightarrow p(x_1^2 + rx_2^2)$$
$$x_1^2 + px_2^2 \longrightarrow r(x_1^2 + px_2^2)$$
$$x_1^2 + prx_2^2 \longrightarrow r(x_1^2 + prx_2^2)$$

However, this is obvious. □

2.9. Transformations of orthogonal bases. Let Φ be a non-degenerate orthogonal form. Let $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_n\}$ be orthogonal bases. We say that they are *adjacent*, if they have $\leqslant 2$ different basic elements.

Lemma 2.13 *Fix a form Φ. For each two orthogonal bases U, V there is a finite sequence $W_1, \dots, W_M$ of orthogonal bases, such that $W_0 = U$, $W_M = V$ and W_{j+1} is adjacent with W_j for each j.*

PROOF OF LEMMA 2.13. It suffices to joint the basis $\{u_1, \dots, u_n\}$ with some basis of the form $\{v_1, z_2, \dots, z_n\}$. After this, we pass to the orthogonal complement $v_1^\perp$ and come to the same problem for bases $\{z_2, \dots, z_n\}$ and $\{v_2, \dots, v_n\}$.

Expand $v_1 = \sum_{j=1} s_j u_j$.

Case 1. If only coefficient s_j is non-zero, then there is nothing to discuss.

Case 2. Let precisely two coefficients, say s_1, s_2, be nonzero. In other words, $v_1 = s_1 u_1 + s_2 u_2$. Consider a vector $h = t_1 u_1 + t_2 u_2$ orthogonal to v_1. Then v_1, h, $u_3, \dots, u_n$ is a desired basis.

Case 3. Let $m > 2$ coefficients s_j be non-zero. To be definite, assume s_1, s_2, $s_3 \neq 0$. Consider the projections of v_1 to planes spanned by the vectors $\{u_1, u_2\}$, $\{u_1, u_3\}$, $\{u_2, u_3\}$:

$$r_{11} = s_1 u_1 + s_2 u_2, \qquad r_{13} = s_1 u_1 + s_3 u_3, \qquad r_{23} = s_2 u_2 + s_3 u_3,$$

At least one of these vectors, say r_{11}, is not isotropic. Otherwise, we obtain a system of equations of the form

$$a_1 s_1^2 + a_2 s_2^2 = 0, \qquad a_1 s_1^2 + a_3 s_3^2 = 0, \qquad a_2 s_2^2 + a_3 s_3^2 = 0$$

which has only a zero solution.

Thus r_{11} is non isotropic. Let h be a vector in the plane $\{u_1, u_2\}$ orthogonal to r_{11}. Consider the new basis $\{r_{11}, h, u_3, \dots, u_n\}$. Now

$$v_1 = r_{11} + \sum_{j \geqslant 3} s_j u_j$$

Only $(m-1)$ coefficients of this sum are nonzero.

Now we can repeat one of the previous steps. $\square$

2.10. The invariance of Ξ. By Lemma 2.12, Ξ is an invariant of binary forms. By Lemma 2.13, it is an invariant in a general case. Indeed Ξ is the same for adjacent bases.

2.11. Completeness of the system of invariants. Let Φ, Ψ be quadratic forms over $\mathbb{Q}_p^n$ with

$$\mathrm{Dis}(\Phi) = \mathrm{Dis}(\Psi), \qquad \Xi(\Phi) = \Xi(\Psi)$$

If $n = 2$, then these forms are equivalent.

Let $n \geqslant 3$. Consider the form $\Phi \ominus \Psi$ on $\mathbb{Q}_p^n \oplus \mathbb{Q}_p^n$,

$$\Phi \ominus \Psi(x \oplus y) = \Phi(x) - \Psi(y)$$

By Corollary 2.9, this form has an isotropic vector. In other words, there are vectors x, and y such that $\Phi(x) = \Phi(y)$. Let $\overline{\Phi}$, be the restriction of Φ to $x^{\perp}$, and $\overline{\Psi}$ be the restriction of Ψ to $\overline{y}^{\perp}$. Evidently,

$$\mathrm{Dis}(\overline{\Phi}) = \mathrm{Dis}(\overline{\Psi}), \qquad \Xi(\overline{\Phi}) = \Xi(\overline{\Psi})$$

and we reduce the problem to $(n-1)$-dimensional spaces.

This completes the proof of Theorem 2.3. $\square$

2.12. Classical groups. We define the following classical groups.

— $\mathrm{GL}(n, \mathbb{Q}_p)$ is the group of all invertible matrices over $\mathbb{Q}_p$

— $\mathrm{Sp}(2n, \mathbb{Q}_p)$ is the group of matrices preserving a nondegenerate skew symmetric bilinear form.

— $\mathrm{O}(n, \mathbb{Q}_p; \Phi)$ is the group of matrices preserving a non-degenerate quadratic form Φ on an n-dimensional $\mathbb{Q}_p$-linear space[2]. For $n \geqslant 5$ all the such groups are non-compact[3].

In real case, there are 10 series of classical groups. In p-adic case there is a lot of classical groups, because we have a possibility consider finite algebraic extensions of K and also central division algebras L over K. Immediately there arises a huge number of groups $\mathrm{GL}(n, K)$ and $\mathrm{GL}(n, L)$ (as analogs of $\mathrm{GL}(n, \mathbb{R})$,

[2] For $n \geqslant 4$, there are precisely 8 nonequivalent quadratic forms (16 for $p = 2$). However, forms Φ and $\lambda\Phi$ produce the same orthogonal groups. For instance, for odd n, we get only two different orthogonal groups

[3] Proof. By Corollary 2.9, the form has an isotropic vector. We represent Φ as (2.6) and observe that the group of isometries contains hyperbolic rotations, $x_1 \mapsto \lambda x_1$, $x_2 \mapsto \lambda^{-1} x_2$.

$\mathrm{GL}(n,\mathbb{C})$, $\mathrm{GL}(n,\mathbb{H})$). We can repeat the same for symplectic and orthogonal groups. Below this zoo does not appear.

There is an another distinction of between real and p-adic case that is more important for us. In the real case, any classical group has a maximal compact subgroup which also is a classical group. In p-adic case, maximal compact subgroups have another nature.

We define $\mathrm{GL}(n,\mathbb{O}_p)$ as the group of matrices g such that g and g^{-1} have integer elements. This group is compact.

Problem 2.5 $g \in \mathrm{GL}(n,\mathbb{O}_p)$ iff g has integer elements and $|\det g| = 1$.

For $j = 1, 2, \dots$, we define the *congruence subgroup* GL^j consisting of matrices having the form $1 + p^j A$, where A is a matrix with integer elements.

Problem 2.6 a) GL^j are normal subgroups in $\mathrm{GL}(n,\mathbb{O}_p)$.

b) $\mathrm{GL}(n,\mathbb{O}_p)/\mathrm{GL}^1 \simeq \mathrm{GL}(n,\mathbb{F}_p)$

c) $\mathrm{GL}^j/\mathrm{GL}^{j+1}$ is the Abelian group $\simeq \mathbb{Z}_p^{n^2}$

2.13. Digression. The case $p = 2$. Let $a = 2^k(1 + a_1 \cdot 2 + a_2 \cdot 2^2 + \dots)$. Define

$$\xi_2(a) = \begin{cases} \frac{1}{\sqrt{2}}(1 + (-1)^{a_1} i) & \text{if } k \text{ is even} \\ \frac{1}{\sqrt{2}}(1 + i) i^{a_1} (-1)^{a_2} & \text{if } k \text{ is odd} \end{cases}$$

Our main Theorem 2.3 survives literally, but a proof is more tedious.

2.14. The Hilbert symbol. We want to link our considerations with the classical approach to the problem.

Let a, $b \in \mathbb{Q}_p^*$. The *Hilbert symbol* is

$$(a,b)_p := \begin{cases} 1 & \text{if } ax^2 + by^2 - z^2 \text{ has an isotropic vector} \\ -1 & \text{otherwise} \end{cases}$$

Equivalently $(a,b)_p = 1$ iff the form $ax^2 + by^2$ 'represents a square', i.e., $ax^2 + by^2 = c^2$ for some x, y. Theorem 2.8 describs ranges of all the binary forms, therefore we know values of the Hilbert symbol for all a, b.

Theorem 2.14 *a) For $p \neq 2$,*

$$(a,b)_p = (-1)^{v(a)v(b)\cdot(p-1)/2} \eta(a)^{v(b)} \eta(b)^{v(a)}$$

b) For $p = 2$, decompose $a = 2^s m$, $b = 2^s l$. Denote $\varepsilon(u) := (u-1)/2$, $\omega(u) := (u^2-1)/8$. Then

$$(a,b)_2 = (-1)^{\varepsilon(u)\varepsilon(v) + \omega(u)\varepsilon(v) + \omega(v)\varepsilon(u)}$$

Corollary 2.15 *The Hilbert symbol is bilinear in the following sense*

$$(a_1 a_2, b)_p = (a_1, b)_p (a_2, b)_p \,, \qquad (a, b_1 b_2)_p = (a, b_1)_p (a, b_2)_p \tag{2.7}$$

Corollary 2.16

$$(a,b)_p = \frac{\xi(a)\xi(b)}{\xi(ab)} \tag{2.8}$$

2.15. Hasse–Minkowski invariant. Let Φ be a non-degenerate quadratic form, let $\Phi(x) = \sum_{j=1}^{n} a_j x_j^2$. The *Hasse–Minkowski invariant* $c_p(\Phi)$ is given by

$$c_p(\Phi) = \prod_{i<j} (a_i, a_j)_p \tag{2.9}$$

Theorem 2.17 *a) c_p does not depend of the choice of an orthogonal basis.*

b) $\mathrm{Dis}(\Phi)$ *and* $c_p(\Phi)$ *is a complete system of invariants of a quadratic form.*

The following lemma show that the Hasse–Minkowski invariant is equivalent to the invariant Ξ defined above. In particular, Theorems and 2.3 and 2.17 are equivalent.

Lemma 2.18

$$c_p(\Phi) = \frac{\Xi(\Phi)}{\xi[\det(\Phi)]}$$

Proof of Lemma. We apply (2.8),

$$\begin{aligned}\Xi(\Phi) = \xi(a_1)\xi(a_2)\xi(a_3)\xi(a_4)\cdots &= (a_1,a_2)_p\,\xi(a_1a_2)\xi(a_3)\xi(a_4)\cdots = \\ &= (a_1,a_2)_p\,(a_1a_2,a_3)_p\,\xi(a_1a_2a_3)\xi(a_4)\cdots = \\ &= (a_1,a_2)_p\,(a_1a_2,a_3)_p\,\ldots(a_1a_2\ldots a_{n-1},a_n)_p\,\xi(a_1a_2\ldots a_n)\end{aligned}$$

Since the Hilbert symbol is bilinear (2.7), the last expression is $c_p(\Phi)\xi(a_1\ldots a_n)$. $\square$

2.16. Digression. Hasse–Minkowski principle. Recall (without a proof) the famous *Hasse–Minkowski principle* Consider a non-degenerate real quadratic form $\Phi_{\mathbb{Q}}(x) = \sum c_{ij}x_ix_j$ with rational coefficients c_{ij}. We can regard this expression as a quadratic form $\Phi_{\mathbb{R}}$ on the linear space $\mathbb{R}^n$ or quadratic forms $\Phi_{\mathbb{Q}_p}$ on the spaces $\mathbb{Q}_p^n$.

Theorem 2.19 *Let $\Phi_{\mathbb{Q}}$ and $\Psi_{\mathbb{Q}}$ be nondegenerate quadratic forms on $\mathbb{Q}^n$. These forms are $\mathbb{Q}$-equivalent iff*

$$\Phi_{\mathbb{R}} \sim \Psi_{\mathbb{R}}, \qquad \Phi_{\mathbb{Q}_p} \sim \Psi_{\mathbb{Q}_p}, \quad \textit{for all } p$$

10.3 Lattices

In what follows the term 'linear space' means a finite-dimensional linear space.

The set $\mathbb{Q}_p$ is a totally disconnected. Hence linear spaces $V \simeq \mathbb{Q}_p^n$ over $\mathbb{Q}_p$ are also totally disconnected. For each basis $e_j \in V$ the subset $\bigoplus \mathbb{O}_p e_j$ is open and closed in V. The subsets $p^k \bigoplus \mathbb{O}_p e_j$ form a fundamental system of neighborhoods of 0.

3.1. $\mathbb{O}_p$-Submodules. An $\mathbb{O}_p$-submodule M in a p-adic linear space V is a subset satisfying the conditions

– if v, $w \in M$, then $v + w \in M$

– if $v \in M$, $\lambda \in \mathbb{O}_p$, then $\lambda v \in M$.

Theorem 3.1 *Each $\mathbb{O}_p$-submodule admits a representation*

$$\mathbb{Q}_p f_1 \oplus \cdots \oplus \mathbb{Q}_p f_k \oplus \mathbb{O}_p g_1 \oplus \cdots \oplus \mathbb{O}_p g_l$$

where $f_1, \ldots, f_k, g_1, \ldots, g_l$ are linear independent vectors.

Problem 3.1 Each closed subgroup (= closed $\mathbb{Z}$-submodule) in $\mathbb{R}^n$ has the form $\mathbb{Z} f_1 \oplus \cdots \oplus \mathbb{Z} f_k \oplus \mathbb{R} g_1 \oplus \cdots \oplus \mathbb{R} g_l$, where $f_1, \ldots, f_k, g_1, \ldots, g_l$ are linear independent vectors.

Proof of Theorem. Let $L \subset V$ be $\mathbb{O}_p$-submodule. Without loss of generality, we can assume that the linear span of L is the whole space M. Otherwise, we pass to a subspace spanned by L.

Next, consider a maximal linear subspace $Y \subset L$. Let $W \subset V$ be a complementary subspace. Denote $M := L \cap W$. Evidently, $L = W \oplus M$.

Now we get the $\mathbb{O}_p$-submodule M in the linear space W. This submodule does not contain lines and its linear span is the whole W. The remaining part of the proof is an examination of the submodule $M \subset W$.

Consider the quotient module M/pM. The subring $p\mathbb{O}_p \subset \mathbb{O}_p$ acts in this module in trivially. Therefore the quotient ring $\mathbb{O}_p/p\mathbb{O}_p = \mathbb{F}_p$ acts in M/pM. In another words, M/pM is an $\mathbb{F}_p$-linear space. Consider a basis in the $\mathbb{F}_p$-linear space M/pM, and choose representatives of the basis elements $e_1, \ldots, e_n \in M$.

We are going to show that *e_j is a basis in W and $M = \bigoplus \mathbb{O}_p e_j$* (this implies the theorem).

1) *The elements e_j generate M as $\mathbb{O}_p$-module.* Let $h \in M$. Then h can be represented as

$$h = \sum a_j^{(1)} e_j + h_1, \qquad a_j^{(1)} \in \mathbb{O}_p,\ h_1 \in pM$$

For the same reason, one can represent $p^{-1}h$ as

$$p^{-1}h_1 = \sum a_j^{(2)} e_2 + h_2, \qquad a_j^{(1)} \in \mathbb{O}_p,\ h_2 \in pM$$

Repeating the same arguments, we get

$$h = \sum a_j^{(1)} e_j + p \sum a_j^{(2)} e_j + \cdots + p^m \sum a_j^{(m)} e_j + h_{m+1}, \qquad h_{m+1} \in p^{m+1} M$$

Passing to the limit, we represent h as a sum of a convergent series

$$h = \sum_{j=1}^{k} \Big(\sum_{m=1}^{\infty} p^{m-1} a_j^{(m)} \Big) e_j$$

Therefore, h is contained in $\mathbb{O}_p$-span of e_j.

2) *The vectors e_j generate the whole linear space W.* Indeed, they generate M and M generates W.

3) *Vectors e_j form a basis in W.* Choose a basis $\zeta_1, \dots, \zeta_k$ of W consisting of elements of M. Put $Y := \oplus \mathbb{O}_p \zeta_j$. We claim that for sufficiently large α, β

$$p^\alpha Y \subset M \subset p^{-\beta} Y$$

Indeed, we can express ζ_i via e_j and e_j via ζ_i.

Therefore the group M/pM is a subquotient of the Abelian p-group

$$p^{-\beta} Y / p^\alpha Y \simeq (\mathbb{Z}_{p^{\alpha+\beta}})^{\dim W}$$

Hence the number of generators of W/pW does not exceed $\dim W$, therefore e_j is a basis in W. □

Observation 3.2 *A subset L in a linear space is an $\mathbb{O}_p$-submodule iff L is a closed subgroup.*

PROOF. The arrow $\Longrightarrow$ follows from the theorem. On the other hand, for v lying in a subgroup L, nv also is contained in L for every integer n. But $\mathbb{Z}$ is dense in $\mathbb{O}_p$, hence the set $\{nv\}$ is dense in $\mathbb{O}_p v$. Therefore, L is a $\mathbb{O}_p$-module.□

3.2. Lattices. A *lattice* L in a linear space V is a subset having the form

$$\mathbb{O}_p f_1 \oplus \dots \oplus \mathbb{O}_p f_n, \qquad \text{where } f_n \text{ is a basis in } V.$$

For instance $\mathbb{O}_p^n \subset \mathbb{Q}_p^n$ is a lattice.

Obviously, lattices are open compact subgroups in the additive group of a linear space. The previous considerations imply the following statement.

Proposition 3.3 *Any open compact subgroup in a linear space is a lattice.*

3.3. Bases in lattices. Let V be an n-dimensional linear space over $\mathbb{Q}_p$.

In proof of Theorem 3.1 we note the following fact.

Observation 3.4 *For a lattice L, the quotient L/pL is an n-dimensional $\mathbb{F}_p$-linear space.*

Theorem 3.5 *Vectors $f_1, \dots, f_n$ of a lattice L generate L iff the images of the f_j in the $\mathbb{F}_p$-linear space $(\mathbb{O}_p/p\mathbb{O}_p)^n \simeq \mathbb{F}_p^n$ consitute a basis.*

PROOF. This was proved during the proof of Theorem 3.1.

3.4. The space of lattices.

Theorem 3.6 *a) The group of linear operators preserving the lattice $\mathbb{O}_p^n \subset \mathbb{Q}_p^n$ is the group $\mathrm{GL}(n, \mathbb{O}_p)$.*

b) The set of all the lattices in $\mathbb{Q}_p^n$ is the homogeneous (discrete countable) space

$$\mathrm{Lat}_n = \mathrm{GL}(n, \mathbb{Q}_p)/\mathrm{GL}(n, \mathbb{O}_p)$$

PROOF. a) Denote by e_j the standard basis in $\mathbb{Q}_p^n$. Let $g \in \mathrm{GL}(n, \mathbb{Q}_p)$ preserves $\mathbb{O}_p^n$. Then $ge_j \in \mathbb{O}_p^n$, i.e., all the matrix elements of g are integer. The same holds for g^{-1}.

b) The homogeneity is a corollary of Theorem 3.1.

The subgroup $\mathrm{GL}(n, \mathbb{O}_p) \subset \mathrm{GL}(n, \mathbb{Q}_p)$ is open, therefore points of $\mathrm{Lat}_n = \mathrm{GL}(n, \mathbb{Q}_p)/\mathrm{GL}(n, \mathbb{O}_p)$ are open, i.e., the space is discrete. Since the group $\mathrm{GL}(n, \mathbb{Q}_p)$ is separable, the space Lat_n is separable; hence it is countable. □

In fact, *the spaces* Lat_n *are* p*-adic analogs of matrix balls.* They are the main objects of our interest in this chapter.

3.5. Lattices and norms. A *norm* $\|\cdot\|$ on a linear space V over $\mathbb{Q}_p$ is a function taking values p^j, where $j \in \mathbb{Z}$, and 0 and satisfying the conditions

$$\|v + w\| \leqslant \max(\|v\|, \|w\|), \qquad \|\lambda v\| = |\lambda| \cdot \|v\|, \qquad \|v\| \neq 0 \text{ whenever } v \neq 0$$

Problem 3.2 The set $\|v\| \leqslant 1$ is a lattice.

EXAMPLE. The function $\|v\| = \max_j |v_j|$ is a norm on $\mathbb{Q}_p^n$. The corresponding lattice is $\mathbb{O}_p^n$. More generally, for fixed a_j

$$\|v\| = \max_j p^{a_j} |v_j|$$

is a norm. The corresponding lattice is $\oplus p^{-a_j} \mathbb{O}_p$. More generally, for any collection of ℓ_j of linear independent linear functionals, the following function is a norm

$$\|v\| = \max_j |\ell_j(v)| \tag{3.1}$$

Proposition 3.7 *For a lattice* $M \subset V$*, the following function is a norm*

$$\|v\| = \min_{k:\, p^k M \ni v} p^k$$

PROOF. Without loss of generality, we can take $M = \mathbb{O}_p^n$. □

Problem 3.3 Each norm has the form (3.1).

3.6. Pairs of lattices.

Theorem 3.8 *Let* M*,* L *be two lattices in an* n*-dimensional space. Then there is a basis* $f_1, \dots, f_n$ *such that*

$$M = \bigoplus \mathbb{O}_p \cdot f_j, \qquad L = \bigoplus p^{-\alpha_j} \mathbb{O}_p \cdot f_j$$

The collection

$$\alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_n \tag{3.2}$$

is uniquely determined by lattices L*,* M*.*[4]

[4] A basis f_j is not canonical

We call the collection of numbers $\alpha_j = \alpha_j(L, M)$ by *complex distance*.

PROOF. Without loss of generality, we can assume $M \supset L$ (otherwise, we consider a lattice $p^{-\alpha}M$ with large α. Then M/L is a finite Abelian p-group with n generators. We represent M/L in the from

$$M/pL = \bigoplus \mathbb{Z}_{p^{m_j}}$$

Evidently, the numbers m_j are invariants of the pair (M, L).

Next, consider the group

$$M/pL \simeq \bigoplus \mathbb{Z}_{p^{m_j+1}}$$

and choose a standard system of its generators f_j. Then $p^{m_j+1}f_j = 0$ and $p^{m_j}f_j$ generate the subgroup L/pL. Take representatives f_j° of f_j in M. By Theorem 3.5, f_j° generate M and $p^{m_j}f_j^\circ$ generate L. □

ANOTHER PROOF. Consider the the space $\mathrm{Lat}_n = \mathrm{GL}(n, \mathbb{Q}_p)/\mathrm{GL}(n, \mathbb{O}_p)$. Put L to the reference point $\mathbb{O}_p^n$. Now we can move M by elements of $\mathrm{GL}(n, \mathbb{O}_p)$. We must describe orbits of $\mathrm{GL}(n, \mathbb{O}_p)$ on $\mathrm{GL}(n, \mathbb{Q}_p)/\mathrm{GL}(n, \mathbb{O}_p)$. In other words, our problem can be reformulated as follows

— *To describe double cosets* $\mathrm{GL}(n, \mathbb{O}_p) \setminus \mathrm{GL}(n, \mathbb{Q}_p)/\mathrm{GL}(n, \mathbb{O}_p)$.

Let $g \in \mathrm{GL}(n, \mathbb{Q}_p)$. We can transform it as $g \mapsto h_1 g h_2$, where h_1, h_2 range in $\mathrm{GL}(n, \mathbb{O}_p)$. In particular, we can do the following operations

a) Left and right multiplications by diagonal matrices, whose entries are in $\mathbb{O}_p \setminus p\mathbb{O}_p$. Equivalently, we can multiply rows and columns of matrices by numbers $\in \mathbb{O}_p \setminus p\mathbb{O}_p$.

b) Left multiplication by matrices of the form[5] $1 + zE_{ij}$, where $i \neq j$ and z ranges in $\mathbb{O}_p$. This is equivalent to the usual operation of addition of rows, for instance

$$\begin{pmatrix} 1 & 0 & \dots \\ z & 1 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} g_{11} & g_{21} & \dots \\ g_{21} & g_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} g_{11} & g_{21} & \dots \\ g_{21} + z g_{11} & g_{22} + z g_{21} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

c) Right multiplications by matrices having the same form. This corresponds to addition of columns.

d-e) Left and right multiplications by operators permutating the standard basis in $\mathbb{Q}_p$. This correspond to permutation of rows or columns respectively.

Now we imitate the 'Gauss algorithm' for solving linear systems. Choose a matrix element g_{ij} having a maximal absolute value $|g_{ij}|$. Move it to the position $\{11\}$ by operations c) and d). Annihilate all the elements of the first column but g_{11} by the operation b). Annihilate all the elements of the first row but g_{11} by the operation d). Make $g_{11} = p^l$ by operation a).

We get a matrix $\begin{pmatrix} 1 & 0 \\ 0 & \widetilde{g} \end{pmatrix}$. Apply the same algorithm to the $(n-1) \times (n-1)$ matrix $\widetilde{g}$. □

[5]Here E_{ij} a matrix having a unique nonzero entry $= 1$ at the place ij

Problem 3.4 $\alpha_j(M,L) = -\alpha_{n-j+1}(M,L)$

Problem* 3.5 Find the number of lattices M such that the complex distance between M and $\mathbb{O}^n$ is a prescribed collection (3.2; this is the analog of Theorems 7.??–7.?? about distributions of eigenvalues.

3.7. Minimax characterization of invariants of a pair of lattices. The following statement is an analog of the minimax characterization of the eigenvalues and angles, see Section 2.5.

Theorem 3.9 *Let W range in the set of all the j-dimensional subspaces in $\mathbb{Q}_p^n$,*

$$p^{\alpha_j} = \min_{W:\dim W=j} \max_{w\in W} \frac{\|w\|_L}{\|w\|_M} \tag{3.3}$$

$$p^{\alpha_{n-j+1}} = \max_{W:\dim W=j} \min_{w\in W} \frac{\|w\|_L}{\|w\|_M} \tag{3.4}$$

Proof. Prove the first statement. In notation of Theorem 3.8, consider the subspace $Y := \mathbb{Q}_p f_j \oplus \dots \oplus \mathbb{Q}_p f_n$. For each $y \in Y$, we have $\|y\|_L/\|y\|_M \geqslant p^{\alpha_j}$. Since $W \cap Y \neq 0$, we get

$$\max_{w\in W} \|w\|_L/\|w\|_M \geqslant p^{\alpha_j}$$

The minimal value p^{α_j} of $\max(\dots)$ is achieved on the $W^\circ := \mathbb{Q}_p f_1 \oplus \dots \oplus \mathbb{Q}_p f_j$. $\square$

3.8. Dual lattices. Let V be a linear space, let V° be the dual linear space, i.e., the space of linear functionals on V. For a lattice $L \subset V$ define the *dual lattice* $L^\lozenge \subset V^\circ$ consisting of linear functionals ℓ such that

$$\ell(v) \in \mathbb{O}_p \qquad \text{for all } v \in L$$

Problem 3.6 a) Show that $L^\lozenge$ is a lattice

b) Find the complex distance between $L^\lozenge$, $M^\lozenge$.

10.4 Bruhat–Tits trees

4.1. Bruhat–Tits trees. Denote by

$$\mathrm{Lat}_2^* = \mathrm{PGL}(2,\mathbb{Q}_p)/\mathrm{PGL}(2,\mathbb{O}_p)$$

the space[6] of all the lattices in $\mathbb{Q}_p^2$ defined to within the equivalence $L \sim \lambda L$, where $\lambda \in \mathbb{Q}_p^*$ (λ actually can be chosen in the form p^β).

Construct a graph J_p. Its vertices are points of Lat_2^*. Two vertices L, M are connected by an edge if, for some k

$$p^k L \supset M, \qquad p^k L/M \simeq \mathbb{Z}_p$$

We also say that L, M are *neighbors*.

[6] PGL is the quotient group of GL with respect to the center, i.e., the subgroup of scalar matrices.

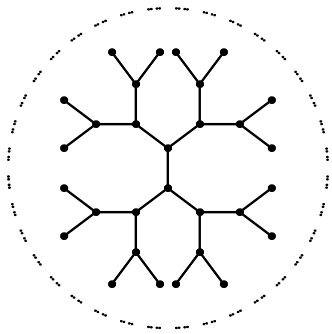


Fig. 10.2: A piece of the tree J_2

Problem 4.1 a) This relation is symmetric.

b) Two lattices are neighbors if the complex distance has a form $(\alpha+1,\alpha)$

Fix a *vertex* (a lattice) L. Let M be its neighbor. We can assume that $p^{-1}L\supset M\supset L$, and $M/L\simeq\mathbb{Z}_p$. The group $p^{-1}L/L$ is the linear space $\mathbb{F}_p^2$ and M/L is a line in ${\mathbb{F}_p}^2$. We get the following statement.

Observation 4.1 *The set of neighbors of a given lattice is in one-to-one correspondence with the projective line* $\mathbf{P}\mathbb{F}_p^1$ *over the field* $\mathbb{F}_p$.

Theorem 4.2 *The graph* J_p *is a tree. Each vertex of this tree is adjacent to* $(p+1)$ *edges.*

This tree is called the *Bruhat–Tits tree.*

Since $\mathrm{PGL}(2,\mathbb{Q}_p)$ acts on Lat_n^*, it acts on the Bruhat–Tits tree J_p.

4.2. Another definition of Bruhat-Tits trees. Now we are going to construct the same tree again. Until the identification of the constructions, we denote a 'new' tree by I_p.

Vertices of I_p are enumerated by balls $B(a,p^k)$. We connect two balls $B(a,p^k)$ and $B(c,p^{k-1})$ with an edge iff $c\in B(a,p^k)$. It is completely evident that we get a tree (this was noted above in Subsection 1.5).

4.3. A rephrasing. We wish to describe the second construction in a more intricate method.

We say that a quasiball in $\mathbb{Q}_p$ is a ball $B(a,p^k)$ or the complement $\mathbb{Q}_p\setminus B(a,p^k)$ to a ball. It is reasonable to regard quasiballs as subsets in the p-adic projective line $\mathbf{P}\mathbb{Q}_p^1\supset\mathbb{Q}_p$.

The group $\mathrm{PGL}(2,\mathbb{Q}_p)$ acts on the projective line by projective transformations $z\mapsto(a+zc)^{-1}(b+zd)$.

Lemma 4.3 *Projective transformations take quasiballs to quasiballs.*

PROOF. The group of projective transformations is generated by affine maps $z \mapsto b + zd$ and the 'inversion' $z \mapsto z^{-1}$. Evidently, affine maps take quasiballs to quasiballs.

Examine the inversion. If a ball contains 0, then the statement is trivial. Otherwise, we represent the ball B as $a(1+p^m\mathbb{O}_p)$, where $a \neq 0$ is a point of ball, and $m > 0$. The image of B under the inversion is $a^{-1}(1+p^m\mathbb{O}_p)$, i.e., it is a ball again. □

Paraphrase a definition of I_p in the terms of quasiballs.

Vertices. Consider a ball $B(a,p^k)$, i.e., a vertex of the tree I_p. Consider p subballs of $B(c_j,p^{k-1}) \subset B(a,p^k)$ of radius p^{k-1} and also the complement $\mathbb{Q}_p \setminus B(a,p^k)$. Thus, to each vertex of I_p we assign a partition of the $\mathbf{P}\mathbb{Q}_p$ into a disjoint union of $(p+1)$ quasiballs.

Problem 4.2 Each partition of $\mathbf{P}\mathbb{Q}_p$ into a disjoint union of $(p+1)$ quasiballs has such the form.

Edges. For each ball B consider the following two subsets of p-adic line: B and $\mathbb{Q}_p \setminus B$. Such partitions are in one-to-one correspondence with edges of I_p.

Thus, we described the tree I_p in the terms of partitions of $\mathbf{P}\mathbb{Q}_p^1$ into quasiballs.

Observation 4.4 *The group* $\mathrm{PGL}(2,\mathbb{Q}_p)$ *acts on* I_p.

PROOF. We refer to Lemma 4.3. □

4.4. Identification of two constructions. Fix a lattice M. Recall that $M/pM \simeq \mathbb{F}_p^2$.

Let $\ell \in \mathbf{P}\mathbb{Q}_p^1$ be a line in $\mathbb{Q}_p^2$. The image of $\ell \cap M$ in $M/pM \simeq \mathbb{F}_p^2$ is a line in $\mathbb{F}_p^2$. Thus, we get a map $\delta_M : \mathbf{P}\mathbb{Q}_p^1 \to \mathbf{P}\mathbb{F}_p^1$.

Observation 4.5 *Preimages* $\delta_M^{-1}(\cdot)$ *of points* $\in \mathbf{P}\mathbb{F}_p^1$ *are quasiballs* $\subset \mathbf{P}\mathbb{Q}_p^n$.

PROOF. Since our considerations are $\mathrm{PGL}(2,\mathbb{Q}_p)$-invariant, it suffices to consider the lattice $M = \mathbb{O}_p^2$. □

Thus each lattice determines a partition of a projective line into a disjoint union of $(p+1)$ quasiballs.

The coincidence of the two constructions is observed. This also proves Theorem 4.2. □

4.5. Digression. Absolute of a tree. We say that a *ray* on J_p is a sequence of vertices $L_1, L_2, \dots,$ such that L_i and L_{i+1} are adjacent and $L_i \neq L_{i+2}$. We say that two rays $\{L_i\}$, $\{M_j\}$ are equivalent if $L_i = M_{i+\alpha}$ for some α starting some moment $i = i_0$. The *absolute* is the space, whose points are rays defined up to equivalence.

Problem 4.3 a) Convince yourself that absolute of J_p is the p-adic projective line $\mathbf{P}\mathbb{Q}_p^1$ (see Fig.10.1

b) Invent a definition of a convergence of lattices $\in \mathrm{Lat}_2^*$ to a point of $\mathbf{P}\mathbb{Q}_p^1$.

10.5 Bruhat–Tits buildings

Now consider n-dimensional p-adic linear space and the set

$$\mathrm{Lat}_n^* := \mathrm{GL}(n, \mathbb{Q}_p)/\mathrm{GL}(n, \mathbb{O}_p)$$

of lattices defined to within a dilatation, $L \sim \lambda L$. We are construct an $(n-1)$-dimensional simplicial complex, whose vertices are points of Lat_n^*.

5.1. Construction of buildings. Fix a lattice M and consider all the lattices N such that

$$M \subset N \subset p^{-1}M$$

Such lattices are in one-to-one correspondence with linear subspaces $\mathfrak{n} := N/M$ in $\mathbb{F}_p^n \simeq p^{-1}M/M$

Consider a nonempty subset

$$I : i_1 < i_2 < \cdots < i_k$$

in the set $\{1, 2, \ldots, n-1\}$. Consider a flag of subspaces

$$\mathfrak{n}_1 \subset \cdots \subset \mathfrak{n}_k \subset \mathbb{F}_p{}^n, \qquad \dim \mathfrak{n}_\alpha = i_\alpha \tag{5.1}$$

For such a flag we 'draw' a k-dimensional simplex (a *facet*) $\Delta_M(\mathfrak{n}_1, \ldots, \mathfrak{n}_k)$, whose vertices correspond to the lattices N_1, ..., N_k, and M.

To each subflag $\mathfrak{n}_{j_1} \subset \cdots \subset \mathfrak{n}_{j_\beta}$ of (5.1 our rule assigns the facet $\Delta_M(\mathfrak{n}_{j_1}, \ldots, \mathfrak{n}_{j_\beta})$. We regard it as a face of the simplex $\Delta_M(\mathfrak{n}_1, \ldots, \mathfrak{n}_k)$.

We repeat this operation for each lattice $M \in \mathrm{Lat}_n^*$. As a result we obtain an infinite simplicial complex $\mathcal{BT}[\mathrm{PGL}(n, \mathbb{Q}_p)]$ called *Bruhat–Tits building.*

Maximal simplices (*chambers*) correspond to complete flags in $\mathbb{F}_p^n$.

REMARK. We can say definition as follows. If $M \subset N \subset p^{-1}M$, then we connect M and N by an edge. If for some collection N_1, ..., N_k of vertices all the edges $[N_\alpha, N_\beta]$ exist, then we 'draw' the corresponding $(k-1)$-dimensional simplex with given vertices.

This definition is shorter, however starting with this definition, we must describe all simplices. □

Proposition 5.1 *The definition is self-consistent. More precisely, let $[M, N]$ be an edge. Then we can draw simplices containing $[M, N]$ referring to M or referring to N. In both cases, we obtain the same result.*

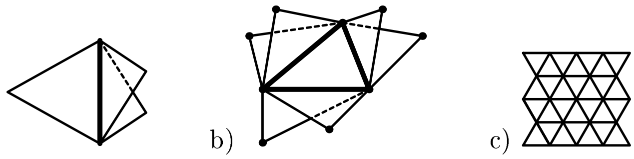
PROOF. Consider a simplex drawn referring M, i.e, a 'flag' of lattices

$$M \subset U_1 \subset \cdots \subset U_\alpha \subset N \subset V_1 \subset \cdots \subset V_\beta \subset p^{-1}M$$

We multiply M and U_j by p^{-1} and get the new chain

$$N \subset V_1 \subset \cdots \subset V_\beta \subset p^{-1}M \subset p^{-1}U_1 \subset \cdots \subset p^{-1}U_\alpha$$

It is the same simplex. However it is drawn referring to N. □



a) b) c)

Fig. 10.3: Sketches for building of $\mathrm{PGL}(3, \mathbb{Q}_2)$

By definition, the group $\mathrm{GL}(n, \mathbb{Q}_p)$ acts on the building preserving its simplicial structure.

5.2. Fine arts. We present several sketches to portrait of the building $\mathcal{BT}[\mathrm{PGL}(3, \mathbb{Q}_2)]$. It is assembled from equilateral[7] triangles.

A) Each edge is contained in 3 triangles, see Fig.10.3.a).

B) Immediate neighbors of a given triangle are shown on Fig.10.3.b).

C) Fix a basis e_1, e_2, $e_3 \in \mathbb{Q}_2^3$. Consider all the lattices $2^{l_1}\mathbb{O}_2 e_1 \oplus 2^{l_2}\mathbb{O}_2 e_2 \oplus 2^{l_4}\mathbb{O}_2 e_3$. Drawing edges we get the 'sheet'[8] shown on Fig.10.3.c).

By definition, the whole building $\mathcal{BT}[\mathrm{PGL}(3, \mathbb{Q}_2)]$ can be assembled by gluing of a countable collection of such sheets. An intersection of two sheets forms a convex hexagon on each sheet (this is not obvious).

D) Try to draw a neighborhood of a vertex. There are 7 lines and 7 planes in $\mathbb{F}_2^3$. Also there are 21 flags.

On Fig. 10.4.a), the referring vertex is O. Lines in $\mathbb{F}_2^3$ are shown by black vertices, planes by gray vertices. The two-dimensional simplices are 14 lateral faces of the pyramid and 7 'diagonal' sections through O and dotted segments in the base.

The group $\mathrm{PGL}(3, \mathbb{F}_2)$ of order $(8-1)(8-2)(8-4) = 168$ acts on this picture. Only a Sylow 7-group of $\mathrm{PGL}(3, \mathbb{F}_2)$ is visually observable on the Figure[9]

The same combinatorial picture[10] is shown on Fig. 10.4.b). Now the referring vertex is in the center of the antiprism, the lines in $\mathbb{F}_2^3$ correspond to vertices on the upper base, the planes correspond to vertices on the lower base.

E) Now the reader can try to imagine the whole building[11]. The reader have to keep in mind that number of vertices in a ball of radius R has an exponential growth.

Some other attempts to draw buildings are contained in Garrett's book [48].

5.3. Iwahoric and parahoric subgroups.. We define some important subgroups in $\mathrm{PGL}(n, \mathbb{Q}_p)$ having no analogs in $\mathrm{PGL}(n, \mathbb{R})$

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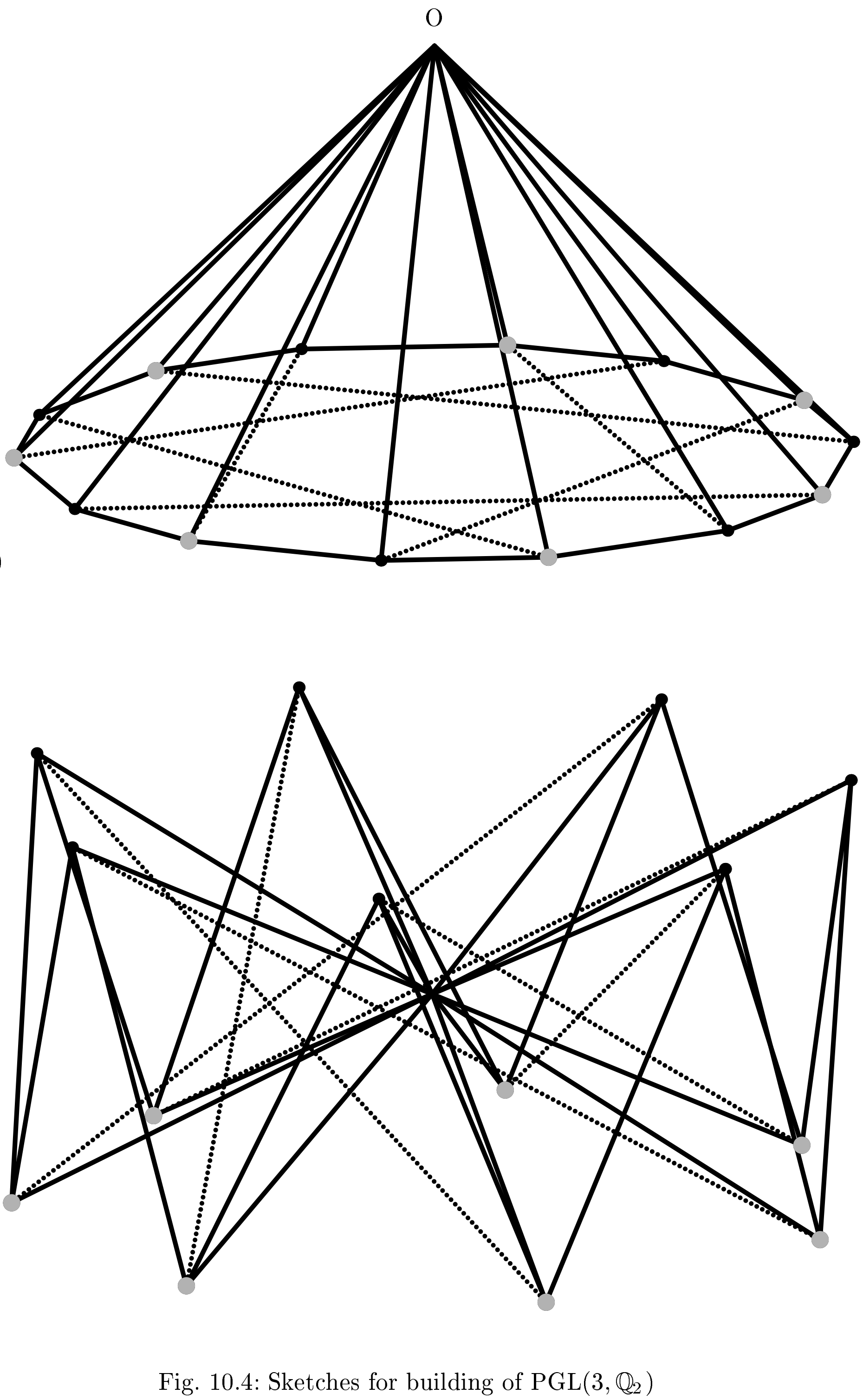
[7] In this moment the term 'equilateral' makes no sense. However the group $\mathrm{GL}(3, \mathbb{Q}_p)$ acts on the set of edges transitively. For $n > 3$ this is not true.

[8] apartments, see below

[9] The combinatorial group of symmetries of this graph is even larger than $\mathrm{PGL}(3, \mathbb{F}_2)$, because $\mathrm{PGL}(3, \mathbb{F}_2)$ preserves black and gray vertices; but they have the same rights on Figure. See below Subsection **??**

[10] It is more appropriate from the point of view of the impressionism.

[11] The author unable to do this



a)

b)

Fig. 10.4: Sketches for building of $\mathrm{PGL}(3, \mathbb{Q}_2)$

The *standard Iwahori subgroup* $\mathrm{Iw}(n, \mathbb{Q}_p)$ in $\mathrm{PGL}(n, \mathbb{Q}_p)$ is the group of invertible matrices of the form

$$\begin{pmatrix} z_{11} & z_{12} & z_{13} & \dots \\ pz_{21} & z_{22} & z_{23} & \dots \\ pz_{31} & pz_{32} & z_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad z_{ij} \in \mathbb{O}_p,\, z_{ii} \in \mathbb{O}_p \setminus p\mathbb{O}_p \tag{5.2}$$

Denote by GL^1 the group of matrices

$$\begin{pmatrix} 1 + p\zeta_{11} & p\zeta_{12} & p\zeta_{13} & \dots \\ p\zeta_{21} & 1 + p\zeta_{22} & p\zeta_{23} & \dots \\ p\zeta_{31} & p\zeta_{32} & 1 + p\zeta_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \zeta_{ij} \in \mathbb{O}_p,$$

Evidently, $\mathrm{GL}^1 \subset \mathrm{Iw}(n)$, and the quotient group $\mathrm{Iw}(n)/\mathrm{GL}^1$ is the upper-triangular matrix group $B(n, \mathbb{F}_p)$ over $\mathbb{F}_p$.

Standard parahoric subgroups are intermediate subgroups H:

$$\mathrm{PGL}(n, \mathbb{O}_p) \supset H \supset \mathrm{Iw}(n, \mathbb{Q}_p)$$

(in particular, $\mathrm{PGL}(n, \mathbb{Q}_p)$, $\mathrm{Iw}(n, \mathbb{Q}_p)$ are parahoric). A standard parahoric subgroup consists of block $(k_1 + k_2 + \dots) \times (k_1 + k_2 + \dots)$ matrices

$$\begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & \dots \\ pZ_{21} & Z_{22} & Z_{23} & \dots \\ pZ_{31} & pZ_{32} & Z_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where Z_{ij} are integer matrices, and Z_{ii}^{-1} are also integer.

A subgroup in $\mathrm{PGL}(n, \mathbb{Q}_p)$ is called *iwahoric* (resp. *parahoric*) if it is conjugate to the standard Iwahori subgroup (resp., a standard parahoric) subgroup.

Proposition 5.2 *a) Fixed points $L_j \in \mathrm{Lat}_n^*$ of an iwahoric subgroup form a flag of lattices*

$$L = L_0 \subset L_1 \subset \dots \subset L_{n-1} \subset p^{-1}L$$

such that all the inclusions are proper.

b) The stabilizer of each flag of such type is an Iwahoric subgroup.

PROOF. a) Since all the iwahoric subgroups are conjugate, it suffices to verify our statement for the standard Iwahori subgroup $\mathrm{Iw}(n)$.

Let $K \subset \mathbb{Q}_p^n$ be a lattice fixed by $\mathrm{Iw}(n)$. Since our lattices are defined to within dilatations, we can assume $K \subset p^{-1}\mathbb{O}^n$ but $K \not\subset \mathbb{O}_n$. Let $u \notin \mathbb{O}_p^n$ be an element of K.

The GL^1-orbit of u consists of vectors $u + pAu$, where A is an arbitrary integere matrix. Therefore this orbit is the set $u + \mathbb{O}_p^n$ and hence $K \supset \mathbb{O}_p^n$. Thus,

$$\mathbb{O}_p^n \subset K \subset p^{-1}\mathbb{O}_p^n$$

Such lattices are enumerated by linear subspaces in $\mathbb{F}_p^n$. Since the group GL^1 stabilizes all the lattices of this form, we actually get an action of the quotient group $\mathrm{Iw}(n)/\mathrm{GL}^1$. The latter group is the upper triangular matrix group over $\mathbb{F}_p$, it stabilizes only the flag $0 \subset \mathbb{F}_p^1 \subset \mathbb{F}_p^2 \subset \dots$ of the coordinate subspaces.

b) Indeed, all such flags are $\mathrm{PGL}(n, \mathbb{Q}_p)$-equivalent. $\square$

5.4. Another definition of buildings. Now we present an alternative definition of the same simplicial complex.

Vertices of the complex are enumerated by subgroups in $\mathrm{PGL}(n, \mathbb{Q}_p)$ of the form $g^{-1} \cdot \mathrm{PGL}(n, \mathbb{Q}_p) \cdot g$. Simplices Δ_H are enumerated by parahoric subgroups H, moreover $\Delta_H \supset \Delta_{H'}$ whenever $H \subset H'$.

In particular, $(n-1)$-dimensional simplices (chambers) are enumerated by iwahoric subgroups.

Lemma 5.3 *Two definitions of the Bruhat–Tits building are equivalent.*

Proof. This follows from Proposition 5.2.

$\square$

5.5. Apartments. Fix a basis $e_1, \dots, e_n \in \mathbb{Q}_p^n$ and consider the space of all the lattices of the form $\bigoplus p^{k_j}\mathbb{O}_p e_j$. Consider all the simplices having vertices at these lattices. We obtain a simplicial subcomplex Ξ called by *apartment*[12].

Vertices of the apartment are enumerated by integer vectors $(k_1, \dots, k_n)$ modulo the equivalence relation $(k_1, \dots, k_n) \sim (k_1 + l, \dots, k_n + l)$.

It is convenient to realize the apartment in hyperplane

$$t_1 + \dots + t_n = 0 \tag{5.3}$$

in $\mathbb{R}^n$. Projecting integer points $(k_1, \dots, k_n)$ on this hyperplane

$$(k_1, \dots, k_n) \mapsto (k_1, \dots, k_n) - \frac{1}{n}\Big(\sum k_j\Big)(1, \dots, 1)$$

we get vertices of the complex. The hyperlanes $t_i - t_j = n$ divide Ξ into equal simplices.

Problem 5.1 a) Prove the last statement.

b) Show that the picture is symmetric with respect to each plane $t_i - t_j = n$.

c) Find the dihedral angles of the simplex.

[12] Apartments are analogs of flat subspaces in matrix balls described in Theorem 2.12.2

For $n = 2$ the apartment is drawn on Fig. 10.3c). For $n = 3$ a chamber (a simplex) is shown on Fig. 10.9 a.

REMARK. We also can realize the apartment as the quotient of $\mathbb{R}^n$ modulo the equivalence

$$(\xi_1, \dots, \xi_n) \sim (\xi_1 + \eta, \dots, \xi_n + \eta) \tag{5.4}$$

Again, the vertices correspond to the integer points, the hyperplanes of symmetries are $\xi_i - \xi_j = n$. □

5.6. The group acting on an apartment. Changing a basis $e_j \in \mathbb{Q}_p^n$ we (generally speaking) get an another apartment. However only lines $\ell_j = \mathbb{Q}_p e_j$ are essential; also the order of the collection $\{\ell_l\}$ has no matter.

Observation 5.4 *a) Apartments in the building are enumerated by collections $\ell_1, \dots \ell_n$ of lines spanning the space $\mathbb{Q}_p^n$.*

b) The stabilizer $G(\Xi)$ of an apartment Ξ is a semidirect product $S_n \ltimes (\mathbb{Q}_p^*)^n$, where S_n is the symmetric group.

We wish to discuss in details transformations of the apartment Ξ induced by its stabilizer $G(\Xi)$.

Denote by $\widetilde{W} \subset \mathrm{PGL}(n, \mathbb{Q}_p)$ the group of matrices such that

a) each column and each row contains precisely one nonzero element

b) All the non-zero elements have form p^k

Obviously, our group is a semidirect product

$$\widetilde{W} = S_n \ltimes \mathbb{Z}^{n-1}$$

where the $\mathbb{Z}^{n-1}$ consists of diagonal matrices with elements p^k, and the symmetric group S_n consists of matrices, whose entries are 0 or 1.

Evidently, the group $\widetilde{W}$ preserves the standard apartment. The subgroup $\mathbb{Z}^{n-1}$ acts by shifts in the coordinates (5.4) by

$$(\xi_1, \dots, \xi_n) \mapsto (\xi_1, \dots, \xi_n) + (k_1, \dots, k_n)$$

The group S_n acts by permutations of $(\xi_1, \dots, \xi_n)$.

Problem 5.2 a) The group $\widetilde{W}$ acts transitively on the set of chambers.

b) The stabilizer of a chamber is a cyclic group $\mathbb{Z}_n$ (see Fig. 10.3c), Fig. 10.9a)[13].

c) The group $\widetilde{W}$ contains all the reflections with respect to 'walls' $\xi_i - \xi_j = k$.

5.7. Euclidean metric in buildings.

Observation 5.5 *An apartment has a natural structure of an Euclidean space.*

[13]This corresponds to cyclic permutations of a complete flag of lattices

We endow an apartment Ξ by the Euclidean distance

$$d_\Xi(t,t') := (\sum (t_j - t'_j)^2)^{1/2}$$

It is a unique to within a scalar $\widetilde{W}$-invariant Euclidean metrics.

Lemma 5.6 *Let Ξ and Ξ' be apartments, let a, $b \in \Xi \cap \Xi'$. Then there is an isometric simplicial map $\Xi \to \Xi'$ fixing both points.*

PROOF. Denote by A, B minimal facets containing a, b. Denote by α_k and β_j their vertices. Then $d_\Xi(\alpha_k,\beta_j) = d_{\Xi'}(\alpha_k,\beta_j)$ because this value is expressed in the terms of the complex distance on Lat^*_n.

5.8. Canonical forms of pairs of chambers.

Theorem 5.7 *a) For each two chambers, there is an apartment containing both of them.*

b) Equivalently, for each pair C, D of chambers, there is an element $g \in \mathrm{PGL}(n,\mathbb{Q}_p)$ such that gC and gD are contained in the standard apartment.

This is the problem of a description of orbits of $\mathrm{PGL}(n,\mathbb{Q}_p)$ on the space of pairs of chambers, i.e., on

$$\mathrm{PGL}(n)/\mathrm{Iw}(n) \times \mathrm{PGL}(n)/\mathrm{Iw}(n)$$

We use the standard algebraic remark:

– *The set of orbits of a group G on $G/H \times G/H$ is in one-to-one correspondence with double cosets $H \setminus G/H$.*

Thus it suffices to describe the double cosets $\mathrm{Iw}(n) \setminus \mathrm{PGL}(n)/\mathrm{Iw}(n)$. This description is given in the following theorem, which also implies Theorem 5.7.

Theorem 5.8 *Each double coset in* $\mathrm{Iw}(n)\backslash\mathrm{PGL}(n,\mathbb{Q}_p)/\mathrm{Iw}$ *contains an element of the group $\widetilde{W}$.*

PROOF. We wish to find a canonical form of $g \in \mathrm{GL}(n,\mathbb{Q}_p)$ under the transformations $g \mapsto h_1 g h_2$, where h_1, $h_2 \in \mathrm{Iw}(n)$. A similar problem for $\mathrm{GL}(n,\mathbb{O}_p) \setminus \mathrm{GL}(n,\mathbb{Q}_p)/\mathrm{GL}(n,\mathbb{O}_p)$ was solved in Theorem 3.8 and we refer to its proof. Now we can apply the following operations:

a) Left and right multiplications by diagonal matrices, whose entries are in $\mathbb{O}_p \setminus p\mathbb{O}_p$.

b) Left multiplication by matrices of the form[14] $1 + zE_{ij}$, $i \neq j$. If $i > j$, then arbitrary $z \in \mathbb{O}_p$ is allowed; if $i < j$, then $z \in p\mathbb{O}_p$.

c) Right multiplications by matrices having the same form.

For a given $g \in \mathrm{GL}(n)$, consider all the matrix elements having maximal possible $|g_{ij}|$. Take an element element g_{kl} of this kind, for which $k + l$ is

[14]Here E_{ij} a matrix having a unique nonzero entry $= 1$ at the place ij

maximal. The operation b) allows to annihilate all the matrix elements in l-th row but g_{kl}. Applying c) we annihilate all the elements in k-th column.

Further, we forget about k-th column and l-th row and repeat the same operation. Etc.

As a result, we come to a matrix having a unique nonzero element in each column and each row. Multiplying it by a diagonal matrix, we obtain a desired representative of the given double coset. □

Problem 5.3 (*Bruhat cell decomposition*) a) Denote by $B(n) \subset \mathrm{GL}(n, \mathbb{C})$ the group of upper triangle matrices (with arbitrary elements on the diagonal) Show that double cosets $B_n \setminus \mathrm{GL}(n, \mathbb{C})/B_n$ are in one-to-one correspondence with the group S_n.

b) Denote by $\mathbb{C}[[\varepsilon]]$ the field of formal Laurent series in the variable ε. Consider the group $\mathrm{GL}(n, \mathbb{C}[[\varepsilon]])$ and its 'Iwahori' subgroup $\mathrm{Iw}(n)$ consisting of matrices of the form $R + O(\varepsilon)$, where $R \in B(n) \subset \mathrm{GL}(n, \mathbb{C})$. Show that $\mathrm{Iw}(n) \setminus \mathrm{GL}(n, \mathbb{C}[[\varepsilon]])/\mathrm{Iw}(n) \simeq \widehat{W}_n$

5.9. Automorphisms of buildings. The group of automorphism of the Bruhat–Tits tree is greatly larger than $\mathrm{PGL}(n, \mathbb{Q}_p)$. Indeed, by construction, the set $\mathcal{X}_v$ of edges entering to a given vertex is one-to-one correspondence with the projective line $\mathbf{P}\mathbb{F}_p^1$. The group $\mathrm{PGL}(2, \mathbb{Q}_p)$ regards this structure (i.e., it induces projective maps $\mathcal{X}_v \to \mathcal{X}_w$).

It can be shown that, for $n > 2$ the group $\mathrm{PGL}(n, \mathbb{Q}_p)$ has index 2 in the group of automorphisms of the building. To observe an additional automorphism, consider an arbitrary non-degenerate bilinear form on $\mathbb{Q}_p^n$. It identifies V and the dual space, and we consider the map assign the dual lattice $L^\diamond$ to each lattice L.

10.6 Buildings related to symplectic groups

6.1. Self-dual modules. Consider a space $\mathcal{V}_{2n} \simeq \mathbb{Q}_p^{2n}$ endowed with a non-degenerate skew-symmetric bilinear form $B(\cdot, \cdot)$. Such a form identifies the $\mathcal{V}_{2n}$ and the dual space (namely to each vector v we assign the linear functional $\ell_v(w) = B(v, w))$). For each $\mathbb{O}_p$-submodule L we define *the dual submodule* $L^\diamond$ as the set of vectors w such that

$$B(v, w) \in \mathbb{O}_p \qquad \text{for all } w \in L$$

An $\mathbb{O}_p$-submodule $Y \subset V_{2n}$ is called *self-dual* if $Y = Y^\diamond$.

Problem 6.1 a) For each self-dual module Y there exists a canonical basis[15] $e_1, \dots,$ e_α, $E_{\alpha+1}, \dots, E_n$, $f_1, \dots f_\alpha$, $F_{\alpha+1}, \dots, F_n$, such that

$$Y = \Big(\bigoplus_{j=1}^{\alpha} \mathbb{Q}_p E_j\Big) \oplus \Big(\bigoplus_{j=\alpha+1}^{n} \big[\mathbb{O}_p e_j \oplus \mathbb{O}_p f_j\big]\Big)$$

b) For each self-dual module, there is a decomposition of $\mathcal{V}_{2n}$ into an orthogonal direct sum $\mathcal{V}_{2n} = W_1 \oplus W_2$ such that $Y = L \oplus R$, where $L \subset W_1$ is an isotropic subspace, and $R \subset W_2$ is a self-dual lattice.

[15] See (3.1.2).

6.2. Pairs of self-dual lattices.

Theorem 6.1 *a) Let R, S be self-dual lattices in $\mathcal{V}_{2n}$. Then there is a canonical basis e_j, f_j in $\mathcal{V}_{2n}$ such that*

$$R = \bigoplus_{j=1}^{n} [\mathbb{O}_p\, e_j \oplus \mathbb{O}_p\, f_j] \qquad S = \bigoplus_{j=1}^{n} [p^{k_j}\mathbb{O}_p\, e_j \oplus p^{-k_j}\mathbb{O}_p\, f_j]$$

b) The collection $k_j(R,S)$ is uniquely determined modulo permutations and transformations $k_j \mapsto \pm k_j$

Problem 6.2 Adapt the second proof of Theorem 3.8 for our case.

We say that the collection

$$k_1(R,S) \geqslant k_2(R,S) \geqslant \ldots \geqslant k_n(R,S) \geqslant 0 \tag{6.1}$$

is the *complex distance* between R and S.

6.3. Almost self-dual lattices. We say that a lattice $L \subset \mathcal{V}_{2n}$ is *almost self-dual* iff L contains a self-dual lattice M and for each v, $w \in L$, $B(v,w) \in p^{-1}\mathbb{O}_p$.

Let us describe such objects in more explicitly. Fix a self-dual lattice M. Consider the quotient

$$p^{-1}M/M \simeq \mathbb{F}_p^{2n}.$$

If v, $w \in p^{-1}M$, then $B(v,w) \in p^{-2}\mathbb{O}$; if also $v \in M$, then $B(v,w) \in p^{-1}\mathbb{O}$.

Observation 6.2 *We get a well-defined form $b_M(v,w)$ on $p^{-1}M/M$ taking values in $p^{-2}\mathbb{O}_p/p^{-1}\mathbb{O}_p \simeq \mathbb{F}_p$.*

Lemma 6.3 *Let M be a self-dual lattice. A lattice $L \supset M$ is almost self-dual iff $L \subset p^{-1}M$ and $L/M \subset p^{-1}M/M \simeq \mathbb{F}_p^{2n}$ is b_M-isotropic*

A volume of a self-dual lattice is 1; volumes of almost self-dual lattices take values 1, p,..., p^n.

6.4. Symplectic buildings. Now, we wish to construct a simplicial complex $\mathcal{BT}(\mathrm{Sp}(2n,\mathbb{Q}_p))$.

Vertices of the complex are enumerated by almost self-dual lattices. Two vertices L_1, L_2 are connected by a edge, iff $L_1 \supset L_2$ or $L_2 \supset L_1$.

Simplices (*facets*) correspond to flags

$$L_{i_1} \subset \cdots \subset L_k$$

of almost self-dual lattices. Maximal simplices (*chambers*) are enumerated by maximal flags of almost self-dual lattices

$$M = L_0 \subset L_1 \subset \cdots \subset L_n, \quad M \text{ is self-dual, } \mathrm{vol}(L_j) = p^j \tag{6.2}$$

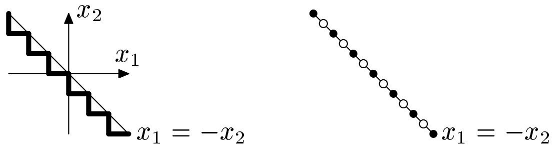


Fig. 10.5: To Problem 6.3, $n = 1$

6.5. Definition of buildings in the terms of iwahoric subgroups. If a symplectic transformation g stabilizes an almost self-dual lattice L, then g also stabilizes the dual lattice $L^{\diamondsuit}$. If g stabilizes a flag (6.2) of almost self-dual lattices, then g stabilizes also the following long flag of lattices

$$L_n^{\diamondsuit} \subset L_{n-1}^{\diamondsuit} \subset \cdots \subset L_1^{\diamondsuit} \subset L_0^{\diamondsuit} = M = L_0 \subset L_1 \subset \cdots \subset L_n \tag{6.3}$$

An iwahoric subgroup is a subgroup stabilizing a maximal flag (6.2) of almost self-dual lattices. Equivalently it stabilizes the long flag (6.3).

Consider a basis $e_j \in \mathcal{V}_{2n}$ satisfying $B(e_i, e_{2n-i}) = 1$ for $i \leqslant n$; all other scalar products are 0. The *standard Iwahori subgroup* is the group of symplectic matrices that have the structure (5.2) in our basis.

A *parahoric subgroup* is a stabilizer of a noncomplete flag of almost self-dual matrices.

Now we present a definition of the building in the terms of iwahoric subgroups (as in Subsection 5.3). Vertices of the complex $\mathcal{BT}[\mathrm{Sp}(2n, \mathbb{Q}_p)]$ are enumerated by maximal parahoric subgroups (i.e., stabilizers of almost self-dual lattices). A collection of vertices is united into a simplex (*facet*) if there is a parahoric subgroup that fixes all these vertices. Chambers are enumerated by iwahoric subgroups.

Since parahoric subgroups and flags of almost self-dual lattices are in one-to-one correspondence, our new definition is only a rephrasing of the previous one.

6.6. Apartments. Fix a canonical basis e_j, f_j in $\mathcal{V}_{2n}$. Consider the set of all the almost self-dual lattices of the form

$$\Big(\bigoplus p^{l_j} \mathbb{O}_p e_j\Big) \oplus \Big(\bigoplus p^{k_j} \mathbb{O}_p f_j\Big)$$

(in fact, $l_j + k_j$ is 0 or (-1)). All the simplices of the buillding having vertices at these points form a subcomplex called an *apartment*

Problem 6.3 Show, that we obtain an Euclidean space equipped with the Coxeter tilling (for instance, see Subsection 8.1) of the type $\widetilde{C}_n$. It is reasonable to argue as follows. Consider admissible points $(k_1, \dots, k_n, l_1 \dots, l_n) \in \mathbb{R}^{2n}$. and draw the corresponding simplicial n-dimensonal survace in $\mathbb{R}^{2n}$. Project this surface orthogonally to the plane $H : x_1 + x_{n+1} = 0$, $x_2 + x_{n+2} = 0$, ... in $\mathbb{R}^n$; see Fig. 10.5.

For $n = 2$ the picture is present on Fig.10.8,a. (we mark by different colors different types of almost self-dual lattices), see Fig.10.9b for $n = 3$.

10.7 The Nazarov category

The Nazarov category is the p-adic analog of the symplectic category **Sp**. It is also equivalent to the category of p-adic Gaussian operators discussed in the next Chapter.

7.1. Nazarov category. An object of the Nazarov category $\mathbf{Sp}(\mathbb{Q}_p)$ is p-adic linear space endowed with a non-degenerate skew-symmetric bilinear form. For two objects Y, Z, we define the skew-symmetric bilinear form

$$B^{\ominus}(y \oplus z, y' \oplus z') := B_Y(y \oplus y') - B_Z(z, z') \tag{7.1}$$

Morphisms $Y \to Z$ are $B^{\ominus}$-self-dual $\mathbb{O}_p$-submodules $P \subset Y \oplus Z$ such that

$$\ker P := P \cap Y \quad \text{and} \quad \operatorname{indef} P := P \cap Z \quad \text{are compact}$$

A product of morphisms is a product of relations.

Theorem 7.1 *a) Product of morphisms is a morphism*

b) Group of automorphisms of $\mathcal{V}_{2n}$ is $\mathrm{Sp}(2n, \mathbb{Q}_p)$.

c) For a morphism P,

$$\ker P = (\operatorname{dom} P)^{\Diamond}, \qquad \operatorname{indef} P = (\operatorname{im} P)^{\Diamond} \tag{7.2}$$

Theorem 7.2 *Let $P : Y \rightrightarrows Z$ be a morphism. For a self-dual submodule L, the submodule PL is self-dual.*

7.2. A preliminary lemma. Let R be a submodule in some $\mathcal{V}_k$. Assume $R \supset R^{\Diamond}$. By definition, if $v \in R^{\Diamond}$, $w \in R$, then $B(v, w) \in \mathbb{O}_p$. Hence we can regard B as a $\mathbb{Q}_p/\mathbb{O}_p$-valued form B^* on $R/R^{\Diamond}$,

$$B^*(x_1 + x_2, y) = B^*(x_1, y) + B^*(x_2, y), \quad B^*(x, y) = -B^*(y, x),$$
$$B^*(\lambda x, y) = \lambda B^*(x, y), \quad \lambda \in \mathbb{O}_p \setminus p\mathbb{O}_p$$

Lemma 7.3 *Let L be a self-dual $\mathbb{O}_p$-module in a p-adic linear space. Let R be an $\mathbb{O}_p$-submodule such that $R^{\Diamond} \subset R$ Then $(L \cap R)/(L \cap R^{\Diamond})$ is a B^*-self-dual submodule of $R/R^{\Diamond}$.*

PROOF. We must show that $(L \cap R) + R^{\Diamond}$ is self-dual in the usual sense. Since $R^{\Diamond} \subset R$,

$$(L \cap R) + R^{\Diamond} = (L + R^{\Diamond}) \cap R$$

Next,

$$\big[(L + R^{\Diamond}) \cap R\big]^{\Diamond} = (L + R^{\Diamond})^{\Diamond} + R^{\Diamond} = (L \cap R) + R^{\Diamond}$$

and this is equivalent to self-duality. □

7.3. Proof of Theorem 7.2. The statement (7.2) is similar to Lemma 2.9.2, we omit a proof.

First of all,

$$P \subset \operatorname{dom} P \oplus \operatorname{im} P, \qquad P \supset \ker P \oplus \operatorname{indef} P$$

Denote by $\widetilde{P}$ the image of P in the quotient group $\operatorname{dom} P/\ker P \oplus \operatorname{im} P/\operatorname{indef} P$.

Lemma 7.4 *a) The groups* $\operatorname{dom} P/\ker P$ *and* $\operatorname{im} P/\operatorname{indef} P$ *are isomorphic. Moreover, the group* $\widetilde{P}$ *is a graph of an isomorphism, say* ι_P*.*

b) The isomorphism ι_P *is an isometry of forms* B^**.*

This statement is obvious.

Proof of theorem. Let $L \subset Y$ be an $\mathbb{O}_p$-submodule, $P : Y \rightrightarrows Z$ a morphism of the Nazarov category. The $\mathbb{O}_p$-submodule PL can be obtained by the following chain of operations.

1^*. We project $L \cap \operatorname{dom} P \subset$ to the quotient group $\operatorname{dom} P/\ker P$. Denote by L_1 the result. By Lemma 7.3, $L_1 1$ is B^*-self-dual

2^*. We push forward L_1 by the map ι_P. Denote by $L_2 \subset \operatorname{Im} P/\operatorname{indef} P$ the result. By Lemma 7.4, L_2 is B^*-self-dual.

3^*. Consider the preimage L_3 of L_2 under the map $\operatorname{Im} P/\operatorname{indef} P$. Obviously, $L_3 \subset Z$ is self-dual.

However $L_3 = PL$. □

7.4. Proof of Theorem 7.1. The only non-obvious statement is a). It is a special case of Theorem 7.2. Indeed, let $R : X \rightrightarrows Y$, $T : Y \rightrightarrows Z$ be morphisms. Denote $R^\square$ be the pseudoinverse relation[16]. Consider the relation

$$R^\square \oplus T : Y \oplus Y \rightrightarrows X \oplus Z$$

(compare with Subsection 2.9.4). Evidently $R^\square \oplus T$ is a morphism of $\mathbf{Sp}(\mathbb{Q}_p)$. Take $L \subset Y \oplus Y$ being the graph of the identity operator.

Then the relation $TR \subset X \oplus Z$ is a result of application of the linear relation $P := R^\square \oplus T$ to the self-dual submodule' L. It remains to apply Theorem 7.2

7.5. Action of Nazarov category on buildings. Compressivity.

Proposition 7.5 *Let* $P : Y \rightrightarrows Z$ *be a morphism of* $\mathbf{Sp}(\mathbb{Q}_p)$*,* $L \subset Y$ *be an almost self-dual lattice. Then the lattice* $PL \subset Z$ *is almost self-dual.*

Proof. Let $M \subset L$ be a self-dual lattice, $p^{-1}M \supset L$.

Then $L/M \simeq (\mathbb{Z}_p)^j$. Following the operations 1^*-3^* in the proof of Theorem 7.2, we obtain that $PL/PM \simeq (\mathbb{Z}_p)^k$, where $k \leqslant j$. In particular, $p^{-1}PM \supset PL$.

The lattice PM is self-dual by Theorem 7.2. Further, let y, $y' \in L$, $y \oplus z$, $y' \oplus z' \in P$. Then

$$B(y,y') - B(z,z') \in \mathbb{O}, \quad B(y,y') \in p^{-1}\mathbb{O} \quad \Longrightarrow \quad B(z,z') \in p^{-1}\mathbb{O}$$

Hence PL is almost self-dual. □

[16] i.e, the same subspace in $Y \oplus X$.

Therefore a morphism of the Nazarov category takes vertices of buildings to vertices of buildings.

Let $\mathcal{B}_1$, $\mathcal{B}_2$ be simplicial complexes, let $\mathrm{Vert}(\mathcal{B}_1)$, $\mathrm{Vert}(\mathcal{B}_2)$ be sets of their vertices. Consider a map $\gamma : \mathrm{Vert}(\mathcal{B}_1) \to \mathrm{Vert}(\mathcal{B}_2)$. We say that γ is *simplicial* if for a collection of vertices $x_1,\dots,x_k$ of any simplex of $\mathrm{Vert}(\mathcal{B}_1)$, their images $\gamma(x_1),\dots,\gamma(x_k)$ are vertices of some simplex in $\mathcal{B}_2$ (we allow $\gamma(x_i) = \gamma(x_j)$). We say that γ is *strictly simplicial* if $\gamma(x_1),\dots,\gamma(x_k)$ are pairwise distinct.

A simplicial map admits a canonical extension to a map $\mathcal{B}_1 \to \mathcal{B}_2$ that is affine on each simplex of $\mathcal{B}_1$.

Proposition 7.6 *Let $P : Y \rightrightarrows Z$ be a morphism of the Nazarov category. Let L range in the set of almost self-dual lattices. Then the map $L \mapsto PL$ is a simplicial map of the corresponding buildings.*

PROOF. If $L \supset M$, then $PL \supset PM$, therefore P takes flags to flags. □

Problem 7.1 a) Let L, $M \subset Y$ be self-dual lattices, let $P : Y \rightrightarrows Z$ be a morphism. Then complex distance (see (**??**) satisfies

$$k_j(L, M) \geqslant k_j(PL, PM) \text{ for all } j$$

7.6. Canonical forms. We want to classify morphisms $P : Y \rightrightarrows Z$ of the category $\mathrm{Sp}(\mathbb{Q}_p)$ modulo the equivalence

$$P \sim hPg, \qquad h \in \mathrm{Sp}(X),\, g \in \mathrm{Sp}(Y)$$

(this is used in proof of Theorem 11.2.10).

We say that a morphism $P : Y \rightrightarrows Z$ is *decomposable* if there are decompositions $Y = Y^{(1)} \oplus Y^{(2)}$, $Z = Z^{(1)} \oplus Z^{(2)}$ into orthogonal sums and morphisms $P_1 : Y^{(1)} \to Z^{(1)}$, $P_2 : Z^{(2)} \to Z^{(2)}$, such that $P = P_1 \oplus P_2$.

Theorem 7.7 *There are only 5 following types of indecomposable morphisms (modulo the equivalence).*

1^*. $Y = 0$, $Z = \mathcal{V}_2$, and $L \subset Z$ is a self-dual lattice in the two-dimensional space $\mathcal{V}_2$ (we can take $P = \mathbb{Q}_p e \oplus \mathbb{Q}_p f$, where e, f is a canonical basis).

2^*. $Y = \mathcal{V}_2$, $Z = 0$, and $L \subset Y$ is a self-dual lattice

3^*. $Y = \mathcal{V}_2$, $Z = \mathcal{V}_2$ and L is the graph of the identical operator

4^*. $Y = \mathcal{V}_2$, $Z = \mathcal{V}_2$; let e, f be a canonical basis in Y and e', f' be a canonical basis in Z; let $k > 0$. We take

$$P := \mathbb{O}_p(e + e') \oplus p^{-k}\mathbb{O}_p(f + f') \oplus p^k\mathbb{O}_p e' \oplus \mathbb{O}_p f \tag{7.3}$$

5^*. Y, Z are the same,

$$P := \mathbb{O}_p(e + e') \oplus \mathbb{Q}_p(f + f') \oplus \mathbb{O}_p f \tag{7.4}$$

We omit a proof, see [112].

REMARK. a) We can regard the canonical form 5^* as a form 4^* with $k = \infty$. On the other hand the form e) with $k = 0$ is a sum of 1^* and 2^*.

b) The canonical form 4^* admits the following transparent description. We consider a basis e, $f \in \mathcal{Y}$ such that $B(e, f) = p^{-k}$ and similar basis e', $f' \in \mathcal{Z}$. In this notation

$$P := \Big(\mathbb{O}_p e \oplus \mathbb{O}_p e' \oplus \mathbb{O}_p f \oplus \mathbb{O}_p f'\Big) + p^{-k}\Big(\mathbb{O}_p(e + e') + \mathbb{O}_p(f + f')\Big) \subset$$
$$\subset p^{-k}\Big(\mathbb{O}_p e \oplus \mathbb{O}_p e' \oplus \mathbb{O}_p f \oplus \mathbb{O}_p f'\Big)$$

The corresponding map $\operatorname{dom} P/\ker P \to \operatorname{im} P/\operatorname{indef} P$ (see Lemma 7.4) is the identical map $(\mathbb{Z}_p)^2 \to (\mathbb{Z}_p)^2$. □

7.7. Relation with the symplectic category over $\mathbb{F}_p$.

Problem 7.2 Equip each object $\mathcal{V}_{2k}$ of the category $\mathrm{Sp}(\mathbb{Q}_p)$ with a basis e_j, f_j satisfying

$$B(f_i, f_j) = B(e_i, e_j) = 0, \qquad B(e_i, f_j) = p^{-1}\delta_{ij}$$

Consider the subcategory $\mathbf{Sp}(\mathbb{F}_p)$ of $\mathbf{Sp}(\mathbb{Q}_p)$ consisting of morphisms P satisfying the conditions

$$\ker P \supset (\mathbb{O}_p)^{2n} \qquad \operatorname{dom} P \subset p^{-1}(\mathbb{O}_p)^{2n}$$
$$\operatorname{indef} P \supset (\mathbb{O}_p)^{2n} \qquad \operatorname{im} P \subset p^{-1}(\mathbb{O}_p)^{2n}$$

Then $\mathbf{Sp}(\mathbb{F}_p)$ is the category of symplectic linear relations over field $\mathbb{F}_p$ (see Section 9.3).

10.8 Buildings. Generalities

This section is not used below.

8.1. Coxeter groups. Consider a simplex C in an k-dimensional Euclidean space. Reflect C with respect to all the $(k-1)$-dimensional faces. Repeat the same operation for all the 'new' simplices etc. The simplex C is called a *Coxeter simplex* iff this procedure produces a tilling of the Euclidean space by non-overlapping simplices. A *Coxeter group* is a group generated by reflections with respect to all the faces of a Coxeter simplex; automatically it contains all the reflections with respect all the faces of the Coxeter tilling.

Problem 8.1 A simplex is Coxeter iff all its dihedral angles α_j have form π/m_j, where m_j are integer

Problem 8.2 A simplex is determined by its dihedral angles uniquely to within a similarity.

In the same way, we define *spherical Coxeter simplices*, we only consider the sphere S^{n-1} instead of $\mathbb{R}^n$. Equivalently, we can regard spherical simplices as simplicial cones in $\mathbb{R}^n$.

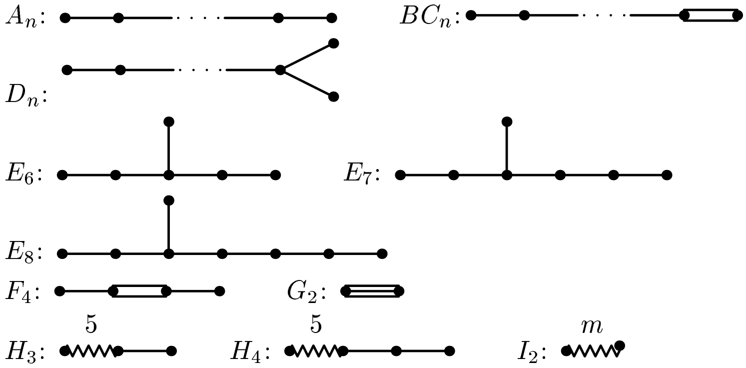


Fig. 10.6: Spherical Coxeter simplices

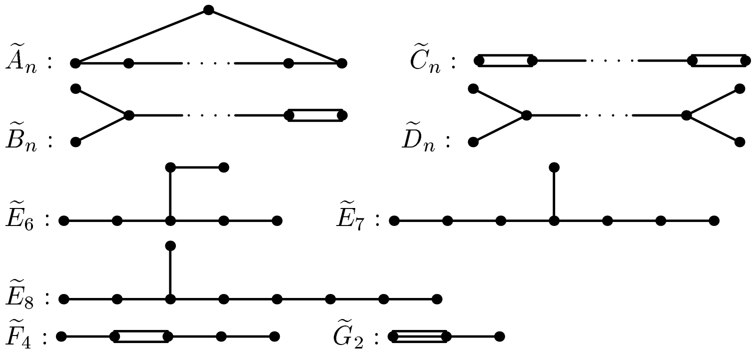


Fig. 10.7: List of Euclidean Coxeter simplices

A classification of Coxeter simplices is well-known (see, for instance, [22]). The list of spherical simplices is given on Fig. 10.6, the list of affine simplices is on Fig. 10.7.

Explain the notation. For each Coxeter simplex C, we draw a graph. Its vertices are enumerated by faces of C; we connect two vertices if the angle between them is not $90°$. We draw one segment if an angle is $60°$, two segments if an angle is $45°$, and 3 segments for an angle $= 30°$; we design a serrated segment and write a number k if the angle is π/k with $k \neq 2, 3, 4, 6$; the subscript in A, B,..., I coincides with n mentioned above. Finally, A, B, ... are generally accepted notation.

Each Coxeter simplex generates a (*Coxeter*) tiling of Euscliden space or sphere by simplices. Reflections with respect to faces of simplices generate a (*Coxeter*) group of isometries of the sphere or Euclidean space.

In the spherical case, this group is finite.

In the Euclidean case $\mathbb{R}^n$, the group consists of affine isometries of Euclidean spaces $x \mapsto U_\alpha x + v_\alpha$. Linear parts of such isometries range in a finite (spherical) Coxeter group having the same notation in our list without $\sim$. Also, this group contains a subgroup $\simeq \mathbb{Z}^n$ consisting of translations.

Example. Tilings of $\mathbb{R}^2$ corresponding to the Coxeter groups $\widetilde{A}_3$, $\widetilde{C}_3$, $\widetilde{G}_3$ are shown on Fig. 10.3.c and 10.8. Three-dimensional Coxeter simplices are on Fig.10.9. $\square$

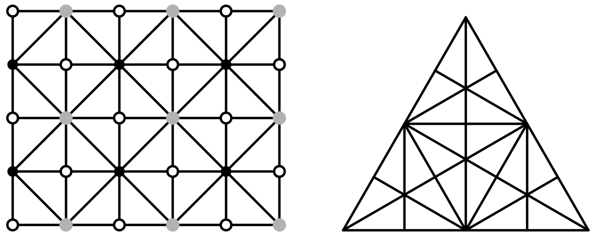


Fig. 10.8: Coxeter tilings of the plane of types $\widetilde{C}_2$ and $\widetilde{G}_2$.

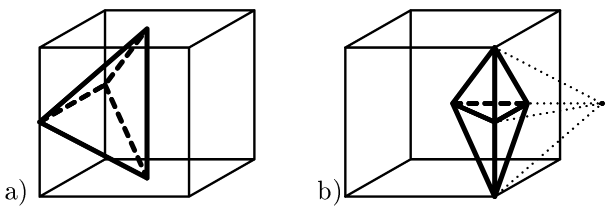


Fig. 10.9: Fundamental simplices for the affine Coxeter groups of the types $\widetilde{A}_3$ (Fig. a)), $\widetilde{C}_3$ (a small simplex on Fig. b)), $\widetilde{B}_3$ (two-pieces simplex on Fig. b)). The 4-pices simplex on Fig. b) is the simplex of the type $\widetilde{A}_3$.

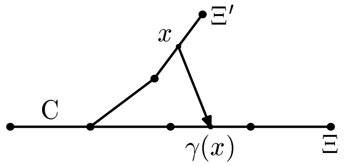


Fig. 10.10: Retraction to an apartment

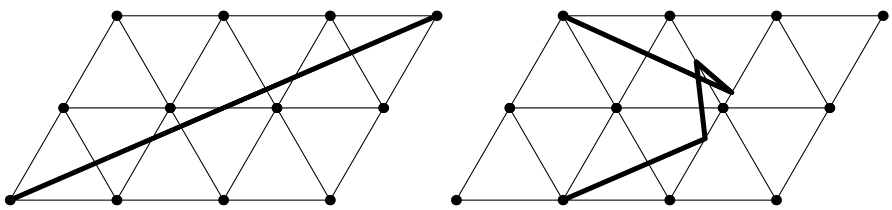


Fig. 10.11: A segment and its (possible) image under a retraction. We have a 'ray' that can intersect a mirror or can be reflected from a mirror

Problem 8.3 Draw all the Coxeter tilings of the 2-dimensional sphere.

EXAMPLE 1. For $\mathrm{GL}(n, \mathbb{Q}_p)$, an apartment of building is the Eucldean space $\mathbb{R}^{n-1}$ endowed with the Coxeter tilling of the type $\widetilde{A}_{n-1}$. The group $\widetilde{W}_n$ defined in Subsection 5.6 contains the Coxeter group $\widetilde{A}_{n-1}$ as a subgroup of index n[17]. □

EXAMPLE 2. For $\mathrm{Sp}(2n, \mathbb{Q}_p)$ an apartment is $\mathbb{R}^n$ equipped with a Coxeter tilling of the type $\widetilde{C}_n$. □

8.2. Axioms of buildings. Let us consider a Euclidean space or a sphere with Coxeter tiling as a simplicial complex. Let us call such ojects as *Coxeter complexes.* We call simplices of the tilling by *chambers.* A *facet* is a face (of dimension 0,1, ..., n) of a chamber.

Fix a Coxeter group G_n. An *abstract building* is a simplicial complex $\mathcal{B}$ such that

B0. $\mathcal{B}$ is a union of Coxeter complexes, they are called *apartments.*

B1. For any two chambers there is an apartment containing both of them.

B2. Let Ξ, Ξ' be apartments containing facets A and B. Then there is an isomorphism $\Xi \to \Xi'$ of simlicial complexes fixing A and B pointwise.

According two types of Coxeter groups, we obtain two types of buildings, namely *spherical buildings* and *Euclidean buildings.*

The axiom **B2** can be replaced by

B2'. Let apartments Ξ, Ξ' contain a chamber A and a facet B. Then there is an isomorphism $\Xi \to \Xi'$ fixing A and B pointwise.

Lemma 8.1 *Axioms* **B0**, **B1**, **B2'** *imply* **B2**.

PROOF. Take a chamber of Ξ containing A and a chamber D of Ξ' containing B. There is an apartment Ξ'' containing C, D. We consider isometries $\Xi \to \Xi'' \to \Xi$. □

Proposition 8.2 *The building $\mathcal{BT}(\mathrm{PGL}(n, \mathbb{Q}_p)$ is a building, i.e., it satisfies the building axioms.*

PROOF. Axiom **B1** is Theorem 5.7. Let us prove **B2'**. Euclidean distances between vertices of an apartment are determined by the complex distance. Therefore pairwise distances between vertices of A and B are same in Ξ and Ξ'.

Thus, $A \cup B \subset \Xi$ and $A \cup B \subset \Xi'$ are isometric figures in the Euclidean spaces Ξ and Ξ'. Therefore there is an isometry $\Xi \to \Xi'$ fixing $A \cup B$. Since $\dim A = n$, this isometry is unique. It is more-or-less evident, that this isometry is a symplical map.□

An Euclidean building can be constructed for an arbitrary classical group over a field of discrete valuation (our example $\mathcal{B}(\mathrm{Sp}(2n, \mathbb{Q}_p)$ is representative). or semisimple noncompact p-adic group.

For $n > 8$ all the Euclidean buildings have such a form. For $n \leqslant 8$ there appear buildings related to exceptional algebraic groups; for $n \leqslant 3$ there is a lot of exotic constructions; for $n = 2$ there are also buildings related to reflection groups on the Lobachevski plane.

8.3. Some properties of Euclidean buildings. For a given apartment Ξ and a chamber $C \subset \Xi$, we construct the standard retraction[18] $\varphi_{\Xi,C} : \mathcal{B} \to \Xi$. For $x \in \mathcal{B}$,

[17] Because $\widetilde{W}_n$ contains transformation preserving a simplex.

[18] Let X be a topological space, let $Y \subset X$ be a closed subspace. A retraction $X \to Y$ is a continuous map that fixes X pointwise.

consider an apartment Ξ' containing x and C, and consider the unique isometric $\gamma : \Xi' \to \Xi$ map fixing the chamber C. We put $\varphi(x) := \gamma(x)$. Thus we obtain a strictly simplicial map $\mathcal{B} \to \Xi$ in the sense of Subsection 7.5.

Axiom **B2** provides an independence of the image of a chamber on a choice of Ξ'. We emphasis the following fact

Observation 8.3 *The preimage of the chamber C under $\varphi_{\Xi,C}$ is the chamber C itself.*

Since any two points x, $y \in \mathcal{B}$ are contained in one apartment, we can measure the distance $d(x, y)$ as a distance in an apartment. By the virtue of Axiom **B2**, the value $d(x, y)$ does not depend on Ξ.

Theorem 8.4 *An Euclidean building is a metric space.*

Theorem 8.5 *Intersection of two apartments $\Xi \cup \Xi'$ is a convex set.*

The last theorem implies the following corollary.

Corollary 8.6 *For each two points there is a unique segment connecting these points*

Proposition 8.7 *Standard retractions contract the distance, i.e.,*

$$d(\varphi(x), \varphi(y)) \leqslant d(x, y)$$

PROOFS. 1. *Contractivity.* Consider two points y, $z \in \mathcal{B}$, an apartment containing them, and the segment connecting x, y in the apartment. The image of $[y, z]$ under any standard retraction is a polygonal path, whose pieces have the same lengths as pieces of $[y, z]$ in the Coxeter tilling of Σ, see Fig.10.11. This proves contractivity.

2) *A building is a metric space.* Indeed, let x, y, z be points of a building. Consider a chamber C containing x and apartments Ξ, Ξ' connecting x with y and z. Consider an apartment Σ containing y, z and the segment $[y, z]$ connecting them. Consider the retraction $\varphi_{\Sigma,C}$. The segment $[x, y]$ remains fixed, The image of $[x, z]$ is a segment and hence

$$d(x, y) + d(y, z) \geqslant d(x, y) + d(y, \varphi(z)) \geqslant d(x, \varphi(z)) = d(x, z)$$

3) *Convexity.* Let x, y lie in an intersection $\Xi \cap \Xi'$ of two apartments. Consider the corresponding segments $[x, y]_\Xi$ and $[x, y]_{\Xi'}$. By Axiom **B2**, their lengths equal. Consider the first moment when $[x, y]_{\Xi'}$ leaves Ξ. Let $C \subset \Xi$ be a chamber visited by $[x, y]_\Xi$ in the next moment. By the Observation 8.3, the $\varphi_{\Xi,C}$-image of $[x, y]_{\Xi'}$ differs from $[x, y]_\Xi$. Thus the retraction $\varphi_{\Xi,C}$ strictly diminish the distance between two x, y; and this contradiction completes the proofs. $\square$

Problem 8.4 For $\mathcal{B}(\mathrm{PGL}(3, \mathbb{Q}_p))$ consider the apartments associated wit bases e_1, e_2, e_3 and $e_1 + p^{10} e_2$, e_2, e_3. Find their intersection.

8.4. Nonpositive curvature. We keep on a discussion of Euclidean buildings.

Theorem 8.8 *Let x, y, $z \in \mathcal{B}$, let $0 < t < 1$. Denote by u_t the point dividing the segment $[x, y]$ in the ratio $(1 - t) : t$. Then*

$$d^2(z, u_t)^2 \leqslant (1 - t) d^2(z, x) + t d^2(z, y) - (1 - t) t d^2(x, y) \tag{8.1}$$

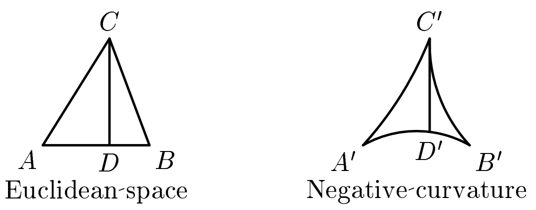


Fig. 10.12: To the nonpositive curvature property. If $AB = A'B'$, $AC = A'C'$, $BC = B'C'$, $AD/DB = A'D'/D'B'$, then $C'D' \leqslant CD$.

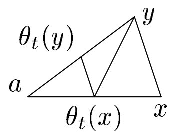


Fig. 10.13: To continuity of the homotopy.

REMARK. If x, y, z are points in Euclidean space, then we have $=$ in the last row. Thus the theorem claims that the segment $[x, u_t]$ is shorter than in Euclidean geometry, see Fig.10.12. Inequality (8.1) is more pleasant for $t = 1/2$:

$$4d^2(z, u_t)^2 \leqslant 2d^2(z, x) + 2d^2(z, y) - d^2(x, y) \tag{8.2}$$

PROOF. Consider an apartment including $[y, z]$ and a chamber C including u_t. Consider the corresponding retraction φ. It preserves the distances xy, xu_t, yu_t, u_tz and diminish xz, yz. □

Theorem 8.9 *Euclidean buildings are contractible spaces.*

PROOF. Indeed, fix a point $a \in \mathcal{B}$. For $x \in \mathcal{B}$ and $0 \leqslant t \leqslant 1$, we define the point $\theta_t(x)$ by

$$\theta_t(x) \in [a, x], \qquad d(a, \theta_t(x)) = t \cdot d(a, x)$$

Thus, we obtain a family of maps $\theta_t : \mathcal{B} \to \mathcal{B}$, $\theta_1(x) = x$, $\theta_0(x) = a$.

It remains to watch the continuity of maps θ_t. See Fig. 10.13 The segment $[y, \theta_t(x)]$ is shorter than the Euclidean geometry forecasts. Therefore the $[\theta_t(x), \theta_t(x)]$ also is shorter than the corresponding segment in the Euclidean geometry. □

Theorem 8.10 *An affine building is a complete metric space.*

SKETCH OF PROOF. Let D be a simplex of the Coxeter tilling of $\mathbb{R}^n$. All the chambers of a building $\mathcal{B}$ are canonically isometric to D, this defines a canonical continuous map $\eta : \mathcal{B} \to D$. Evidently this map diminish distances (since the image of an segment is a billiard trajectory in C). It is easy to show that the preimage of a point $\in D$ is a discrete subset in $\mathcal{B}$.

Consider a fundamental sequence $x_j \in \mathcal{B}$. Then $\eta(x_j)$ converges to some $y \in D$. For each j choose a point y_j such that

— $\eta(y_j) = y$

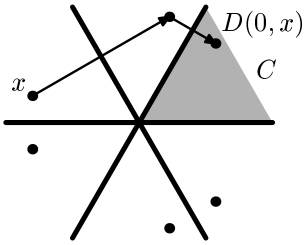


Fig. 10.14: To the definition of complex distance.

— y_j, x_j lie in one chamber.

Then y_j also is a fundamental sequence, therefore it is eventually constant, hence it has a limit. But $d(x_j, y_j) \to 0$ and hence the limit of x_j is the same.

□

Let $Q \subset \mathcal{B}$ be a subset. We say that a is a circumcenter of Q if Q is contained in a closed ball with radius r center at a and is not contained in any ball of a smaller radius $< r$.

Lemma 8.11 *Each compact subset of an affine building has a unique circumcenter.*

PROOF. The existence is a standard general topological exercise. Let a, b be two circumcenters (therefore Q is contained in two balls of the same radius, say r). Let c be the midpoint of the segment $[a, b]$. By virtue of the inequality (8.2), we observe that Q is contained in a ball having center at c and radius $< r$. □

Theorem 8.12 (Bruhat–Tits fixed-Point Theorem) *A compact group acting on a bulding has a fixed point.*

PROOF. Consider an arbitrary orbit of the group. Then its circumcenter is a fixed point. □

Let ξ be a fixed point of a compact group K. Then K leaves invariant the minimal facet containing ξ.

Observation 8.13 *Classical p-adic group can have non-conjugate maximal compact subgroups*

For instance, the group $\mathrm{PGL}(n, \mathbb{O}_p)$ is a maximal compact subgroup in $\mathrm{PGL}(n, \mathbb{Q}_p)$. However a stabilizer of a fixed chamber also is a maximal compact subgroup (an element stabilizing a chamber can move vertices of the chamber preserving their cyclic order).

8.5. Complex distance in Euclidean buildings and the triangle inequality. First, consider a space $\mathbb{R}^n$ equipped with a an action of a Coxeter group $\widetilde{W}$. Put the point of origin 0 to a vertex of the tilling. Denote by W the stabilizer of the vertex 0, recall that W is a finite Coxeter group. Mirrors of reflections $r \in H$ separate $\mathbb{R}^n$ into simplicial cones. Fix one of such cones, say C. For $x \in \mathbb{R}^n$, we define the complex distance $\mathfrak{d}(0, x) \in C$ being the point of C lying in the orbit $W \cdot x$ (see Fig 10.14). We define $D(x, y) := D(0, y - x)$.

Observation 8.14 *The complex distance is invariant with respect to the Coxeter group.*

For points x, y of a building $\mathcal{B}$ consider an apartment connecting x, y and denote by $D(x, y)$ the distance between x, y in the sense of the apartment. Evidently, $D(x, y)$ does not depend on the choice of an apartment.

Observation 8.15 $D(x, y)$ *is invariant under the group of automorphisms of the building*

As in Section 2.12, we can formulate the triangle inequality.

For instance, for the space of lattices in $\mathbb{Q}_p^n$ it is equivalent to the following problem.

— *Let A be a finite Abelian p-group, $B \subset A$ a subgroup, $C := A/B$ the quotient group,*

$$A \simeq \bigoplus p^{m_j}, \qquad B \simeq \bigoplus p^{k_j}, \qquad C \simeq \bigoplus p^{l_j}$$

Find the domain of possible values of triples $\{m_j\}$, $\{k_j\}$, $\{l_j\}$.

Indeed, for a triple of lattices L, M, $N \in \mathrm{Lat}_n$, we can consider another triple $L' := p^\alpha L$, $M' := p^\beta M$, $N' := N$ such that $p^\alpha L \subset p^\beta M \subset N$ (definitely, the problem does not change) and the quotient groups

$$A := N'/L'; \qquad B := M'/L', \qquad C := N'/M' = A/B$$

The triangle inequality in buildings is known, it is interesting that it is the same as in Riemannian symmetric spaces (and it depend only on the Coxeter group).

8.6. Classical groups and spherical buildings. Now we wish to construct a spherical building of the type A_{n-1}.

Consider a field k and a linear space k^n. Vertices of the building are enumerated by proper subspaces in k^n. A collection of vertices assembles a simplex if these subspaces are elements of a flag.

Thus, a simplicial complex is constructed, it remains to present an apartment. Fix a basis $e_1, \ldots, e_n$; consider all the proper subspaces spanned by collections $e_{j_1}, \ldots e_{j_k}$ of basis vectors; consider all the simplices having such vertices.

Problem 8.5 Verify axioms of building.

For $k = \mathbb{F}_p$ we already saw this structure above when we described a link of a vertex in $\mathcal{BT}(\mathrm{PGL}(n, \mathbb{Q}_p))$.

For $k = \mathbb{R}$, we will observe this structure in the following two subsections.

8.7. Tilings of symmetric spaces by Weyl chambers. Consider the space $\mathcal{P}_n = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n, \mathbb{R})$ of real positive definite matrices with the determinant $=1$. The invariant Riemannian metric on this space is given by the formula

$$ds^2 = \mathrm{tr}(T^{-1} \cdot dT \cdot T^{-1} \cdot dT)$$

We define the standard *Cartan subspace* $\mathcal{X}$ as space of diagonal matrices

$$X(t) = \begin{pmatrix} t_1 & 0 & \ldots \\ 0 & t_2 & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix}, \qquad t_j \in \mathbb{R} \tag{8.3}$$

This subspace is flat with respect to Riemannian metric (see proof of Theorem 2.12.2). *Cartan subspaces* in $\mathcal{P}$ are images of the standard Cartan subspaces with respect to various isometries of $\mathcal{P}_n$.

Denote by $\mathcal{E}$ the set of all the Cartan subspaces containing the point $1 \in \mathcal{P}$. The group $\mathrm{SO}(n)$ fixes 1 and hence it acts on $\mathcal{E}$.

The following remarks are obvious.

Observation 8.16 *The set $\mathcal{E}$ is in one-to-one correspondence with the set of n-plets of pairwise orthogonal lines in $\mathbb{R}^n$*

Observation 8.17 *If a symmetric matrix has pairwise distinct eigenvalues, the it is contained in a unique Cartan subspace,*

Observation 8.18 *For any $\mathcal{Y} \in \mathcal{E}$, the intersection $\mathcal{Y} \cap \mathcal{X} \subset \mathcal{X}$ is contained in some hyperplane $t_k = t_l$.*

Problem 8.6 Moreover, $\mathcal{Y} \cap \mathcal{X} \subset \mathcal{X}$ is an intersection of several hyperplanes of the type $t_i = t_j$.

The hyperplanes $t_i = t_j$ separate the space $\mathcal{X} \simeq \mathbb{R}^{n-1}$ into $n!$ pieces which are called *Weyl chambers.*

Theorem 8.19 *There is a canonical one-to-one correspondence between the set of all the Weil chambers in all the Cartan subspaces $\mathcal{Y} \in \mathcal{E}$ and the set of all the complete flags*

$$0 \subset V_1 \subset \cdots \subset V_{j-1} \subset \mathbb{R}^n, \qquad \dim V_j = j$$

To construct this correspondence, consider a Cartan subspace $\mathcal{Y}$ and a Weil chamber Δ in it. Consider an interior point $T \in \Delta$. It has pairwise distinct eigenvalues $t_1 > t_2 > \cdots > t_n$. Let $e_1, \ldots, e_n$ be the corresponding eigenbasis, it is uniquely defined to within transformations $e_j \mapsto \pm e_j$. We put

$$V_j = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_j \qquad \square$$

Thus we get a tiling of $\mathrm{SL}(n,\mathbb{R})/\mathrm{O}(n)$ *by Weil chambers.*

Discuss this tiling in more details. Consider an intersection Σ of several hyperplanes $t_i = t_j$ in a Cartan subspace $\mathcal{Y}$ (let call such intersections by *facets*). Consider a generic element T of this intersection. Denote by

$$s_1 > s_2 > \cdots > s_\mu$$

pairwise distinct eigenvalues of T. Let $E_1, \ldots E_k$ be the corresponding eigenspaces. We construct a incomplete flag

$$0 \subset E_1 \subset E_1 \oplus E_2 \subset E_1 \oplus E_2 \oplus E_3 \subset \ldots$$

Thus we obtain a stronger variant of the previous theorem.

Theorem 8.20 *a) There is a canonical one-to-one correspondence*

$$\left\{\begin{matrix}\text{the set of all the facets}\\ \text{in all the Cartan subspaces}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{the set of all the}\\ \text{flags in } \mathbb{R}^n\end{matrix}\right\}$$

b) Intersection of two facets corresponds to an intersection of two flags (i.e., we select subspaces that are contained in the both flags).

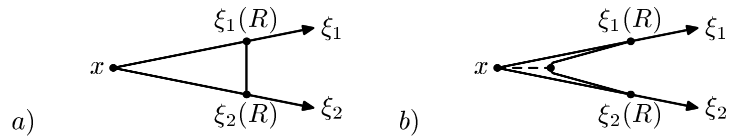


Fig. 10.15: The Tits metric. Difference between Euclidean and Lobachevsky plane. In the second case the distance between x and the segment $[\xi_1(R), \xi_2(R)]$ remains bounded as $R \to \infty$. Hence length of the segment $[\xi_1(R), \xi_2(R)]$ is $\simeq 2R$ for large R.

Thus we represented our space $\mathrm{SL}(n, \mathbb{R})/\mathrm{O}(n, \mathbb{R})$ as a complex assembled from simplicial cones (facets).

Intersecting this picture with a sphere centered at the origin, we get a spherical building drawn on the sphere.

8.8. Tits metric at infinity. Consider a Riemannian manifold M of non-positive curvature. Denote by Σ_∞ its sphere at infinity. Recall this concept.

Fix a point $a \in M$. The sphere Σ_a is in one-to-one correspondence with the set of all (naturally parameterized) geodesics starting at a.

For a point x denote by γ_x a unique geodesic connecting a and x. We say that a sequence $x_j \in M$ converges to $\sigma \in \Sigma_\infty$ if $\mathrm{dist}(a, x_j)$ tends to ∞ and $\gamma(x)$ tends to σ.

Consider geodesics γ_1, γ_2 starting at x and coming to σ_1, $\sigma_2 \in \Sigma_\infty$. Let $\sigma_1(R)$, $\sigma_2(R)$ be the intersections of these geodesics with the sphere $S_R(x)$ of radius R with center at x. The *Tits metrics* at Σ_∞ is defined as

$$\mathrm{dist}(\sigma_1, \sigma_2) = \lim_{R\to\infty} \frac{1}{R}\, \rho\big(\sigma_1(R), \sigma_2(R)\big) \tag{8.4}$$

Example. a) If $M = \mathbb{R}^2$ is Euclidean plane, then we get the standard metric on the circle: the distance between two points is $2\sin(\varphi - \psi)/2$.

b) If M is Lobachevsky plane, then the distance between any two distinct points is 2. I.e., we get a discrete metric space.

Now, let $M = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$. We present the final result without proof.

Theorem 8.21 *The metric* $\mathrm{dist}(\cdot, \cdot)$ *on* $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ *is the intrinsic metric of the spherical building constructed in the previous subsection.*

Remark. In particular, a distant geodesics connecting $\sigma_1(R)$, $\sigma_2(R)$ looks from a as a polygonal curve on the sphere whose segments are located in Weyl chambers. □

Remark. We emphasis that the Tits metrics is nonseparable. □

8.9. Tits metric for Bruhat–Tits buildings. The previous construction can be adapted to Bruhat–Tits buildings.

Fix a vertex a of an Euclidean building. Denote by S_R the sphere of center a and radius R. For $R' > R$ there is a natural projection $S_{R'} \to S_R$; namely, for $x \in S_{R'}$, we take the intersection of $[a, x]$ with S_R.

We define the sphere Σ_∞ at infinity as the inverse limit of the spheres Ξ_R and define the Tits metric on Σ_∞ by same formula (8.4).

Theorem 8.22 *For* $\mathcal{B}(\mathrm{PGL}(n, \mathbb{Q}_p))$ *we get the spherical building of* $\mathbb{Q}_p$.

8.10. Digression. Problems of harmonic analysis. Harmonic analysis on buildings has mysterious parallels with harmonic analysis on Riemannian symmetric spaces. This analogy discovered by Macdonald [101], 1972, was an important standpoint for further development of the theory of multi-variant special functions.

Recall some problems in harmonic analysis on buildings.

1) Decompose the space l_2 on the set of vertices with respect to action of the group G (this is the subject of Macdonald's book)

2) Decompose the space l_2 on the set of simplices of a given type. As far as I know, this problem is not solved.

3) There is an analog of Berezin kernels (see 2.12.6 and Chapter 7) for the space of lattices $\mathrm{GL}(n,\mathbb{Q}_p)/\mathrm{GL}(n,\mathbb{O}_p)$ (here it reasonable to consider GL instead of PGL).

Fix $\alpha \geqslant 0$. Consider a Hilbert space H_α with a system of vectors e_L enumerated by self-dual lattices; let their in inner products be[19]

$$\langle e_L, e_M\rangle = \Bigl(\frac{\mathrm{vol}(L\cap M)^2}{\mathrm{vol}\, L\cdot \mathrm{vol}\, M}\Bigr)^{\alpha/2} \tag{8.5}$$

Theorem 8.23 *A Hilbert space H_α exists iff α ranges in a set of the form*

$$\alpha = 0, 1, \dots, n-1, \qquad \textit{or } \alpha > n-1$$

Example. Consider the space $L^2(\mathbb{Q}_p^n)$. For each lattice $M \subset \mathbb{Q}_p^n$ denote by e_M the functions, that is equal $(\mathrm{vol}\, M)^{-1/2}$ on M and 0 outside M. Then the vectors e_L satisfy the condition (8.5) with $\alpha = 1$. By Theorem 7.1.14, Hilbert spaces H_α exist for all positive integer α. □

The problem is

– *to decompose H_α with respect to action of the group* $\mathrm{GL}(n,\mathbb{Q}_p)$.

In this case the problem is solved; however nothing is known for over buildings.

[19] In fact, this is a special case of a Hilbert space determined by a reproducing kernel.