Chapter 11

Weil representation over p-adic field

[Draft] 30.06.07 /users/neretin/public_html/lectures/lectures.htm

11.1 Gaussian integrals over p-adic field

For simplicity, we assume $p \neq 2$.

We are interested in integrals of the form

$$\int_{r+L} \exp\Big\{2\pi i(zAz^t + bz^t)\Big\}\, dz \tag{1.1}$$

where $z \in \mathbb{Q}_p^n$ is a matrix-row, A is a symmetric $n \times n$-matrix, $b \in \mathbb{Q}^n$ is a matrix-row, L is an $\mathbb{O}_p$-submodule in $\mathbb{Q}_p^n$, and $r \in \mathbb{Q}_p^n$ is a vector.

Applying a shift $z \mapsto z - r$, we can reduce such integrals to the case $r = 0$. Also, if A is invertible, then the shift $z \mapsto z + \frac{1}{2}bA^{-1}$ allows to 'kill' b (but we can not remove both b, r).

Recall that $\exp\big\{2\pi iz\big\}$ is constant on every ball $a + \mathbb{O}_p$, therefore our integrand is a locally constant function. Hence our integrals can be reduced to sums (if L is compact, then the sum is finite).

1.1. One-dimensional Gaussian integrals over compact sets. Each integral of this type can be reduced to one of the three following propositions.

Proposition 1.1

$$\int_{\mathbb{O}_p} \exp\{2\pi ibz\}\, dz = \begin{cases} 1 & \textit{if } |b| \leqslant 1 \\ 0 & \textit{if } |b| > 1 \end{cases}, \tag{1.2}$$

Note, that the first variant holds iff the integrand is constant function.

PROOF. If $|b| \leqslant 1$, then the integrand $= 1$.

Let $b = \theta \cdot p^{-\mu}$, $|\theta| = 1$, $\mu > 0$. We divide the ball $|z| \leqslant 1$ into subballs $k + p^{\mu}\mathbb{O}_p$ of radii $p^{-\mu}$, where k ranges in $\{0, 1, \dots, p^{\mu} - 1\}$. The integrand is constant on every subball. Thus we come to the sum

$$p^{-\mu} \sum_{0 \leqslant k \leqslant p^{\mu}-1} \exp\Bigl\{2\pi i \frac{k\theta}{p^{\mu}}\Bigr\}$$

Since $|\theta| = 1$, the map $k \mapsto k\theta$ induces a bijection of the group $\mathbb{Z}_{p^{\mu}}$ to itself and we get the sum of all the roots of unity of power p^{μ}, i.e., 0. □

Corollary 1.2 *Functions* $\exp\{2\pi i b z\}$, *where* b *ranges in* $\mathbb{Q}_p/\mathbb{O}_p$, *form an orthonormal basis in* $L^2(\mathbb{O}_p)$.

Problem 1.1 Show that this orthogonal system is complete.

REMARK. Certainly, the last corollary (and hence also the previous proposition) are special cases of the following statement: *characters of a compact Abelian group A constitute a basis in $L^2(A)$.* □

Proposition 1.3 *Let a ball $\xi + p^{-\mu}\mathbb{O}_p$ do not contain 0 (i.e., $|\xi| > p^{\mu}$). Then*

$$\int_{z \in \xi + p^{-\mu}\mathbb{O}_p} \exp\bigl\{2\pi i a z^2\bigr\}\, dz = \begin{cases} \exp\{2\pi i a \xi^2\} p^{\mu} & \text{if } |\xi a| \leqslant p^{-\mu} \\ 0 & \text{otherwise} \end{cases}$$

Note that the first variant holds iff the integrand is constant.

PROOF. After a shift we get

$$\int_{p^{-\mu}\mathbb{O}_p} \exp\bigl\{2\pi i a(\xi + z)^2\bigr\}\, dz$$

Dividing the ball $p^{-\mu}\mathbb{O}_p$ into p smaller balls $kp^{-\mu} + p^{-\mu+1}\mathbb{O}_p$, we obtain

$$= \sum_{k=0}^{p-1} \int_{p^{-\mu+1}\mathbb{O}_p} \exp\bigl\{2\pi i a(\xi + kp^{-\mu} + v)^2\bigr\}\, dv \tag{1.3}$$

Let us examine the integrands

$$\{\dots\} = 2\pi i\bigl(a\xi^2 + 2a\xi k p^{-\mu} + akp^{-\mu} \cdot kp^{-\mu} + 2a\xi v + av^2 + 2akp^{-\mu} \cdot v\bigr) \tag{1.4}$$

The first term in the brackets is constant. Since $|\xi| > |p^{-\mu}| > |v|$, the term $2a\xi k p^{-\mu}$ has maximal norm among nonconstant terms. Consider 3 cases.

Let $|\xi a p^{-\mu}| \leqslant 1$. Then all the summands in (1.4) except $a\xi^2$ are integer. Therefore $\exp\{\dots\} = \exp\{2\pi i a \xi^2\}$ (for all k and v), i.e. the integrand in the whole integral is constant.

If $|\xi a p^{-\mu}| = p$, then

$$\exp\{2\pi a(\xi + kp^{-\mu+1} + v)^2\} = \exp\Big\{2\pi a(\xi^2 + 2\xi k p^{-\mu})\Big\}$$

(because other terms in(1.4) are integer). Hence the sum (1.3) is

$$\exp\{2\pi i a\xi^2\} \sum_{k=0}^{p-1} \exp\{2\pi i(a\xi p^{-\mu+1})\frac{k}{p}\}$$

Since $|a\xi p^{-\mu+1}| = 1$, we get a sum of roots of unity, i.e., 0.

If $|a\xi p^{-\mu}| = p^s \geqslant p^2$, then we divide our ball into p^{s-1} smaller balls. In each subball the preceding case is realized. $\square$

Proposition 1.4 *Denote* $|a| = p^{-k}$

$$\int_{|z|\leqslant 1} \exp\{2\pi i a z^2\}\, dz = \begin{cases} 1 & \textit{if } k \leqslant 0 \\ p^{-k/2}, & \textit{if } k \textit{ is even positive} \\ \Gamma_p((p-1)/2)\eta(a)p^{-k/2} & \textit{if } k \textit{ is odd positive} \end{cases}$$

We recall that $\eta(\cdot)$ is the Legendre character of $\mathbb{Q}_p^*$, see (10.2.2), and $\Gamma_p((p-1)/2)$ is given by (9.2.9).

Proof. 1. If $k \leqslant 0$, then the integrand is constant

2. Let $k = 1$. We divide the ball $|z| \leqslant 1$ into subballs $m + p\mathbb{O}_p$, where $m = 0,1, \dots p-1$. The integrand is constant on each subball, and hence

$$\int_{|z|\leqslant 1} = \sum_{m=0}^{p-1} \int_{m+p\mathbb{O}_p} = p^{-1} \sum_{m=0}^{p-1} \exp\{2\pi i a m^2/p\}$$

We get a Gaussian quadratic sum evaluated in Theorem 9.2.10.

3. Let $k \geqslant 2$. We carry out the same transformation,

$$\int_{|z|\leqslant 1} = \sum_{m=0}^{p-1} \int_{m+\mathbb{O}_p} = \int_{\mathbb{O}_p}$$

We leave only one term, because in other cases the integrands are non-constant; therefore the integrals are zero by Proposition 1.3. After the substitution $z = up$ we come to

$$p^{-1} \int_{|u|\leqslant 1} \exp\{2\pi i a p^2 u^2\}\, du$$

We obtain the integral of the same type but a is changed by ap^2. We come one of the cases 1, 2, 3 again. $\square$

1.2. Gaussian integral over $\mathbb{Q}_p$.

Proposition 1.5

$$\int_{\mathbb{Q}_p} \exp\{2\pi i(az^2 + bz)\}\, dz = \xi(a)|a|^{-1/2} \exp\Big\{-2\pi i \frac{b^2}{4a}\Big\} \tag{1.5}$$

Here the (divergent) integral is defined as

$$\lim_{N\to\infty} \int_{|z|\leqslant p^N} \exp\{2\pi i(az^2 + bz)\}\, dz \tag{1.6}$$

and $\xi(a)$ is defined above (10.2.4). In particular

$$\int_{\mathbb{Q}_p} \exp\{2\pi i a z^2\}\, dz = \xi(a)|a|^{-1/2} \tag{1.7}$$

PROOF. First, let $b = 0$. Proposition 1.4 provides us an explicit expression for $c_n := \int_{|z|\leqslant p^N}$ and shows that this sequence is eventually constant.

If $b \neq 0$, then we consider the shift $z \mapsto z - b/2a$. □

Problem 1.2 Why a shift of the arguments in the regularized integral (1.6). is admissible?

1.3. Gaussian integrals over $\mathbb{Q}_p^n$.

Theorem 1.6 *Let A be a non-degenerate symmetric matrix*

$$\int_{\mathbb{Q}_p^n} \exp\Big\{2\pi i(zAz^t + bz^t)\Big\}\, dz = \Xi(A)|\det(A)|^{-1/2} \exp\Big\{-2\pi i \frac{1}{4} bAb^t\Big\} \tag{1.8}$$

where Ξ is the invariant (10.2.5) of a quadratic form. In particular,

$$\int_{\mathbb{Q}_p^n} \exp\Big\{2\pi i zAz^t\Big\}\, dz = \Xi(A)|\det(A)|^{-1/2} \tag{1.9}$$

PROOF. Without loss of generality we can assume $b = 0$, otherwise we consider a shift $z \mapsto z - \frac{1}{2}bA^{-1}$.

Next, assume that for a some A equality (11.1) is valid. Substituting $z = uR$ we obtain

$$\int \exp\Big\{2\pi i zRAR^t z^t\Big\}\, dz = \det(R)^{-1}\Xi(A)|\det(A)|^{-1/2} = \Xi(A)|\det(RAR^t)|^{-1/2}$$

i.e., the identity holds for RAR^t. Therefore it is sufficient to prove the identity for a diagonal matrix A. In this case, our integral is a product of one-dimensional integrals evaluated in the previous theorem. □

REMARK. This theorem explains the invariant $\Xi(\cdot)$ of quadratic forms.

REMARK. (Analogy with $\mathbb{R}$). Let A be a real nondegenerate symmetric matrix, having p positive and q negative eigenvalues. Define

$$I(A) = \int_{\mathbb{R}^n} \exp\Big\{-\frac{i}{2} xAx^t\Big\}\, dx := \lim_{\varepsilon\to+0} \int_{\mathbb{R}^n} \exp\Big\{-\frac{1}{2}x(iA+\varepsilon)x^t\Big\}\, dx$$

Let $A = hRh^t$, where h is an orthogonal matrix and R is diagonal with elements r_k. Then

$$(2\pi)^{-n/2} I(A) = \lim_{\varepsilon \to +0} \det(\varepsilon + iA)^{-1/2} = \lim_{\varepsilon \to +0} \det(\varepsilon + ihRh^t)^{-1/2} =$$
$$= \lim_{\varepsilon \to +0} \prod (\varepsilon + ir_k)^{-1/2} = \prod e^{\pi i \cdot \mathrm{sgn}(r_k)/2} r_k^{-1/2} = e^{\pi i k(p-q)/2} |\det A|^{-1/2}$$

In this sense, $e^{\pi i(q-p)/2}$ is an analog of the modified Hasse–Minkowski invariant Ξ.

1.4. One two-dimensional Gaussian integral. The following integral concludes the list of the simplest Gaussian integrals.

Proposition 1.7

$$\int_{\mathbb{Q}_p} \int_{p^{-\mu}\mathbb{O}_p} \exp\{2\pi i(xy + bx + cy)\}\, dy\, dx = \begin{cases} \exp\{-2\pi i bc\} & \text{if } |b| \leqslant p^{-\mu} \\ 0 & \text{otherwise} \end{cases}$$

Proof. We write the iterated integral

$$\int_{\mathbb{Q}_p} \exp\{2\pi i bx\} \left[\int_{p^{-\mu}\mathbb{O}_p} \exp\{2\pi i (x + c)y)\}\, dy \right] dx =$$

and integrate in y with (1.2), the result is non-zero iff the integrand is constant, i.e., $x + c \in p^{\mu}\mathbb{O}_p$ We obtain

$$p^{-\mu} \int_{-c+p^{\mu}\mathbb{O}_p} \exp\{2\pi i bx\}\, dx = p^{-\mu} \exp\{-2\pi i bc\} \int_{p^{\mu}\mathbb{O}_p} \exp\{2\pi i bx\}\, dx$$

It remains to apply (1.2) again. □

1.5. Quadratic forms and lattices. Recall that $p \neq 2$. We are going to present a classification of p-adic quadratic forms under the action of the group $\mathrm{GL}(n, \mathbb{O}_p)$.

First, reformulate the problem. Let V be a linear space over $\mathbb{Q}_p$, B be a symmetric bilinear form on V, L be a lattice. For a subspace $W \subset V$, denote by $B\big|_W$ the restriction of B to W. We say that a triple (V, B, L) is *decomposable* if there is a decomposition $V = V_1 \oplus V_2$ such that

$$B = B\Big|_{V_1} \oplus B\Big|_{V_2}, \qquad L = (L \cap V_1) \oplus (L \cap V_2)$$

Theorem 1.8 *The only indecomposable triples are one-dimensional triples. In other words, each quadratic form can be diagonalized.*

More precise variant of the theorem is the following

Theorem 1.9 *Fix $r \in \mathbb{Q}_p$ satisfying $|r| = 1$, $\eta(r) = -1$. Each nondegenerate quadratic form on $\mathbb{Q}_p^n$ is $\mathrm{GL}(n, \mathbb{O}_p)$-equivalent to a unique form of the type*

$$p^{m_1}(x_1^2 + \dots + x_{\alpha_1 - 1}^2 + \varepsilon_1 x_{\alpha_1}^2) + p^{m_2}(x_{\alpha_1+1}^2 + \dots + x_{\alpha_2 - 1}^2 + \varepsilon_2 x_{\alpha_2}^2) + \dots \quad (1.10)$$

where $m_1 > m_2 > \dots$ and ε_1, ε_2,... take values 1 or r.

The statement is not difficult, see the book of Cassels [29].

1.6. Quadratic forms and $\mathbb{O}_p$-submodules. Now, let $M \subset V = \mathbb{Q}_p^n$ be a $\mathbb{O}_p$-submodule such that $\mathbb{Q}_p \cdot M = V$. Let B be a symmetric bilinear form on V (it can be degenerate). Again, consider indecomposable triples (V, B, M).

Theorem 1.10 *The only indecomposable triples are one-dimensional triples and the following two-dimensional objects:*

$$V = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2, \quad M = \mathbb{Q}_p e_1 \oplus p^m \mathbb{O}_p e_2, \quad B(xe_1 \oplus ye_2, x'e_2 \oplus y'e_2) = xy' + yx'$$

PROOF. Let V be a p-adic linear space, $\mathcal{M}$ be a submodule, $\mathcal{M} = N \oplus L$, where N is a linear subspace, and L is a lattice. First, we reject trivial variants.

Let B be degenerate on V, let $\ker B$ be its kernel. Denote $Y = \ker B \cap \mathcal{M}$. Take a basis e_i, f_j in $\ker B$ such that $Y = (\oplus \mathbb{Q}_p e_i) \oplus (\oplus \mathbb{O}_p f_j)$. It is more-or-less obvious that this basis can be extended to a basis e_i, f_j, E_k, F_m in the whole submodule $\mathcal{M}$,

$$\mathcal{M} = (\oplus \mathbb{Q}_p e_i) \oplus (\oplus \mathbb{Q}_p E_k) \oplus (\oplus \mathbb{O}_p f_k) \oplus (\oplus \mathbb{O}_p F_l)$$

Now we split V into a sum of the subspace with basis e_i, f_j (i.e., $\ker B$) and the subspace with basis E_k, F_l.

Therefore we suppose that B *is nondegenerate on* V.

If the form B is nondegenerate on N. Then there is nothing to discuss. We simply split

$$\mathcal{M} = N \oplus (N^\perp \cap \mathcal{M})$$

We get a quadratic form on the linear space N and a quadratic form on the lattice $N^\perp \cap \mathcal{M}$. It remains to refer to Proposition 2.2 and Theorem 1.9.

It remains the case when B *is nondegenerate on* V *and degenerate on* N. Let $v \in N$ be a vector in the kernel of B on N. Consider the B-orthocomplement $v^\perp$; by definition, $v^\perp \supset N$. Consider the one-dimensional space $V/v^\perp$. Project our $\mathbb{O}_p$-module L to $V/v^\perp$. Since $N \subset v^\perp$, the image is compact. Take an element of the image having the maximal absolute value. Choose its representative $u \in M$. Denote by H the subspace spanned by u, v. Then $H \cap \mathcal{M} = \mathbb{Q}_p u \oplus \mathbb{O}_p v$.

Decompose $V = H \oplus H^\perp$. Let us prove that

$$\mathcal{M} = (H \cap \mathcal{M}) \oplus (H^\perp \cap \mathcal{M})$$

The inclusion $\supset$ is evident. Decompose $h\mathcal{M}$ as $h = \xi + \eta$, where $\xi \in H$, $\eta \in H^\perp$. Then $\xi \in \mathcal{M}$, therefore $\eta \in \mathcal{M}$. $\square$

1.7. General multivariate Gaussian integrals. Consider an arbitrary integral of the form

$$\int_{r+L} \exp\Big\{2\pi i(zAz^t + bz^t)\Big\}\, dz \qquad (1.11)$$

where $L \subset V$ is $\mathbb{O}_p$-submodule.

Without loss of generality, we can assume that $r = 0$. Also, we can assume that $\mathbb{Q}_p L \simeq V$.

Next, decompose the module L into a direct sum of one-dimensional and two-dimensional pairwise orthogonal submodules as it was done in Theorem 1.10 Now the integral is reduced to a product of one-dimensional and two-dimensional Gaussian integrals.

Thus, ideologically, our problem is simple. I do not know a nice presentation of the final result.

11.2 Weil representation, p-adic case

We prefer to think, that $p \neq 2$. However main theorems and their proofs survive also in this case.

2.1. Test-functions and distributions. Let $V \simeq \mathbb{Q}_p^n$ be a p-adic linear space. For a subset $S \subset V$ denote by $\mathcal{I}_S(z)$ the *indicator function* of S,

$$\mathcal{I}(z) = \begin{cases} 1 & z \in S \\ 0 & z \notin S \end{cases}$$

Fix a point $a \in \mathbb{Q}_p^n$ and $k \in \mathbb{Z}$. Consider a 'ball' $B(a, p^{-k})$

$$z: \ |z_j - a_j| \leqslant p^{-k} \forall j$$

A *test function* on V is a finite linear combination of indicator functions of such balls. The *Bruhat space* $\mathcal{B}(V)$ is the space of all the test functions.

Problem 2.1 $f \in \mathcal{B}(V)$ iff f is locally constant compactly supported function.[1]

Let $M \subset L \subset V$ be lattices. Denote by $\mathcal{B}(L|M)$ the space of functions supported by L and invariant under shifts $z \mapsto z + h$, where $h \in M$. Evidently,

$$\dim \mathcal{B}(L|M) = \text{order of the group } L/M.$$

Observation 2.1

$$\mathcal{B}(V) = \bigcup_{M \subset L \subset V} \mathcal{B}(L|M)$$

A *p-adic distribution* is a linear functional on the space $\mathcal{B}(V)$. By $\mathcal{B}'(V)$ we denote the space of all the distributions on V . in what follows we write distributions as integrals.

Let $\varphi(x)$ be a distribution on V, $\psi(y)$ be a distribution on W. We define their *tensor product* as a distribution on $V \oplus W$ given by

$$f \mapsto \int_{x \in V} \varphi(x) \Big\{ \int_{y \in W} \psi(y) f(x, y) \Big\}$$

[1] A function f is called locally constant if for each z there is an open neighborhood U of z such that $f(z) = \text{const}$ in U.

We denote such a tensor product by $\varphi(x)\psi(y)$.

2.2. Kernel theorem. For a distribution (*kernel*) $K(x,y)$ on $V \oplus W$, consider the bilinear form S on $\mathcal{B}(V) \times \mathcal{B}(W)$ given by

$$S(f,g) := \int_{(x,y)\in V\oplus W} K(x,y)f(x)g(y)$$

For a fixed $f \in \mathcal{B}(W)$, we define a linear functional $g \mapsto \mathfrak{B}(f,g)$. In other words we get a map $\mathcal{B}(W) \mapsto \mathcal{B}'(V)$; we denote it by

$$Ag(x) = \int_{y\in W} K(x,y)g(y) \tag{2.1}$$

The following analog of the kernel theorem is trivial.

Theorem 2.2 *Each operator $\mathcal{B}(W) \to \mathcal{B}'(V)$ has form (2.1).*

2.3. Fourier transform. For a function f on $\mathbb{Q}_p^n$, we define its *Fourier transform* by

$$\mathcal{F}f(x) = \int_{\mathbb{Q}_p^n} f(y)\exp\{2\pi i x y^t\}\,dy$$

Theorem 2.3 *a) If $f \in \mathcal{B}(\mathbb{Q}_p^n)$, then $\mathcal{F}f \in \mathcal{B}(\mathbb{Q}_p^n)$.*

b) For $f_1,\ f_2 \in \mathcal{B}(\mathbb{Q}_p^n)$,

$$\int_{\mathbb{Q}_p^n} f_1(x)\overline{f_2(x)}\,dx = \int_{\mathbb{Q}_p^n} (\mathcal{F}f_1)(x)\overline{\mathcal{F}f_2(x)}\,dx \qquad \textit{(the Plancherel formula)} \tag{2.2}$$

c) The Fourier transform $\mathcal{B}(\mathbb{Q}_p^n) \to \mathcal{B}(\mathbb{Q}_p^n)$ can be extended by continuity to a unitary operator $L^2(\mathbb{Q}_p^n) \to L^2(\mathbb{Q}_p^n)$.

Proof. To simplify notation, assume $n = 1$. Fix an integer k. Consider all the balls of the form $|x-a| \leqslant p^k$, they are enumerated by elements of the quotient-group $\mathbb{Q}_p/p^{-k}\mathbb{O}_p$. Their indicator functions $\mathcal{I}_{|x-a|\leqslant p^k}$ are pairwise orthogonal. Further,

$$\mathcal{F}\mathcal{I}_{|z-a|\leqslant p^k}(x) = \int_{|z-a|\leqslant p^k} \exp\{2\pi i x z\}\,dz =$$
$$\exp\{2\pi i a x\}\int_{|u|\leqslant p^k} \exp\{2\pi i u x\}\,du = \exp\{2\pi i a x\}\int_{|v|\leqslant 1} \exp\{2\pi i v x p^{-k}\}\,p^{-k}dv =$$
$$= p^{-k}\exp\{2\pi i a x\}\mathcal{I}_{|z|\leqslant p^{-k}}(x) \tag{2.3}$$

(here we substitute $u = z - a$, $v = up^{-k}$) and apply (1.2). The right-hand side is an element of $\mathcal{B}(\mathbb{Q}_p)$ and this proves a).

b) The identity (2.2) for

$$f_1 = \mathcal{I}_{|z-a_1|\leqslant p^k}, \qquad f_2 = \mathcal{I}_{|z-a_2|\leqslant p^k},$$

is reduced to the integral (1.2).

Having arbitrary $f_1,\ f_2 \in \mathcal{B}(\mathbb{Q}_p)$, we represent them as linear combinations of $\sum c_j \mathcal{I}_{|z-a_j|\leqslant p^{-N}}$ with sufficiently large N. It remains to refer to the previous paragraph.

c) follows from b). □

2.4. Weil representation of the group $\mathrm{Sp}(2n,\mathbb{Q}_p)$**.** We repeat the considerations of Section 1.1.2 and follow the notation of that Section. The p-adic Heisenberg group $\mathrm{Heis}_n(\mathbb{Q}_p)$ is the group of $(1+n+1)\times(1+n+1)$-matrices

$$R\big[v_+ \oplus v_-;\alpha\big] = R\big[v;\alpha\big] = \begin{pmatrix} 1 & v_+ & \alpha + \frac{1}{2}v_+v_-^t \\ 0 & 1 & v_- \\ 0 & 0 & 1 \end{pmatrix}$$

with coefficients in $\mathbb{Q}_p$; here $v_\pm \in \mathbb{Q}_p^n$, $\alpha \in \mathbb{Q}_p$.

This group acts in $L^2(\mathbb{Q}^n)$ by the unitary transformations

$$U\big[v_+ \oplus v_-;\alpha\big] = f(x+v_+)\exp\big\{2\pi i(xv_-^t + \frac{1}{2}v_+v_-^t)\big\}$$

For $g \in \mathrm{Sp}(2n,\mathbb{R})$, we define the automorphism of $\mathrm{Heis}_n(\mathbb{Q}_p)$ by

$$\gamma_g : \big[v;\alpha\big] \mapsto \big[vg;\alpha\big]$$

Theorem 2.4 *a) For each* $g \in \mathrm{Sp}(2n,\mathbb{Q}_p)$ *there is a unique up to a scalar factor bounded invertible operator* $\mathrm{We}(g) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *such that*

$$U\big[vg;\alpha\big] = \mathrm{We}(g)^{-1}U\big[v;\alpha\big]\,\mathrm{We}(g) \tag{2.4}$$

for each $\big[v;\alpha\big] \in \mathrm{Heis}_n(\mathbb{Q}_p)$

b) Operators $\mathrm{We}(g)$ *are unitary to within a scalar factor and satisfy*

$$\mathrm{We}(g_1)\,\mathrm{We}(g_2) = c(g_1,g_2)\,\mathrm{We}(g_1g_2) \tag{2.5}$$

where $c(g_1,g_2) \in \mathbb{C}^*$.

c) The operators $\mathrm{We}(g)$ *preserve the subspace* $\mathcal{B}(\mathbb{Q}_p^n) \subset L^2(\mathbb{Q}_p^n)$.

Proof of Theorem 1.2.5 survives in our case literally, we simply write analogs of formulae (2.14)–(2.16),

$$\mathrm{We}\begin{pmatrix} A & 0 \\ 0 & A^{t-1} \end{pmatrix} f(x) = f(xA^{-1})|\det A|^{-1/2} \tag{2.6}$$

$$\mathrm{We}\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} f(x) = f(x)\exp\big\{\pi i x B x^t\big\} \tag{2.7}$$

$$\mathrm{We}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f(x) = \int_{\mathbb{R}^n} f(y)\exp\{2\pi i x y^t\}\,dy \tag{2.8}$$

Problem 2.2 For $g \in \mathrm{Sp}(2n, \mathbb{O}_p)$,

$$\mathrm{We}(g)\mathcal{I}_{\mathbb{O}_p^n} = \lambda \cdot \mathcal{I}_{\mathbb{O}_p^n}, \qquad \lambda \in \mathbb{C}^*$$

There are two ways of a proof, to use the definition (2.4) or to watch generators of $\mathrm{Sp}(2n, \mathbb{O}_p)$, see (1.2.13). For $g \in \mathrm{Sp}(2n, \mathbb{O}_p)$ we can normalize $\mathrm{We}(g)$ by the assumption $\lambda = 1$. Automatically, the factor $c(g_1, g_2)$ in (2.5) is 1, i.e., we obtain a linear representation of $\mathrm{Sp}(2n, \mathbb{O}_p)$. Is it correct for $p = 2$?

2.5. Gaussian distributions. A *Gaussian distribution* on a p-adic linear space $X = \mathbb{Q}_p^n$ is a linear functional on $\mathcal{B}(X)$ given by

$$\varphi \mapsto \int_L \exp\{2\pi i x A x^t\}\varphi(x)\, dx$$

where L is an $\mathbb{O}_p$-submodule in X.

Proposition 2.5 *The Fourier transform of a Gaussian distribution is a Gaussian distribution*

PROOF. Let χ be a Gaussian distribution. By virtue Theorem 1.10, we can decompose it into a tensor product of one-dimensional and two-dimensional distributions.

Now we must verify the statement in one-dimensional cases and in the indecomposable two-dimensional case. The corresponding Gaussian integrals are evaluated in Propositions 1.1–1.5, 1.7. Straightforward calculations are rather simple, however for the function

$$f(x) = \exp\{2\pi i a x^2\}\mathcal{I}_{|x|\leqslant 1} \tag{2.9}$$

a final result is not immediately visible. To attain a self-evidence, we note that the Fourier transform takes products to convolutions, i.e., we get the function

$$\widehat{f}(y) = \xi(a)|a|^{-1} \int_{y+\mathbb{O}_p} \exp\{2\pi i a^{-1} x^2/4\}\, dx \tag{2.10}$$

In formula (2.9), we have $|a| > 1$, otherwise $\exp\{\cdot\}$ is constant on $\mathbb{O}_p$. Therefore $|a^{-1}| < 1$. Therefore $\widehat{f}(0) = 1$ (this is rather important because we do not refer to Proposition 1.4). However, we refer to Proposition 1.3 and get a Gaussian supported by $|y| \leqslant |a|$. □

2.6. Stabilizers of a Gaussian distributions in the Heisenberg group. Consider the space $\mathcal{V}_{2n} := \mathbb{Q}_p^n \oplus \mathbb{Q}_p^n$ equipped with a standard symplectic form. Consider the corresponding Heisenberg group.

For a Gaussian distribution $\chi(x)$ denote by $P(\chi)$ the set of all the $v \in \mathcal{V}$ such that

$$U\,[h; 0]\,\chi = \chi$$

Theorem 2.6 *a) $P(\chi)$ is self-dual $\mathbb{O}_p$-submodule.*

b) The map $\chi \mapsto P(\chi)$ is a bijection between Gaussian distributions defined to within a scalar factor and self-dual $\mathbb{O}_p$-submodules in $\mathcal{V}_{2n}$.

PROOF. a) It suffices to verify the statement, for indecomposable distributions (see Theorem 1.10), i.e., for the one-dimensional distributions

$$\chi_1(x) = \delta(x), \qquad \chi_2(x) = 1,$$
$$\chi_3(x) = \exp\{2\pi i a x^2\}, \qquad \chi_4(x) = \exp\{2\pi i a x^2\}\mathcal{I}_{|z|\leqslant p^k}(x)$$

and the two-dimensional distribution

$$\chi_5(x_1, x_2) = \exp\{2\pi i x y\}\mathcal{I}_{|z|\leqslant p^k}(x)$$

It seems that in all the cases the result is sufficiently obvious.

Problem 2.3 Examine the case χ_4.

For the proof of b), we need in a following lemma

Lemma 2.7 *Operators of the Weil representation take Gaussian distributions to Gaussian distributions*

PROOF OF THE LEMMA. We must verify the statement for the generators (2.6)–(2.8) of the symplectic group. In the first two cases, the statement is obvious; the Fourier transform was examined in Proposition 2.5. □

PROOF OF THEOREM 2.6.B. Now let $L \subset \mathcal{V}_{2n}$ be a self-dual module. We wish to show that the system of equations

$$\Big\{ U(v \oplus 0)\chi = \chi, \qquad v \in L \tag{2.11}$$

has a unique solution in the class of distributions and this solution is a Gaussian.

Applying automorphisms of Heis_n, we can reduce L to a canonical form. Thus we assume that L consists of vectors

$$v = v_+ \oplus v_- = (a_1, \dots, a_k, 0, \dots, 0) \oplus (b_1, \dots, b_k, c_1, \dots, c_{n-k})$$

where a_j, $b_j \in \mathbb{O}_p$, and $c_l \in \mathbb{Q}_p$. The unique solution of the system (2.11) is

$$\mathrm{const} \cdot \prod_{j=1}^{k} \mathcal{I}_{|x_j|\leqslant 1}(x_j) \cdot \prod_{j=k+1}^{n} \delta(x_j)$$

This completes the proof of b). □

Problem 2.4 Let χ be a Gaussian distribution. Show that

$$\chi \in \mathcal{B}(\mathbb{Q}_p^n) \Longleftrightarrow \text{submodule } V(\chi) \text{ is compact} \Longleftrightarrow \chi \in L^2(\mathbb{Q}_p^n)$$

2.7. The Weil representation of the Nazarov category. Now, let $\mathcal{V}_{2n} \simeq \mathbb{Q}_p^{2n}$, $\mathcal{V}_{2m} \simeq \mathbb{Q}_p^{2m}$ be p-adic symplectic spaces. Consider their direct sum $\mathcal{V}_{2m} \oplus \mathcal{V}_{2n}$ and equip it with the difference of the symplectic forms. Let

L be a self-dual module in $\mathcal{V}_{2m} \oplus \mathcal{V}_{2n}$. Let χ_L be the corresponding Gaussian distribution on $\mathcal{V}_{2m} \oplus \mathcal{V}_{2n}$. Define a Gaussian operator

$$\mathrm{We}(L) : \mathcal{B}(\mathbb{Q}_p^n) \mapsto \mathcal{B}'(\mathbb{Q}_p^m)$$

by

$$\mathrm{We}(L)f(x) = \int_{\mathbb{Q}_p^n} \chi(x,y) f(x)\, dx$$

Theorem 2.8 *The operator* $\mathrm{We}(L)$ *satisfies the identity*

$$U\big[h;0\big]\,\mathrm{We}(L) = \mathrm{We}(L) U\big[r;0\big] \tag{2.12}$$

for all $r \oplus h \in L$. *Moreover,* $\mathrm{We}(L)$ *is determined by this identity to within a scalar.*

PROOF. This is a reformulation of Theorem 2.6. □

Proposition 2.9 *Let* $M \subset \mathcal{V}_{2n}$ *be a compact self-dual submodule. Let* $\mathrm{we}(M) \in \mathcal{B}(\mathbb{Q}_p^n)$ *be the corresponding Gaussian distribution. Then*

$$\mathrm{We}(L)\,\mathrm{we}(M) = \lambda(L,M)\,\mathrm{we}(LM), \qquad \lambda \in \mathbb{C}^*$$

We leave this statement as an exercise, see Theorem 1.9.6.

Theorem 2.10 *a) An operator* $\mathrm{We}(L)$ *is a bounded operator* $L^2(\mathbb{Q}_p^n) \to L^2(\mathbb{Q}_p^m)$ *iff the modules*

$$\ker(L) = L \cap \mathcal{V}_{2n}, \qquad \mathrm{indef}(L) = L \cap \mathcal{V}_{2m}$$

are compact. In other words, L is a morphism of the Nazarov category.

b) Under the same conditions $\mathrm{We}(L)$ *is an operator* $\mathcal{B}(\mathbb{Q}_p^n) \to \mathcal{B}(\mathbb{Q}_p^m)$.

c) For morphisms $L : \mathcal{V}_{2n} \to \mathcal{V}_{2m}$, $M : \mathcal{V}_{2m} \to \mathcal{V}_{2k}$, *we have*

$$\mathrm{We}(M)\,\mathrm{We}(L) = \lambda(M,L)\,\mathrm{We}(LM), \qquad \lambda \in \mathbb{C}^*$$

PROOF. a) Let $\mathrm{We}(L)$ be bounded. Consider a self-dual submodule $K \subset \mathcal{V}_{2n}$. By Proposition 2.9, $\mathrm{We}(L)\,\mathrm{we}(K) = \sigma\,\mathrm{we}(LK)$. If $\mathrm{indef}\, L$ is noncompact, then, by Problem 2.4, $\mathrm{we}(LK) \notin L^2(\mathbb{Q}_p^n)$.

Applying the same reasoning to the adjoint operator $\mathrm{We}(L)^*$, we obtain that $\ker L$ is compact. Thus, L is a morphism of the Nazarov category.

Conversely, let L be a morphism of the Nazarov category We can reduce it to the canonical form given in Theorem 10.7.7 by transformations

$$L \mapsto gLh, \qquad h \in \mathrm{Sp}(2n, \mathbb{Q}_p), g \in \mathrm{Sp}(2m, \mathbb{Q}_p)$$

Observation 2.11 *For morphisms* $L_1 : \mathcal{X}_1 \rightrightarrows \mathcal{Y}_1$, $L_2 : \mathcal{X}_2 \rightrightarrows \mathcal{Y}_2$ *of Nazarov category, consider their direct sum* $L_1 \oplus L_2 : \mathcal{X}_1 \oplus \mathcal{X}_2 \rightrightarrows \mathcal{Y}_1 \oplus \mathcal{Y}_2$. *We have*

$$\mathrm{We}(L_1 \oplus L_2) = \mathrm{We}(L_1) \otimes \mathrm{We}(L_2)$$

Thus, we must watch the boundedness for the canonical morphisms a)-e) from Theorem 7.7. For the cases a)-c), there is nothing to discuss. For the case d), we get respectively

$$Af(x) = \mathcal{I}_{|z|\leqslant 1}(x) \int_{x+p^k\mathbb{O}_p} f(y)\,dy$$

$$Af(x) = \mathcal{I}_{|z|\leqslant 1}(x) f(x)$$

In the case d), we have the operator

$$Bf(x) = \int_{|y-x|\leqslant p^k} \mathcal{I}_{|y|\leqslant 1} f(y)\,dy, \qquad \text{where } k>0$$

In both cases the boundedness is obvious. This completes a proof of a).

The same examination of canonical forms proves b).

The statement c) follows automatically from the defining equation (2.12), see the proof of Theorem 9.3.3. □

2.8. Central extension.

Theorem 2.12 *The operators of the Weil representation of* $\mathrm{Sp}(2n, \mathbb{Q}_p)$ *admit a normalization such that*

$$\mathrm{We}(g_1)\,\mathrm{We}(g_2) = \pm\,\mathrm{We}(g_1 g_2)$$

See, [178], [96].

On another hand, for an arbitrary morphism $L : \mathcal{V}_{2n} \rightrightarrows \mathcal{V}_{2m}$, we can normalize the operators $\mathrm{We}(L)$ by

$$\langle \mathrm{We}(L)\mathcal{I}_{\mathbb{O}_p^n}, \mathcal{I}_{\mathbb{O}_p^m}\rangle = 1$$

Then in the formula $\mathrm{We}(L)\,\mathrm{We}(M) = \lambda(L,M)\,\mathrm{We}(LM)$, the factor $\lambda(L,M)$ is canonically defined. This cocycle was investigated in [112].

11.3 Adeles

3.1. Adeles. An *adele* is a sequence

$$a = (a_\infty, a_2, a_3, a_5, a_7, a_{11}, \dots)$$

where $a_\infty \in \mathbb{R}$, $a_p \in \mathbb{Q}_p$ for prime p, and $|a_p|_p \leqslant 1$ for all but finite number of p.

We prefer to omit the term a_∞. Our basic object is the space $\mathbb{A}$ of *finite adeles* (below we refer to them as *adeles*). A (finite) adele

$$a = (a_2, a_3, a_5, a_7, a_{11}, \dots)$$

where $a_p \in \mathbb{Q}_p$, and $|a_p|_p \leqslant 1$ for all but finite number of p.

The space $\mathbb{A}$ is a commutative ring with respect to the coordinate-wise addition and multiplication. We define also the ring of *integer adeles*

$$\mathbb{O}\mathbb{A} = \prod_{p \text{ is prime}} \mathbb{O}_p \subset \mathbb{A}$$

3.2. Adelic convergence. A sequence $a^{(j)} \in \mathbb{A}$ converges to a if

1) there is a finite set S of primes such that $|a_p^{(j)}|_p \leqslant 1$ for all the $p \notin S$.

2) For each p, the sequence $a_p^{(j)}$ converges to a_p.

We also say that a *neighborhood* of a point $a = \{a_p\} \in \mathbb{A}$ consists of all $b \in \mathbb{A}$ such that $|a_p - b_p|_p \leqslant p^{-j(p)}$, where $j(p) = 0$ for all prime p except a finite set.

REMARK. For the ring $\mathbb{O}\mathbb{A}$ of integer adeles we get the usual Tikhonov product-topology. We also obtain the product-topology on each subring

$$\prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{O}_p \subset \mathbb{A} \qquad \text{where a set } S \text{ is finite}$$

However, *the topology on the whole space $\mathbb{A}$ is not a product-topology.* □

3.3. The image of $\mathbb{Q}$ in $\mathbb{A}$. Each rational number $r \in \mathbb{Q}$ can be regarded as an element of $\mathbb{Q}_p$. Therefore we get the diagonal embedding $\mathbb{Q} \mapsto \mathbb{A}$,

$$r \mapsto \mathbf{r} := (r, r, r, \dots)$$

Proposition 3.1 *The map $r \mapsto \mathbf{r}$ induces an isomorphism of the quotient groups $\mathbb{Q}/\mathbb{Z} \simeq \mathbb{A}/\mathbb{O}\mathbb{A}$.*

PROOF. Each rational number r admits a unique representation as a *finite* sum

$$r = n + \sum_{p \text{ is prime}} \frac{m_p}{p^{N_p}}, \qquad \text{where } n \text{ and } m_p \text{ are integer and } 0 \leqslant m_p < p^{N_p}$$

The image of r in $\mathbb{Q}_p/\mathbb{O}_p$ is $p^{-N_p} m_p$, since all the other summands are p-adic integers. Therefore we get the isomorphism

$$\mathbb{Q}/\mathbb{Z} \to \bigoplus \mathbb{Z}_{p^\infty} = \mathbb{A}/\mathbb{O}_p\mathbb{A}$$

(see (1.1). □

Proposition 3.2 *a) The image of $\mathbb{Q}$ is dense in $\mathbb{A}$.*

b) The image of $\mathbb{Z}$ is dense in $\mathbb{O}\mathbb{A}$.

PROOF. b) We want to approximate an element $a = (a_2, a_3, \dots) \in \mathbb{O}\mathbb{A}$ by a sequence $n^{(j)} \in \mathbb{Z}$. Consider $n^{(1)} \in \mathbb{Z}$ such that $|a_2 - n^{(1)}|_2 \leqslant 1/2$. Next, consider integer α, $\beta \in \mathbb{Z}$ such that

$$|\alpha - a_2|_2 \leqslant 1/4, \qquad |\beta - a_3|_3 \leqslant 1/9$$

By the Chinese residue Theorem, there exists $n^{(2)}$ such that

$$n^{(2)} \equiv \alpha \mod 2^2, \qquad n^{(2)} \equiv \beta \mod 3^2$$

hence

$$|n^{(2)} - a_2|_2 \leqslant 1/4, \qquad |n^{(2)} - a_3|_3 \leqslant 1/9$$

By the same arguments, there is $n^{(3)}$ such that

$$|n^{(3)} - a_2|_2 \leqslant 2^{-3}, \qquad |n^{(3)} - a_3|_3 \leqslant 3^{-3}, \qquad |n^{(3)} - a_5|_5 \leqslant 5^{-3}$$

etc. In this way, we obtain a sequence $n^{(j)}$ convergent to a.

a) An element $b \in \mathbb{A}$ has a form a/l, where $a \in \mathbb{OA}$ and $l \in \mathbb{Z}$. If $n^{(j)}$ converges to a, then $n^{(j)}/l$ converges to b. $\square$

Theorem 3.3 *Consider the map $\mathbb{Q} \to \mathbb{R} \times \mathbb{A}$ given by $r \mapsto (r, r, r, \dots)$.*

a) Its image is a discrete set.

b) Moreover, $(\mathbb{R} \times \mathbb{A})/\mathbb{Q}$ is compact.

Proof. a) Let a non-eventually constant sequence $r_j = u_j/v_j \in \mathbb{Q}$ converge in $\mathbb{R}$. Then its denominators $v_j = \prod p^{N_p(j)}$ tend to ∞. Hence either the number of distinct prime factors in all u_j is infinite or for some p the sequence $N_p(j)$ is unbounded. In both cases, there is no adelic convergence.

b) Let us show that each coset of $(\mathbb{R}\times\mathbb{A})/\mathbb{Q}$ contains an element of $[0,1]\times\mathbb{OA}$. Indeed, let $\overline{a} = (a_\infty, a_2, a_3, a_5, \dots) \in \mathbb{R} \times \mathbb{A}$. Let S be the set of all primes, for which $|a_p| > 1$. For each $p \in S$ consider the fraction $b_p = m_p/p^{N_p} \in \mathbb{Q}$ such that $|a_p - b_p| \leqslant 1$. Consider $r := \sum b_p$. Then all the $\mathbb{Q}_p$-entries of $\overline{a} - \mathbf{r}$ are p-adic integer. We correct $r \to r + n$ by $n \in \mathbb{Z}$ and put the $\mathbb{R}$-entry into $[0,1)$.

However, the space $[0,1]\times\mathbb{OA}$ is compact, therefore every sequence contains a convergent subsequence. $\square$

3.4. Rational lattice. Consider the linear space $\mathbb{Q}^n$ over $\mathbb{Q}$. Recall that a *lattice* in $\mathbb{Q}^n$ is an additive subgroup of the form $\mathbb{Z}f_1 \oplus \dots \oplus \mathbb{Z}f_n$, where f_j is a basis in $\mathbb{Q}^n$.

Observation 3.4 *Let L, M be two lattices in $\mathbb{Q}^n$. Then there is a basis v_j in $\mathbb{Q}^n$ such that $L = \oplus\mathbb{Q}v_j$, $M = \oplus r_j\mathbb{Q}v_j$ with $r_j \in \mathbb{Q}$.*

Indeed, choosing an appropriate $a \in \mathbb{Z}$, we can achieve $aM \subset L$. Now the question is reduced to canonical forms of subgroups in finitely generated Abelian groups (see textbooks on algebra).

3.5. Adelic lattices. Denote by $\mathbb{A}^n$ the product of n copies of $\mathbb{A}$ endowed with a product topology. This space is a module over $\mathbb{A}$, the multiplication of a vector by a scalar is a continuous operation.

We say that a *lattice* in $\mathbb{A}^n$ is a set

$$L_2 \times L_3 \times L_5 \times \dots$$

where L_p is a lattice in $\mathbb{Q}_p^n$ and $L_p = \mathbb{O}_p^n$ for all but finite number of p.

We denote the set of all lattices in $\mathbb{A}^n$, $\mathbb{Q}^n$, $\mathbb{Q}_p^n$ by

$$\mathrm{Lat}_n(\mathbb{A}), \qquad \mathrm{Lat}_n(\mathbb{Q}), \qquad \mathrm{Lat}_n(\mathbb{Q}_p)$$

The group $\mathrm{GL}(n,\mathbb{Q})$ acts on the spaces $\mathrm{Lat}_n(\mathbb{Q})$. Evidently,

$$\mathrm{Lat}_n(\mathbb{Q}) = \mathrm{GL}(n,\mathbb{Q})/\mathrm{GL}(n,\mathbb{Z})$$

where $\mathrm{GL}(n,\mathbb{Z}) = \mathrm{SL}^{\pm}(n,\mathbb{Z})$ is the group of matrices with integer elements, whose inverse matrix also have integer elements, the determinant of such matrix is ± 1. We claim that this space coincides with

$$\mathrm{Lat}_n(\mathbb{A}) = \mathrm{GL}(n,\mathbb{A})/\mathrm{GL}(n,\mathbb{OA})$$

Theorem 3.5 *a) For a lattice $L \subset \mathbb{Q}^n$, its closure $\overline{L} \subset \mathbb{A}^n$ is a lattice in $\mathbb{A}^n$.*

b) The map $L \mapsto \overline{L}$ is a bijection $\mathrm{Lat}_n(\mathbb{Q}) \to \mathrm{Lat}_n(\mathbb{A})$.

c) For two lattices $L \supset M$ in $\mathbb{Q}^n$, the quotient groups L/M and $\overline{L}/\overline{M}$ are isomorphic.

3.6. p-adic closures of rational lattices. Denote by $\mathbb{Z}(p^{-1})$ the ring of p-rational numbers, i.e., the numbers u/p^N, where $u \in \mathbb{Z}$. By $\mathrm{GL}(n,\mathbb{Z}(p^{-1}))$ we denote the group of matrices g with p-rational entries such that $\det g = p^k$. In other words, g and g^{-1} have p-rational entries.

A lattice in $\mathbb{Q}^n$ is *p-rational* if it is contained in $\left[\mathbb{Z}(p^{-1})\right]^n$ and its covolume[2] has the form p^k. By $\mathrm{Lat}_n(\mathbb{Z}(p^{-1}))$ we denote the set of all p-rational lattices,

$$\mathrm{Lat}_n(\mathbb{Z}(p^{-1})) = \mathrm{GL}(n,\mathbb{Z}(p^{-1}))/\mathrm{GL}(n,\mathbb{Z})$$

For a p-rational lattice L, consider its closure $\overline{L} \subset \mathbb{Q}_p^n$.

Theorem 3.6 *The map $L \mapsto \overline{L}$ is a bijection* $\mathrm{Lat}_n(\mathbb{Z}(p^{-1})) \to \mathrm{Lat}_n(\mathbb{Q}_p)$

PROOF. If $g \in \mathrm{GL}(n,\mathbb{Z}(p^{-1}))$ is not contained in $\mathrm{GL}(n,\mathbb{Z})$, then $g \notin \mathrm{GL}(n,\mathbb{O}_p)$. Hence

$$\overline{g\mathbb{Z}^n} = g\overline{\mathbb{Z}^n} = g\mathbb{O}_p^n \neq \mathbb{O}_p^n = \overline{\mathbb{Z}^n}$$

Since our map is $\mathrm{GL}(n,\mathbb{Z}(p^{-1}))$-equivariant, it is injective.

Next, let us show that each p-adic lattice K is a closure of a p-rational lattice. Choose a basis $v_1,\ldots,v_n$ in K. Take α such that $p^\alpha\mathbb{O}_p \subset K$. We choose p-rational vectors w_j such that

$$w_j = v_j \mod p^{\alpha+1}\mathbb{O}_p^n$$

Then w_j form a basis in K, see Theorem 10.3.5. On another hand, p-rational vectors $w_j \in \mathbb{Q}^n$ generate a lattice L in $\mathbb{Q}^n$; by construction, $\overline{L} = K$.

[2]A covolume of a lattice $L \subset \mathbb{R}^n$ is the volume of the quotient $\mathbb{R}^n/L$. If $v_j = \sum s_{ij}e_j$ is a basis of L, then the covolume is $\det\{s_{ij}\}$

However, the covolume of L is not necessary p^k. Consider a basis $h_n \subset \mathbb{Z}^n$ such that $L = \oplus \frac{a_j}{b_j}\mathbb{Z}h_j$. We write this in the form

$$L = \oplus \frac{a'_j}{b'_j} p^{s_j}\mathbb{Z}h_j, \qquad \text{where } a'_j,\ b'_j \text{ are relatively prime with } p$$

Consider the p-rational lattice $L' := \oplus p^{s_j} h_j$. Since $a'_j/b'_j \in \mathbb{O}_p \setminus p\mathbb{O}_p$, the closures of L and L' coincide. □

3.7. Proof of Theorem 3.5. a) By Proposition 3.2, the closure of $\mathbb{Z}^n$ is $\mathbb{O}\mathbb{A}^n$. For $g \in \mathrm{GL}(n, \mathbb{Q})$, we have $\overline{g\mathbb{Z}^n} = g\overline{\mathbb{Z}^n}$, and this proves a).

b) The injectivety follows from the same argument as in Theorem 3.6.

Let us prove the surjectivity. Consider a lattice $K \subset \mathbb{A}^n$, i.e., a collection of lattices $K_p \subset \mathbb{Q}_p^n$ such that $L_p = \mathbb{O}_p$ for all p but a finite set S. Take $p \in S$, and choose matrix $g \in \mathrm{GL}(n, \mathbb{Z}(p^{-1})$ such that $g\mathbb{O}_p = K_p$. Consider the lattice $g^{-1}K$. For all primes $q \neq p$ we have $g \in \mathrm{GL}(n, \mathbb{O}_q)$, therefore g^{-1} fixes $\mathbb{O}_q^n$. Thus the new set S for $g^{-1}K$ is smaller than for K.

We repeat the same trick for 'next' p. Etc. □

c) The statement is obvious if $n = 1$. For general case we refer to Observation 3.4.

11.4 Adelic Weil representation

4.1. Adelic measure. We define the measure da on $\mathbb{A}^n$ by the assumptions:

1. On the space $\mathbb{O}\mathbb{A}^n \simeq \prod_p \mathbb{O}_p^n$, our measure is the product measure. In particular, we require $\mu(\mathbb{O}\mathbb{A}^n) = 1$.

2. The measure is translation-invariant.

Recall that $\mathbb{A}^n$ is a union of a countable collection of disjoint copies of $\mathbb{O}\mathbb{A}^n$.

4.2. Tensor products of Hilbert spaces. Recall a definition of the *tensor product* of a countable family of Hilbert spaces. Let H_j be Hilbert spaces, $\xi^{(j)} \in H_j$ be distinguished unit vectors. Choose an orthonormal basis $e_1^{(j)} = \xi_{(j)}$, $e_2^{(j)}$, ..., in each H_j. Consider the Hilbert space

$$\bigotimes (H_j, \xi^{(j)})$$

with an orthonormal basis

$$e_{k_1}^{(1)} \otimes e_{k_2}^{(2)} \otimes e_{k_3}^{(3)} \otimes \dots$$

where $e_{k_j}^{(j)} = \xi^{(j)}$ for all but j finite number of j[3].

[3] This is a crucial phrase. If we omit it, then we get a continual set of basic vectors and (semi)-pathological object. We also emphasis the importance of ξ_j in this definition. However, the construction is independent on the choice of a basis.

Let $v_{(j)} \in H_j$ be vectors satisfying

$$\sum \bigl|\langle v^{(j)}, v^{(j)}\rangle_{H_j} - 1\bigr| < \infty, \qquad \sum \bigl|\langle v^{(j)}, \xi^{(j)}\rangle_{H_j} - 1\bigr| < \infty \tag{4.1}$$

We define $\otimes v^{(j)} \in \otimes H_j$ in the evident way (however, the conditions (4.1) are necessary for such definition).

Consider a family of bounded operators $A_j : H_j \to H_j$ such that $\prod \|A_j\|$ converges (i.e., the series $\sum \bigl| \|A\| - 1\bigr|$ converges) and $v^{(j)} := A_j \xi^{(j)}$ satisfy the condition (4.1). Then we define the tensor product of operators

$$\otimes A_j : \otimes\bigl(H_j, \xi^{(j)}\bigr) \to \otimes\bigl(H_j, \xi^{(j)}\bigr)$$

by

$$\otimes A_j \cdot \bigl(\otimes v^{(j)}\bigr) = \otimes\bigl(A_j v^{(j)}\bigr)$$

4.3. Adelic L^2. We define the space $L^2(\mathbb{A}^n)$ in the usual way. Evidently,

$$L^2(\mathbb{A}^n) = \bigotimes \bigl(L^2(\mathbb{Q}_p^n), \mathcal{I}_{\mathbb{O}_p^n}\bigr)$$

4.4. Test functions and distributions. A *test function* on $\mathbb{A}^n$ is a finite linear combination of functions $f(x) = \mathcal{I}_L(x + a)$, where L is a lattice and $a \in \mathbb{A}^n$. We denote $\mathcal{B}(\mathbb{A}^n)$ the space of all the test functions.

Proposition 4.1 *a) A function f is a test function iff it is a compactly supported locally constant function.*

b) For lattices $M \subset L \subset \mathbb{A}^n$, we denote by $\mathcal{B}(L|M)$ the set of all M-invariant functions supported by L. Then

$$\mathcal{B}(\mathbb{A}_n) = \bigcup_{M \subset L \subset \mathbb{A}^n} \mathcal{B}(L|M)$$

4.5. The Fourier transform. Define the adelic exponent by

$$\exp\{2\pi i a\} = \prod_p \exp\{2\pi i a_p\}$$

(all the factors but a finite number are 1).

Problem 4.1 The group of characters of the additive group $\mathbb{Q}$ coinsides with the group $(\mathbb{R} \times \mathbb{A})/\mathbb{Q}$ mentioned in Theorem 3.3. For $(a_\infty, a_2, a_3, \dots) \in \mathbb{R} \times \mathbb{A}$, we assume

$$\chi(r) = \exp\bigl\{-2\pi i a_\infty r\bigr\} \cdot \prod_p \exp\bigl\{2\pi i a_p r\bigr\}, \qquad r \in \mathbb{Q}$$

The *Fourier transform* is given by the usual formula

$$\mathcal{F} f(x) = \int_{\mathbb{A}^n} \exp\{2\pi i x y\}\, f(y)\, dy$$

It has the usual properties.

Proposition 4.2 *a) $\mathcal{F}$ is a bijection $\mathcal{B}(\mathbb{A}^n) \to \mathcal{B}(\mathbb{A}^n)$.*

b) $\mathcal{F}$ is a unitary operator $L^2(\mathbb{A}^n) \to L^2(\mathbb{A}^n)$.

In fact, $\mathcal{F}$ is the tensor product of p-adic Fourier transforms, hence there is nothing to prove. □

4.6. Adelic Weil representation. We define the adelic symplectic group $\mathrm{Sp}(2n, \mathbb{A})$ as the group of sequences

$$g = (g_2, g_3, g_5, \dots)$$

where $g_p \in \mathrm{Sp}(2n, \mathbb{Q}_p)$ for all $g_p \in \mathrm{Sp}(2n, \mathbb{O}_p)$ but a finite number of p. The product is component-wise.

Equivalently, $\mathrm{Sp}(2n, \mathbb{A})$ is the group of symplectic matrices over the ring $\mathbb{A}$.

For each p, we normalize the operators $\mathrm{We}(g)$, where $g \in \mathrm{Sp}(2n, \mathbb{O}_p)$, by the condition (see Problem 2.2),

$$\mathrm{We}(g)\mathcal{I}_{\mathbb{O}_p^n} = \mathcal{I}_{\mathbb{O}_p^n}$$

Now, we can define the Weil representation of $\mathrm{Sp}(2n, \mathbb{A})$ as

$$\mathrm{We}(g) = \bigotimes_p \mathrm{We}(g_p)$$

11.5 Group $\mathrm{Sp}(2n, \mathbb{Q})$ and the real-adelic correspondence

5.1. The space $\mathcal{M}(\mathbb{Q}^n)$. Let $K \subset L \subset \mathbb{Q}^n$ be lattices. Denote by $\mathcal{M}(L|K) = \mathcal{M}(L|K, \mathbb{Q})$ the space of functions on $\mathbb{Q}^n$ that are K-invariant and supported by L. Evidently, $\dim \mathcal{M}(L|K)$ equals the order of the group L/K.

Next, we define the space

$$\mathcal{M}(\mathbb{Q}^n) := \bigcup_{K \subset L \subset \mathbb{Q}^n} \mathcal{M}(L|K, \mathbb{Q})$$

Remark. The space $\mathcal{M}(\mathbb{Q}^n)$ is generated by functions $\varphi(x) = \mathcal{I}_K(x + a)$, where $K \subset \mathbb{Q}^n$ is a lattice, $a \in \mathbb{Q}^n$. □

We define the *rational Heisenberg group* $\mathrm{Heis}_n(\mathbb{Q})$ as the group of matrices

$$R\big[v_+ \oplus v_-; \alpha\big] = R\big[v; \alpha\big] = \begin{pmatrix} 1 & v_+ & \alpha + \frac{1}{2}v_+v_-^t \\ 0 & 1 & v_-^t \\ 0 & 0 & 1 \end{pmatrix} \tag{5.1}$$

where v_+, $v_- \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. As above, we define the operators

$$U\big[v_+ \oplus v_-; \alpha\big] f(x) = f(x + v_+) \exp\{2\pi i (xv^+ + \alpha + \frac{1}{2}v_+v_-^t)\}$$

Proposition 5.1 *a) The operators* $U\big[v_+ \oplus v_-;\alpha\big]$ *preserve the space* $\mathcal{M}(\mathbb{Q}^n)$

b) The representation of $\mathrm{Heis}_n(\mathbb{Q})$ *in* $\mathcal{M}(\mathbb{Q}^n)$ *is irreducible.*

c) Each linear operator $S : \mathcal{M}(\mathbb{Q}^n) \to \mathcal{M}(\mathbb{Q}^n)$ *commuting with all the operators* $U\big[v_+ \oplus v_-;\alpha\big]$ *is a scalar operator*

PROOF. THE STATEMENT A. Obvious. □

For a lattice K, denote by $L^\diamond$ its dual lattice[4]. The following statement is also obvious.

Lemma 5.2 *a) The space* $\mathcal{M}(L|K)$ *is precisely the space of functions that are fixed by all the operators*

$$A(w)f(x) = f(x+w),\ w \in K,$$
$$B(r)f(x) = f(x)\exp\{2\pi i x r^t\},\ r \in L^\diamond$$

b)The space $\mathcal{M}(L|K)$ *is invariant with respect to the operators*

$$A(w)f(x) = f(x+w),\ w \in L, \qquad \textit{and } B(r)f(x) = f(x)\exp\{2\pi i x r^t\},\ r \in K^\diamond$$

Therefore, for $\widetilde{w} \in L/K$, we have well-defined operators $A(\widetilde{w})$ in $\mathcal{M}(L|K)$; for $r \in K^\diamond/L^\diamond$ we have well-defined operators $B(\widetilde{r})$. The operators $A(\widetilde{w})$, $B(\widetilde{r})$ generate a certain finite group $\mathrm{Heis}(L|K)$ of transformations of $\mathcal{M}(L|K)$.

Lemma 5.3 *a) The representation of* $\mathrm{Heis}(L|K)$ *in* $\mathcal{M}(L|K)$ *is irreducible.*

b) There is no proper $\mathrm{Heis}(L|K)$*-invariant subspace in* $\mathcal{M}(L|K)$.

This is a simplified variant of Theorem 1.2.2. We leave both statement as exercises. □

PROOF OF PROPOSITION 5.1.B. Consider two chains of lattices

$$L_1 \subset L_2 \subset \dots, \qquad K_1 \supset K_2 \supset \dots$$

such that $\cup\mathcal{M}(L_j|K_j) = \mathcal{M}(\mathbb{Q}^n)$. By definition,

$$\mathcal{M}(L_{j+1}|K_{j+1}) \supset \mathcal{M}(L_j|K_j)$$

By virtue of irreducibility, if $V \subset \mathcal{M}(\mathbb{Q}^n)$ is a Heis_n-invariant subspace, then each intersection $V \cap \mathcal{M}(L|K)$ is 0 or the whole space $\mathcal{M}(L|K)$. This implies $V = 0$ or $\mathcal{M}(\mathbb{Q}^n)$. □

PROOF OF PROPOSITION 5.1.C. Let $f \in \mathcal{M}(L|K)$. Then for $w \in K, r \in L^\diamond$,

$$A(w)Sf = HA(w)f = Sf, \qquad B(r)Sf = HB(r)f = Sf$$

Therefore S takes $\mathcal{M}(L|K)$ to itself. By the last lemma, S is a scalar operator on $\mathcal{M}(L|K)$. □

5.2. Adelization of $\mathcal{M}(\mathbb{Q}^n)$.

[4]The lattice $L^\diamond$ consists of $y \in \mathbb{Q}^n$ such that $wy^t \in \mathbb{Z}$ for each $w \in L$.

Proposition 5.4 *a) Each function $f \in \mathcal{M}(\mathbb{Q}^n)$ admits a unique continuous extension $\overline{f}$ to a function defined on the space $\mathbb{A}^n$.*

b) The map $f \mapsto \overline{f}$ is a bijection $\mathcal{M}(\mathbb{Q}^n) \to \mathcal{B}(\mathbb{A}^n)$. This map also induces a bijection $\mathcal{M}(L|K) \to \mathcal{B}(\overline{L}|\overline{K})$).

c) The map $f \mapsto \overline{f}$ commutes with the action of the group $\mathrm{Heis}_n(\mathbb{Q})$.

PROOF. Let $K \subset \mathbb{Q}^n$ be a lattice, $a \in \mathbb{Q}^n$; let $\overline{K}$ be the adelic closure of K. Then the continuous extension of $\mathcal{I}_{K+a}$ is $\mathcal{I}_{\overline{K}+a}$.

All functions $f \in \mathcal{B}(\mathbb{Q}^n)$ are linear combinations of such functions. $\square$

5.3. Realification of $\mathcal{M}$. For a function $f \in \mathcal{M}(\mathbb{Q}^n)$ consider the distribution on $\mathbb{R}^n$ given by

$$\sum_{h\in\mathbb{Q}^n} f(h)\delta(x-\sqrt{2\pi}h)$$

We denote the linear space of all such distributions by $\mathcal{P}(\mathbb{R}^n)$.

The space $\mathcal{P}(\mathbb{R}^n)$ is spanned by the functions

$$\sum_{k_1,\dots,k_n\in\mathbb{Z}} \delta\Big(x-\sqrt{2\pi}(\sum k_j a_j - b)\Big)$$

where $a_1,\dots,a_n$ is a basis in $\mathbb{Q}^n$, $b \in \mathbb{Q}^n$.

Consider the standard action of the Heisenberg group $\mathrm{Heis}_n(\mathbb{R})$ in $\mathcal{S}(\mathbb{R}^n)$, but we rather modify it. For a matrix (5.1) define the operators.

$$U\big[v_+ \oplus v_-;\alpha\big]f(x) = f(x+\sqrt{2\pi}v_+)\exp\big\{\sqrt{2\pi}ixv^+ + 2\pi i(\alpha+\frac{1}{2}v_+v_-^t)\big\}$$

Recall that the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ acts in $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ via the Weyl representation.

Theorem 5.5 *a) The space $\mathcal{P}(\mathbb{R}^n)$ is invariant under the action of the subgroup* $\mathrm{Heis}_n(\mathbb{Q}) \subset \mathrm{Heis}_n(\mathbb{R})$.

b) The space $\mathcal{P}(\mathbb{R}^n)$ is invariant under the action of the subgroup $\mathrm{Sp}(2n,\mathbb{Q}) \subset \mathrm{Sp}(2n,\mathbb{R})$.

PROOF. a) To simplify notation, consider the case $n=1$. Obviously, $\mathcal{P}(\mathbb{R}^n)$ is invariant with respect to shifts by rational vectors.

Next, consider

$$\exp(\sqrt{2\pi}iwx)\sum_{k\in\mathbb{Z}}\delta\big(x-\sqrt{2\pi}(ka-c)\big) = \\ = \exp\{2\pi iwc\}\sum_{k\in\mathbb{Z}}\exp\{2\pi ikaw\}\delta\big(x-\sqrt{2\pi}(ka-c)\big) \quad (5.2)$$

Since wa is rational, the factor $\exp\{2\pi ikaw\}$ is periodic in k. Now the statement becomes obvious.

b) In Subsection 1.2.7, we discussed the generators of $\mathrm{Sp}(2n,\mathbb{R})$, see (1.2.12). Matrices of the same types with rational entries generate $\mathrm{Sp}(2n,\mathbb{Q})$. Thus, we must verify the statement for the operators (1.2.14)–(1.2.16) with rational matrices A, B.

1) For the operators,

$$f(x)\mapsto f(xA), \qquad A\in \mathrm{GL}(n,\mathbb{Q})$$

the statement is obvious.

2) The Fourier transform. We refer to the Poisson summation formula, see above Corollary 8.2.3.

3) Examine the operators

$$f(x)\mapsto \exp\bigl\{\frac{i}{2}xCx^t\bigr\}f(x)$$

To simplify notation, assume $n=1$,

$$\exp\bigl\{\frac{i}{2}cx^2\bigr\}\sum_{k\in\mathbb{Z}}\delta\bigl(x-\sqrt{2\pi}(ka-b)\bigr)= \\ =\exp(\pi icb^2)\sum_k \exp\{\pi i(k^2\,ca^2+2k\,bac)\}\delta\bigl(x-\sqrt{2\pi}(ka-b)\bigr)$$

The factor $\exp\{\pi i(k^2\,ca^2+2k\,bac)\}$ is periodic in k. This completes a proof. □

5.4. Canonical embedding of the Schwartz space to the space of adelic distributions. We have the canonical bijections

$$\mathcal{P}(\mathbb{R}^n)\longleftrightarrow \mathcal{M}(\mathbb{Q}^n)\longleftrightarrow \mathcal{B}(\mathbb{A}^n)$$

Theorem 5.6 *The canonical map* $\mathcal{P}(\mathbb{R}^n)\to\mathcal{B}(\mathbb{A}^n)$

a) commutes with the action of the Heisenberg group $\mathrm{Heis}_n(\mathbb{Q})$.

b) commutes with the action of $\mathrm{Sp}(2n,\mathbb{Q})$

PROOF. The first statement is evident.

We have two representations of $\mathrm{Sp}(2n,\mathbb{Q})$ in $\mathcal{M}(\mathbb{Q}^n)$, one is transferred from $\mathcal{P}(\mathbb{R}^n)$, the second one is transferred from $\mathcal{B}(\mathbb{A}^n)$. They satisfy the same commutation relations

$$U\bigl[vg;\alpha\bigr]=\mathrm{We}_1(g)^{-1}U\bigl[v;\alpha\bigr]\,\mathrm{We}_1(g), \qquad U\bigl[vg;\alpha\bigr]=\mathrm{We}_2(g)^{-1}U\bigl[v;\alpha\bigr]\,\mathrm{We}_2(g)$$

Hence

$$\mathrm{We}_2(g)^{-1}\,\mathrm{We}_1(g)$$

is an operator in $\mathcal{M}(\mathbb{Q}^n)$ commuting with the Heisenberg group $\mathrm{Heis}_n(\mathbb{Q})$. By Proposition 5.1, this operator is scalar. □

By duality, we get $\mathrm{Sp}(2n,\mathbb{Q})$-equivariant embeddings

$$\mathcal{S}(\mathbb{R}^n)\mapsto\mathcal{B}'(\mathbb{A}^n), \qquad \mathcal{B}(\mathbb{A}^n)\mapsto\mathcal{S}'(\mathbb{R}^n)$$

5.5. Weyl distribution.

Theorem 5.7 *For $f \in \mathcal{S}(\mathbb{R}^n)$, the corresponding distribution on $\mathbb{A}^n$ is given by*

$$\sum_{r \in \mathbb{Q}^n} f(r)\delta(a - \mathbf{r})$$

where $\mathbf{r} := (r, r, \dots) \in \mathbb{A}^n$ and δ is a delta-function on adeles.

We leave this statement as an exercise.

We define a '*test functions*' on $\mathbb{R} \times \mathbb{A}$ as a function of the form

$$\varphi(x, a) = \sum_{j=1}^{N} f_j(x)\mathcal{I}_{L^{(j)}+b_j}(a),$$

where $f_j \in \mathcal{S}(\mathbb{R}^n)$ and $L^{(j)}$ are lattices.

The *Weil distribution* $\mathfrak{W}$ on $\mathbb{R} \times \mathbb{A}$ is given by

$$\ll \mathfrak{W}, \varphi \gg = \sum_{p/q \in \mathbb{Q}} \varphi(r, \mathbf{r})$$

Then our map is nothing but the map $\mathcal{S}(\mathbb{R}^n) \to \mathcal{B}'(\mathbb{A}^n)$, whose kernel is the Weil distribution.

5.6. p-adic variant of the same construction. Return to the situation discussed in Subsection 3.6. Fix prime p. Denote

$$\mathcal{M}_p(\mathbb{Q}^n) = \bigcup_{K \subset L \text{ are } p\text{-rational lattices}} \mathcal{M}(L|K)$$

Denote by $\mathcal{P}_p(\mathbb{R}^n)$ the corresponding subspace in $\mathcal{P}(\mathbb{R}^n)$.

On another hand, each function $f \in \mathcal{M}(\mathbb{Q}^n)$ admits a unique continuous extension to $\mathbb{Q}_p^n$ and we obtain the bijection $\mathcal{M}_p(\mathbb{Q}_n)$ with the Bruhat space $\mathcal{B}(\mathbb{Q}_p^n)$.

Observation 5.8 *We obtain an embedding $\mathcal{S}(\mathbb{R}^n) \to \mathcal{B}'(\mathbb{Q}_p^n)$ which commutes with actions of $\mathrm{Heis}_n\big(\mathbb{Z}(p^{-1})\big)$ and $\mathrm{Sp}\big(2n, \mathbb{Z}(p^{-1})\big)$.*

11.6 Constructions of modular forms

In this section we explain the construction of modular forms through theta-functions.

6.1. Congruence subgroups. Fix an integer positive N. A *principal congruence subgroup* $\Gamma_N \subset \mathrm{Sp}(2n, \mathbb{Z})$ consists of matrices g such that N divides all the entries $g_{ij} - \delta_{ij}$.

A *congruence subgroup* in $\mathrm{Sp}(2n, \mathbb{Z})$ is a subgroup containing some Γ_N.

Denote by Γ_l° the subgroup in $\mathrm{Sp}(2n, \mathbb{R})$ generated by matrices

$$\begin{pmatrix} a & 0 \\ 0 & a^{t-1} \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \tag{6.1}$$

where l divides all the entries of matrices $a - 1$, b, c.

We give the following theorem (see [172]) without a proof.

Theorem 6.1 *The group Γ_l° is a congruence subgroup.*

6.2. Definition of modular forms. Denote by P_n the Siegel wedge of symmetric matrices with positive imaginary part. Let $n > 1$. A holomorphic function on P_n is a *modular function* of a weight k with respect to a congruence subgroup Γ if

$$f\big((a+zc)^{-1}(b+zd)\big)\det(a+zc)^{-k} = f(z) \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

Actually, we consider the representations T_k of $\mathrm{Sp}(2n,\mathbb{R})$ in the space $\mathrm{Hol}(\mathrm{P}_k)$ of holomorphic functions on P_k examined in Chapter 7; *modular functions* are functions fixed by a congruence subgroup.

REMARK. We emphasis that these functions are not elements of the Hilbert spaces $\mathcal{H}_k$ examined in Chapter 7. □

If $n \geqslant 2$, then a *modular form* is a modular function.

However, for $n = 1$ (in this case, P_1 is the usual upper half-plane on $\mathbb{C}$) some extra conditions are necessary. A modular function f is a *modular form* if

a) f is bounded in some domain $\operatorname{Im} z > \sigma$

b) For each rational point $\xi \in \mathbb{R}$ there is $h > 0$ such that

$$|f(z)| \leqslant \mathrm{const}\, |z-\xi|^{-k} \qquad \text{for } |z - \xi + ih| < h \tag{6.2}$$

Let us try to explain the sence of these conditions. Our group Γ contains some subgroup of shifts $z \mapsto z + jl$, where j ranges in $\mathbb{Z}$ and l is a constant depending on Γ. Our function f is invariant under such shifts. The condition a) claims that f can be expanded in a Taylor series $f(u) = \sum_{k\geqslant 0} c_k u^k$ in the variable $u := \exp\{2\pi i z/l\}$.

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\mathrm{SL}(2,\mathbb{Z}) = \mathrm{Sp}(2,\mathbb{Z})$ taking r to ∞. Then condition b) is condition a) written for the transformed function.

If $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, then the condition b) follows from a) by the last remark. Generally, the group Γ has several orbits on the set $\mathbb{Q} \cup \infty$, and for each orbit we have its own condition b).

For $n > 1$ there is the so-called Koecher effect: *a modular function is automatically bounded in each domain of the form $z_0 + \mathrm{P}_n$, where z_0 is a fixed interior point of the Siegel wedge.*

6.3. Linearization of the Weil representation on $\Gamma_{1,2}$. The Igusa subgroup $\Gamma_{1,2}$ was defined above in 8.3.1–8.3.2. A proof of the following theorem is contained in [110].

Theorem 6.2 *The group $\Gamma_{1,2}$ is generated by elements*

$$\begin{pmatrix} a & 0 \\ 0 & a^{t-1} \end{pmatrix}, \qquad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \tag{6.3}$$

where matrices a, a^{t-1}, b, c are integer, b, c are symmetric and have even diagonals.

Denote

$$\Delta(x) = \sum_{k_1,\dots,k_n} \prod_j \delta(x - \sqrt{2\pi}k_j)$$

Proposition 6.3 *The restriction of the Weil representation of* $\mathrm{Sp}(2n,\mathbb{R})$ *to* $\Gamma_{1,2}$ *is equivalent to a linear representation. Moreover, we can choose the operators* $\mathrm{We}(g)$, $g \in \Gamma_{1,2}$ *such that*

$$\mathrm{We}(g)\Delta = \Delta \tag{6.4}$$

PROOF. First, change the collection of generators (6.3) by (1.2.12). By Theorem 8.2.1, Δ is invariant under the Fourier transform. Obviously, Δ is an eigenfunction of

$$\mathrm{We}\begin{pmatrix} a & 0 \\ 0 & a^{t-1} \end{pmatrix} f(x) = f(xa)|\det a|^{-1/2}$$

Next,

$$\exp\Big\{\frac{i}{2}xbx^t\Big\}\Delta(x) = \sum_{k\in\mathbb{Z}^n}\Big(\exp\Big\{\pi i \sum b_{\sigma,\tau}k_\sigma k_\tau\Big\}\prod_j \delta(x-\sqrt{2\pi}k_j)\Big)$$

Since the diagonal of b is even, the sum $\sum b_{\sigma\tau}k_\sigma k_\tau$ is even, hence $\exp\{\dots\} = 1$.

Since Δ is an eigenfunction for all the generators, it is also an eigenfunction for each $g \in \Gamma_{1,2}$, i.e.,

$$\mathrm{We}(g)\Delta = \lambda(g)\Delta, \qquad \lambda \in \mathbb{C}^*$$

We can replace $\mathrm{We}(g) \mapsto \lambda(g)^{-1}\,\mathrm{We}(g)$. Then we have (6.4), and the representation becomes linear. □

6.4. Congruence subgroups and the space $\mathcal{P}(\mathbb{R}^n)$. Now, we consider the linear action of $\Gamma_{1,2}$ in $\mathcal{P}(\mathbb{R}^n)$.

Theorem 6.4 *The stabilizer of each element of* $\mathcal{P}(\mathbb{R}^n)$ *is a congruence subgroup.*

PROOF. Fix an integer N. Consider the subspace $H_N \subset \mathcal{P}(\mathbb{R}^n)$ consisting of distributions supported by $N^{-1}\sqrt{2\pi}\mathbb{Z}^n$ and invariant under $N\sqrt{2\pi}\mathbb{Z}^m$. Evidently, $\cup H_N = \mathcal{P}(\mathbb{R}^n)$.

The theorem follows from the next lemma.

Lemma 6.5 *The space* $\mathcal{H}_N$ *is invariant with respect to the group* $\Gamma^\circ_{2N^2}$.

PROOF. The group $\Gamma^\circ_{2N^2}$ is generated by matrices of the form

$$S(\alpha) = \begin{pmatrix} 1+2N^2\alpha & 0 \\ 0 & (1+2N^2\alpha)^{-1}\end{pmatrix}, \quad T(\beta) = \begin{pmatrix} 1 & 2N^2\beta \\ 0 & 1\end{pmatrix},$$
$$R(\gamma) = \begin{pmatrix} 1 & 0 \\ 2N^2\gamma & 1\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} 1 & -2N^2\gamma \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}^{-1} \tag{6.5}$$

We use the formulae for generators given above.

Let $f \in H_N$.

1) First,

$$\mathrm{We}\big(S(\alpha)\big) f(x) = f(x + 2N^2 x\alpha) = \mathrm{const}$$

Since $x \in N^{-1}\mathbb{Z}^n$, we have $2N^2 x\alpha \in (2N)\mathbb{Z}^N$, hence $f(x + 2N^2 x\alpha) = f(x)$.

2) The function $\exp\{2\pi i N^2 x\beta x^t\}$ is constant on $N^{-1}\mathbb{Z}^n$, hence

$$\mathrm{We}(T(\beta)) f(x) = \exp\{2\pi i N^2 x\beta x^t\} f(x) = f(x)$$

3) The Fourier transform $\mathrm{We}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ leaves the space H_N invariant.

By (6.5), $\mathrm{We}(R(\gamma)) f = f$. □

6.5. Modular forms of the weight $1/2$**.** Now we apply the construction of Section 7.6. Namely, we consider the operator

$$J\chi(z) = \int_{x \in \mathbb{R}^n} \exp\{\frac{1}{2} x z x^t\} \chi(x)$$

from $\mathcal{S}'(\mathbb{R}^n)$ to the space $\mathrm{Hol}(\mathrm{P}_n)$.

Recall that the group $\mathrm{Sp}(2n, \mathbb{R})$ acts in $\mathrm{Hol}(\mathrm{P}_n)$ by transformations

$$T_{1/2}\begin{pmatrix} A & B \\ C & D \end{pmatrix} f(x) = f\big((A + zC)^{-1}(B + zD)\big) \det(A + zC)^{-1/2} \tag{6.6}$$

By Proposition 6.3, the action of the subgroup $\Gamma_{1,2} \subset \mathrm{Sp}(2n, \mathbb{R})$ is equivalent to a linear representation, we denote the corresponding operators by

$$T'_{1/2}(g) = \sigma(g) T_{1/2}(g)$$

(for explicit form of the factor $\sigma(g)$, see [110], it ranges in 8-th roots of unity).

The operator J intertwines actions of $\mathrm{Sp}(2n, \mathbb{R})$ in $\mathcal{S}'(\mathbb{R}^n)$ and $\mathrm{Hol}(\mathrm{P}_n)$. Therefore it takes distributions invariant under a congruence subgroup Γ to Γ-invariant holomorphic functions.

By the virtue of Theorem 6.4, we get the following statement

Theorem 6.6 *For each $\chi \in \mathcal{P}(\mathbb{R}^n)$, the function $J\chi$ is a modular form with respect to some congruence subgroup.*

Observation 6.7 *Jf is a θ-function in the sense of Section 8.4.*

6.6. Modular forms of arbitrary weights. The representation theory provides tools for making numerous copies of the previous construction. We consider only the simplest variant of this industry.

Define the embedding $\mathrm{Sp}(2n,\mathbb{R}) \to \mathrm{Sp}(2nm,\mathbb{R})$ given by

$$\pi_m : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & \dots & b & 0 & \dots \\ 0 & a & \dots & 0 & b & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ c & 0 & \dots & d & 0 & \dots \\ 0 & c & \dots & 0 & d & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{pmatrix}$$

Define also the embedding $\iota_m : \mathrm{P}_n \mapsto \mathrm{P}_{nm}$,

$$\iota_m : z \mapsto \begin{pmatrix} z & 0 & \dots \\ 0 & z & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Consider the action of $\mathrm{Sp}(2nm,\mathbb{R})$ on the Siegel wedge P_{nm} and the corresponding action $T_{1/2}$ in the space $\mathrm{Hol}(\mathrm{P}_{mn})$.

The image $\iota_m(\mathrm{P}_n)$ is a $\mathrm{Sp}(2n,\mathbb{R})$-invariant submanifold in P_{mn}; therefore we get an action of $\mathrm{Sp}(2n,\mathbb{R})$ in holomorphic functions on $\iota_m(\mathrm{P}_n) \simeq \mathrm{P}_n$.

It is easy to see that this action is given by the formula

$$T_{m/2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = f((a+zc)^{-1}(b+zd)) \det(a+zc)^{-m/2}$$

i.e., this is an action discussed in Chapter 7.

For the congruence subgroup $\Gamma_{1,2}$, there are the corresponding corrected operators

$$T'_{m/2}(g) = \sigma^m(g) T_{m/2}(g)$$

Thus we get a map

$$\mathcal{S}'(\mathbb{R}^{mn}) \to \text{holomorphic functions on } \mathrm{P}_n$$

given by

$$J_m\chi(z) = \int_{x\in\mathbb{R}^{mn}} \exp\{\tfrac{1}{2} x\, \iota_m(z) x^t\} \chi(x), \qquad z \in \mathrm{P}$$

Theorem 6.8 *For any $\chi \in \mathcal{P}(\mathbb{R}^{2mn})$, there is a congruence subgroup Γ such that*

$$T'(g)\Psi = \Psi$$

for all $g \in \Gamma$.

PROOF. Obvious. □

11.7 Bibliographical remarks to Chapters 10-11

7.1. Historical remarks. The p-adic numbers were discovered by K.Hensel in 1897. The classification of quadratic forms over $\mathbb{Q}$ was obtained by H.Minkowski, 1885. Its nice form (Hasse–Minkowski principle) was discovered by H.Hasse in 1920; this approach admits wide generalizations. Adeles were introduced by C.Chevalley. Investigation of unitary representations of p-adic groups was initiated at the end of 1950s by F.Mautner and F.Bruhat. The Bruhat–Tits buildings were invented by J.Tits and F.Bruhat in 1960 in series papers that look as hard for reading. The Weil representation of $\mathrm{Sp}(2n, \mathbb{Q}_p)$ was introduced by A.Weil [178].

7.2. p-adic numbers. For systematic exposition, see for instance, [20], [29], [162], [179].

In several topics, we avoided the case $p = 2$; from the point of view of the Hasse–Minkowski principle and the adelic ideology this is inadmissible. The case $p = 2$ is not difficult, however theorems and proofs became longer. See the books cited above, for Gaussian integrals with $p = 2$, see [174].

7.3. Buildings. See books M.A.Ronan [150], K.S.Brown [25], P.Garrett [48], see also surveys [155], [149], and original works of J.Tits and F.Buhat [26]. On buildings at infinity, see P.B.Eberlein [36]. A nice introduction to trees is contained in J.P.Serre [163] The triangle inequality for p-groups was found by Ph.Hall, see [102].

7.4. Exotic buildings. There is also a business on abstract buildings of non-p-adic origin, see [150], [9], ???

7.5. Harmonic analysis on buildings. Basic works are I.G.Macdonald [101], [108], see also [102], but buildings remain invisible in this book. The analog of Berezin kernels is introduced in [124].

7.6. Nazarov semigroup and its Weil representation were constructed in M.L.Nazarov [112].

7.7. Constructions of modular forms through theta-functions and the Weil representation are very old (elder than Weil representation, I do not know an origin), see [157], [110], [96].