Chapter 4

Segal–Bargmann transform

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The principal topics of this chapter are contained in Sections 4.1–4.2 (only these two sections are used below). In Sections 4.3–4.5 we briefly discuss the Segal–Bargmann transform as a tool of the time-frequency analysis and the micro-local analysis. Our purpose is only a demonstration of logical links. In the remaining Sections 4.6–4.9 we discuss some exotic inversion formulae.

4.1 Fock space

1.1. The Fock space $\mathbf{F}_n$. Let $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. Let $z_k = x_k + iy_k$. We denote by $\lambda(z)$ the Lebesgue measure on $\mathbb{C}^n$ normalized as

$$d\lambda(z) = \pi^{-n} \prod\nolimits_{j=1}^{n} dx_j\, dy_j$$

Consider the space $\mathbf{F}_n$ consisting of holomorphic functions $f(z)$ on $\mathbb{C}^n$ satisfying the condition

$$\int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2} d\lambda(z) < \infty \tag{1.1}$$

We define the inner product in $\mathbf{F}_n$ by

$$\langle f, g \rangle = \int_{\mathbb{C}^n} f(z)\overline{g(z)} e^{-|z|^2} d\lambda(z) \tag{1.2}$$

Theorem 1.1 *a) The space $\mathbf{F}_n$ is a Hilbert space, i.e., it is complete with respect to the inner product (1.2).*

b) Monomials $z_1^{k_1}\dots z_n^{k_n}$ form an orthogonal basis in $\mathbf{F}_n$, and

$$\|z_1^{k_1}\dots z_n^{k_n}\|^2 = k_1!\dots k_n!$$

PROOF. To simplify the notation, we consider the case $n=1$.

1) Let us verify the orthogonality of the monomials,

$$\int_{\mathbb{C}} z^k \overline{z}^m d\lambda(z) = \frac{1}{\pi}\int_0^{2\pi}\int_0^\infty r^k e^{ik\varphi} r^m e^{-im\varphi} e^{-r^2} r dr\, d\varphi = \\ = \frac{1}{2\pi}\int_0^{2\pi} e^{i(k-m)\varphi} d\varphi \cdot \int_0^\infty r^{k+m} e^{-r^2} dr^2$$

If $m \neq k$, then the first factor is 0. For $m=k$, we obtain

$$\int_0^\infty r^{2k} e^{-r^2} dr^2 = \int_0^\infty \rho^k e^{-\rho} d\rho = k!$$

2) Next, we are going to show that each function $f(z) \in \mathbf{F}_1$ orthogonal to all the z^k is 0 (and hence z^k is a basis).

Expand $f \in \mathbf{F}$ into a Taylor series,[1] $f(z) = \sum_{k=0}^\infty c_k z^k$. Consider its Fourier coefficient $\langle f, z^l\rangle$. Since both functions f, z^n are contained in L^2, this integral absolutely converges,

$$\langle f, z^l\rangle = \int_{\mathbb{C}} (\sum c_k z^k)\overline{z}^l e^{-|z|^2}\, d\lambda(z) = \lim_{R\to\infty}\int_{|z|\leqslant R} (\sum c_k z^k)\overline{z}^l e^{-|z|^2}\, d\lambda(z)$$

Since the Taylor series converges uniformly on disks $|z| \leqslant R$, we can transpose the summation and the integration,

$$\lim_{R\to\infty} \sum c_k \int_{|z|\leqslant R} z^k \overline{z}^l e^{-|z|^2}\, d\lambda(z) = \lim_{R\to\infty}\Big(0+\dots+0+c_n\int_0^R r^{2l} e^{-r^2} dr^2 + 0 + \dots\Big)$$

(to evaluate integrals, we pass to the polar coordinates as above). Finally, $\langle f, z^l\rangle = c_l \cdot l!$. Hence each function f orthogonal to the vectors z^l is zero.

3) Denote by $\overline{\mathbf{F}}_1$ the completion of $\mathbf{F}_1$. Since z^k are pairwise orthogonal,

$$\sum a_k z^k \in \overline{\mathbf{F}}_1 \quad \Longleftrightarrow \quad \sum |a_k|^2 k! < \infty$$

But the last condition provides a convergence of the series $\sum a_k z^k$ for all $z \in \mathbb{C}$, i.e., $\mathbf{F}_1 = \overline{\mathbf{F}}_1$. □

1.2. The reproducing property. Denote by $(\cdot,\cdot)$ the standard inner product in $\mathbb{C}^n$, i.e., $(u,v) = \sum u_j \overline{v}_j$. For $u \in \mathbb{C}^n$, define the function

$$\varphi_u(z) = \exp(z,u) = \exp\Big\{\sum z_k \overline{u}_k\Big\} \tag{1.3}$$

[1] Definitely, this is also the expansion of f with respect to the basis z^k, but we must prove this.

Theorem 1.2 *a)* $\varphi_u \in \mathbf{F}_n$*, and*

$$\langle \varphi_u, \varphi_v \rangle = \exp(v, u) \tag{1.4}$$

b) For each $f \in \mathbf{F}_n$*,*

$$\langle f, \varphi_u \rangle = f(u) \qquad \textit{(reproducing property)} \tag{1.5}$$

PROOF. For simplicity, assume $n = 1$. Taking in account the orthogonality of z^k, we obtain

$$\langle \varphi_u, \varphi_v \rangle \;=\; \langle \sum \frac{\overline{u}^k}{k!} z^k, \sum \frac{\overline{v}^k}{k!} z^k \rangle \;=\; \sum \frac{1}{(k!)^2} (v\overline{u})^k \langle z^k, z^k \rangle \;=\; \exp(v, u)$$

Hence $\|\varphi_u\|^2 = \exp(|u|^2)$, therefore $\varphi_u \in \mathbf{F}_1$.

Next,

$$\langle f, \varphi_u \rangle = \langle \sum c_k z^k, \sum \frac{\overline{u}^k}{k!} z^k \rangle = \sum \frac{c_k u^k}{k!} \langle z^k, z^k \rangle = \sum c_k u^k = f(u)$$

In fact, a) is a special case of b). □

REMARK. Emphasis the fundamental equation (1.5). It shows that φ_u is something similar to a delta-function. There are several terms for φ_u, namely *supercomplete system*, *overfilled basis*, *system of coherent states*, for this formalism, see Section 7.1. □

1.3. Growth of functions $f \in \mathbf{F}_n$.

Proposition 1.3 *a) For* $f \in \mathbf{F}_n$*,*

$$|f(u)| \leqslant \|f\| \cdot e^{|u|^2/2} \tag{1.6}$$

b) If an entire function on $\mathbb{C}^n$ *satisfies the condition*

$$|f(z)| \leqslant \text{const} \cdot (1 + |z|)^{-n-1} e^{|z|^2/2} \tag{1.7}$$

then $f \in \mathbf{F}_n$

PROOF. a) Indeed, $f(u) = \langle f, \varphi_u \rangle \leqslant \|f\| \cdot \|\varphi_u\|$.

b) immediately follows from the definition (1.1). □

1.4. Operators. For a bounded operator

$$A : \mathbf{F}_n \to \mathbf{F}_m$$

we define its *kernel* (or *symbol*) by

$$K_A(u, \overline{v}) = \langle A\varphi_v, \varphi_u \rangle_{\mathbf{F}_m} \tag{1.8}$$

REMARKS. 1. The kernel is a function of the variables $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^n$. The symbol $\overline{v}$ indicates that K is antiholomorphic with respect to the second variable. Also K is holomorphic with respect to the first variable.

2. Consider an integral operator in a space of function on $\mathbb{R}^n$,

$$Af(x) = \int K(x,y) f(y)\, dy$$

Then

$$K(y,z) = \ll A\delta_z, \delta_y \gg = A\delta_z(y)$$

where δ_y is the delta function supported by y. We observe a formal analogy with (1.8) and the next formula (1.9). □

We also can represent the kernel in the forms

$$K_A(u, \overline{v}) = A\varphi_v(u) = \overline{A^*\varphi_u(v)} \tag{1.9}$$

Theorem 1.4 *For each $f \in \mathbf{F}_n$,*

$$Af(u) = \int K(u,\overline{v}) f(v) e^{-|v|^2} d\lambda(v) \tag{1.10}$$

The integral in the right-hand side absolutely converges.

PROOF.

$$Af(u) = \langle Af, \varphi_u \rangle = \langle f, A^*\varphi_u \rangle$$

But

$$A^*\varphi_u(v) = \langle A^*\varphi_u, \varphi_v \rangle = \langle \varphi_u, A\varphi_v \rangle = \overline{K_A(u,\overline{v})}$$

Therefore $Af(u)$ is the inner product of $f = f(v)$ and the function $s_u(v) := \overline{K_A(u,\overline{v})}$. But this is precisely formula (1.10). □

Corollary 1.5 *A kernel K_A of a bounded operator A satisfies*

$$|K(u,\overline{v})| \leqslant \|A\| e^{-|u|^2/2} e^{-|v|^2/2} \tag{1.11}$$

PROOF. By (1.8),

$$K(u,v) = \langle A\varphi_v, \varphi_u \rangle \leqslant \|A\| \cdot \|\varphi_u\| \cdot \|\varphi_v\|$$ □

Proposition 1.6 *The kernel of an operator A is a generating function for its matrix coefficients, namely*

$$K_A(u,\overline{v}) = \sum_{\alpha_1,\dots,\alpha_m} \sum_{\beta_1,\dots,\beta_n} \frac{u_1^{\alpha_1}}{\alpha_1!} \cdots \frac{u_n^{\alpha_n}}{\alpha_n!} \cdot \frac{\overline{v}_1^{\beta_1}}{\beta_1!} \cdots \frac{\overline{v}_1^{\beta_m}}{\beta_m!} \langle A \prod_j z_j^{\beta_j}, \prod_i Z_i^{\alpha_i} \rangle \tag{1.12}$$

where $\prod_j Z_j^{\beta_j}$ and $\prod_i z_i^{\alpha_i}$ are the standard bases in $\mathbf{F}_n$ and $\mathbf{F}_m$.

PROOF. We expand the defining equation (1.8) into a series in u, $\overline{v}$. □

Problem 1.1 a) Verify (1.10) using the last formula.

b) Find kernels of the operators $Af(z) = f(z)$, $Bf(z) = f(2z)$, $Cf(z) = f(0)$.

REMARK. The last proposition allows to define the kernels in a more general situation than our basic definition (1.8). Indeed, the formal series (1.12) makes sense for an arbitrary operator acting from the space polynomials in $z_1, \dots, z_n$ to the space of formal series in $Z_1, \dots, Z_m$. □

1.5. Products of operators.

Theorem 1.7 *Let $A : \mathbf{F}_n \to \mathbf{F}_m$, $B : \mathbf{F}_m \to \mathbf{F}_k$ be bounded operators. Then*

$$K_{BA}(u, \overline{z}) = \int_{\mathbb{C}^m} K_B(u, \overline{w}) K_A(w, \overline{z}) e^{-|w|^2} d\lambda(w) \tag{1.13}$$

In particular, the integral is absolutely convergent for all u, z.

PROOF. By definition,

$$K_{BA}(u, \overline{z}) = \langle BA\varphi_z, \varphi_u \rangle = \langle A\varphi_z, B^*\varphi_u \rangle$$

By the reproducing property,

$$A\varphi_z(w) = \langle A\varphi_z, \varphi_w \rangle = K_A(w, z) \tag{1.14}$$

$$B^*\varphi_u(w) = \langle B^*\varphi_u, \varphi_w \rangle = \langle \varphi_u, B\varphi_w \rangle = \overline{\langle B\varphi_w, \varphi_u \rangle} = \overline{K_B(u, \overline{w})} \tag{1.15}$$

But the right-hand side of the desired formula (1.13) is the inner product of (1.14) and (1.15). □

Problem 1.2 Prove the product formula starting Proposition 1.6.

1.6. Standard orthogonal decomposition of $\mathbf{F}_n$. Denote by S_n^j the subspace in $\mathbf{F}_n$ consisting of homogeneous polynomials of power j. Obviously, we get orthogonal decomposition

$$\mathbf{F}_n = \bigoplus_{j=0}^{\infty} \mathsf{S}_n^j$$

1.7. Operators of change of variables. Let L be an operator $\mathbb{C}^m \to \mathbb{C}^n$ with norm $\leqslant 1$ (we consider $\mathbb{C}^n$ and $\mathbb{C}^m$ as Euclidean spaces). Then we have a well-defined operator $\mathbf{F}_n \to \mathbf{F}_m$ given by

$$R(L)f(z) = f(zL)$$

Obviously,

$$R(L)\mathsf{S}_n^j \subset \mathsf{S}_m^j$$

The kernel of $R(L)$ is

$$K(z, \overline{u}) = \exp\{zLu^*\}$$

Problem 1.3 Show that $\|R(L)\| \leqslant 1$ and the vector $f(z) = 1$ is a maximizer. Find the spectrum of $R(L)$.

1.8. Adjoint operators. The following statement is obvious.

Proposition 1.8 $$K_{A^*}(z, \overline{u}) = \overline{K_A(\overline{u}, z)}$$

1.9. Creation-annihilation operators. We define the *creation operators* $\widehat{a}_j$ and the *annihilation operators* $\widehat{a}_j^*$ in $\mathbf{F}_n$ by

$$\widehat{a}_j f(z) = z_j f(z); \qquad \widehat{a}_j^* = \frac{\partial}{\partial z_j} f(z) \tag{1.16}$$

Obviously,

$$[\widehat{a}_k, \widehat{a}_l] = 0; \qquad [\widehat{a}_k^*, \widehat{a}_l^*] = 0; \qquad [\widehat{a}_k, \widehat{a}_l^*] = -\delta_{kl}$$

REMARK. These operators are unbounded. Indeed, for $f(z) = \sum c_k z^k$,

$$\|f\|^2 = \sum |c_k|^2, \qquad \|\widehat{a} f\|^2 = \sum |(k+1)c_k|^2, \qquad \|\widehat{a}^* f\|^2 = \sum |(k-1)c_k|^2$$

Proposition 1.9 *The operator a_j^* is adjoint to a_j*

PROOF. It suffices to write the matrices of a_j and a_l^* in the standard basis. However, the following formal calculation explains the phenomenon better

$$\int_{\mathbb{C}^n} \Big[\frac{\partial}{\partial z_j} f(z)\Big] \overline{g(z)} e^{-z\overline{z}} d\lambda(z) = -\int_{\mathbb{C}^n} f(z) \Big[\frac{\partial}{\partial z_j} \overline{g(z)} e^{-z\overline{z}}\Big] d\lambda(z) =$$
$$= -\int_{\mathbb{C}^n} f(z)\overline{g(z)} \Big[\frac{\partial}{\partial z_j} e^{-z\overline{z}}\Big] d\lambda(z) = \int_{\mathbb{C}^n} f(z) \overline{g(z) z_j}\, e^{-z\overline{z}} d\lambda(z) \qquad \square$$

We wish to formulate several simple remarks. Demote by K_B the kernel of the operator B. Then

$$\widehat{a}_j B f(z) = \int_{\mathbb{C}^n} z_j K(z, \overline{u}) f(u) e^{-|u|^2} d\lambda(u) \tag{1.17}$$

$$\widehat{a}_j^* B f(z) = \int_{\mathbb{C}^n} \frac{\partial K(z, \overline{u})}{\partial z_j} f(u) e^{-|u|^2} d\lambda(u) \tag{1.18}$$

Integrating by parts we obtain

$$B\, \widehat{a}_j f(z) = \int_{\mathbb{C}^n} \frac{\partial K(z, \overline{u})}{\partial \overline{u}_j} f(u) e^{-|u|^2} d\lambda(u) \tag{1.19}$$

$$B\, \widehat{a}_j^* f(z) = \int_{\mathbb{C}^n} \overline{u}_j K(z, \overline{u}) f(u) e^{-|u|^2} d\lambda(u) \tag{1.20}$$

1.10. The action of the Heisenberg group. For $a \in \mathbb{C}^n$, define the operator

$$T(a) f(z) = f(z+a) \exp\{-(z,a) - (a,a)/2\} \tag{1.21}$$

Proposition 1.10 *The operators $T(a)$ are unitary and satisfy the identity*

$$T(a)T(b) = \exp\{-i\,\mathrm{Im}(a,b)\}T(a+b) \tag{1.22}$$

PROOF. Verify the first statement

$$\langle T(a)f, T(a)g\rangle = \int f(z+a)\overline{g(z+a)}\exp\{-(z,a)-(a,z)-(a,a)-(z,z)\}d\mu(z) =$$
$$= \int f(z+a)\overline{g(z+a)}e^{(z+a)(\overline{z}+\overline{a})}\,d\mu(z+a) = \langle f, g\rangle$$

Next,

$$T(a)T(b)f(z) = T(b)f(z+b)\cdot\exp\{-(z,b)-(b,b)/2\} =$$
$$= f(z+a+b)\cdot\exp\{-(z+a,b)-(b,b)/2\}\cdot\exp\{-(z,a)-(a,a)/2\} =$$
$$= f(z+a+b)\cdot\exp\{-(z,a+b)\}\cdot\exp\{-\big[-(a,a)-(b,b)-2(a,b)\big]/2\} \qquad \square$$

REMARK. Thus, the proof is trivial, however we wish to comment it. Consider the measure $e^{-|z|^2}d\lambda(z)$ and the shifted measure $e^{-|z+a|^2}d\lambda(z+a)$. Denote by $\gamma_a(z)$ the density of the latter measure with respect to the former. The operators $T(a)$ are unitary, because the correcting factor in (1.21) satisfies $|\exp\{-(z,a)-(a,a)/2\}| = \sqrt{\gamma_a(z)}$. The operators

$$\mathfrak{T}(a)f(z) = f(z+a)\sqrt{\gamma_a(z)}$$

are unitary in $L^2(\mathbb{C}^n, e^{-|z|^2})$ and satisfy $\mathfrak{T}(a)\mathfrak{T}(b) = \mathfrak{T}(a+b)$. Nevertheless, they are not well defined in $\mathbf{F}_n$, because the square root is not holomorphic; we have to choose a holomorphic function with a given absolute value. A price for our choice is an appearance of the additional factor in (1.22). $\square$

By (1.22), $T(a)$ is a projective unitary representation of the additive group of $\mathbb{C}^n$. Also we can treat T as a linear representation of a central extension of $\mathbb{C}^n$. Precisely, consider the space $\mathbb{C}^n \oplus \mathbb{R}$ equipped with the multiplication

$$(a,x)\circ(b,y) = (a+b, x+y+\mathrm{Im}(a,b))$$

The corresponding operators of linear representation are

$$(a,x) \mapsto e^{ix}T(a) \tag{1.23}$$

This group is nothing but the Heisenberg group defined in 1.1.2. Our construction produces not only the same group, but also the same representation; the intertwining operator is the Segal–Bargmann transform discussed in the next section.

Problem 1.4 Recall that a_k, a_k^* denote the creation-annihilation operators (1.16) Find $\exp\{a_k - a_k^*\}$ and $\exp\{i(a_k - a_l^*)\}$. Show, that for each anti-self-adjoint linear combination of a_k, a_k^*, its exponent has the form (1.23).

1.11. Digression. Wick symbols. To simplify notation, consider the case $n = 1$. Let

$$L(z, \xi) = \sum_{kl} c_{kl} z^k \xi^l$$

be a formal series of two variables. Define the operator $A : \mathbf{F}_1 \to \mathbf{F}_1$ by

$$A = L\Big(z, \frac{\partial}{\partial z}\Big) := \sum_{kl} c_{kl} z^k \frac{\partial^l}{\partial z^l} \tag{1.24}$$

We say that $L(z, \xi)$ is the *Wick symbol* of an operator A.

Problem 1.5 The operator

$$\sum\nolimits_{j=0}^{\infty} \frac{(-1)^k}{k!} z^k \frac{\partial^k}{\partial z^k}$$

is the projection on the one-dimensional space consisting of constants.

Proposition 1.11 *The kernel of the operator (1.24) is*

$$K_A(z, \overline{v}) = L(z, \overline{v}) \exp(z\overline{v})$$

PROOF. By (1.9), it follows that the kernel of (1.24) is

$$\Big(\sum_{kl} c_{kl} z^k \frac{\partial^l}{\partial z^l}\Big) e^{z\overline{v}} = \Big(\sum_{kl} c_{kl} z^k \overline{v}^l\Big) e^{z\overline{v}} = L(z, \overline{v}) e^{z\overline{v}} \qquad \square$$

Thus, Wick symbols can be reduced to kernels. On another hand, the Wick symbols exist only for operators acting from $\mathbf{F}_n$ to itself. For this reason, we prefer kernels.

1.12. Weak convergence in $\mathbf{F}_n$.

Proposition 1.12 *For $f_j \in \mathbf{F}_n$, $f \in \mathbf{F}_n$, the following conditions are equivalent*

A. f_j weakly converges to f

B. The norms $\|f_j\|$ are uniformly bounded and $f_j(z)$ converges to $f(z)$ for each z.

C. The norms $\|f_j\|$ are uniformly bounded and $f_j(z)$ converges to $f(z)$ uniformly on each ball $|z| < R$

PROOF. A $\Leftrightarrow$ B. We apply the criterion of weak convergence (Theorem 1.1.1) to the system φ_z.

B $\Rightarrow$ C. Since the norms $\|f_j\|$ are uniformly bounded, then the family of functions $f_j(z)$ is uniformly bounded on any ball $|z| < R$, see (1.6). But a pointwise convergence of uniformly bounded sequence of holomorphic functions is the uniform convergence. $\square$

1.13. Weak convergence of operators.

Proposition 1.13 *Let A_j, A be bounded operators in $\mathbf{F}_n$. The following conditions are equivalent.*

A) A sequence A_j of operators weakly converges to A.

B) The norms of A_j are uniformly bounded and $K_{A_j}(z, \overline{u})$ converges to $K(z, u)$ pointwise.

C) The norms of A_j are uniformly bounded and $K_{A_j}(z, \overline{u})$ converges to $K(z, u)$ uniformly on each domain $|z| < R_1$, $|u| < R_2$.

PROOF. We apply the criterion of weak operator convergence (Theorem 1.1.2) to the family of vectors φ_z. □

1.14. Exercises.

Problem 1.6 For any $f \in \mathbf{F}_n$,

$$f^{(n)}(a) = \langle f(z), z^n \exp(z, a) \rangle$$

4.2 Segal-Bargmann transform

2.1. Definition. For $f \in L^2(\mathbb{R}^n)$ we define the holomorphic function $\mathcal{B}f(z)$ (*Segal–Bargmann transform* of f) on $\mathbb{C}^n$ by

$$\mathcal{B}f(z) = (\pi)^{-n/4} \int_{\mathbb{R}^n} f(x) \exp\{-z^2/2 + \sqrt{2}zx - x^2/2\} dx \tag{2.1}$$

here $z^2 := \sum z_j^2$, $xz := \sum x_j z_j$.

2.2. Unitarity property. Denote

$$\Phi_z(x) = \pi^{-n/4} \exp\{-\overline{z}^2/2 + \sqrt{2}\overline{z}x - x^2/2\} \tag{2.2}$$

In this notation,

$$\mathcal{B}f(z) = \langle f, \Phi_z \rangle_{L^2(\mathbb{R}^n)}$$

Theorem 2.1

$$\langle \Phi_u, \Phi_v \rangle = \exp(v, u) \tag{2.3}$$

PROOF. $\langle \Phi_u, \Phi_v \rangle =$

$$= \pi^{-n/2} \int_{\mathbb{R}^n} \exp\{-\overline{u}^2/2 + \sqrt{2}\overline{u}x - x^2/2\} \exp\{-v^2/2 + \sqrt{2}vx - x^2/2\} dx =$$

$$= \pi^{-n/2} \exp\{v\overline{u}\} \int_{\mathbb{R}^n} \exp\{-\frac{1}{2}(v + \overline{u})^2 + \sqrt{2}x(v + \overline{u}) - x^2\}\, dx =$$

$$= \pi^{-n/2} \exp\{v\overline{u}\} \int_{\mathbb{R}^n} \exp\{-\big(x - (v + \overline{u})/\sqrt{2}\,\big)^2\}\, dx =$$

$$= (2\pi)^{-n/2} \exp\{v\overline{u}\} \int_{\mathbb{R}^n} \exp\{-\tfrac{1}{2}y^2\}\, dy = \exp\{v\overline{u}\} \qquad \square$$

Problem 2.1 a) Let A be a set. Let H, H' be Hilbert spaces. Let $\xi_\alpha \in H$, $\xi'_\alpha \in H'$ be total systems of vectors enumerated by points $\alpha \in A$. Assume

$$\langle \xi_\alpha, \xi_\beta \rangle = \langle \xi'_\alpha, \xi'_\beta \rangle \qquad \text{for all } \alpha, \beta \in$$

Then there exists a unique unitary operator $U : H \to H'$ such that $U\xi_\alpha = \xi'_\alpha$.

b) Show that linear combinations of systems of vectors $\Phi_u \in L^2(\mathbb{R}^n)$ and $\varphi_u \in \mathbf{F}_n$ are dense in the corresponding spaces.

Theorem 2.2 *a) The Segal–Bargmann transform $\mathcal{B}$ is an (invertible) unitary operator operator $L^2(\mathbb{R}^n) \to \mathbf{F}_n$.*

b) $\mathcal{B}\,\Phi_z = \varphi_z$

PROOF. Comparing (1.4) and (2.3), we get

$$\langle \Phi_u, \Phi_v \rangle_{L^2(\mathbb{R}^n)} = \langle \varphi_u, \varphi_v \rangle_{\mathbf{F}_n}$$

Thus, there is a unitary operator

$$U : L^2(\mathbb{R}^n) \to \mathbf{F}_n, \qquad U\Phi_u = \varphi_u$$

Next,

$$Uf(z) = \langle Uf, \varphi_z \rangle_{\mathbf{F}_n} = \langle f, U^{-1}\varphi_z \rangle_{L^2(\mathbb{R}^n)} = \langle f, \Phi_z \rangle_{L^2(\mathbb{R}^n)}$$

But the last expression is the Segal-Bargmann transform. □

Problem 2.2 For the operators (1.2.6) of Heisenberg group acting in $L^2(\mathbb{R}^n)$ find their push-forwards under $\mathcal{B}$ and identify them with operators (1.23). In fact, the Segal–Bargmann transform was originally introduced from this reasoning.

2.3. Differential operators. Obviously, we get the following correspondence of differential operators under the Segal–Bargmann transform

$$\begin{cases} z_j \longleftrightarrow \dfrac{1}{\sqrt{2}}\Bigl(x_j - \dfrac{\partial}{\partial x_j}\Bigr), \\ \dfrac{\partial}{\partial z_j} \longleftrightarrow \dfrac{1}{\sqrt{2}}\Bigl(x_j + \dfrac{\partial}{\partial x_j}\Bigr), \end{cases} \qquad \begin{cases} x_j \longleftrightarrow \dfrac{1}{\sqrt{2}}\Bigl(z_j + \dfrac{\partial}{\partial z_j}\Bigr), \\ \dfrac{\partial}{\partial x_j} \longleftrightarrow \dfrac{1}{\sqrt{2}}\Bigl(-z_j + \dfrac{\partial}{\partial z_j}\Bigr), \end{cases} \tag{2.4}$$

(for instance, the formula for $\partial/\partial x_j$ follows from the differentiation by parts in (2.1), the formula for x_j is obtained the differentiation of integral (2.1) with respect to parameter z_j.

Corollary 2.3 *The map $D \mapsto \mathcal{B}D\mathcal{B}^{-1}$ is a bijection of the ring of polynomial differential operators in $L^2(\mathbb{R}^n)$ onto the ring of polynomial differential operators in $\mathbf{F}_n$.*

Next, $\mathcal{B}$ takes the function $e^{-x^2/2}$ to 1.

Corollary 2.4 *The Segal–Bargmann transform is a bijection of the space of functions $p(x)e^{-x^2/2}$, where p is a polynomial, onto the space of all polynomials on $\mathbb{C}^n$.*

2.4. The $\mathcal{B}$-images of smooth functions and tempered distributions. The Segal–Bargmann transform makes sense for any tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$. Denote by $\mathcal{S}\mathbf{F}_n$ and $\mathcal{S}'\mathbf{F}$ the $\mathcal{B}$-images of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and its dual $\mathcal{S}'(\mathbb{R}^n)$ under the Segal-Bargmann transform.

Proposition 2.5 *a) An entire function $f(z) = \sum c_\alpha z^\alpha$ is contained in $\mathcal{B}(\mathcal{S})$ iff $L(z, \partial/\partial z)g \in \mathbf{F}_1$ for each polynomial differential operator L.*

b) The space $\mathcal{S}\mathbf{F}_n$ consists of entire functions satisfying the condition

$$\text{for each } M \qquad \sup_\alpha \bigl(|c_\alpha| \cdot \prod \alpha_j^M \sqrt{\alpha_j!}\bigr) < \infty \tag{2.5}$$

PROOF. a) To simplify notation, assume $n = 1$. First, we characterize the Schwartz space. A function $h \in L^2(\mathbb{R}^n)$ is contained in $\mathcal{S}(\mathbb{R}^1)$ iff for each polynomial differential operator $D = D(x, \partial/\partial x)$ the distribution Dh is contained in L^2. Corollary 2.3 translates this characterization to $\mathbf{F}_1$.

b) In particular, $f \in \mathcal{B}(\mathcal{S})$ must satisfy $\bigl(z\frac{\partial}{\partial z}\bigr)^p f(z) \in \mathbf{F}_1$ i.e.,

$$\sum |c_\alpha|^2 \alpha^{2p} \alpha! < \infty \qquad \text{for all } p$$

the terms of the series must pass to 0, and this is the necessity of (2.5).

Conversely, the condition (2.5) implies $f \in \mathbf{F}_1$, condition (2.5) is also stable under an application of any operator $L(z, \partial/\partial z)$; this is the sufficiency of (2.5). □

Proposition 2.6 *a)The space $\mathcal{S}'\mathbf{F}_1$ consists of entire functions $g(z) = \sum c_\alpha z^\alpha$ admitting a representation $g = L(z, \partial/\partial z)f$, where f ranges in $\mathbf{F}_1$ and L is a polynomial differential operator.*

b) The space $\mathcal{S}'\mathbf{F}_n$ consists of entire functions $f(z) = \sum c_\alpha z^\alpha$ satisfying

$$\sup_\alpha \bigl(|c_\alpha| \cdot \prod_j \alpha_j^{-M} \sqrt{\alpha_j!}\bigr) < \infty \qquad \text{for some } M \tag{2.6}$$

PROOF. a) The space $\mathcal{S}'(\mathbb{R}^1)$ consists of functions that can be represented in the form $D(x, \partial/\partial x)h$, where $h \in L^2(\mathbb{R}^1)$ and D is a polynomial differential operator. It remains to refer to Corollary 2.3.

b) Evidently, functions $f = L(z, \partial/\partial z)g$, $g \in \mathbf{F}_1$, satisfy (2.6). Further, for f satisfying (2.6), the function

$$f_{[p]} := \sum \frac{c_\alpha}{(\alpha+1)^p} z^k$$

is contained in $\mathbf{F}_1$ for sufficiently large p. On another hand, $(1+z\partial/\partial z)^p f_{[p]} = f$. □

Corollary 2.7 *Both spaces $\mathcal{S}\mathbf{F}_n$, $\mathcal{S}'\mathbf{F}_n$ are invariant with respect to the operators of a change of variable $f(z) \mapsto f(zU)$ with unitary U.*

PROOF. The characterizations of $\mathcal{S}\mathbf{F}_n$ and $\mathcal{S}'\mathbf{F}_n$ given in the statements a) of Propositions 2.6–2.6 regard the action of the unitary group $\mathrm{U}(n)$. $\square$

2.5. The Mehler semigroup again. Relations (2.4) yield the following correspondence of operators under $\mathcal{B}$

$$-\frac{1}{2}\Big(\frac{\partial^2}{\partial x^2} + x^2\Big) \quad \longleftrightarrow \quad z\frac{\partial}{\partial z} + \frac{1}{2}$$

Therefore

$$\exp\Big\{\frac{t}{2}\Big(\frac{\partial^2}{\partial x^2} + x^2\Big)\Big\} \quad \longleftrightarrow \quad \exp\Big\{-t\big(z\frac{\partial}{\partial z} + \frac{1}{2}\big)\Big\}$$

We evaluated the left-hand side expression in Theorem 1.3.5. An evaluation of the exponent in the right-hand side is trivial,

$$\exp\Big\{-t\big(z\frac{\partial}{\partial z} + \frac{1}{2}\big)\Big\} f(z) = e^{-t/2} f(e^{-t} z)$$

Thus we proved the following theorem

Theorem 2.8 *a) Mehler operators $U(t)$ in $L^2(\mathbb{R}^n)$ (see (3.11)) correspond to the operators*

$$\widetilde{U}(t) f(z) = e^{-t/2} f(e^{-t} z) \tag{2.7}$$

b) In particular, the Fourier transform $e^{-\pi i/4} U(\pi i/2)$ corresponds to the 'rotation' $f(z) \to f(-iz)$.

Problem 2.3 Deduce the Mehler Theorem calculating $\mathcal{B}^{-1}\widetilde{U}(t)\mathcal{B}$.

2.6. The Hermite polynomials. We define the Hermite functions H_n by

$$H_n(x) = 2^{-n/2}\Big(x - \frac{\partial}{\partial x}\Big)^n e^{-x^2/2}$$

Obviously,

$$H_n(x) = e^{-x^2/2} \cdot \big\{\text{a polynomial of degree } n\big\}$$

Such polynomials are called the *Hermite polynomials*[2].

Theorem 2.9 *a) $(\mathcal{B}H_n)(z) = z^n$*

b) The system $H_n(x)$ is an orthogonal basis in $L^2(\mathbb{R}^n)$ and $\|H_n\|^2 = n!$.

c) $$2^{-1/2}\Big(x + \frac{\partial}{\partial x}\Big) = nH_{n-1}, \qquad 2^{-1/2}\Big(x - \frac{\partial}{\partial x}\Big) = H_{n+1}$$

[2]There are several normalizations of these polynomials. Our normalization is not the most common.

d) $2^{1/2} x H_n = n H_{n-1} - H_{n+1}$

$$2^{1/2} \frac{\partial}{\partial x} H_n(x) = n H_{n-1}(x) - H_n(x)$$

e) The Hermite polynomials are eigenfunctions of the oscillator operator,

$$\frac{1}{2}\Big(-\frac{\partial^2}{\partial x^2} + x^2\Big) H_n(x) = \Big(n + \frac{1}{2}\Big) H_n(x)$$

f) $$\sum \frac{1}{k!} \overline{u}^k H_k = \Phi_u(x) = \exp\{-\overline{u}^2/2 + \sqrt{2} x \overline{u} - x^2/2\}$$

g) The following Mehler bilinear formula holds

$$\sum_{k=0}^{\infty} \frac{1}{k!} e^{-(k+1/2)t} H_k(x) H_k(y) =$$
$$= \frac{1}{\sqrt{2\pi \sinh t}} \cdot \exp\Big\{-\frac{1}{2} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -\cosh t/\sinh t & 1/\sinh t \\ 1/\sinh t & -\cosh t/\sinh t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\Big\} \tag{2.8}$$

h) $$\mathcal{B} f(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \int_{\mathbb{R}} f(x) H_k(x)\, dx$$

PROOF. a) The Segal–Bargmann transform takes the function $\Phi_0 = e^{-x^2/2}$ to $\varphi_0 = 1$. Applying the operator $\widehat{a}^n$ to 1 and using the correspondence formula (2.4), we obtain our statement.

b) corresponds to the orthogonality relations for z^k.

c) corresponds to $\widehat{a}^* z^n = n z^{n-1}$, $\widehat{a} z^n = z^{n+1}$ and d) follows from c).

e) corresponds to $z \frac{\partial}{\partial z} z^n = n z^n$

f) corresponds to $e^{z\overline{u}} = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \overline{u}^k$.

g) The operator $\widetilde{U}(t)$ given by (2.7) can be represented as

$$\widehat{U}(t) f(z) = \sum \frac{1}{k!} e^{-(k+1/2)t} \langle f, z^k \rangle z^k$$

Hence the kernel of the corresponding operator $U(t)$ in $L^2(\mathbb{R}^n)$ is the left-hand side of (2.8). But the explicit expression for this kernel obtained in Theorem 1.3.5 gives the right-hand side.

h) is a reformulation of the expansion $f = \sum \frac{1}{n!} \langle f, z^n \rangle z^n$. □

Problem 2.4 Show that the Schwartz space $\mathcal{S}(\mathbb{R})$ consists of series $\sum c_k H_k(x)$, where $\max |c_k| \cdot k^M \sqrt{k!} < \infty$ for all M.

2.7. Inversion formula. I. Obviously, for

$$g(z) = \sum_{k_1,\dots,k_n} c_{k_1,\dots,k_n} z_1^{k_1} \dots z_n^{k_n}$$

we have

$$\mathcal{B}^{-1} g(x) = \sum_{k_1,\dots,k_n} c_{k_1,\dots,k_n} H_{k_1}(x_1) \dots H_{k_n}(x_n)$$

(the series in the right-hand side converges in L^2-sence).

2.8. Inversion formula. II. Since $\mathcal{B}$ is unitary operator, it follows that $\mathcal{B}^{-1} = \mathcal{B}^*$. Further, we can regard $\mathcal{B}$ as an isometric embedding $L^2(\mathbb{R}) \to L^2(\mathbb{C}^n)$. It remains to write a kernel of the adjoint operator $L^2(\mathbb{C}^n) \to L^2(\mathbb{R})$. This gives

$$\mathcal{B}^{-1} g(x) = \int_{\mathbb{C}^n} g(z) \exp\{-\overline{z}^2/2 + \sqrt{2}\overline{z}x - x^2/2\} e^{-|z|^2} d\lambda(z)$$

Denoting

$$\xi_x(z) = \overline{\Phi_z(x)}$$

we write this formula as

$$\mathcal{B}(x) = \langle g, \xi_x \rangle_{\mathbf{F}_n}$$

The function ξ_a is the $\mathcal{B}$-image of $\delta(x-a)$, hence $\xi_a \in \mathcal{S}'\mathbf{F}_n$. Therefore, the last inversion formula holds at least $g \in \mathcal{S}\mathbf{F}_n$, see (2.5).

2.9. Inversion formula. III. Represent formula (2.1) as

$$g(z) = \pi^{-n/4} 2^{-n/2} e^{-z^2/2} \int_{\mathbb{R}^n} e^{yz} \left[e^{-y^2/4} f(y/\sqrt{2}) \right] dy$$

Substituting $z = it$ and applying the inversion formula for the Fourier transform, we obtain

$$f(x) = \pi^{-3n/4} 2^{-n/2} e^{x^2/2} \int_{-\infty}^{\infty} g(it) e^{-i\sqrt{2}tx - t^2/2} dt$$

In the same way, we can write the inversion formula from any line $z = a + it$. Apparently, many other inversion formulae can be written, one exotic formula is discussed below in Section 4.9.

4.3 Spectral analysis of signals

This section is purely heuristic. We discuss the Segal–Bargmann transform as a tool of the time-frequency analysis.

3.1. Recognition of signals. Statement of the problem. Let we have a signal $\gamma(t)$ (a sound, a radio signal, etc.) that is a superposition of waves,

$$\gamma(t) = \sum_{j=1}^{N} \alpha_j(t) e^{i\nu_j t}, \tag{3.1}$$

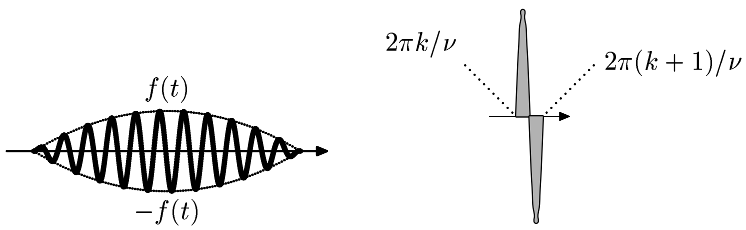


Fig. 4.1: To the proof of the Riemann Lemma

We wish to restore frequencies ν_j and amplitudes $\alpha_j(t)$ starting from $\gamma(t)$. One of variants of this problem is to separate a noise and a useful signal.

This problem is not well posed; for instance

$$e^{i\nu t} + e^{\mu t} = 2\cos(\mu - \nu)t/2e^{i(\mu+\nu)t/2}$$

however it is important.

3.2. Some remarks on Fourier transform. The idea to apply the Fourier transform is based on simple and well-known arguments, we are going to refresh them for a reader since we must keep in the mind rather proofs than statements.

Theorem 3.1 (Riemann Lemma) *a) Let f be a compactly supported continuous function on $\mathbb{R}$. Then*

$$\int_{-\infty}^{\infty} f(t)\sin\nu t\,dt, \qquad \int_{-\infty}^{\infty} f(t)\cos\nu t\,dt, \qquad \int_{-\infty}^{\infty} f(t)e^{i\nu t}\,dt$$

tend to 0 *as* $\nu \to \pm\infty$.

b) If f has N continuous derivatives, then these integrals are $o(\nu^{-N})$ as $\nu \to \infty$.

c) If $f(t) \geqslant 0$, then the maximum $\left|\int f(t)e^{i\nu t}\,dt\right|$ is achieved at $\nu = 0$.

PROOF. a), see Fig. 4.1,

$$\int_{-\infty}^{\infty} f(t)\sin(\nu t)\,dt = \sum_k \Big(\int_{2k\pi/\nu}^{(2k+1)\pi/\nu} f(t)\sin(\nu t)\,dt + \int_{(2k+1)\pi/\nu}^{(2k+2)\pi/\nu} f(t)\sin(\nu t)\,dt \Big) =$$

$$= \sum_k \int_{2k\pi/\nu}^{(2k+1)\pi/\nu} \Big[f(t) - f(t+\pi/\nu) \Big] \sin\nu t\,dt$$

We estimate this expression as

$$\Big|\ldots\Big| \leqslant \sum_k \int_{2k\pi/\nu}^{(2k+1)\pi/\nu} \Big| f(t) - f(t+\pi/\nu) \Big|\,dt$$

Since f is continuous and ν is large, the integrands are uniformly small.

b) We integrate by parts

$$\int_{-\infty}^{\infty} f(t)e^{i\nu t}\,dt = -\frac{1}{i\nu}\int_{-\infty}^{\infty} f'(t)e^{i\nu t}\,dt = \cdots = (-1)^N \frac{1}{(i\nu)^N} \int_{-\infty}^{\infty} f^{(N)}(t)e^{i\nu t}\,dt$$

By the statement a), it follows that each integral tends to 0 as $\nu \to \infty$. $\square$

Also recall what is happened if f has a singularity.

EXAMPLE. Let f be a function supported by a segment $[0, M]$ and smooth everywhere but 0. Assume f has the following asymptotic expansion

$$f(t) = at^\alpha + bt^\beta + \dots + zt^\omega + o(t^N) \tag{3.2}$$

in the right half-neighborhood of 0. Consider the function $\widetilde{f}(t) = f(t) - at^\alpha e^{-t} = f(t) - at^\alpha(1 - 1/t + \dots)$. It has a decomposition of the same type (3.2), but the term at^α is replaced by

$$-a \sum\nolimits_{k \geqslant 1,\, \mathrm{Re}\,\alpha + k \leqslant N} \frac{(-1)^k}{k!} t^{\alpha+k}$$

Then we repeat the same operation over and over again. As a result, we obtain a representation

$$f(t) = \sum A_j t^{r_j} e^{-t} + f^\circ(t)$$

where A_j, r_j are some expicit constants, the sum is finite, $r_j \leqslant N$, and f° has N continuous derivatives at 0. New f° is not compactly supported but the conclusion of the Riemann Lemma holds. Further,

$$\int_0^\infty t^r e^{-t} e^{it\nu}\, dt = \frac{\Gamma(r+1)}{(1 - i\nu)^{r+1}}$$

Finally, we get

$$\int f(t) e^{i\nu t} dt = \sum A_j \Gamma(r_j + 1)(1 - i\nu)^{-r_j - 1} + O(\nu)^N, \qquad \nu \to \infty$$

The parameters α, β, $\dots$, a, b, $\dots$ can be reconstructed from A_j, r_j. Thus, the Fourier transform 'see' (or 'hear') singularities of our function.

3.3. The Fourier transform and a problem of separation of waves. It is clear that the global Fourier transform is not a perfect tool for solution of this problem. We illustrate this with some examples.

Let $\gamma = Ae^{i\alpha t} + Be^{i\beta t}$. Then the Fourier transform $\widehat{\gamma}(\xi)$ is the distribution $\delta(\xi + \alpha) + \delta(\xi + \beta)$, and the harmonics of γ are observable. Now let amplitudes depend on time, for instance $\gamma = A(t)e^{i\alpha t} + B(t)e^{i\beta t}$, where $A(t)$, $B(t)$ are positive compactly supported functions (whose supports are long as compared with $1/\alpha$, $1/\beta$). Then $|\gamma(t)|$ has two peaks near α and β, the heights of these peaks are approximately $\int A(t)\, dt$, $\int B(t)\, dt$. However it is impossible to understand how to restore A, B. See Fig.4.2.

Explain this in another way. The Fourier transform represents a sound as a superposition

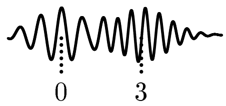
$$\gamma(t) = \int c(s) e^{ist} dt$$

of globally existing waves e^{ist}. But an actual signal (for instance, a sound, radio, radar-location, seismic oscillation, etc.) usually is a superposition of shortly living waves; in this context, the Fourier transform seems to be violence.

Consider the function

$$f(x) = \exp(-x^2/2 + 8ix) + \exp(-(x-3)^2/2 + 12ix)$$

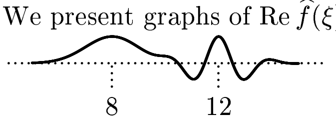
Re $f(x)$:

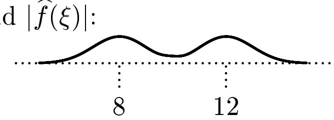


Its Fourier transform is

$$\widehat{f}(\xi) = \exp(-(\xi-8)^2/2) + \exp(-(\xi-12)^2/2) \cdot \exp(3(12-\xi)i)$$

We present graphs of Re $\widehat{f}(\xi)$ and $|\widehat{f}(\xi)|$:





Certainly, they have peaks near $\xi = 8$ and $\xi = 12$ and this indicates the presence of harmonics $\exp(8ix)$ and $\exp(12ix)$ in $f(x)$. However, one does not observe the intervals of the line $x \in \mathbb{R}$, where these harmonics are present. A height of a peak is $\simeq$ the product of the intensity and the duration of a harmonic.

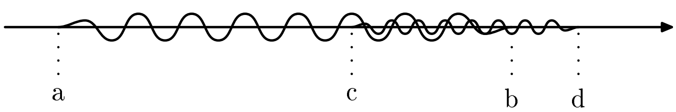
Fig. 4.2: To Subsection 3.1.

Fig. 4.3: Consider a function (a signal) $\gamma(t)$ of the form

$$\gamma(t) = A(t)\exp(i\omega t) + B(t)\exp(i\nu t),$$

where $A(t)$, $B(t)$ are supported by segments $[a,b]$, $[c,d]$ respectively; Re$(\gamma(t)$ is a sum of two summands having approximately following form

$\gamma(t)$:



Consider $r(t)$ of a form

$r(t)$

It is more-or less clear, that $|G(\tau,\xi)|$ has ridges along the following two segments in $\mathbb{R}^2$: $\big[(a,-\omega),(b,-\omega)\big]$, $\big[(c,-\nu),(d,-\nu)\big]$ and we can reconstruct our signal. Now we know frequencies of the signals, their intervals, and intensities.

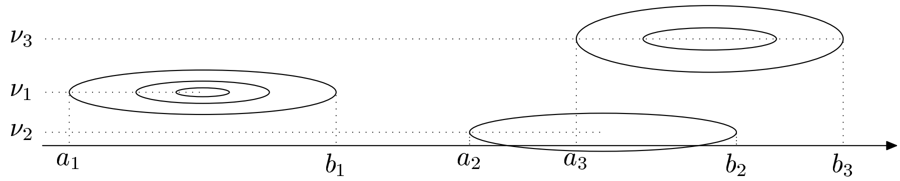


Fig. 4.4: An idealized picture of level lines of $|G(\tau,\xi)|$. Signals correspond to ridges, whose longitudes indicate durations of signals. The latitudes are frequencies, altitudes are intensities.

3.4. Short time Fourier transforms. Fix a function $r(t)$ supported by a finite interval $[-A, A]$. For a function $\gamma(t)$, define a function of two variables

$$\mathcal{G}(\tau,\xi) = \int_{-\infty}^{\infty} \gamma(t) r(t-\tau) e^{it\xi}\, dt, \qquad \tau \in \mathbb{R},\ \xi \in \mathbb{R}$$

For a given τ, the function $\mathcal{G}_r(\tau,\xi)$ is the Fourier transform $\widehat{\Psi}_\tau$ of the function

$$\Psi_\tau(t) := \gamma(t) r(t-\tau)$$

This function is supported by the interval $[-A+\tau, A+\tau]$; the Fourier transform allows to observe harmonics and their total intensities for Φ_τ. However, if γ is actually a superposition of harmonics with periods $\ll A$, then the same harmonics are present in $\Psi_\tau(\xi)$, and they produce peaks of $\widehat{\Psi}_\tau$.

New possibilities are explained in Fig. 4.4–4.5. An idealized picture for $|G(\tau,\xi)|$ is a topographic map with a family of ridges in longitude directions. Latitudes of the ridges indicate the frequencies, longitudes indicate intervals of existence of signals, and the altitudes of ridges show intensities.

Now we must choose $r(t)$. The common way is $r(t) = e^{-t^2/2}$ (or, more generally, $r(t) = \exp\{-t^2/2\lambda\}$). This produces the transform

$$\mathcal{GBS}f(\tau,\xi) = \int_{-\infty}^{\infty} f(t) e^{-(t-\tau)^2/2} e^{i\xi t} dt = e^{-\tau^2} \int_{-\infty}^{\infty} f(t) e^{-t^2/2 - t(\tau+i\xi)}\, dt$$

However, this is a slightly modified Segal–Bargmann transform: we put $\sqrt{2}z := \tau + i\xi$, and write an additional factor in front of the integral.

3.5. Choice of $r(t)$ and the Heisenberg uncertainty. First remark. The immediate idea to take

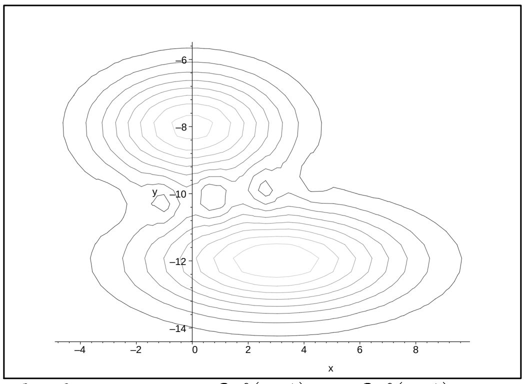
$$r(t) = \begin{cases} 1, & \text{if } -1 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

must be immediately rejected. Indeed, the functions $\Psi_\tau(t) = \gamma(t) r(t-\tau)$ have two singularities (jumps), and this implies slow decreasing of Fourier transforms

a) Put $r(t) := e^{-t^2/2}$. Consider the short time Fourier transform $\mathcal{G}\, f(\tau,\xi)$ of

$$f(t) = \exp(-t^2/8 + 8it) + \exp(-(t-3)^2/18 + 12it)$$

On the topographic map we present the line curves of $|\mathcal{G}(t,\xi)|$.



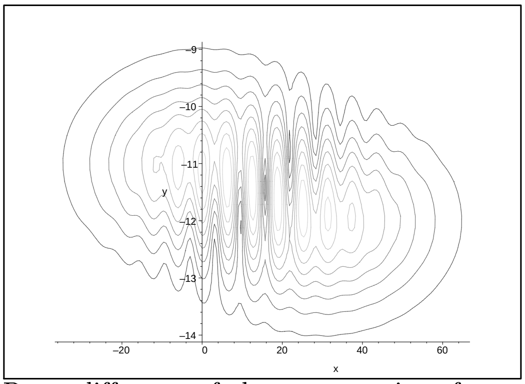
The functions $\mathrm{Re}\,\mathcal{G}\, f(\tau,\xi)$ $\mathrm{Im}\,\mathcal{G}\, f(\tau,\xi)$ are oscillating (they are not depicted). Instead of smooth ridges, they have a sequence of sharp peaks and deep depressions.

Here the frequencies $=8$ and $=12$ are relatively far one from another.

b) We depict $|\mathcal{G}\, f(\tau,\xi)|$ for

$$f(t) = \exp(-t^2/600 + 11it) + \exp(-(t-30)^2/600 - 12it)$$

Now we have close frequencies $=12$ and $=11$ on two intersecting interval.



Due a difference of phases, a meeting of two close ridges produces a sequence of peaks.

Two additional remarks. The same phenomenon of interference is present on Fig. a) in the valley between ridges. Next, the west and east ends of the mountain system b) are smooth because near each end only one frequency is present

Fig. 4.5:

$\widehat{\Psi}_\tau(\xi)$ at infinity ($\sim 1/\xi$). Therefore, contributions of actually existing high frequencies will be confused with a strong noise of artificial singularities.

This argument shows that we need functions with rapidly decreasing Fourier transform.

SECOND REMARK. We promised a compactly supported function $r(t)$ and finally imposed $e^{-t^2/2\lambda}$. However, $\exp\{-20^2\} = 0$ in any realistic computations. In fact, our reasonings do not require a compact support, we need only in rapidly decreasing function.

After all, it seems that $e^{-x^2/2}$ is the best of functions. Indeed, a function f and its Fourier transform $\widehat{f}$ can not simultaneously decrease too rapidly, and the function $e^{-x^2/2}$ is singled out from this point of view. Recall two classical theorems.

Theorem 3.2 (Heisenberg inequalities) *a) Let A, B be self-adjoint matrices (or operators in a Hilbert space). Then, for each vector v,*

$$\|Av\| \cdot \|Bv\| \geqslant \big|\langle (AB - BA)v, v\rangle\big| \tag{3.3}$$

The equality holds iff $(\lambda A + i\mu B)v = 0$ for some real λ, μ.

b) Let $f \in L^2(\mathbb{R})$, $\widehat{f}$ be its Fourier transform. Then

$$\int_{-\infty}^{\infty} x^2 |f(x)|^2\, dx \cdot \int_{-\infty}^{\infty} x^2 |\widehat{f}(x)|^2\, dx \geqslant \Big(\int_{-\infty}^{\infty} |f(x)|^2\, dx\Big)^2 \tag{3.4}$$

The equality holds iff $f(x) = e^{-\sigma x^2/2}$ for some $\sigma > 0$.

PROOF. a) Let λ, $\mu \in \mathbb{R}$. Then

$$\begin{aligned} 0 \leqslant \|(\lambda A + i\mu B)v\|^2 = \langle (\lambda A + i\mu B)v, (\lambda A + i\mu B)v\rangle = \\ = \lambda^2 \langle Av, Av\rangle + \lambda\mu \langle i(BA - AB)v, v\rangle + \mu^2 \langle Bv, Bv\rangle \end{aligned}$$

The inequality (3.3) is precisely the condition of positivity of the quadratic trinomial. Equality holds iff $\|\cdot\| = 0$.

b) We put $Af(x) = xf(x)$, $B(x) = i\frac{\partial}{\partial x} f(x)$, hence $AB - BA = i$. Next, we rewrite $\|Bx\|^2$ by the Plancherel formula for the Fourier transform. The equality holds for solutions of differential equations

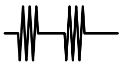
$$\big(\lambda x + \mu \frac{\partial}{\partial x}\big) f(x) = 0 \qquad \square$$

Problem 3.1 Under the same conditions, $\|Av\| \cdot \|Bv\| \geqslant |\langle (AB + BA)v, v\rangle$.

REMARK. It is no wonder that the Heisenberg inequality which is known in quantum mechanics appears in our situation. Indeed, in both cases there arises a problem of localization of wave packets. $\square$

$r(t)$:

This $r(t)$ does not allow to separate waves of law frequency.



$r(t)$

and this $r(t)$ does not allow to localize signals having a short duration.

Fig. 4.6:

Theorem 3.3 (Hardy) *Let f be a function, $\widehat{f}$ its Fourier transform*

a) $\quad f(x) = O(e^{-x^2/2}),\ \widehat{f}(x) = O(e^{-x^2/2}) \Longrightarrow f(x) = \mathrm{const}\cdot e^{-x^2/2}$

b) $f(x) = O(x^N\cdot e^{-x^2/2}),\ \widehat{f}(x) = O(x^N\cdot e^{-x^2/2}) \Longrightarrow f(x) = \{\text{polynomial}\}\cdot e^{-x^2/2}$

For a proof, see [3], Ex.6.51.

3.6. An extension of the Segal-Bargmann transform. Obviously, a short-time Fourier transform with a fixed $r(t)$ does not allow to recognize too long waves and also waves of too short duration, see Fig. 4.6

We define the *Iagolnitzer–Bros transform* as

$$\mathcal{IB}f(\lambda,\tau,\xi) = \lambda^{-1/2}\int_{-\infty}^{\infty} f(t)\exp\Big\{-\frac{(t-\tau-i\xi)^2}{2\lambda}\Big\}\,dt = \tag{3.5}$$

$$= \lambda^{-1/2}e^{\xi^2/2\lambda}e^{-i\tau\xi}\int_{-\infty}^{\infty} f(t)\exp\Big\{-\frac{(t-\tau)^2}{2\lambda}\Big\}e^{i\xi t/\lambda}\,dt \tag{3.6}$$

where $\lambda > 0$, τ, $\xi \in \mathbb{R}$.

It is more-or-less short time Fourier transform, we only write an additional factor in front of the integral[3] and also change $\xi \to \xi\lambda$.

A function $g := \mathcal{IB}f$ satisfies the heat equation

$$\Big(\frac{\partial}{\partial\lambda} - \frac{\partial^2}{\partial z^2}\Big)g(\lambda,z) = 0, \qquad z := \tau + i\xi$$

Moreover, expression (3.5) is the Poisson formula (1.3.9). We only permit the space z coordinate to be complex (we can also admit complex λ with $\mathrm{Re}\,\lambda > 0$, but this is not useful in the context of this and next sections).

Problem 3.2 Let

$$\gamma(t) := Ae^{i\omega t} + Be^{(i\omega+\varepsilon)t} = e^{i(\omega+\frac{\varepsilon}{2})t}(Ae^{-i\varepsilon t/2} + Be^{i\varepsilon t/2})$$

[3]It may be that it is better to omit this factor

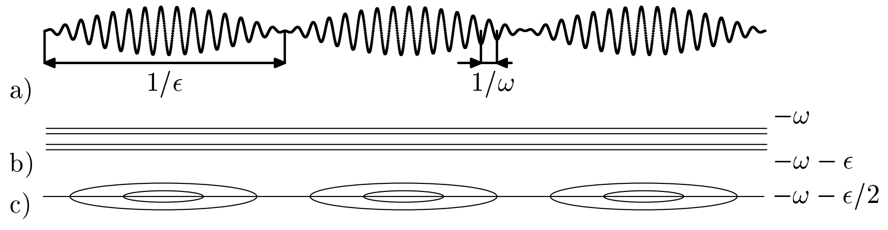


Fig. 4.7: To Problem 3.2. a) The function $\gamma(t)$;
b), c) $|H|$ for very large and for intermediate values of λ.

and $H(\lambda,\tau,\xi) := e^{-\xi^2/2}\mathcal{I}\mathcal{B}(\lambda,\tau,\xi)$. Convince yourself in the following behavior of H on the planes $\lambda = \text{const}$.

1°. If λ is very large with respect to ε^{-1}, then the graph $|H(\lambda,\tau,\xi)|$ looks like two infinite parallel ridges along the lines $\xi = -\omega\lambda$, $\xi = -(\omega+\varepsilon)\lambda$.

2°. If $\lambda > 0$ is very small, then $H \simeq \gamma$.

3°. An interesting phenomenon appears for an (appropriate) intermediate values of λ, namely $\varepsilon \ll \lambda \ll \omega$; level lines of $|H(\lambda,\tau,\xi)|$ are shown on Fig. 4.7.b. We have an infinite chain of ridges supported near line $\xi = -(\omega+\varepsilon/2)\lambda$, their outlines are controlled by the function $Ae^{-i\varepsilon t/2} + Be^{i\varepsilon t/2}$.

The reader can observe the results of solution of Problem c) in our every-day life. First, a radio-receiver produces a short-time Fourier transform of electromagnetic waves near a certain frequency $\xi = \omega$ with a certain λ (we ourselves choose ξ by tuning, $\lambda = \lambda(\xi)$ is chosen by engineers). We receive 'outline of ridges' as sound waves; next, our ears and brain produce a new short-time Fourier transform in certain interval of frequencies.

3.7. Inversion formulae.

Problem 3.3 Verify the following inversion formulae for a short time Fourier transform,

$$\gamma(t)r(t-\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{G}_r(\tau,\xi)e^{-i\tau\xi}\,d\xi$$
$$\gamma(t)\cdot\int_{-\infty}^{\infty} r(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{G}_r(\tau,\xi)e^{-i\tau\xi}\,d\xi\,d\tau$$
$$\widehat{\gamma}(\xi)\cdot\int_{-\infty}^{\infty} r(t)\,dt = \int_{-\infty}^{\infty} \mathcal{G}_r(\tau,\xi)e^{-i\tau\xi}\,d\tau$$

Problem 3.4 Let r_1, r_2 be even and supported by intervals $|t| \leqslant \alpha_1$, $|t| \leqslant \alpha_2$ respectively, $\int r_1\,dt = \int r_2\,dt = 1$. Representing $f_1\cdot f_2$ in the form

$$f_1(t)f_2(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(t)f_2(t)r_1(t-\tau)r_2(t-\tau)\,d\tau\,d\sigma$$

show that

$$\widehat{f_1f_2}(\xi) = \iint_{\sigma,\tau:\,|\sigma-\tau|\leqslant\alpha_1+\alpha_2} \mathcal{G}_{r_1}(\tau,\xi) *_\xi \mathcal{G}_{r_2}(\sigma,\xi)\,d\sigma\,d\tau \tag{3.7}$$

where $*$ denotes the convolution with respect to ξ. This is not the shortest way of evaluation of the Fourier transform. However, sometimes this is a good method to calculate products of functions $f_1 \cdot f_2$ (see the next section).

4.4 Spectral analysis of singularities

In this section, we discuss wave fronts of distributions and visualizations of wave fronts in the terms of short time Fourier transforms and the Iagolnitzer–Bros transform.

4.1. Lemmae on convolutions. Let f, g be functions on $\mathbb{R}^n$. Recall that their *convolution* $f * g$ is the function defined by

$$(f * g)(x) = \int_{y_1+y_2=x} f(y_1)g(y_2)\, dy_1 := \int_{\mathbb{R}^n} f(y_1)g(x-y_1)\, dy_1 \tag{4.1}$$

This integral can be divergent, it can be readily checked that it converges a.s when f, g are integrable on $\mathbb{R}^n$. In this subsection, we discuss some cases of existence of convolutions for functions admitting growth at infinity.

Recall that a convolution is a commutative associative operation and the Fourier transform takes a convolution to a product

$$\widehat{f * g} = \widehat{f} * \widehat{g}$$

A *cone* $\Gamma \subset \mathbb{R}^n$ is a subset invariant under to homotheties, i.e., $x \in \Gamma$ implies $tx \in \Gamma$ for all $t > 0$. By $\overline{\Gamma}$ we denote the closure of the cone Γ.

For two cones Γ_1, Γ_2, define their *sum* as the set of vectors $x_1 + x_2$, where x_1 ranges in Γ_1 and x_2 ranges in Γ_2.

Problem 4.1 If Γ_1, Γ_2 are closed cones, then $\Gamma_1 + \Gamma_2$ is a closed cone. If they are also convex, then $\Gamma_1 + \Gamma_2$ is the closed convex hull of $\Gamma_1 \cup \Gamma_2$.

We say that a smooth function f on $\mathbb{R}^n$ is *tempered* if all its partial derivatives (including f itself) admit estimates

$$\left|\frac{\partial^\alpha}{\partial x^\alpha} f(x)\right| \leqslant C(1+|x|)^N \qquad \text{for some } N > 0 \text{ and } C$$

A tempered function *rapidly decreases* in an open cone Γ if for each multi-index α and for each $N > 0$ there is a constant C such that

$$\left|\frac{\partial^\alpha}{\partial x^\alpha} f(x)\right| \leqslant C(1+|x|)^{-N}, \qquad x \in \Gamma \tag{4.2}$$

For nonzero $y \in \mathbb{R}^n$ consider the corresponding ray, i.e., the set $\{t \cdot y\}$, where t ranges in positive numbers. A tempered function *rapidly decreases along a ray* if there is an open cone Γ containing the ray such that f rapidly decreases in this cone. For a function f denote by $\Gamma(f)$ the (closed) cone that is complement to the set of all the directions of rapid decreasing.

Problem 4.2 In any closed sub-cone Ω in the complement of $\Gamma(f)$, the estimate (4.2) holds. HINT. Apply the Heine–Borel Lemma about finite subcoverings.

Lemma 4.1 *a) A convolution of rapidly decreasing functions rapidly decreases.*

b) If f rapidly decreases in an open cone Δ and g rapidly decreases in $\mathbb{R}^n$, then $f * g$ rapidly decreases in Δ. In other words, $\Gamma(f * g) \subset \Gamma(f)$.

c) Let f, g be supported by open cones Δ and Ξ respectively, and $\overline{\Delta} \cap (-\overline{\Xi}) = 0$. Then $f * g$ is a tempered function supported by the cone $\Delta + \Xi$.

d) Let f, g be tempered function satisfying $\Gamma(f) \cap (-\Gamma(g)) = 0$. Then the convolution $f * g$ is well defined. Moreover, it is tempered and

$$\Gamma(f * g) \subset \Gamma(f) + \Gamma(g)$$

PROOF. a) We must estimate the convolution (4.1). For this aim, we write

$$|f(y_1)| \leqslant \text{const} \cdot (1 + |y_1|)^{-N-M}, \qquad |g(y_2)| \leqslant \text{const} \cdot (1 + |y_2|)^{-M}$$

and estimate (4.1) as

$$|f * g(x)| \leqslant \text{const} \int_{y_1 + y_2 = x} (1 + |y_1|)^{-N-M} (1 + |y_2|)^{-M} \, dy_1$$

Since $y_1 + y_2 = x$, it follows that either $|y_1| \geqslant |x|/2$ or $|y_2| \geqslant |x|/2$. Therefore,

$$(1 + |y_1|)^{-M} (1 + |y_2|)^{-M} \leqslant (1 + |x|/2)^{-M},$$

(we 'forget' one of the factors). Thus,

$$|f * g(x)| = \left| \int \right| \leqslant (1 + |x|/2)^{-M} \int (1 + |y_1|)^{-N} dy_1$$

If $N > n + 1$, then the last integral converges. Thus, the decreasing of $f * g$ is proved.

Partial differentiations commute with convolutions, $\partial^\alpha (f * g) = (\partial^\alpha f) * g$ and this implies the decreasing of derivatives.

b) Consider a closed subcone Δ^* in our open cone Δ. If $x \in \Delta^*$, $y \notin \Delta$, then for some (small) constants ε, δ independent on x, y

$$|x| \leqslant \varepsilon^{-1} \cdot |x - y|, \qquad |y| \leqslant \delta^{-1} \cdot |x - y| \tag{4.3}$$

Now we are ready to estimate (4.1) for $x \in \Delta^*$. We split the integral into two summands corresponding $y_1 \in \Delta$ and $y_1 \notin \Delta$. Obviously (due to the proof of a), the integral over Δ rapidly decreases as function of x. Therefore, let $y_1 \notin \Delta$. Applying estimates (4.3), we get

$$|f(y_1)| \leqslant \text{const}(1 + |y_1|)^k,$$
$$|g(y_2)| \leqslant \text{const}(1 + |y_2|)^{-N-M} \leqslant \text{const}(1 + \delta|y_1|)^{-N} \cdot (1 + \varepsilon|x|)^{-M}$$

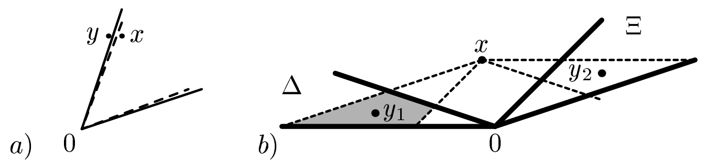


Fig. 4.8: a) To proof of Lemma 4.1.b. The cone Δ and its subcone Δ^*.
b) To proof of Lemma 4.1.c. The domain of integration. If $\overline{\Xi} \cap (-\overline{\Delta}) \neq 0$, then this domain is non-compact.

Now rapid decreasing of (4.1) becomes obvious.

c) Since $\overline{\Delta} \cap (-\overline{\Xi}) = 0$, the domain of integration in (4.1) is compact (see Fig. 4.8.b). Further, if $x = y_1 + y_2$ with $y_1 \in \overline{\Delta}$, $y_2 \in \overline{\Xi}$, then for some constant ε,

$$|y_1| \leqslant \varepsilon^{-1}|x|, \qquad |y_2| \leqslant \varepsilon^{-1}|x|$$

(this is an exercise in the elementary topology). Hence the domain of integration (4.1) is contained in the ball $|y_1| \leqslant \varepsilon^{-1}|x|$ and its volume $\leqslant \mathrm{const}\, \varepsilon^{-n}|x|^n$. Also,

$$|f(y_1)| \leqslant (1 + |y_1|)^m \leqslant (1 + \varepsilon^{-1}|x|)^m, \quad |g(y_2)| \leqslant (1 + |y_2|)^k \leqslant (1 + \varepsilon^{-1}|x|)^k,$$

Therefore (4.1) is $\leqslant \mathrm{const} \cdot |x|^{n+m+k}$.

d) We decompose $f = f_1 + f_2$, $g = g_1 + g_2$ with f_1, g_1 supported by respective cones and f_2, g_2 being rapidly decreasing. Further, we apply a), b), c). □

4.2. The Localization Lemma.

Theorem 4.2 (Localization Lemma) *Let u be a distribution on $\mathbb{R}^n$ with a compact support*[4]*. Let φ be a smooth function. Then*

$$\Gamma(\widehat{\varphi u}) \subset \Gamma(\widehat{u})$$

Proof. This is a rephrasing of Lemma 4.1.b. □

4.3. Wave fronts. Let u be a distribution on $\mathbb{R}^n$, let $x \in \mathbb{R}^n$. Choose a sequence of compactly supported smooth functions φ_j such that $\varphi_j(x) \neq 0$ and the diameters of their supports tend to 0. Then the *wave front* of the distribution u at x is

$$\mathrm{WF}_x(u) := \bigcap_j \Gamma(\widehat{\varphi_j u})$$

It is convenient to exclude the point $x = 0$ from considerations and to regard the wave front as a conic subset in $\mathbb{R}^n \setminus 0$.

Lemma 4.3 *The result does not depend on a choice of a sequence φ_j.*

[4] And hence $\widehat{u}$ is holomorphic function in $\mathbb{C}^n$

PROOF. Let ψ_k be another such sequence. Fix φ_l. Starting some j, one has $\psi_l = \varphi_j \theta_{jl}$, where θ_{jl} are a smooth compactly supported functions. By the Localization Lemma, $\Gamma(\widehat{\psi_j u}) \subset \Gamma(\widehat{\varphi_l u})$. Now the statement becomes obvious.□

The *total wave front* $\mathrm{WF}(u)$ is the subset in $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ defined by

$$(x, \xi) \in \mathrm{WF}(u) \qquad \text{whenever} \qquad \xi \in \mathrm{WF}_x(u)$$

Lemma 4.4 $\Gamma(\widehat{u}) = \bigcup \mathrm{WF}_x(u)$, *where x ranges in the support*[5] $\mathrm{supp}(u)$ *of* u.

PROOF. By definition, $\mathrm{WF}_x(u) \subset \Gamma(\widehat{u})$.

Further, let $\xi \in \Gamma(\widehat{u})$. Take small $\rho > 0$ and collection of smooth positive functions $\varphi_1, \dots, \varphi_N$, whose supports are contained in balls of radii $\leqslant \rho$, and $\sum \varphi_j = 1$ on $\mathrm{supp}(u)$. Then $\xi \in \Gamma(\widehat{\varphi_k u})$ for some k. Next, we consider smaller ρ and repeat the same operation with 'new' distribution $\varphi_k u$, etc. In this way, we obtain a sequence $\theta_\alpha u$ of distributions such that

1. Diameters of their supports tend to 0
2. $\mathrm{supp}(\theta_\alpha u) \supset \mathrm{supp}(\theta_{\alpha+1} u)$.

Let x be the unique point of intersection of the supports. Then $\xi \in \mathrm{WF}_x(u)$.
□

We can also reformulate our definition as follows:

– *a point* $(a, \xi) \notin \mathrm{WF}(u)$ *if there is a smooth compactly supported function* φ *such that* $\varphi(a) \neq 0$ *and the ray* ξ *is a ray of rapid decreasing of* $\widehat{\varphi u}$.

Thus we define $\mathrm{WF}(u)$ as the complement to a union of open sets. Therefore, *wave front is a closed subset in* $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$.

Problem 4.3 a) Find wave fronts of the following distributions on $\mathbb{R}$:

$$\ll u, \varphi \gg = \varphi(0), \quad \ll u, \varphi \gg = \int_0^\infty x^\lambda f(x)\, dx, \qquad \ll u, \varphi \gg = \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} \frac{\varphi(x)}{x}\, dx$$

$$\ll u_\pm, \varphi \gg = \lim_{\varepsilon \to \pm 0} \int_{-\infty}^\infty \frac{\varphi(x)\, dx}{x + i\varepsilon}, \quad \ll u, \varphi \gg = A \int_0^\infty x^\lambda f(x)\, dx + B \int_{-\infty}^0 (-x)^\lambda f(x)\, dx$$

b) Find the wave fronts of the following 2-dimensional distributions

$$\ll u, \varphi \gg = \int_0^{2\pi} \varphi(\cos t, \sin t)\, dt, \qquad \ll u, \varphi \gg = \int_{-1}^1 \varphi(t, -t)\, dt$$

c) Find the wave fronts of the following distributions on the space of Hermitian matrices

$$\ll u, \varphi \gg = \int_{X > 0} \det(X)^\lambda \varphi(X)\, dX, \quad \ll u, \varphi \gg = \lim_{\varepsilon \to +0} \int \det(X + i\varepsilon)^\lambda \varphi(X)\, dX$$

Problem 4.4 Show that the projection of $\mathrm{WF}(u)$ on the coordinate space $\mathbb{R}^n$ is the singular support of u. Recall the definition: $x \in \mathbb{R}^n$ does not contained in the *singular support* of u, if one can find a smooth function h such that $x \notin \mathrm{supp}(u - h)$.

[5] Recall the definition. A point x is not contained in the *support* $\mathrm{supp}(u)$ of a distribution u, if there is a neighborhood U of x such that $\ll u, f \gg = 0$ for each f supported by U.

4.4. Wave fronts and short time Fourier transforms. Let r be a smooth non-negative function with compact support, let $r(0) > 0$. Consider the short time Fourier transform of a distribution u,

$$\mathcal{G}(\lambda, \tau, \xi)u = \int_{x\in\mathbb{R}^n} u(x)r\Big(\frac{x-\tau}{\lambda}\Big)e^{i\xi x} :=\ll u(x), r\Big(\frac{x-\tau}{\lambda}\Big)e^{i\xi x} \gg, \qquad \lambda > 0$$

Fix λ. Define the set $\Omega_\lambda = \Omega_\lambda(f) \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ as:

– $(\tau, \xi) \in \Omega_\lambda$ *if the function* $\gamma(\xi) := \mathcal{G}f(\lambda, \tau, \xi)$ *rapidly decreases along the direction* ξ.

The set Ω_λ is open in $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$. If $\lambda' < \lambda$, then $\Omega'_\lambda \subset \Omega_\lambda$.

We define $\mathrm{WF}(u)$ as

$$\mathrm{WF}(u) = \bigcap \Omega_\lambda$$

This is nothing but a rephrasing of the previous definition.

Theorem 4.8 extends this construction to functions r that are not compactly supported. However, the generalization is not straightforward.

4.5. Digression. Products of distributions. Now let u, v be compactly supported distributions satisfying

$$(x, \xi) \in \mathrm{WF}(u) \quad \Longrightarrow \quad (x, -\xi) \notin \mathrm{WF}(v) \tag{4.4}$$

Then we can define their product $u \cdot v$ as follows. Fix small $\varepsilon > 0$. Choose a finite collection of non-negative smooth functions φ_j such that

1. φ_j are supported by balls of radii $\leqslant \varepsilon$
2. $\sum \varphi_j = 1$ on the supports of u and v.

Lemma 4.5 *Under (4.4), for sufficiently small* $\varepsilon > 0$

$$\operatorname{supp}\varphi_k \cap \operatorname{supp}\varphi_l \neq 0 \qquad \Longrightarrow \qquad \Gamma(\widehat{\varphi_k u}) \cap \big(-\Gamma(\widehat{\varphi_l v})\big) = 0 \tag{4.5}$$

We omit a proof (it is a topological manipulation with Lemma 4.4).

Next, we write

$$u \cdot v = (\sum \varphi_j u) \cdot (\sum \varphi_k v) = \sum_{k,l} (\varphi_k u) \cdot (\varphi_l v)$$

It is natural to think, that a product of distributions is zero when their supports have empty intersection. We write out

$$u \cdot v = \sum_{k,l:\, \operatorname{supp}\varphi_k \cap \operatorname{supp}\varphi_l \neq \varnothing} (\varphi_k u) \cdot (\varphi_l v)$$

For all the summands, we can calculate their Fourier transforms by Lemma 4.1

$$\mathcal{F}(\varphi_k u \cdot \varphi_l v) := \widehat{\varphi_k u} * \widehat{\varphi_l v}$$

and define $(\varphi_k u) \cdot (\varphi_l v)$ as the inverse Fourier transform. Thus, we know $u \cdot v$.

Problem 4.5 a) The result of the calculation does not depend on a choice of φ_k (satisfying (4.5)).

b) If $(x, \xi) \in \mathrm{WF}(u \cdot v)$, then

$$\xi \in \big(\mathrm{WF}_x(u) + \mathrm{WF}_x(v)\big) \cup \mathrm{WF}_x(u) \cup \mathrm{WF}_x(v)$$

(the last expression is the closure of $\mathrm{WF}_x(u) + \mathrm{WF}_x(v)$; we recall that $0 \notin \mathrm{WF}_x(\cdot)$, for this reason the sum of cones is not closed).

4.6. Digression. A rephrasing of the definition of products. Let u, v be the same as above, let r be a smooth function supported by a ball $|x| \leqslant \varepsilon$, and $\int r\,dx = 1$, $r(0) \neq 0$. We write

$$u = \int u(x)r(x-\tau)\,d\tau, \qquad v = \int v(x)r(x-\sigma)\,d\sigma$$

$$u \cdot v = \int\int \big[u(x)r(x-\tau)\big] \cdot \big[v(x)r(x-\sigma)\big]\,d\sigma\,d\tau$$

We *assume* that the integrand is 0 outside the strip $|\sigma - \tau| < 2\varepsilon$ (because $r(x-\sigma)r(x-\tau) = 0$ outside the strip).

$$u \cdot v := \iint_{|\sigma-\tau|\leqslant 2\varepsilon} \big[u(x)r(x-\tau)\big] \cdot \big[v(x)r(x-\sigma)\big]\,d\sigma\,d\tau$$

Further, we take the Fourier transform of both sides; in the right-hand side we represent Fourier transforms of product as convolutions

$$\widehat{u \cdot v}(\xi) = \iint_{|\sigma-\tau|\leqslant 2\varepsilon} \mathcal{G}_r(\tau,\xi) *_\xi \mathcal{G}_r(\sigma,\xi)\,d\sigma\,d\tau$$

If ε is sufficiently small, then the convolutions are well defined and we can evaluate $\widehat{u \cdot v}$ and $u \cdot v$.

4.7. Wave fronts and the Iagolnitzer–Bros transform. Let us apply the Iagolnitzer–Bros transform $\mathcal{IB}$, see Subsection 3.6, to a compactly supported distribution $u(x)$ on $\mathbb{R}^n$,

$$\mathcal{IB}\,u(\lambda, z) := \lambda^{-1/2} \cdot \ll u(y), \exp\big\{-(y-z)^2/2\lambda\big\} \gg, \qquad z = x + i\xi \in \mathbb{C}^n$$

Lemma 4.6 *Fix x. If $\xi \notin \mathrm{WF}_x(u)$, then for η in a neighborhood of ξ*

$$\mathcal{IB}u(\lambda, x, \eta) = e^{|\eta|^2/2\lambda} \cdot O(\lambda^N) \qquad \textit{for all } N, \qquad \lambda \to +0 \tag{4.6}$$

PROOF. We write

$$\lambda^{-n/2} e^{\xi^2/2\lambda} \cdot e^{-ix\xi}\mathcal{IB}u(\lambda, x, \xi) = \int_{y\in\mathbb{R}^n} u(y) e^{-(y-x)^2/2\lambda} e^{i\xi y/\lambda} \tag{4.7}$$

We are going to examine a behavior of the last integral. For this aim, we represent $u = v + w$, where $\operatorname{supp} w$ does not contain x and $\Gamma(\widehat{v})$ does not contain ξ. Write our integral as

$$\int v(y) e^{-(y-x)^2/2\lambda} e^{i\xi y/\lambda} + \int w(y) e^{-(y-x)^2/2\lambda} e^{i\xi y/\lambda} \tag{4.8}$$

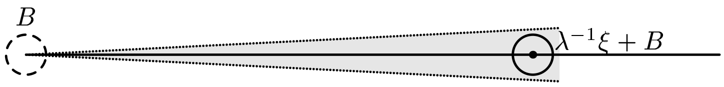


Fig. 4.9: To proof of Lemma 4.6. A cone of rapid decriasing of $\widehat{v}(\zeta)$ around the ray $\mathbb{R}_+\xi$. Since $\lambda^{-1} \gg \lambda^{-1/2}$, a shifted ball is contained in the cone for small λ.

The second summand of (4.8). Since w is a continuous linear functional in some C^N-topology, one has

$$| \ll w, \varphi \gg | \leqslant \sum_j \sum_{|\alpha|\leqslant N} C_\alpha \max_{\mathrm{supp}(w)} |\partial^\alpha \varphi|$$

For our $\varphi = e^{-(y-x)^2/2\lambda} e^{i\xi y/\lambda}$, partial derivatives have the form *polynomial* $\cdot$ $e^{-(y-x)^2/2\lambda}$. Since $x \notin \mathrm{supp}(w)$, the integral can be estimated as $\leqslant \mathrm{const} \cdot e^{-\lambda^{-1}\delta}$.

The first summand of (4.8). Applying the Plancherel formula, we get

$$\lambda^{1/2} \int \widehat{v}(\zeta) \exp\Big\{ \frac{\lambda}{2}(\zeta - \xi/\lambda)^2 + ix(\zeta - \xi/\lambda) \Big\}\, d\zeta$$

This expression is the convolution of $\widehat{v}(\zeta)$ and $e^{-\lambda\zeta^2/2 - ix\zeta}$, the value of the convolution is taken at a point $\zeta = \xi/\lambda$.

The function $\widehat{v}$ rapidly decreases in the direction ξ. The function $e^{-\lambda\zeta^2/2+ix\zeta}$ is very small (and very rapidly decreases) outside some ball $B : |\zeta| \leqslant R \cdot \lambda^{-1/2}$, Hence the shifted ball $\lambda^{-1}\xi + B$ is contained in a cone of rapid decreasing of $\widehat{v}(\zeta)$, therefore the convolution approximately equals to the integral over the shifted ball, and hence it is $O(\lambda^{-N})$. We omit precise estimates. □

Theorem 4.7 (A.Cordoba, C.Fefferman) *Let u be a tempered distribution. Then $(x, \xi) \notin \mathrm{WF}(u)$ iff the estimate (4.6) holds.*

We omit a proof. Next, we formulate (also without a proof) a more general statement (see Folland, [40]) which clarifies the Cordoba–Feferman theorem.

Let $r \in \mathcal{S}(\mathbb{R}^n)$ be an arbitrary nonzero *even* function, $r(0) \neq 0$. For a tempered distribution u, we write the expression

$$\mathcal{P}(s; \xi, x) := s^{n/4} \int e^{is\xi y} r\big(\sqrt{s}(x-y)\big)\, u(y)$$

Theorem 4.8 $(\xi_0, x_0) \notin \mathrm{WF}(u)$ *iff in some neighborhood of* (ξ_0, x_0) *the following estimate holds*

$$\mathcal{P}(s; \xi, x) = O(s^{-N}), \qquad s \to \infty$$

for all N

In the previous theorem, $\varphi(x) = e^{-x^2/2}$, $s = \lambda^{-1}$. The factor $e^{|\eta|^2/2\lambda}$ arises from another normalization.

4.8. The Iagolnitzer–Bros transform and analyticity.

Theorem 4.9 *Let u be a compactly supported distribution. Then u is a real analytic function in a neigborhood of x_0 iff for some $\varepsilon > 0$, $K > 0$, and C*

$$|\mathcal{IB}u(\lambda, x, \xi)| \leqslant C \cdot \exp\{|\xi|^2/2\lambda\} \cdot \exp\{-\varepsilon/\lambda\}, \qquad |\xi| > K$$

We prove $\Rightarrow$, it a trivial part of the theorem. To simplify notation, put $n = 1$. We must examine a behavior of the integral

$$\int_{y\in\mathbb{R}} u(y)e^{-s(y-x)^2/2}e^{is\xi y}$$

as $s \to \infty$. Let u be real analytic on a segment $[x_0 - \nu, x_0 + \nu]$. Let us choose an appropriate smooth nonnegative function $\psi(y)$ such that

1. $\psi(y) = 0$ outside our segment $[x_0 - \nu, x_0 + \nu]$
2. $\psi(y)$ is a small constant on some interval $[x_0 - \delta, x_0 + \delta]$.

Moving a contour of integration $y \mapsto y + i\psi(y)$, we get

$$\int_{y\in\mathbb{R}} u(y + i\psi(y))e^{-s(y+i\psi(y)-x)^2/2}e^{is\xi(y+i\psi(y))}$$

Next, represent $u(y + i\psi y) = v(y) + w(y)$, where the support of w does not contain x_0 and v is supported by $[x_0 - \delta, x_0 + \delta]$. Respectively, split the integral into the two summands. For the summand with w we get exponential decreasing as we mentioned in the proof of Lemma 4.6. For the summand with v we have the get a smooth integrand having exponential decreasing in s. $\square$

This theorem is an origin of the following definition of analytic wave fronts of distributions (we only present the definition and stop at this point). Let u be a compactly supported distribution on $\mathbb{R}^n$. A point (x, ξ) is not a point of the *analytic wave front* of u if there are a neighborhood $\mathcal{O}$ of (x, ξ) and positive constants C, K and δ such that

$$\mathcal{IB}u(\lambda, x, \xi) \leqslant C \cdot \exp\{(|\xi|^2 - \delta)/2\lambda), \qquad \text{for } |\xi| > K,\ (x, \xi) \in \mathcal{O}.$$

4.5 Symbols of operators

5.1. *qp*-symbols of polynomial differential operators. As above, define operators

$$Q_k f(x) = x_k f(x), \qquad P_k f(x) = -i\frac{\partial}{\partial x_k} f(x)$$

in the space of functions on $\mathbb{R}^n$.

To each polynomial

$$\sigma(x, \xi) = \sum c_{\alpha\beta} x^\alpha \xi^\beta \tag{5.1}$$

on $\mathbb{R}^n \times \mathbb{R}^n$, we assign the differential operator

$$\widehat{\sigma}(P, Q) = \sum c_{\alpha\beta} P^\alpha Q^\beta \tag{5.2}$$

We say, that σ is the qp-symbol of the differential operator $\widehat{\sigma}$

Conversely, for a differential operator $D(x, -\partial_x)$ we write

$$\sigma(x,\xi) = \Big(D(x, -\partial_y)e^{i(y-x)\xi}\Big)\Big|_{y=x} \tag{5.3}$$

Theorem 5.1 *a) The symbol the adjoint operator $\widehat{\sigma}(Q,P)$ is*

$$\exp\Big\{-i\sum_{k=1}^{n}\frac{\partial^2}{\partial x_k\partial\xi_k}\Big\}\sigma(x,\xi) \tag{5.4}$$

b) The symbol $\sigma_3(x,\xi)$ of a product $\widehat{\sigma}_1\cdot\widehat{\sigma}_2$ is given by the formula

$$\sigma_3(x,\xi) = \exp\Big\{-i\sum_{k=1}^{n}\frac{\partial^2}{\partial y_k\partial\eta_k}\Big\}\sigma_1(x,\eta)\sigma_2(y,\xi)\Big|_{y=x,\eta=\xi} \tag{5.5}$$

Proof. In both cases the question is reduced to a change of order of P and Q. More precisely, we must find an explicit formula for

$$\prod\Big(i\frac{\partial}{\partial x_k}\Big)^{\alpha_k}\prod x_k^{\beta_k}f(x)$$

Since P_k and Q_l commute for $k\neq l$, the question is reduced to the case of one variable. By the Leibnitz formula,

$$\Big(-i\frac{\partial}{\partial x}\Big)^m x^l f(x) =$$
$$= \sum_{k\geqslant 0}(-i)^k\frac{l(l-1)\dots(l-k+1)\cdot m(m-1)\dots(m-k+1)}{k!}x^{m-k}\Big(-i\frac{\partial}{\partial x}\Big)^{m-k}f(x)$$

Thus the symbol of the operator Q^lP^m is

$$\exp\Big\{-i\frac{\partial^2}{\partial x\partial\xi}\Big\}(x^l\xi^m)$$

Both statements easily follow from this observation. □

5.2. ***pq*-symbols.** Let us play at changing rules associating a differential operator to a given polynomial $\sigma(x,\xi)$. The first variant is to change the order of P and Q,

$$\check{\sigma}(P,Q) := \sum c_{\alpha\beta}P^\beta Q^\alpha$$

We say that σ is the *pq-symbol* of the operator $\check{\sigma}$ Evidently,

$$\check{\sigma}(P,Q) = \big(\widehat{\sigma}(P,Q)\big)^*$$

This reduces pq symbols to qp-symbols.

5.3. Weyl symbols. For any collection of linear operators $X_1,\ldots, X_m$ define its symmetric product

$$[X_1 \ldots X_m]_{sym} := \frac{1}{m!}\sum_{\tau\in S_m} X_{\tau(1)}\ldots X_{\tau(m)}$$

where the summation is taken over all the permutations of the set $\{1,\ldots,m\}$. Denote by

$$[X_1^{\alpha_1}\ldots X_l^{\alpha_l}]_{sym}$$

the symmetric product of α_1 copies of X_1, α_2 copies of X_2, etc.

It is useful to keep in mind the following formulae

$$[X_1 \ldots X_m]_{sym} = \frac{\partial^m}{\partial s_1\ldots\partial s_m}\exp\{s_1X_1+\cdots+s_mX_m\}\Big|_{s_1=\cdots=s_m=0} \tag{5.6}$$

$$[X_1^{\alpha_1}\ldots X_m^{\alpha_m}]_{sym} == \Big(\prod \frac{\partial^{\alpha_j}}{\partial s_j^{\alpha_j}}\Big)\exp\Big\{\sum s_jX_j\Big\}\Big|_{s_j=0} \tag{5.7}$$

Further, for a polynomial $\tau(x,\xi)$ we define the operator

$$\hat{\hat{\tau}} := \sum_{\alpha\beta} c_{\alpha\beta}[Q^\alpha P^\beta]_{sym} := \sum_{\alpha\beta} c_{\alpha\beta}\cdot\big[Q_1^{\alpha_1}\ldots Q_m^{\alpha_m}P_1^{\beta_1}\ldots P_m^{\beta_m}\big]_{sym}$$

We say that τ is the *Weil symbol* of the operator $\hat{\hat{\tau}}$.

Since P_k and Q_l with different indices commute, we can write

$$\big[Q_1^{\alpha_1}\ldots Q_m^{\alpha_m}P_1^{\beta_1}\ldots P_m^{\beta_m}\big]_{sym} = \big[Q_1^{\alpha_1}P_1^{\beta_1}\big]_{sym}\cdots\big[Q_m^{\alpha_m}P_1^{\beta_m}\big]_{sym}$$

Theorem 5.2 *a) The Weyl symbol $\tau(x,\xi)$ and qp-symbol $\sigma(x,\xi)$ of the same operator are connected as*

$$\tau(x,\xi) = \exp\Big\{\frac{i}{2}\sum_{k=1}^n \frac{\partial^2}{\partial x_k\partial\xi_k}\Big\}\sigma(x,\xi)$$

b) If $\tau(x,\xi)$ is the Weyl symbol of an operator A, then the Weyl symbol of A^* is $\overline{\tau(x,\xi)}$

c) The product of operators with the symbols τ_1 and τ_2 has the symbol

$$\tau_3(x,\xi) = \exp\Big\{\frac{i}{2}\sum_k\Big(\frac{\partial^2}{\partial x_k\partial\eta_k}-\frac{\partial^2}{\partial y_k\partial\xi_k}\Big)\Big\}\tau_1(x,\xi)\tau_2(y,\eta)\Big|_{y=x,\eta=\xi} \tag{5.8}$$

Lemma 5.3 *Put $n=1$. The qp-symbol of $[Q^mP^l]_{sym}$ is*

$$\exp\Big\{-\frac{i}{2}\frac{\partial^2}{\partial x\partial\xi}\Big\}x^m\xi^l$$

Proof of Lemma. The formula (5.7) suggests us to evaluate the following exponent

$$\exp\Big\{sx - it\frac{\partial}{\partial x}\Big\}f(x) = e^{sx-ist/2}f(x-ist) = e^{-ist/2}\exp\{sx\}\exp\Big\{-it\frac{\partial}{\partial x}\Big\}f(x)$$

To avoid questions about the convergence, we consider these operators in the space of all the entire functions on $\mathbb{C}$. As soon as a formula for exponent is written, it can be easy verified. However, this is a calculation in the Heisenberg group.

It remains to extract the Taylor coefficient at $s^m t^l$ and compare with the desired formula. □

Proof of theorem. a) immediately follows from the lemma, b) is obvious.

To derive c), we evaluate qp-symbols of our operators by the statement a), evaluate qp-symbol of the product by formula (5.5), and finally return to Weyl symbols again. In fact, we must exaluate

$$\exp\Big\{\frac{i}{2}\sum_k\Big(\frac{\partial}{\partial x_k}+\frac{\partial}{\partial y_k}\Big)\cdot\Big(\frac{\partial}{\partial \xi_k}+\frac{\partial}{\partial \eta_k}\Big)\Big\}\cdot\exp\Big\{-i\sum_k\frac{\partial^2}{\partial y_k\partial\xi_k}\Big\}\times$$
$$\times\exp\Big\{-\frac{i}{2}\sum_k\frac{\partial^2}{\partial x_k\partial\xi_k}\Big\}\exp\Big\{-\frac{i}{2}\sum_k\frac{\partial^2}{\partial y_k\partial\eta_k}\Big\}\tau_1(x,\xi)\tau_2(y,\eta)\Big|_{y=x,\eta=\xi} \qquad (5.9)$$

We must explain the first exponential factor that corresponds to returning to the Weil symbols. To apply (**??**), we must differentiate a result of a substitution (5.5). For this aim, we write out the usual Leibnitz formula as

$$\frac{\partial^m}{\partial^m x}f(x)g(x) = \Big(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\Big)^m f(x)g(y)\Big|_{x=y}$$

It remains to multiply the exponents in (5.9). Since all the operators that are present in the formula commute, this step is trivial. We obtain (5.8).

5.4. *qp*-symbols. General case. Now consider an arbitrary integral operator $\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n)$,

$$Af(x,y) = \int_{y\in\mathbb{R}^n} K(x,y)f(y) \qquad (5.10)$$

We define its qp-symbol as

$$\sigma(x,\xi) = \int_{y\in\mathbb{R}^n} K(x,y)e^{i(x-y)\xi^t} \qquad (5.11)$$

Proposition 5.4 *a) σ is a tempered distribution*

b) If $K\in\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n)$, then $K\in\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n)$.

c) The inversion formula is

$$K(x,y) = \frac{1}{(2\pi)^n}\int_{\xi\in\mathbb{R}^n}\sigma(x,\xi)e^{i(x-y)\xi^t} \qquad (5.12)$$

d) For polynomial differential operators $\sigma(x,\xi)$ coincides with qp-symbol defined above.

e) The symbol of A^* is

$$\frac{1}{(2\pi)^n}\iint_{y\in\mathbb{R}^n,\,\eta\in\mathbb{R}^n}\overline{\sigma(y,\eta)}e^{-i(x-y)(\xi^t-\eta^t)} \tag{5.13}$$

PROOF. a),b),c) We write the formula for symbols as

$$\sigma(x,\xi)=e^{ix\xi^t}\int_{y\in\mathbb{R}^n}K(x,y)e^{-iy\xi^t}$$

the integral is a Fourier transform with respect to variable y, it takes tempered distributions to tempered distribution. Also the inversion formula becomes obvious.

d) is formula (5.3).

e) is a straightforward corollary of the definition and the inversion formula. □

We emphasis that formulae (5.11)–(5.13) determine Gaussian integral operators in sense of Chapter 1, these operators are bounded as operators

$$\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n),\ \mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n\times\mathbb{R}^n),$$
$$L^2(\mathbb{R}^n\times\mathbb{R}^n)\to L^2(\mathbb{R}^n\times\mathbb{R}^n)$$

Consider a convolution of kernels of two integral operators as

$$K_3(x,z)=\int_{y\in\mathbb{R}^n}K_1(x,y)K_2(y,z) \tag{5.14}$$

We interpret it in the following (at the first glance, strange) way. Consider the Gaussian integral operator $\mathcal{S}(\mathbb{R}^{4n})\to\mathcal{S}(\mathbb{R}^{2n})$ given by

$$Jf(x,z)=\int_{\mathbb{R}^n}f(x,y,y,z)\,dy$$

The operator J takes the function $K_1(x,y)K_2(y',z)$ to K_3 given by formula (5.14).

Since a composition of Gaussian operators is a Gaussian operator (and we know how to calculate it), a product-formula for symbols must be written as a Gaussian integral operator. For this reason, we present the following result without a proof.

Theorem 5.5 *The symbol of a product is given by*

$$\sigma_3(x,\xi)=\int_{y\in\mathbb{R}^n,\,\eta\in\mathbb{R}^n}\sigma_1(x,\eta)\sigma_2(y,\xi)e^{-i(x-y)(\xi^t-\eta^t)} \tag{5.15}$$

The formula holds at least for σ_1, σ_2 being in the Schwartz space.

The crucial point of our discussion are two product formulas (5.5) and (5.5). It turn out to be that the first formula[6] survives in essential wider generality. This phenomenon is a standpoint of theory of *pseudodifferential operators*. However, we stop our propulsion here.

Problem 5.1 Derive the following formulae (which hold at least for symbols being the Schwartz space)

$$\operatorname{tr} A* = \int_{x\in\mathbb{R}^n,\xi\in\mathbb{R}^n} \sigma(x,\xi)$$
$$\operatorname{tr} A_1 A_2^* = \int_{x\in\mathbb{R}^n,\xi\in\mathbb{R}^n} \sigma_1(x,\xi)\overline{\sigma_2(x,\xi)}$$

5.5. Weyl symbols. General case. For an integral operator A given by (5.10) define its Weyl symbol by

$$\tau(p,q) = \int_{\eta\in\mathbb{R}^n} K(x+\eta/2, y-\eta/2)e^{-ix\eta}$$

This formula extends the construction given above. We omit a detailed discussion and only note the the last integral is Gaussian integral operator and the product formula given by (a Gaussian integral operator)

$$\tau_3(p,q) = \int\limits_{p_1,q_1,p_2,q_2\in\mathbb{R}^n} \tau(p_1,q_1)\tau(p_2,q_2)\exp\Big\{2ih(qp_1+q_1p_2+q_2p-pq_1-p_1q_2-pq_2)\Big\}$$

Weyl symbols provide an alternative formalism for pseudo-differential operators.

5.6. Wick symbols. Define the differential operators

$$a_k f(x) = \frac{1}{\sqrt{2}}\Big(x_k - \frac{\partial}{\partial x_k}\Big) f(x), \qquad a_k^* f(x) = \frac{1}{\sqrt{2}}\Big(x_k + \frac{\partial}{\partial x_k}\Big) f(x)$$

For a polynomial $W(z,\overline{u}) = \sum c_{\alpha\beta} z^\alpha \overline{u}^\beta$ we define the differential operator

$$\widehat{W} := \sum c_{\alpha\beta} a^\alpha (a^*)^\beta$$

We say that W is the *Wick symbol* of the operator $\widehat{W}$[7].

In fact, we do the following. For an operator A in $L^2(\mathbb{R})$ we

1. pass to $\mathbf{F}_n$ and write the kernel of $\widetilde{A} := \mathcal{B}A\mathcal{B}^{-1} : \mathbf{F}_n \to \mathbf{F}_n$
2. write the Wick symbol for the operator with given kernel.

[6] It was derived for polynomial differential operators. Evidently, it holds for arbitrary differential operators.

[7] At first glance, there is no serious difference between qp-symbols and Wick symbols, for instance the algebraic considerations of Subsection 5.1 survive literally. Nevertheless, these theories drastically diverse from analytic point of view.

Both steps are trivial. First, we write the kernel $L(u,v)$ of $\widetilde{A}$ by definition,

$$L(u,\overline{v}) = \langle A\varphi_v, \varphi_u\rangle_{\mathbf{F}_n} = \langle A\Phi_v, \Phi_u\rangle_{L^2(\mathbb{R}^n)}$$

where φ_u is the supercomplete system in $\mathbf{F}_n$ and Φ_u is the corresponding system in $L^2(\mathbb{R}^n)$, see formulae (1.3), (2.2). Next, we pass to the Wick symbol as it was explained in Subsection 1.11. Finally, we come to a general formula for Wick symbols

$$W(z,\overline{u}) = \iint_{x,y\in\mathbb{R}^n} K(x,y)\exp\Bigl\{-\frac{1}{2}\bigl(x^2+y^2-2\sqrt{2}(x\overline{u}+yv)+(u+\overline{v})^2\bigr)\Bigr\}$$

Therefore, for each operator $\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$ its Wick symbol is an entire function on $\mathbb{C}^n\times\mathbb{C}^n$[8].

We stop our discussion here because the difference between Wick symbols and kernels in the sense of Subsection 1.4 is minor.

5.7. Anti-Wick symbols. Now, for a polynomial $W(z,\overline{u}) = \sum c_{\alpha\beta}z^\alpha\overline{u}^\beta$, consider the differential operator

$$\widehat{W}^\circ := \sum c_{\alpha\beta}(a^*)^\alpha a^\beta$$

More generally, for a function $W(z,\overline{u})$ we define the operator in $\mathbf{F}_n$ by

$$\widehat{W}^\circ f(z) = \int_{\mathbb{C}^n} W(u,\overline{u})e^{z\overline{u}}f(u)e^{|u|^2}\,d\lambda(z) \tag{5.16}$$

Problem 5.2 a) If W is a polynomial, then we get the same operator (5.16).

b) Show that the operator in $L^2(\mathbb{C}^n, e^{-|z|^2})$ given by

$$Pf(z) = \int_{\mathbb{C}^n} e^{z\overline{u}}f(u)e^{|u|^2}\,d\lambda(u)$$

is the orthogonal projection on $\mathbf{F}_n$.

In the previous cases, all 'reasonable' operators admit symbols. Now we come to a special class of operators, Namely, we take a function $f(u)\in\mathbf{F}_n$, write the function $W(\overline{u},u)f(u)$ on $\mathbb{C}^n$, and project it on $\mathbf{F}^n$. If W is a bounded function, then the corresponding operator is bounded, it is called a *Toeplitz operator.*

5.8. Remarks on abstract functions of noncommutative variables. Let $\mathfrak{g}$ be a Lie algebra, let $X_1,\dots, X_n$ be its basis. To each monomial $z^\alpha := z_1^{\alpha_1}\dots z_n^{\alpha_n}$ we assign the expression

$$\frac{\prod\alpha_j!}{|\alpha|!}\sum X_{k_1}\dots X_{k_{|\alpha|}}$$

where $|\alpha| = \sum\alpha_j$, the summation is taken over all product containing α_1 entries of X_1, α_2 entries of X_2, etc. Thus, we identify the algebra of non-commutative polynomials in X_j (i.e., the *universal enveloping algebra*) with the space of polynomials in n variables

[8] qp-symbols are distributions

(the 'Poincare–Birkhoff-Witt Theorem'). Multiplication in the universal enveloping algebra induces a product $\circledcirc$ in the algebra of polynomials.

For X, $Y \in \mathfrak{g}$ being in a neighborhood of 0, define

$$\Phi(X,Y) := \ln(e^X e^Y)$$

Problem 5.3 Prove the following formula

$$z^\alpha \circledcirc z^\beta = \frac{1}{\alpha!\beta!}\frac{\partial^\alpha}{\partial t^\alpha}\frac{\partial^\beta}{\partial s^\beta}\Phi\Big(\sum t_j X_j, \sum s_j Y_j\Big)\Big|_{s=0,t=0} \qquad \square$$

The function Φ can be written explicitly for solvable $\mathfrak{g}$. However, for a semisimple (in particular, classical) $\mathfrak{g}$ this way looks as hopeless. Only for $\mathfrak{g} = \mathfrak{sl}(2)$ the function Φ admits an explicit (horrible...) expression in the terms of elementary functions. However, there are numerous explicit in enveloping algebras of semisimple $\mathfrak{g}$.

Problem* **5.4** In notation of the previous problem, express $X^n \circledcirc Y$ in terms of $[X,Y]$, $[X,[X,Y]]$, $[X,[X,[X,Y]]]$, ... in an arbitrary Lie algebra. The solution is given in the terms of Bernoulli numbers.

Now, for each noncommutative monomial $X_1 \dots X_n$ denote

$$\Psi(X_1 \dots X_n) = \frac{1}{n}\Big[X_1, [X_2, \dots [X_{n-1}, X_n] \dots]\Big]$$

Extend Ψ as a linear map to the set of all the noncommutative polynomials in X, Y.

Theorem 5.6 *(Campbell–Hausdorff–Dynkin formula) The following identity of formal series holds*

$$\ln(e^X e^Y) = \Psi(\ln(e^X e^Y))$$

4.6 Perelomov problem

From this point up to the end of the chapter, we discuss the following three questions.

6.1. Perelomov problems. Let N be a closed subset in $\mathbb{C}$.

A) *Is a function $f \in \mathbf{F}_1$ uniquely determined by its values at points $z \in N$?*

B) *Is it possible an explicit reconstruction of f from its values at points $z \in N$?*

C) *Let $g \in L^2(\mathbb{R})$. Is it possible to reconstruct g explicitly from values of its Bargmann–Segal transform $\mathcal{B}g$ at points of N?*

Due to the reproducing property (1.5), we can reformulate the Problem A as follows.

— *Consider the set of functions φ_u, where u ranges in N. Is it total in $\mathbf{F}_1$?*

6.2. Preliminary discussion. The following statement is obvious.

Observation 6.1 *A function $f \in \mathbf{F}_1$ vanishing on N is unique iff one can not find a function $\varphi \in \mathbf{F}_1$ vanishing on N.*

Since elements of $\mathbf{F}_1$ are holomorphic functions, it follows that for a non-discrete N the question A has the affirmative answer.

Our basic example of N is a lattice. Precisely, fix ω_1, $\omega_2 \in \mathbb{C}$ such that $\omega_2/\omega_1 \notin \mathbb{R}$. A *lattice* $L(\omega_1, \omega_2) \subset \mathbb{C}$ is the set $\{m\omega_1 + n\omega_2\}$, where m, n range in $\mathbb{Z}$.

Denote by S the area of the parallelogram with sides ω_1, ω_2.

Theorem 6.2 (Perelomov) *a) If $S < 2\pi$, then $L(\omega_1, \omega_2)$ is a set of uniqueness for $\mathbf{F}_1$.*

b) If $S > 2\pi$, then $L(\omega_1, \omega_2)$ is not a set of uniqueness

c) If $S = 2\pi$, then $L(\omega_1, \omega_2)$ is a set of uniqueness for $\mathbf{F}_1$. It remains to be a set of uniqueness if we remove an arbitrary point and lost this property, if we remove an arbitrary pair of points.

A proof is given in Subsection 7.8.

Return to a general discussion. From this moment, $N \subset \mathbb{C}$ is a sequence a_1, a_2,... tending to ∞. The question is

— *Does exists a function $f \in \mathbf{F}_1$ such that $f(a_j) = 0$.*

The last problem can be reduced to the general theory of entire functions and we wish to discuss this reduction. Theorems formulated below can be found in text-books on complex analysis, for an advanced exposition, see monographs [91], [18], [92].

6.3. Entire functions with prescribed zeros. Consider a sequence $a_j \in \mathbb{C}$ tending to ∞. It is convenient to separate terms that are equal to 0 and to write the sequence in the form

$$\underbrace{0, \dots, 0}_{p \text{ times}},\ a_1,\ a_2, \dots$$

We wish to find an entire function having zeros at these points. The immediate idea is to write

$$f(z) := z^p \prod\nolimits_{j=1}^{\infty} \Bigl(1 - \frac{z}{a_j}\Bigr)$$

However, this product absolutely converges iff $\sum |z/a_j| < \infty$. In other words, our formula is satisfactory iff $\sum |a_j|^{-1} < \infty$.

We can also write an another function

$$f(z) := z^p \prod\nolimits_{j=1}^{\infty} \Bigl(1 - \frac{z}{a_j}\Bigr) \exp\Bigl\{\frac{z}{a_j}\Bigr\}$$

Problem 6.1 Show that this formula is satisfactory iff $\sum |a_j|^{-2} < \infty$

Theorem 6.3 (Weierstrass) *a) There is an entire function having zeros precisely at points of given sequence $a_j \to \infty$.*

b) If $F_1(z)$, $F_2(z)$ are two such functions, then $F_2(z) = F_1(z) \exp\bigl(r(z)\bigr)$, where $r(z)$ is an entire function.

We recall a method of proof, the reader interested in details can reconstruct them himself or to consult with a text-book. Denote

$$Y_k(z) := \sum_{j=1}^{k} z^k/k;$$

This expression is a terminated Taylor expansion expansion of $-\ln(1-z)$; therefore

$$(1 - \frac{z}{a_j}) \exp\{Y_{n_j}(z/a_j)\} = 1 + c \cdot z^{k+1} + \dots$$

One can choose a sequence n_j such that the product

$$F(z) = z^p \prod_{j=1}^{\infty} \Big(1 - \frac{z}{a_j}\Big) \exp\Big\{Y_{n_j}(z/a_j)\Big\}$$

converges (this claim requires a proof). Then $F(z)$ has the desired property.

To prove b), we observe that F_1/F_2 is a holomorphic function without zeros. Therefore its logarithm $\ln(F_2/F_1)$ is weel defined. □

6.4. Growth of an entire function and the density of its zeros. We begin with two definitions.

For a sequence a_j in $\mathbb{C}$, we define the *exponent* ξ *of convergence* as

$$\xi := \Big\{ \text{ infimum of } \zeta \text{ such that } \sum |a_j|^{-\zeta} < \infty \Big\}$$

Problem 6.2 For a sequence $a_k \in \mathbb{C}$, denote by $h(R)$ the number of its terms in the disk $|z| \leqslant R$. Denote by ξ the infimum of s such that $h(R) \leqslant \mathrm{const} \cdot R^s$. Show that this is the same ξ.

The *order* μ of an entire function $f(z)$ is the infimum of ν such that

$$|f(z)| \leqslant \mathrm{const} \cdot e^{|z|^\nu}$$

Theorem 6.4 (Hadamard) *For an entire function f, denote by μ its order and by ξ the exponent of convergence of the sequence of its zeros. Then $\mu \geqslant \xi$. Moreover, if the order is non-integer, then $\xi = \mu$.*

REMARK. If $P(z)$ is a polynomial of degree k, then $e^{P(z)}$ is a function of *integer* order k without zeros. □

Theorem 6.5 (E.Borel) *Let a_j be a sequence having a finite exponent of convergence ξ, let $m := \lfloor \xi \rfloor$ be the integer part of ξ. Then the function*

$$f(z) = z^p \prod_{j=1}^{\infty} \Big(1 - \frac{z}{a_j}\Big) \exp\big(Y_m(z/a_j)\big)$$

has the order ξ.

Theorem 6.6 (Hadamard) *Let f be a function of a finite growth μ, let a_j be its nonzero roots; let $m = \lfloor \mu \rfloor$. Then f admits the representation*

$$f(z) = z^p \exp\{H(z)\} \prod_{j=1}^{\infty} \Big(1 - \frac{z}{a_j}\Big) \exp\big(Y_m(z/a_j)\big)$$

where $H(z)$ is a polynomial of degree $\leqslant m$.

6.5. Corollaries. We know, that a function $f \in \mathbf{F}_1$ satisfies $|f(z)| \leqslant \text{const} \cdot e^{|z|^2}$. On another hand, we know that all the entire functions satisfying $|f(z)| \leqslant \text{const} \cdot (1+|z|)^{-1} e^{|z|/2}$ are contained in $\mathbf{F}_1$. Thus, Theorems 6.4–6.6 imply the following corollary.

Corollary 6.7 *Let ξ be the exponent of convergence of a sequence a_j.*

a) If $\xi > 2$, then a function $f \in \mathbf{F}_1$ is uniquely determined by[9] *its values at a_j.*

b) If $\lambda < 2$, then there is a nonzero function $g \in \mathbf{F}_1$, such that $g(a_j) = 0$.

For the lattice $\{m\omega_1 + n\omega_2\}$, Theorems 6.4–6.6 are not sufficient. Indeed, the exponent is $\xi = 2$, hence a minimal order of a function having zeros at $n\omega_1 + m\omega_2$ is 2. Such a functions has all the rights to be or not to be an element of $\mathbf{F}_1$.

In fact, there are finer general theorems that allow to obtain (incomplete) information in this case. However, we prefer an another way.

4.7 Preliminaries on θ-functions

This section is a preparation to the next two sections.

7.1. Definition. Quasiperiodicity. Fix $q \in \mathbb{C}$, such that $0 < |q| < 1$. Consider the Laurent series[10]

$$\vartheta(u; q) = \sum_{n=-\infty}^{\infty} (-1)^n u^n q^{n(n-1)/2}$$

Evidently, this function satisfies the following quasiperiodicity condition

$$\vartheta(qu; q) = -u^{-1} \vartheta(u; q) \tag{7.1}$$

Iterating this identity, we get

$$\vartheta(q^n u; q) = (-u)^{-n} q^{-n(n-1)/2} \vartheta(u; q) \tag{7.2}$$

Problem 7.1 If a Laurent series $T(u) = \sum_{k=-\infty}^{\infty} c_k u^k$ satisfies the quasiperiodicity condition (7.1), then $T(u) = \text{const} \cdot \vartheta(u, q)$.

[9]If some a_j has a multiplicity s, we assume that $f(a_j)$, $f'(a_j)$, ..., $f^{(s-1}(a_j)$ are given

[10]Here and below (Subsection 7.6) we do not try to be consistent with the standard notation.

7.2. Zeros.

Lemma 7.1 *The function $\vartheta(q,u)$ has zeros precisely at the points $u = q^k$.*

PROOF. Obviously, $u = 1$ is zero. By the quasiperiodicity, it follows that points $z = q^k$ are also zeros. For the same reason, zeros must form several geometric progressions Aq^k, Bq^k, etc.

Take a circle $|z| = R$ free of zeros. Consider the ring $qR \leqslant |z| \leqslant R$. By the Argument Principle, the number of zeros of ϑ inside the ring is

$$\{\text{index of path } \vartheta(Re^{i\varphi};q)\} - \{\text{index of path } \vartheta(Rqe^{i\varphi};q)\}, \quad \varphi \in [0,2\pi]$$

By the quasiperiodicity (7.1), the second path is $-R^{-1}e^{-i\varphi}\vartheta(q, Re^{i\varphi})$. Therefore the difference of indices $= 1$. This completes a proof. $\square$

7.3. The Jacobi triple identity.

Theorem 7.2 (Jacobi)

$$\vartheta(u;q) := (1-u)\prod_{n=1}^{\infty}(1-q^n)(1-uq^n)(1-u^{-1}q^n) \tag{7.3}$$

PROOF. Since $\vartheta(u;q)$ has zeros at $u = q^k$, it is natural to compare it with the product having the same zeros,

$$T(u) := (1-u)\prod_{n=1}^{\infty}(1-uq^n)(1-u^{-1}q^n) = \prod_{m=-\infty}^{\infty}(1-q^{|m|}u^{\operatorname{sgn} m}) \tag{7.4}$$

We must compare θ and T. First, we observe, that $T(u)$ satisfies the same quasiperiodicity condition (7.1). Therefore, $\vartheta(u;q) = AT(u)$. It remains to find a constant A.

We are going to evaluate the partial products

$$T_{\alpha,\beta}(u) := \prod_{m=\alpha}^{\beta}(1-q^{|m|}u^{\operatorname{sgn} m}) =: \sum_k c_{\alpha,\beta}[k]u^k \tag{7.5}$$

Evidently,

$$T_{\alpha,\beta}(qu) := \begin{cases} T_{\alpha+1,\beta+1}(u), & \text{if } \operatorname{sgn}\alpha = \operatorname{sgn}\beta \\ (-u)T_{\alpha+1,\beta+1}(u), & \text{if } \operatorname{sgn}\alpha \neq \operatorname{sgn}\beta \end{cases} \tag{7.6}$$

Theorem 7.3 (q-binomial formula) *$T_{0,h}(u)$ is*

$$\prod_{k=0}^{h}(1-q^k u) = \sum_{m=0}^{h}(-1)^{m+1}q^{m(m+1)/2}\frac{(1-q^{m+1})\dots(1-q^h)}{(1-q)(1-q^2)\dots(1-q^{h-m})}\cdot u^m$$

REMARK. Passing to limit $q \to 1$ we get the usual Newton binomial formula. $\square$

PROOF OF THE LEMMA. Writing out (7.6) for $T_{0,h}$, we get

$$(1-u)T_{0,h}(qu) = T_{0,h+1}(u) = (1-q^{h+1}u)T_{0,h}(u)$$

Equating coefficients in u^n, we obtain the recurrence relation

$$c_{0,h}[k](q^k-1) = -c_{0,h}[k-1](q^{h+1}-q^{k-1})$$

Definition (7.5) yields that the coefficient at the leading term is

$$c_{0,h}[h+1] = (-1)^{h+1}u^{h(h+1)}$$

Further, we calculate all the $c_{0,h}[k]$ from the recurrence relation. $\square$

Let us return to the proof of the triple formula. Since we know $T_{0,\beta}$, the formula (7.6) provides all the $T_{\alpha,\beta}$. In particular, the coefficient at u^0 in $T_{-l,l}(u)$ is

$$\frac{(1-q^{l+2})\dots(1-q^{2l+1})}{(1-q)(1-q^2)\dots(1-q^l)}$$

Passing to limit as $l \to \infty$, we the obtain the desired coefficient A. $\square$

7.4. Derivatives at zeros. As a corollary of the triple identity, we get

$$\vartheta'(1;q) = \frac{d}{du}\vartheta(u;q)\Big|_{u=1} = -\prod(1-q^n)^3 \tag{7.7}$$

Differentiating (7.2) and substituting $u=1$, we obtain

$$\vartheta'(q^n;q) = (-1)^{n+1}q^{-n(n-1)/2}\vartheta'(1;q) \tag{7.8}$$

7.5. Growth of $\vartheta(u;q)$. The function $f(u) = |\vartheta(u;q)|$ satisfies the quasiperiodicity condition $f(qu) = |u|^{-1}f(u)$. It can be easily checked, that the function

$$\eta(u) = \exp\{-\frac{1}{2\ln|q|}\ln^2|u| + \frac{1}{2}\ln|u|\} \tag{7.9}$$

satisfies the same quasiperiodicity condition, $\eta(qu) = |u|^{-1}\eta(u)$. Therefore their ratio $\psi := \theta/\eta$ is periodic. We get the following statement.

Observation 7.4 *The function $|\vartheta(u;q)|$ can be represented in the form*

$$|\vartheta(u;q)| = \eta(u)\psi(u); \qquad \text{where } \psi(qu) = \psi(u) \tag{7.10}$$

7.6. Weierstrass functions. Fix a lattice $L : m\omega_1 + n\omega_2 \subset \mathbb{C}$. For definiteness, we assume

$$\operatorname{Im}\omega_1/\omega_2 > 0$$

(otherwise, we change $\omega_2 \mapsto (-\omega_2)$). Denote by

$$S := \frac{1}{i}(\overline{\omega}_1\omega_2 - \overline{\omega}_2\omega_1) > 0$$

the area of the parallelogram spanned by the vectors ω_1, ω_2.

Define the following function[11]

$$\varsigma(z) = \varsigma(z;\omega_1,\omega_2) = \exp\Big\{\frac{\pi\overline{\omega}_1}{S\omega_1}z^2 - \frac{\pi i}{\omega_1}z\Big\}\cdot\vartheta(e^{2\pi iz/\omega_1}, e^{2\pi i\omega_2/\omega_1}) \tag{7.11}$$

Proposition 7.5 *a) $\varsigma(z)$ has zeros precisely at points $z = m\omega_1 + n\omega_2$*

b) The function $\varsigma(z)$ satisfies the quasi-periodicity conditions

$$\varsigma(z+\omega_j) = -\varsigma(z)\cdot\exp\Big\{\frac{\pi}{S}(2z\overline{\omega}_j + \omega_j\overline{\omega}_j)\Big\}; \tag{7.12}$$

PROOF. a) is a rephrasing of the theorem about zeros of ϑ,

b) For the shift $z\mapsto z+\omega_1$, only the exponential factor in (7.11) changes. For the shift $z\mapsto z+\omega_2$ we apply (7.1)). The calculation is straightforward. □

7.7. Growth of the Weierstrass function.

Lemma 7.6 *a) The function $|\varsigma(z;\omega_1,\omega_2)|$ has the form*

$$|\varsigma(z;\omega_1,\omega_2)| = \exp\Big\{\frac{\pi}{S}|z|^2\Big\}\varphi(z), \tag{7.13}$$

where $\varphi(z)$ is a double-periodic function, $\varphi(z+\omega_1) = \varphi(z+\omega_2) = \varphi(z)$.

b) For $\lambda = m\omega + n\omega_2$ (i.e., for zeros of ς)

$$|\varsigma'(\lambda;\omega,\omega_2)| = \mathrm{const}\cdot\exp\Big\{\frac{\pi}{S}|\lambda|^2\Big\}$$

PROOF. a) By (7.12),

$$|\varsigma(z+\omega_j)/\varsigma(z)| = \exp\Big\{\frac{\pi}{S}(z\overline{\omega}_j + \overline{z}\omega_j + \omega_j\overline{\omega}_j)\Big\} = \\ = \exp\{\frac{\pi}{S}|z+\omega_j|^2\}\Big/\exp\{\frac{\pi}{S}|z|^2\}$$

and this implies the desired statement.

b) Certainly, values ς' at points of L can be written explicitly by (7.7)–(7.8). However, we prefer to differentiate (7.12) and to obtain

$$\varsigma'(\lambda+\omega_j) = -\varsigma'(\lambda)\cdot\exp\Big\{\frac{\pi}{S}(2\lambda\overline{\omega}_j + \omega_j\overline{\omega}_j)\Big\}; \tag{7.14}$$

Hence

$$|\varsigma'(\lambda+\omega_j)/\varsigma'(\lambda)| = \exp\{\frac{\pi}{S}|\lambda+\omega_j|^2\}\Big/\exp\{\frac{\pi}{S}|\lambda|^2\} \qquad \square \tag{7.15}$$

[11] It is more-or-less one of the Weierstrass functions, however our factor $\exp\{\dots\}$ is not standard; we wish to have uniform growth in all the directions (Lemma 7.6). Also, our factor provides a symmetry in $\omega_1 \leftrightarrow \omega_2$ in (7.12).

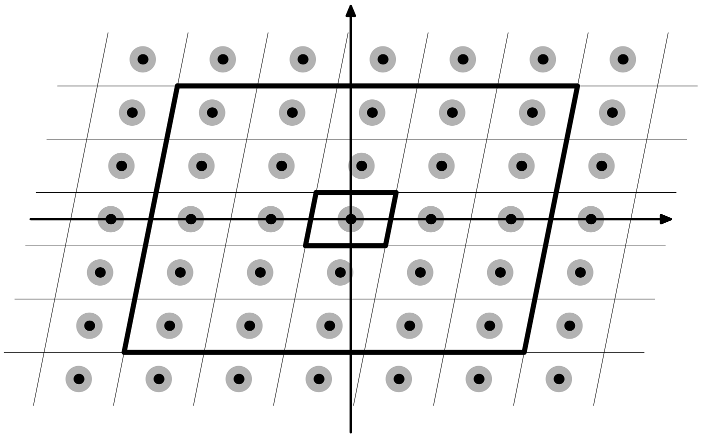


Fig. 4.10: To Lemma 7.8 and Subsection 7.8 The set $\mathcal{E}$ is the grid of thin lines. Zeros of ς are black points.

Corollary 7.7 *There is a constant β such that*

$$\bigl|\varsigma(z)\bigr| \leqslant \beta \cdot \exp\bigl\{\frac{\pi}{S}|z|^2\bigr\} \qquad \textit{for all } z$$

Denote by $\mathcal{E}$ the union of all the lines

$$a_k(t) := (k+1/2)\omega_1 + t\omega_1, \qquad b_l(s) := (l+1/2)\omega_2 + s\omega_1$$

(see thin lines on Fig. 4.10). Consider some periodic closed neighborhood Ω of $\mathcal{E}$ that does not contain points of the lattice L (an example of Ω: the complement to the grey disks on the figure).

Lemma 7.8 *There is constants $\alpha > 0$ such that*

$$\bigl|\varsigma(z)\bigr| \geqslant \alpha \cdot \exp\bigl\{\frac{\pi}{S}|z|^2\bigr\} \qquad \textit{for } z \in \Omega \tag{7.16}$$

PROOF. Denote by Π the parallelogram with vertices $\pm\omega_1/2\pm\omega_2/2$ (the small parallelogram containing 0 on Figure). Let α be the minimum of $|\varsigma(z)|/\exp\{\frac{\pi}{S}|z|^2\}$ over the domain $\Pi\cap\Omega$. An estimate of type (7.16) holds in $\Pi\cap\Omega$. By the quasi-periodicity (Lemma 7.6), it follows that the estimate holds in the whole Ω. □

7.8. Proof of the Perelomov Theorem 6.2. We must understand is there a function $g \in \mathbf{F}_1$ vanishing at points of the lattice L.

THE CASE $S > 2\pi$. By (7.13) and Proposition 1.3 b), it follows that the function $\varsigma(z;\omega_1,\omega_2)$ is contained in $\mathbf{F}_1$.

THE CASE $S < 2\pi$. We must show that any function $Q(z)$ vanishing at points of L is not contained in $\mathbf{F}_1$. First of all, $\varsigma(z;\omega_1,\omega_2) \notin \mathbf{F}_1$. Next, Q can be represented as $Q(z) = h(z)\varsigma(z)$, where $h(z)$ is an entire function.

Denote by D_k the boundary of the parallelogram with the vertices

$$\pm(k+1/2)\omega_1 \pm (k+1/2)\omega_2$$

see Fig. 4.10. Estimate (7.16) holds on D_k Therefore,

$$\max_{z\in D_k} \left|\varsigma(z)h(z)/\exp\left\{\frac{\pi}{S}|z|^2\right\}\right| \geqslant \alpha \cdot \max_{z\in D_k} |h(z)|$$

If $\varsigma h \in \mathbf{F}_1$, then the left-hand side tends to 0 as $k \to \infty$. Hence $h(z) = 0$. □

THE CRITICAL VALUE $S = 2\pi$. The proof is the same. Let a, b be points of the lattice. Then $\varsigma(z)/\big((z-a)(z-b)\big)$ is contained in $\mathbf{F}_1$ (because of Proposition 1.3).

Further, consider $\varsigma(z)/(z-a)$. Let $\Omega \supset \mathcal{E}$ be the same as in Lemma 7.8. Then $|\varsigma(z)| \geqslant \alpha \exp\{|z|^2/2\}$ for $z \in \Omega$. Hence

$$\left|\frac{\varsigma(z)}{z-a}\right| \geqslant \alpha \frac{\exp\{|z|^2/2\}}{|z-a|}, \qquad z \in \Omega$$

and the integral in definition (1.1) of the Fock space diverges. □

4.8 Interpolation and Lagrange formula

8.1. The Lagrange interpolation formula. Consider an open domain $\Omega \subset \mathbb{C}$ and a sequence a_j that has not limit points in Ω. Let $f(z)$ be a function holomorphic in Ω. *Is it possible to restore f from its values $f(a_j)$?*

Let $\nu(z)$ be a holomorphic function, whose zeros are a_j. Consider the function

$$f^*(z) := \sum_{j=0}^{\infty} \frac{f(a_j)\nu(z)}{(z-a_j)\nu'(a_j)} \tag{8.1}$$

This expression is nothing but the Lagrange interpolation formula written for an infinite collection of nodes of interpolation.

Formal calculation gives $f^*(a_j) = f(a_j)$. Indeed,

$$f^*(a_k) = \sum_{j\neq k} \frac{f(a_j)\nu(z)}{(z-a_j)\nu'(a_j)}\Big|_{z=a_k} + f(a_k) \lim_{z\to a_k} \frac{\nu(z)}{\nu'(a_k)(z-a_k)}$$

All the summands of $\sum$ are zeros (because $\nu(a_k) = 0$) and the limit is 1 by the l'Hospital rule.

For two reasons, our success is doubtful. First, we are not sure that the series in (8.1) is convergent. Secondly, $f^*(a_j) = f(a_j)$, nevertheless we are not sure that $f^*(z) = f(z)$. We only know that $f^*(z) - f(z)$ has zeros at points a_k; therefore

$$f^*(z) - f(z) = \nu(z)q(z) \qquad \text{where } q \text{ is holomorphic in } \Omega.$$

However, a problem of reconstruction of a holomorphic function from its values at a countable discrete set is rather important. In particular, such reconstructions are used rather often in explicit calculations, see, for instance, text-book of Andrews, Askey, Roy on special functions [3]('Carlson theorem' and its numerous applications).

8.2. An example of interpolation. The Paley–Wiener space. We begin our discussion with a toy example.

Consider the tautological embedding $L^2[-\pi,\pi] \to L^2(\mathbb{R})$; namely, we extend a function f from $[-\pi,\pi]$ to the whole line assumed that $f=0$ outside $[-\pi,\pi]$.

The *Paley–Wiener space* PW is the push-forward of $L^2[-\pi,\pi] \subset L^2(\mathbb{R})$ under the Fourier transform. Since functions $f \in L^2[-\pi,\pi]$ have compact support, it follows that their Fourier transforms are entire functions. Recall (without proof, see [138])

Theorem 8.1 (Paley–Wiener) *An entire function g is an element of* PW *iff* $|g(z)| \leqslant \mathrm{const}\cdot\exp\{\pi|\operatorname{Im} z|\}$ *and the functions $h_\pm(t) := g(\pm\pi+it)$ are contained in $L^2(\mathbb{R})$.*

Problem 8.1 Denote

$$\varphi_p(z) = \frac{\sin\pi(x-p)}{x-p}, \qquad p \text{ ranges in } \mathbb{C}$$

Then the following *reproducing property* holds:

$$\langle g, \varphi_p\rangle_{\mathrm{PW}} = g(p) \qquad \text{for all } g \in \mathrm{PW}.$$

Theorem 8.2 (Whittaker–Kotelnikov–Shanon) *For a function $g \in$ PW,*

$$g(z) = \sum_{n=-\infty}^{\infty} g(n)\cdot\frac{\sin\pi(z-n)}{z-n} = \sum_{n=-\infty}^{\infty}(-1)^n g(n)\cdot\frac{\sin\pi z}{z-n} \tag{8.2}$$

Remark. This formula is the Lagrange interpolation formula for entire functions $f \in$ PW with nodes $a_n := n$ and $\nu(z) := \sin z$. □

Remark. This statement can also be represented as

$$\frac{f(z)}{\sin\pi z} = \sum_{n=-\infty}^{\infty}\frac{1}{z-n}\cdot\operatorname*{res}_{z=n}\frac{f(z)}{\sin\pi z}$$

where res denotes the residue. □

Proof. Let $g=\widehat{f}$. Expand $f(x)=\sum c_n e^{inx}$. Then $c_n=\widehat{f}(n)$ and $\frac{\sin(x-n)}{x-n}$ is the Fourier transform of $e^{inx} \in L^2[-\pi,\pi]$. Thus, (8.2) is simply the Fourier-transform push-forward of the expansion into a Fourier series. □

8.3. Interpolation in the Fock space.

Theorem 8.3 (Lyubarsky) *Let $m\omega_1 + n\omega_2$ be a lattice in $\mathbb{C}$, let $S < 2\pi$. Then, for each $f \in \mathbf{F}_1$,*

$$f(z) = \varsigma(z;\omega_1,\omega_2)\sum_{m,n} \frac{f(m\omega_1 + n\omega_2)}{(z - m\omega_1 - n\omega_2)\varsigma'(m\omega_1 + n\omega_2;\omega_1,\omega_2)} \tag{8.3}$$

where ς is the Weierstrass function defined in the previous section.

Proof. Let $f \in \mathbf{F}_1$. We write out f^* by (8.1) and get the right-hand side of (8.3). Since $f^*(z) - f(z) = 0$ at points of the lattice,

$$f^*(z) - f(z) = \varsigma(z)h(z)$$

where $h(z)$ is an entire function. Dividing both sides by $\varsigma(z)$, we get

$$\sum_{m,n} \frac{f(m\omega_1 + n\omega_2)}{(z - m\omega_1 - n\omega_2)\varsigma'(m\omega_1 + n\omega_2)} - \frac{f(z)}{\varsigma(z)} = h(z) \tag{8.4}$$

Consider the contour D_k as above (see Fig. 4.10). and tend k to ∞. Let are going to show that for both summands in the left-hand side

$$\lim_{k\to\infty} \max_{z\in D_k}\Big|\ldots\Big| = 0 \tag{8.5}$$

If we believe in this, then

$$\lim_{k\to\infty} \max_{z\in D_k} |h(z)| = 0$$

and the maximum principle implies $h(z) = 0$; therefore $f = f^*$.

Return to (8.4). For the second summand (8.5) is obvious because (see (1.6), (7.16))

$$|f(z)| \leqslant \exp\{|z|^2/2\}; \qquad \varsigma(z) \geqslant \mathrm{const}\cdot\exp\Big\{\frac{\pi}{S}|z|^2\Big\}, \quad z \in D_k$$

and $\pi/S > 1/2$.

To examine the first summand, we observe that (7.15) implies

$$\left|\frac{f(m\omega_1 + n\omega_2)}{\varsigma'(m\omega_1 + n\omega_2)}\right| \leqslant \mathrm{const}\cdot\exp\{-\delta|m\omega_1 + n\omega_2|^2\} \qquad \text{for some } \delta > 0.$$

Therefore,

$$\Big|\sum_{m,n}\ldots\Big| \leqslant \mathrm{const}\cdot\sum_{m,n}\frac{\exp\{-\delta|m\omega_1 + n\omega_2|^2\}}{|z - m\omega_1 - n\omega_2|} \tag{8.6}$$

We wish to estimate the last sum for z being in the grid $\mathcal{E}$ (see Fig. 4.10). For this purpose, fix a contour D_p and consider separately points of the lattice lying inside and outside the contour,

$$\Big|\sum_{m,n}\ldots\Big| \leqslant \mathrm{const}\cdot\sum_{m,n:\, m\omega_1+n\omega_2 \text{ is inside } D_p}\frac{1}{|z - m\omega_1 - n\omega_2|} + \\ + \mathrm{const}\cdot\sum_{m,n:\, m\omega_1+n\omega_2 \text{ is outside } D_p}\exp\{-\delta|m\omega_1 + n\omega_2|^2\}$$

(for the first sum we note that $\exp\{\cdot\} \leqslant 1$, for the second sum we observe that a distance between points of $\mathcal{E}$ and points $n\omega_1 + m\omega_2$ is bounded from below). The second sum is small for large p. The first sum (with a fixed p) tends to 0 as $|z| \to \infty$. Thus (8.5) actually holds and this completes the proof. □

4.9 Inversion of the Segal–Bargmann transform from points of a lattice

Here we restore a function $f \in L^2(\mathbb{R}^n)$ from values of $\mathcal{B}f$ at points of a lattice $k + i\tau m$. To avoid writing $\sqrt{2}$, we slightly change the normalization of $\mathcal{B}$.

9.1. Inversion formula. Fix τ such that $\operatorname{Re}\tau > 0$ (it is pleasant but not necessary to think that $\tau \in \mathbb{R}$). For a function $f \in L^2(\mathbb{R})$, we define the coefficients

$$\gamma_{m,k} = \int_{-\infty}^{\infty} e^{-ikx - \tau m x} f(x) e^{-x^2/4}\, dx$$

where m, k range in $\mathbb{Z}$.

We wish to reconstruct f from $\gamma_{m,k}$. By the Perelomov Theorem 6.2, this is impossible for $\operatorname{Re}\tau > \pi$; for $\operatorname{Re}\tau \leqslant \pi$, the problem is overdetermined. There are many ways for reconstruction of f. For instance, see Janssen [77]); we can also apply the Lyubarsky Theorem 8.3. Here we discuss a formula that is relatively simple and relatively closed.

Suppose $0 < \operatorname{Re}\tau < \pi$. Denote $q := e^{-2\pi\tau}$. Define the coefficients

$$\mathcal{E}_m(\tau) = \frac{(-1)^m q^{m(m-1)/2}}{\prod_{l=1}^{\infty}(1-q^l)^3} \sum_{j \geqslant 0} (-1)^j q^{j(j+2m+1)/2} \tag{9.1}$$

Theorem 9.1 *For $f \in L^2(\mathbb{R})$,*

$$f(x) = \frac{1}{2\pi} e^{x^2/4} \sum_m \Big\{ \mathcal{E}_m(\tau) e^{m\tau x} \sum_k \gamma_{m,k} e^{ikx} \Big\}$$

The interior sum is an L^2-sum of a Fourier series, the exterior sum is a.s. convergent series.

Remark. The series in (9.1) is a terminated series for $\vartheta(q^{m+1}; q)$. Recall that the sum of complete series $\sum_{-\infty}^{\infty}$ is 0. □

9.2. Reduction to an interpolation problem. Denote

$$g(x) := \frac{1}{2\pi} f(x) e^{-x^2/4}$$

Applying the Poisson summation formula

$$\sum_{k=-\infty}^{\infty} h(x + 2\pi k) = \sum_{m=-\infty}^{\infty} \widehat{h}(x + l)$$

(see below Theorem 8.2.1) to the function $g(x)e^{-\tau m x}$, we get

$$\sum_{k=-\infty}^{\infty} \gamma_{m,k} e^{ikx} = e^{-m\tau x} \sum_{j=-\infty}^{\infty} g(x+2\pi j) e^{-2\pi\tau m j} \tag{9.2}$$

Denote

$$A_m = e^{m\tau x} \sum_{k=-\infty}^{\infty} \gamma_{m,k} e^{ikx}$$

We know $\gamma_{k,m}$, therefore we know A_m.

Fix $x \in \mathbb{R}$. Consider the function

$$G_x(z) := \sum_{j=-\infty}^{\infty} g(x+2\pi j) z^j$$

defined in the domain $\mathbb{C} \setminus 0$. In this notation, formula (9.2) converts to

$$G_x(q^m) = A_m$$

We wish to find $G_x(z)$ from these data. For this goal, let us apply the Lagrange interpolation formula (8.1). The function having zeros at points $z = q^n$ is the theta-function $\vartheta(q; z)$. Denote

$$G_x^*(z) = \sum_{n=-\infty}^{\infty} A_n \frac{\vartheta(z;q)}{(z-q^n)\vartheta'(q^n;q)} = \sum_{n=-\infty}^{\infty} A_n \frac{(-1)^{n+1} q^{n(n-1)/2}}{\prod (1-q^j)^3} \frac{\vartheta(z;q)}{(z-q^n)} \tag{9.3}$$

By construction, $G_x(q^n) = G_x^*(q^n)$. Therefore,

$$G_x(z) = G_x^*(z) + \vartheta(z;q)\alpha(z) \tag{9.4}$$

for a certain function $\alpha(z)$ holomorphic in $\mathbb{C} \setminus 0$. Equivalently,

$$\frac{G_x(z)}{\vartheta(z;q)} - \frac{G_x^*(z)}{\vartheta(z;q)} = \alpha(z) \tag{9.5}$$

Lemma 9.2 *The series (9.3) converges uniformly on compact sets in $\mathbb{C} \setminus 0$. Moreover, $\alpha(z) = 0$, i.e., $G_x(z) = G_x^*(z)$.*

Theorem 9.1 is a corollary of this lemma. Indeed, $g(x)$ is the Laurent coefficient of $G_x(z) = G_x^*(z)$ in z^0; it remains to evaluate the Laurent expansion of $G_x^*(z)$. For this purpose, we must write the Laurent series for

$$\frac{\vartheta(z;q)}{z-q^n} = (z-q^n)^{-1} \sum_{l=-\infty}^{\infty} (-1)^l z^l q^{l(l-1)/2}$$

Assuming $|z| > |q|^n$, we get

$$(z^{-1} + z^{-2}q^n + z^{-3}q^{2n} + \dots) \cdot \sum_{l=-\infty}^{\infty} (-1)^l z^l q^{l(l-1)/2}$$

Finally, we obtain (9.1) as a coefficient in the front of z^0.

REMARK. The function $\vartheta(z,q)/(z-q^n)$ has a removable singularity at $z = q^n$. Therefore writing the Laurent expansion in the domain $|z| < |q|^n$, we must obtain the same result. □

9.3. Reduction of Lemma 9.2. For a function $\Phi(z)$ we denote

$$\mathcal{M}_k[\Phi] := \max_{|z|=q^{k+1/2}} |\Phi(z)|$$

Lemma 9.3 *a)* $\lim_{k\to\pm\infty} \mathcal{M}_k\big[G_x(z)/\vartheta(z;q)\big] = 0$

b) $\lim_{k\to\infty} \mathcal{M}_k\big[G_x^*(z)/\vartheta(z;q)\big] = 0$ *and this sequence remains bounded as* $k \to -\infty$.

Assume that the last lemma is proved. Look to (9.5). The function $\alpha(z)$ is bounded, therefore $\alpha(z)$ has a removable singularity at 0. By the Liouville Theorem $\alpha(z) = \text{const}$. Since $\alpha \to 0$ at ∞, it follows that $\alpha = 0$.

Thus, it remains to prove the last lemma.

9.4. Preparation to the proof of Lemma 9.3.

Lemma 9.4 *For* $f \in L^2(\mathbb{R})$, *the quantity*

$$V_x := \sum\nolimits_{j=-\infty}^{\infty} |f(x+2\pi j)|^2$$

is finite for almost all x *and*

$$\int_0^{2\pi} V(x)\,dx = \int_{-\infty}^{\infty} |f(x)|^2\,dx$$

PROOF. We must justify the identity

$$\int_{\mathbb{R}} |f(x)|^2 dx = \int_0^{2\pi} \Big(\sum\nolimits_{j=-\infty}^{\infty} |f(x+2\pi j)|^2\Big)\,dx$$

For this purpose, we identify the spaces $[0,2\pi]\times\mathbb{Z}$ and $\mathbb{R}$ by $(a,n) \mapsto (a+2\pi n)$. Applying the Fubbini Theorem for Lebesgue integral, we reduce an integral over $\mathbb{R}$ to an iterated integral over $[0,2\pi]\times\mathbb{Z}$. □

Lemma 9.5 *The following upper estimate holds*

$$|G_x(z)| \leqslant V_x^{1/2} \cdot \Big[e^{-x^2}\vartheta\Big(-|z|^2 e^{-2\pi x - 2\pi^2}; e^{-4\pi^2}\Big)\Big]^{1/2}$$

PROOF. By the Schwartz inequality,

$$|G_x(z)| = \Big|\sum f(x+2\pi j)e^{-(x+2\pi j)^2/4}z^j\Big| \leqslant$$
$$\leqslant \Big(\sum |f(x+2\pi j)|^2\Big)^{1/2}\Big[\sum e^{-(x+2\pi j)^2/2}|z|^{2j}\Big]^{1/2}$$

We transform the second factor as

$$\Big[\dots\Big] = e^{-x^2/2}\sum_j \big(|z|^2 e^{-2\pi x - 2\pi^2}\big)^j \cdot e^{-4\pi^2 j(j-1)/2} = e^{-x^2}\vartheta\Big(-|z|^2 e^{-2\pi x-2\pi^2}; e^{-4\pi^2}\Big)$$

Corollary 9.6 *For almost all x, the function $|G_x(z)|$ admits an upper estimate of the form*

$$|G_x(z)| \leqslant \exp\big\{\frac{1}{4\pi^2}\ln^2|z| + O(\ln|z|) + O(1)\big\} \tag{9.6}$$

PROOF. We refer to asymptotic behavior of the theta-function (7.9)-(7.10). □

Now we are ready to estimate coefficients in the interpolation formula.

Lemma 9.7 *For some $\varepsilon > 0$ and some C,*

$$|A_m/\vartheta'(q^m;q)| \leqslant C\cdot\exp\{-\varepsilon m^2\}$$

PROOF. By the previous corollary,

$$|A_m| = |G_x(q^m)| \leqslant \exp\big\{\frac{\ln^2|q|}{4\pi^2}\, m^2 + O(m) + O(1)\big\}$$

By (7.8),

$$\vartheta'(q^m;q) = \exp\big\{-\frac{1}{2}m^2\ln|q| + O(m) + O(1)\big\}$$

Since $(-\ln|q|) = 2\pi\tau < 2\pi^2$, we obtain our statement. □

9.5. Proof of Lemma 9.3. Let z lie on some contour $|z| = p^{k+1/2}$

a) We wish to estimate $G_x^*(z)/\vartheta(z;q)$. By (7.9)–(7.10),

$$|\vartheta(z)| \geqslant \exp\big\{-\frac{1}{2\ln q}\ln^2|z| + O(\ln|z|) + O(1)\big\}$$

Comparing with (9.6), we come to the statement a) of the lemma.

b) Now let us estimate $G_x^*(z)/\vartheta(z;q)$

$$\mathcal{M}_k\big[G_x^*(z)/\vartheta(z;q)\big] = \mathcal{M}_k\Big[\sum_m \frac{A_m}{\vartheta'(q^m;q)}\cdot\frac{1}{z-q^m}\Big] \leqslant C\sum\frac{e^{-\varepsilon m^2}}{\big||q|^{k+1/2} - |q|^m\big|}$$

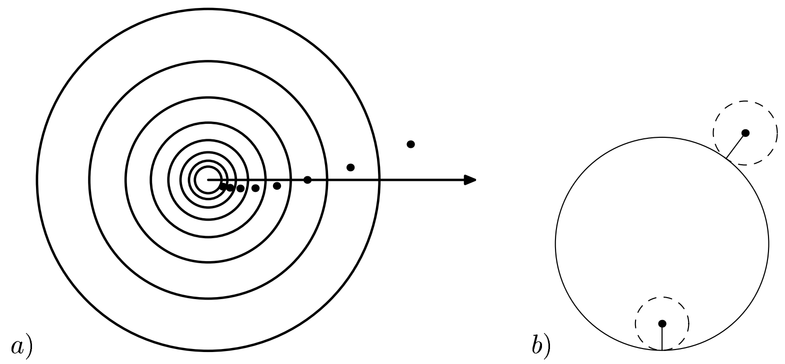


Fig. 4.11: To Subsection 9.3. a) Zeros $u = q^k$ of the function $\theta(u;q)$ and the contours $|u| = p^{k+1/2}$.
b) The distance between a circle and a point.

Here we applied Lemma 9.7, and $|z - q^m| > \big||q|^{k+1/2} - |q|^m\big|$ follows from Fig. 4.11b.

Next, we estimate below the denominators

$$\big||q|^{k+1/2} - |q|^m\big| = \begin{cases} |q|^m \big|1 - |q|^{-m+k+1/2}\big| \geqslant |q|^m(1 - |q|^{1/2}) \\ |q|^{k+1/2}\big|1 - |q|^{m-k/1/2}\big| \geqslant |q|^{k+1/2}(1 - |q|^{1/2}) \end{cases}$$

Applying the first estimate, we get

$$\mathcal{M}_k[\dots] \leqslant C \sum_{m=-\infty}^{\infty} \frac{e^{-\varepsilon m^2}}{|q|^m(1 - |q|^{1/2})}$$

The series is convergent and independent in k. Therefore the sequence $\mathcal{M}_k[\cdot]$ is bounded.

Applying the second estimate, we obtain

$$\mathcal{M}_k[\dots] \leqslant C \sum_{m=-\infty}^{\infty} \frac{e^{-\varepsilon m^2}}{|q|^{k+1/2}(1 - |q|^{1/2})}$$

The right-hand side tends to zero as $k \to \infty$. This completes a proof of Lemma 9.3 and the inversion formula. □

9.6. Digression. The Post–Widder inversion formula for the Laplace transform. We cite the following result due to its exoticism.

Consider the Laplace transform

$$g(t) = \mathcal{L}f(t) := \int_0^\infty e^{-ts} f(s)\, ds$$

Then f can be reconstructed from the asymptotic behavior of $g(t)$ for large *real* t in the following way:

$$f(s) = \lim_{k\to\infty} \frac{(-1)^k}{k!} \left(\frac{k}{s}\right)^{k+1} g^{(k)}\left(\frac{k}{s}\right) = \frac{1}{\sqrt{2\pi}} \lim_{k\to\infty} \frac{(-1)^k k^{1/2} e^k}{s^{k+1}} g^{(k)}\left(\frac{k}{s}\right) \tag{9.7}$$

To prove this, we write

$$(-1)^n g^{(n)}(nt) = \int_0^\infty e^{-nts} s^n f(s)\, ds = \int_0^\infty e^{-n(ts-\ln s)} f(s)\, ds$$

Let us investigate a behavior of the last integral as $b \to \infty$ by the Laplace method. The function $\alpha(s) := st - \ln s$ has a unique minimum at $s_0 = 1/t$. Hence we obtain the asymptotics

$$\sim \int_0^\infty e^{-n(\alpha(s_0)+\frac{1}{2}\alpha''(s_0)(s-s_0)^2)} f(s)\, ds \sim \Bigl(\frac{2\pi}{n\alpha''(s_0)}\Bigr)^{1/2} f(s_0) e^{-n\alpha(s_0)} =$$

$$= \Bigl(\frac{2\pi}{nt^2}\Bigr)^{1/2} f(1/t) e^{-n} t^{-n}$$

Thus (at least for f having a continuous derivative) we get

$$f(1/t) = \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} (-1)^n (et)^n t n^{1/2} g^{(n)}(nt)$$

4.10 Bibliographical remarks

10.1. The (boson) Fock space was introdused by V.A.Fock in 1929 as direct sum of symmetric powers $\bigoplus_{j=0}^\infty \mathsf{S}^n(H)$ of an infinite dimensional Hilbert space H, see our 1.6, [11], [144], [118]. Several functional models of Fock space are known. In particular,

– L^2 on Hilbert space (introduced by I.Segal, a nice readable text is [166], unfortunately it exists only on Russian)

– Bargmann–Segal holomorphic model, i.e. our Section 4.1, see also [11], [118]

– The P.Cartier model, see below Chapter 8.

– L^2 on the Poisson process (introduced independently by G.A.Goldin–J.Grodnik–R.T.Powers–D.H.Sharp, A.M.Vershik–I.M.Gelfand–M.I.Graev, and R.S.Ismagilov in 1974–1975, see [117])

– Fermionic Fock space, that admits a canonical identification with our (boson) Fock space (T.H.R.Skyrme,1971), see [144], [125]

– The space of symmetric function; I do not know who observed this correspondence, see [144], [125]

– The space of holomorphic functionals on the space of univalent functions, see [114].

The last 3 models exist only for an infinite-dimensional H, i.e., they are characteristic phenomena of analysis in an infinite number of variables. The Cartier model is known (or well-understood) only for finite-dimensional H.

10.2. The Segal–Bargmann transform was introduced (dependently) by I.Segal and V.Bargmann near 1960 and also by F.A.Berezin in the same time.

10.3. Short-time Fourier transforms usually is attributed to Gabor (1950-s). I am nor sure that this is correct since the radio communication existed earlier. On this subject, see, for instance [59].

10.4. Wave fronts and microlocal analysis, see books of L.Hermander [69], v.1,3, and G.B.Folland [40].

10.5. Symbolic calculus. On symbolic calculus, see papers of F.A.Berezin [12], [13] and books G.Folland [40]. On pseudo-differential operators see, for instance, L.Hoermander [69], v.3.

10.6. Campbell–Hausdorff–Dynkin formula. For further discussion, see Reutenauer [147], Torrossian [170]. On samples of calculations in enveloping algebras, see [109], [169].

10.7. Interpolation. For a general discussion of distribution of zeros of holomorphic functions and interpolation, see [18], [91], [161]. For Section, see 4.8, see [100]; for Section 4.6, see [77], [130].

10.8. For the E.Post–D.V.Widder formula, see Widder's book [181]. Widder also refers to the J.S.Stiltjets–Ch.Hermit private correspondence, 29 August, 1893.