Chapter 5

Gaussian operators in Fock spaces.

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Gaussian operators in $\mathbf{F}_n$ are $\mathcal{B}$-push-forwards of Gaussian operators in $L^2(\mathbb{R}^n)$. However, our present considerations are independent on Chapter 1.

5.1 Gaussian operators

1.1. Gaussian vectors. For a symmetric $n \times n$ matrix R denote $b[R]$ the Gaussian vector[1]

$$b[R](z) = \exp\{\tfrac{1}{2} z R z^t\}$$

Proposition 1.1 *a) $b[R]$ is an element of the Fock space $\mathbf{F}_n$ iff $\|R\| < 1$.*

b) $b[R]$ is a $\mathcal{B}$-image of a tempered distribution iff $\|R\| \leqslant 1$.

c) Inner products of Gaussian vectors are given by

$$\langle b[T], b[R]\rangle = \det(1 - T\overline{R})^{-1/2} \tag{1.1}$$

Emphasis that the vectors $b[R]$ are parameterized by points T of the matrix Cartan ball B_n, see Chapter 3.

Proof. a) follows immediately from the estimate of growth of functions $f \in \mathbf{F}_n$, see Proposition 4.1.3.

b) For $\mathbf{F}_1$, we must examine functions $\exp\{\lambda z^2\}$; we can refer to Proposition 4.2.6.

[1] This notation is similar to parallel notation from Chapter 1, p.17 but it is not identical

For general n, we consider transformations $\mathbf{F}_n \to \mathbf{F}_n$ given by $f(z) \mapsto f(zU)$, where U is a unitary matrix. Under such transformations, any Gaussian vector can be reduced to $\exp\{\sum \lambda_j z_j^2\}$, $\lambda_j \geqslant 0$, see Lemma 3.3.7. Now the statement becomes obvious.

c) This integral is evaluated in the next lemma. □

Problem 1.1 Find the $\mathcal{B}$-preimage of $\exp\{\frac{1}{2}\sum \lambda_j z_j^2\}$.

Problem 1.2 a) Gaussian vectors in $\mathbf{F}_n$ are precisely the $\mathcal{B}$-images of Gaussian vectors in $L^2(\mathbb{R}^n)$.

b) The $\mathcal{B}$-images of Gaussian distributions $\in \mathcal{S}'(\mathbb{R}^n)$ are precisely the vectors $b[T]$ with $\|T\| \leqslant 1$. HINT. Without loss of generality, we can consider a Gaussian distribution $\exp\{-\frac{1}{2}(\lambda_1 x_1^2 + \dots + \lambda_j x_j^2)\}\delta(x_{k+1})\dots\delta(x_n)$

c) Prove Proposition 1.3.3 (cone of Gaussian distributions is closed in $\mathcal{S}'(\mathbb{R}^n)$).

1.2. Gaussian vectors. Extended variant. Consider more general vectors

$$b[T|\tau] := \exp\left\{\tfrac{1}{2} zTz^t + \tau z^t\right\}, \qquad \text{where } \|T\| < 1 \tag{1.2}$$

Lemma 1.2 *a) A vector $b[T|\tau]$ is an element of $\mathbf{F}_n$,*

$$b)\ \langle b[T|\tau], b[R|\rho]\rangle = \det(1 - T\overline{R})^{-1/2} \cdot \exp\left\{\begin{pmatrix}\tau & \overline{\rho}\end{pmatrix}\begin{pmatrix}1 & -T\\ -\overline{R} & 1\end{pmatrix}^{-1}\begin{pmatrix}\tau\\ \overline{\rho}\end{pmatrix}\right\} \tag{1.3}$$

PROOF. The statement a) is obvious, because $b[T|\tau]$ satisfies the estimate (4.1.6).

Let us evaluate inner products,

$$\begin{aligned}\big\langle b[T|\tau], b[R|\rho]\big\rangle &= \int_{\mathbb{C}^n} \exp\left\{\tfrac{1}{2} zTz^t + \tau z^t\right\} \exp\left\{\tfrac{1}{2}\overline{z}\overline{R}\overline{z}^t + \overline{\rho}\overline{z}^t\right\} \exp\{-z\overline{z}\} d\lambda(z) = \\ &= \int_{\mathbb{C}^n} \exp\left\{\frac{1}{2}\begin{pmatrix}z & \overline{z}\end{pmatrix}\begin{pmatrix}T & -1\\ -1 & \overline{R}\end{pmatrix}\begin{pmatrix}z^t\\ \overline{z}^t\end{pmatrix} + \begin{pmatrix}\tau & \overline{\rho}\end{pmatrix}\begin{pmatrix}z^t\\ \overline{z}^t\end{pmatrix}\right\} d\lambda(z) = \\ &\int_{\mathbb{C}^n} \exp\left\{\frac{1}{2}\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}\begin{pmatrix}T & -1\\ -1 & \overline{R}\end{pmatrix}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}^t\begin{pmatrix}x^t\\ y^t\end{pmatrix} + \right. \\ &\qquad\qquad \left. + \begin{pmatrix}\tau & \overline{\rho}\end{pmatrix}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}\begin{pmatrix}x^t\\ y^t\end{pmatrix}\right\}\frac{dx\,dy}{\pi^n}\end{aligned}$$

and we apply the usual Gauss integral. □

1.3. Gaussian operators. Let

$$S = \begin{pmatrix}K & L\\ L^t & M\end{pmatrix}$$

be a symmetric $(m+n)\times(m+n)$-matrix ($S = S^t$). We define the corresponding Gaussian operator $B[S] : \mathbf{F}_n \to \mathbf{F}_m$ by

$$B[S]f(x) = \int_{\mathbb{C}^n} \exp\left\{\frac{1}{2}\begin{pmatrix}z & \overline{u}\end{pmatrix}\begin{pmatrix}K & L\\ L^t & M\end{pmatrix}\begin{pmatrix}z^t & \overline{u}^t\end{pmatrix}\right\} f(u) e^{-|u|^2} d\lambda(u) \tag{1.4}$$

Proposition 1.3 *If $B[S] : \mathbf{F}_n \to \mathbf{F}_m$ is bounded, then*
1. $\|S\| \leqslant 1$.
2. $\|K\| < 1$, $\|M\| < 1$.

PROOF. The statement a) follows from a priory estimate (4.1.11) of a kernel of a bounded operator.

Further, $b[S] \cdot 1 = b[M]$, $b[S]^* \cdot 1 = b[\overline{K}]$. Therefore $b[M] \in \mathbf{F}_m$, $b[\overline{K}] \in \mathbf{F}_m$ and we refer to Proposition 1.1. □

Theorem 1.4 **(Olshanski)** *Under the conditions 1-2 of the previous proposition, the operator $B[S] : \mathbf{F}_n \to \mathbf{F}_m$ is bounded.*

In fact, Gaussian operators in $\mathbf{F}_n$ are $\mathcal{B}$-images of Gaussian operators in $L^2(\mathbb{R}^n)$. Thus, the last theorem is equivalent to the boundedness Theorem 8.2. However, in this chapter, we present an independent proof.

1.4. Creation-annihilation operators. Let

$$\mathcal{W}_{2n} = \mathcal{W}_n^+ \oplus \mathcal{W}_n^- = \mathbb{C}^n \oplus \mathbb{C}^n$$

be the complexification of a real symplectic space (see Section 3.2). Recall that this space is endowed with symmetric bilinear form Λ having the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the Hermitian form M with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and the operator $J : v^+ \oplus v^- \mapsto \overline{v}^- \oplus \overline{v}^+$ of complex conjugation.

Recall that these spaces are objects of the symplectic category $\mathbf{Sp}$, see Section 1.8.

Denote

$$\widehat{a}(w) = \sum_{j=1}^{n} w_j^+ z_j - \sum_{j=1}^{n} w_j^- \frac{\partial}{\partial z_j}$$

Obviously,

$$\begin{aligned} \widehat{a}(w)^* &= -\widehat{a}(Jw) \\ [\widehat{a}(v), \widehat{a}(w)] &= \Lambda(v, w) \cdot 1 \\ [\widehat{a}(v), \widehat{a}(w)^*] &= M(v, w) \cdot 1 \end{aligned}$$

Certainly, these relations are the same as in (1.6.2)–(1.6.4), the explicit correspondence of operators is given by formulae (4.2.4).

1.5. Main construction. Let $P : \mathcal{W}_{2n} \rightrightarrows \mathcal{W}_{2m}$ be a morphism of the category $\mathbf{Sp}$. Let $S(P) = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ be its Potapov transform, i.e., P consists of vectors $v \oplus w \in \mathcal{W}_{2n} \oplus \mathcal{W}_{2m}$ such that

$$\begin{pmatrix} w_+ & v_- \end{pmatrix} = \begin{pmatrix} w_- & v_+ \end{pmatrix} \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} \tag{1.5}$$

We define the Gaussian operator[2]

$$\mathrm{Be}(P) = B[S(P)] : \mathbf{F}_n \to \mathbf{F}_m$$

Theorem 1.5 *a)* $\mathrm{Be}(P)$ *is bounded as an operator* $\mathbf{F}_n \to \mathbf{F}_m$.

b) $\mathrm{Be}(P)$ *is bounded as an operator* $\mathcal{S}\mathbf{F}_n \to \mathcal{S}\mathbf{F}_m$, *where* $\mathcal{S}\mathbf{F}$ *is the* $\mathcal{B}$*-image of* $\mathcal{S}(\mathbb{R}^n)$.

c) For each $v \oplus w \in P$,

$$\widehat{a}(w)\,\mathrm{Be}(P) = \mathrm{Be}(P)\widehat{a}(v) \tag{1.6}$$

The operator $\mathrm{Be}(P)$ *is a unique to within a scalar operator satisfying these conditions.*

d) Let $P : \mathcal{W}_{2n} \rightrightarrows \mathcal{W}_{2m}$, $Q : \mathcal{W}_{2m} \to \mathcal{W}_{2k}$ *be morphisms of the category* $\mathbf{Sp}$, *let* $\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$, $\begin{pmatrix} \widetilde{K} & \widetilde{L} \\ \widetilde{L}^t & \widetilde{M} \end{pmatrix}$ *be their Potapov transforms. Then*

$$\mathrm{Be}(Q)\,\mathrm{Be}(P) = \det(1 - \widetilde{M}K)^{-1/2}\,\mathrm{Be}(QP) \tag{1.7}$$

e) The operators $\mathrm{Be}(P) : \mathbf{F}_n \to \mathbf{F}_m$ *defined now coincide with the* $\mathcal{B}$*-images of the operators* $\mathrm{Be}(P) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^m)$ *defined above (Section 1.8).*

f) Let $g = \begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R})$. *Then the operators*

$$\mathrm{Be}^\circ(g) := \pm\det(\Phi)^{-1/2}\,\mathrm{Be}(g) = \pm\det(\Phi)^{-1/2} B \begin{bmatrix} -\Phi^{-1}\Psi & \Phi^{-1} \\ \Phi^{t-1} & \overline{\Psi}\Phi^{-1} \end{bmatrix} \tag{1.8}$$

are unitary (there is no canonical choice of sign) and

$$\mathrm{Be}^\circ(g_1)\,\mathrm{Be}^\circ(g_2) = \pm\,\mathrm{Be}^\circ(g_1 g_2) \tag{1.9}$$

g) $\mathrm{Be}(P^\sigma) = \mathrm{Be}(P)^*$

Theorem is proved below in the next two sections (except the statement b), whose proof uses canonical forms discussed in the next chapter).

REMARK. The scalar factor in (1.7) is well defined by (2.3.7). □

REMARK. The expression for the matrix $S(QP)$ (see (1.7) in terms of $S(Q)$, $S(P)$ is the formula of product in the Potapov coordinates, i.e., (2.8.9),

$$S(QP) = \begin{pmatrix} K + L\widetilde{K}(1 - M\widetilde{K})^{-1}L^t & L(1 - \widetilde{K}M)^{-1}\widetilde{L} \\ \widetilde{L}^t(1 - M\widetilde{K})^{-1}L^t & \widetilde{M} + \widetilde{L}^t(1 - M\widetilde{K})^{-1}M\widetilde{L} \end{pmatrix} \tag{1.10}$$

REMARK. The matrix $\begin{bmatrix} -\Phi^{-1}\Psi & \Phi^{-1} \\ \Phi^{t-1} & \overline{\Psi}\Phi^{-1} \end{bmatrix}$ from the formula (1.8) is unitary; indeed, $g \in \mathrm{Sp}(2n, \mathbb{R}) \subset \mathrm{U}(n, n)$ and we apply Theorem 2.8.1. □

[2]Thus, we use a double notation $B[S]$ and $\mathrm{Be}(P)$ for the same operators.

1.6. Remark on normalization of Gaussian operators. The relation (1.6) determines the operators $\mathrm{Be}(P)$ to within a scalar. However, formula (1.4) defines our operators canonically. In fact, our operators $\mathrm{Be}(P) : \mathbf{F}_n \to \mathbf{F}_m$ satisfy the following normalization condition

$$\langle \mathrm{Be}(P) \cdot 1_{\mathbf{F}_n}, 1_{\mathbf{F}_m} \rangle_{\mathbf{F}_m} = 1 \tag{1.11}$$

This normalization produces the scalar factor

$$\det(1 - \widetilde{M}K)^{-1/2} = \exp\Bigl\{\frac{1}{2} \operatorname{tr} \ln(1 - \overline{M}K)\Bigr\}$$

in (1.7). This factor is $\exp\bigl\{-\frac{1}{2}c(P,Q)\bigr\}$, where $c(P,Q)$ is the cocycle given by (3.7.2).

For further discussion, see Chapter 3.

5.2 Proof of the product formula

This section contains the proof of product formula (1.7) for Gaussian operator. The formula itself is more-or-less trivial, we simply calculate the convolution of kernels by (4.1.13). Nevertheless, this is a rigorous proof only modulo the boundedness theorem; on another hand, we need in the product formula for the proof of boundedness.

We also prove all the statements of Theorem 1.5 except boundedness.

2.1. Formal definition of $B[S]$.

Observation 2.1 *For each $f \in \mathbf{F}_n$, the expression $B[S]f$ is a well-defined entire function on $\mathbb{C}^m$.*

Proof. Indeed,

$$B[S]f(z) = \exp\Bigl\{\frac{1}{2} zKz^t\Bigr\} \int_{\mathbb{C}^n} \exp\Bigl\{\frac{1}{2} uM\overline{u}^t + zL\overline{u}^t\Bigr\} \cdot f(z)\, e^{-|z|^2}\, d\lambda(z) =$$
$$= \exp\Bigl\{\frac{1}{2} zKz^t\Bigr\} \cdot \langle f, b[\overline{M}|zL] \rangle_{\mathbf{F}_n} \tag{2.1}$$

The inner product is well defined because f, $b[\overline{K}|zL] \in \mathbf{F}_n$. □

2.2. Hilbert–Schmidt condition.

Lemma 2.2 *Assume $\|S\| < 1$.*

a) The operator $B[S] : \mathbf{F}_n \to \mathbf{F}_m$ is a Hilbert-Schmidt operator. In particular, $B[S]$ is bounded.

b) The operator $B[S]$ transfers $\mathcal{S}\mathbf{F}_n \to \mathcal{S}\mathbf{F}_m$.

PROOF. a) In this case, the kernel

$$S(z,\overline{u}) = \exp\Big\{\frac{1}{2}\begin{pmatrix} z & \overline{u}\end{pmatrix}\begin{pmatrix} K & L \\ L^t & M\end{pmatrix}\begin{pmatrix} z^t \\ \overline{u}^t\end{pmatrix}\Big\}$$

is contained in

$$L^2\big(\mathbb{C}^n \oplus \mathbb{C}^m, \exp(-|z|^2 - |u|^2)\big)$$

Therefore (see Subsection 1.1.2) this kernel determines a Hilbert-Schmidt operator

$$\mathfrak{B}[S] : L^2\big(\mathbb{C}^n, \exp(-|u|^2)\big) \to L^2\big(\mathbb{C}^m, \exp(-|z|^2\big)$$

Our operator $\mathrm{B}[S]$ is the restriction of $\mathfrak{B}[S]$ to the subspace $\mathbf{F}_n$.

b) is proved in the next subsection. □

2.3. Product formula for bounded operators. Now let us verify the formula (1.7) for bounded operators. We must evaluate

$$\int_{\mathbb{C}^m} \exp\Big\{\frac{1}{2}\begin{pmatrix} z & \overline{u}\end{pmatrix}\begin{pmatrix} \widetilde{K} & \widetilde{L} \\ \widetilde{L}^t & \widetilde{M}\end{pmatrix}\begin{pmatrix} z^t \\ \overline{u}^t\end{pmatrix}\Big\}\exp\Big\{\frac{1}{2}\begin{pmatrix} u & \overline{\xi}\end{pmatrix}\begin{pmatrix} K & L \\ L^t & M\end{pmatrix}\begin{pmatrix} u^t \\ \overline{\xi}^t\end{pmatrix}\Big\}e^{-|u|^2}d\lambda(u) =$$

$$= \exp\big\{\frac{1}{2}z\widetilde{K}z^t + \frac{1}{2}\overline{\xi}M\overline{\xi}^t\big\}\cdot\big\langle b\big[\overline{\widetilde{M}}\,|\,z\widetilde{L}\big], b\big[\overline{K}\,|\,\xi\overline{L}^t\big]\big\rangle$$

It remains to apply formula (1.3). □

In particular, the product formula holds for operators $B[S_1]$, $B[S_2]$ such that $\|S_1\| < 1$, $\|S_2\| < 1$. It suggests that the formula can be extended to a general case by continuity considerations. This actually is done in Subsection 2.6 after some preparations.

PROOF OF LEMMA 2.2.B. We decompose $B[S]$ as

$$B\begin{bmatrix} K & L \\ L^t & M\end{bmatrix} = B\begin{bmatrix} 0 & 1-\varepsilon \\ 1-\varepsilon & 0\end{bmatrix} B\begin{bmatrix} (1-\varepsilon)^{-2}K & (1-\varepsilon)^{-1}L \\ (1-\varepsilon)^{-1}L^t & M\end{bmatrix}$$

If $\varepsilon > 0$ is sufficiently small, then the second factor satisfies the condition of the lemma; therefore it is a bounded operator $\mathbf{F}_n \to \mathbf{F}_m$. The first factor equals

$$B\begin{bmatrix} 0 & 1-\varepsilon \\ 1-\varepsilon & 0\end{bmatrix} f(z) = f\big((1-\varepsilon)z\big)$$

and it takes $\mathbf{F}_n$ to $\mathcal{S}\mathbf{F}_m$ (see the characterization of $\mathcal{S}\mathbf{F}_n$ in Proposition 4.2.5). □

2.4. Action on Gaussian vectors.

Proposition 2.3 *Let* $\begin{pmatrix} K & L \\ L^t & M\end{pmatrix}$ *satisfy the conditions of Proposition 1.3, i.e., the hypothetical conditions of boundedness. Let* $\|T\| < 1$. *Then*

$$B\begin{bmatrix} K & L \\ L^t & M\end{bmatrix} b[T|\tau] = \det(1-MT)^{1/2}\, b\big[K + LT(1-MT)^{-1}L^t\;\big|\;L(1-TM)^{-1}\tau^t\big] \tag{2.2}$$

In particular, the vector $b[\cdot]$ *in the right-hand side is contained in* $\mathbf{F}_m$.

PROOF.

$$B\begin{bmatrix} K & L \\ L^t & M \end{bmatrix} b[T|\tau](u) =$$
$$= \exp\{\tfrac{1}{2} zKz^t\} \cdot \int_{\mathbb{C}^n} \exp\Big\{\frac{1}{2}\overline{u}M\overline{u}^t + zL\overline{u}^t\Big\} \cdot \exp\Big\{\frac{1}{2}uTu^t + u\tau^t\Big\} e^{-u\overline{u}}\, d\lambda(u)$$

The last integral is an inner product of form (1.3). This implies explicit formula (2.2).

Finally, we must show that $\|K + LT(1 - MT)^{-1}L^t\| < 1$, this is necessary for $b[\cdot] \in \mathbf{F}_n$. This is not obvious, however we can refer to our discussion of the Krein-Schmullian functor, see (2.9.2). $\square$

In particular,

$$B\begin{bmatrix} K & L \\ L^t & M \end{bmatrix} b[T] = \det(1 - MT)^{1/2}\, b\big[K + LT(1 - MT)^{-1}L^t\big] \tag{2.3}$$

Observation 2.4 *An operator* $\mathrm{Be}(P)$ *takes a Gaussian vector* $b[T]$ *to a vector of the form* const $\cdot b[T']$, *where* T' *is the image of* T *under the Krein–Schmulian map corresponding to* P.

2.5. An invariant dense subspace. Denote by $\mathfrak{F}_n$ the subspace in $\mathbf{F}_n$ consisting of vectors of the form

$$p(z)\exp\{\tfrac{1}{2}zTz^t + \tau z^t\}, \qquad p(z) \text{ is a polynomial, and } \|T\| < 1$$

Proposition 2.5 *For each* $f \in \mathfrak{F}_n$ *and each* $B[S]$ *satisfying the conditions of the boundedness theorem, the vector* $B[S]f$ *is contained in* $\mathfrak{F}_m$.

PROOF. Denote $\varepsilon := (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{C}^n$. Consider the following $f \in \mathfrak{F}_n$,

$$f = z_1^{p_1} \dots z_n^{p_n} \exp\{\tfrac{1}{2}zTz^t + \tau z^t\} = \frac{d^{p_1}}{d\varepsilon_1^{p_1}} \cdots \frac{d^{p_1}}{d\varepsilon_n^{p_1}} \exp\{\tfrac{1}{2}zTz^t + \tau z^t + \varepsilon z^t\}\Big|_{\varepsilon=0} \tag{2.4}$$

Then $B\begin{bmatrix} K & L \\ M & N \end{bmatrix} f$ is

$$\det(1-MT)^{-1/2}\frac{d^{p_1}}{d\varepsilon_1^{p_1}} \cdots \frac{d^{p_1}}{d\varepsilon_n^{p_1}} b\big[K+LT(1-MT)^{-1}L^t \,\big|\, L(1-TM)^{-1}(\tau+\varepsilon)^t\big]\Big|_{\varepsilon=0} \tag{2.5}$$

and this vector is contained in $\mathfrak{F}_n$. It is also necessary to watch an differentiation with respect to parameters in (2.1). $\square$

2.6. Proof of the product formula (1.7). General case.

Lemma 2.6 *For* $f \in \mathfrak{F}_n$ *and* $\mathrm{Be}(Q)$, $\mathrm{Be}(P)$ *satisfying the conditions of the boundedness theorem*

$$\mathrm{Be}(Q)\,\mathrm{Be}(P)f = \lambda \cdot \mathrm{Be}(QP)f \tag{2.6}$$

and λ *is the same as in (1.7).*

PROOF. This is obvious for the following reason.

Let S, $\widetilde{S}$ be Potapov transforms of P and Q, let $f(z) = \prod z_j^{p_j} \exp\{\frac{1}{2} zTZ^t + z\tau^t\}$. If $\|S\| < 1$, $\|\widetilde{S}\| < 1$, then our operators are bounded and we can refer to Lemma 2.2.

Further, both sides of (2.6) are explicit expressions depending real-analytically in S, $\widetilde{S}$, T, τ[3].

Since the left-hand side and the right-hand side coincide for $\|S\| < 1$, $\|\widetilde{S}\| < 1$, then they coincide always. □

2.7. Verification of the commutation relations (1.6).

Lemma 2.7 *Let $P : \mathcal{W}_{2n} \rightrightarrows \mathcal{W}_{2m}$ be a morphism of the category* **Sp**. *For each $f \in \mathfrak{F}_n$ and $v \oplus w \in P$*

$$\widehat{a}(w)\,\mathrm{Be}(P) f = \mathrm{Be}(P)\widehat{a}(v) f \tag{2.7}$$

PROOF. First of all, by Observation 2.1 both sides are well defined.

Denote the kernel of the operator $\mathrm{Be}(P)$ by $S(z, \overline{u})$,

$$S(z, \overline{u}) = \exp\Bigl\{\frac{1}{2}\begin{pmatrix} z & \overline{u}\end{pmatrix}\begin{pmatrix} K & L \\ L^t & M\end{pmatrix}\begin{pmatrix} z^t \\ \overline{u}^t\end{pmatrix}\Bigr\} \tag{2.8}$$

Let us apply formulae (4.1.17)-(4.1.20). The kernel of the operator $\widehat{a}(w)\,\mathrm{Be}(P)$ is

$$\Bigl(\sum w_j^+ z_j - \sum w_j^- \frac{\partial}{\partial z_j}\Bigr) S(z, \overline{u}) = \Bigl[w^+ z^t - w^- K z^t - w^- L \overline{u}^t\Bigr] S(z, \overline{u}) \tag{2.9}$$

The kernel of the operator $\mathrm{Be}(P)\widehat{a}(v)$ is

$$\Bigl(-\sum v_j^+ \frac{\partial}{\partial \overline{u}_j} + \sum v_j^- \overline{u}\Bigr) S(z, \overline{u}) = \Bigl[v^- \overline{u}^t - v^+ K \overline{u}^t - v^+ L^t z^t\Bigr] S(z, \overline{u}) \tag{2.10}$$

The factors in the square brackets in (2.9) and (2.10) coincide iff $v \oplus w \in P$, see (1.5). □

2.8. Adjoint operators. The statement g) of our Theorem 1.5 is obvious.

2.9. Unitary operators. Now we are going to prove statement f).

For $g \in \mathrm{Sp}(2n, \mathbb{R})$, we have $g^\sigma g = 1$. By Subsection 2.8,

$$\mathrm{Be}(g)\,\mathrm{Be}(g)^* = \lambda \cdot \mathrm{Be}(g g^\sigma) = \lambda \cdot 1, \qquad \lambda \in \mathbb{C}^*$$

The product formula (1.7) provides

$$\lambda = \sqrt{\det \Phi\Phi^*}$$

Since $\sqrt{\lambda} = |\det \Phi|$, it follows that the operators $\mathrm{Be}^\circ(g) := \pm \det(\Phi)^{-1/2}\,\mathrm{Be}(g)$ are unitary

[3]It is not too difficult (and not too pleasant) to write these expressions explicitly; however this is not a necessary element of the proof.

It remains to evaluate products of the operators $\mathrm{Be}^\circ(g)$. Return to the notation of Section 3.3.6. For $g \in \mathrm{Sp}(2n, \mathbb{R})$ denote $g = \begin{pmatrix} \Phi(g) & \Psi(g) \\ \overline{\Psi}(g) & \overline{\Phi}(g) \end{pmatrix}$. The product formula provides

$$\mathrm{Be}(g_1)\,\mathrm{Be}(g_2) = \det\bigl(1 + \Phi(g_1)^{-1}\Psi(g_1)\overline{\Psi(g_2)})\Phi(g_2)^{-1}\bigr)^{-1/2}\,\mathrm{Be}(g_1 g_2)$$

We write out the determinant factor as

$$\pm \det \Phi(g_1)^{1/2} \det\bigl(\Phi(g_1)\Phi(g_2) + \Psi(g_1)\overline{\Psi(g_2)}\bigr)^{-1/2}\Phi(g_2)^{1/2} = \\ = \pm \det \Phi(g_1)^{1/2} \det \Phi(g_1 g_2)^{-1/2}\Phi(g_2)^{1/2} \quad (2.11)$$

we have to write $\pm$ since the $\det \Phi(g)^{1/2}$ is not canonically defined. In other words, $B^\circ(g_1)B^\circ(g_2) = \pm B^\circ(g_1 g_2)$. Q.E.D. $\square$

Thus all the statements of Theorem 1.5 except the boundedness are proved.

2.10. Reduction of boundedness to a self-adjoint case. An operator $\mathrm{Be}(P)$ is bounded iff the following function is bounded on the unit ball $\|f\| \leqslant 1$ in $\mathfrak{F}_n$

$$\gamma(f) = \langle \mathrm{Be}(P)f, \mathrm{Be}(P)f\rangle = \langle f, \mathrm{Be}(P)^*\,\mathrm{Be}(P)f\rangle = \mathrm{const}\cdot\langle f, \mathrm{Be}(P^\sigma P)f\rangle$$

Thus the problem is reduced to boundedness of the symmetric operator $\mathrm{Be}(P^\sigma P)$.

5.3 Boundedness. Spectra of self-adjoint Gaussian operators

Here we prove Boundedness Theorem 1.5. Recall that it was reduced to self-adjoint operators.

3.1. Fix-point argument. Let $P : \mathcal{W}_{2n} \rightrightarrows \mathcal{W}_{2n}$ be a morphism of the category **Sp**, let

$$S(P) = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} \quad (3.1)$$

be its Potapov transform.

Theorem 3.1 *a) There exists a function $b[T]$ with $\|T\| < 1$ such that*

$$\mathrm{Be}(P)b[T] = \lambda b[T]$$

moreover

$$\lambda = \det(1 - MT)^{-1/2} \quad (3.2)$$

b) If $\|S(P)\| \leqslant 1$, then $\|T\| < 1$, i.e., $b[T] \in \mathbf{F}_n$

PROOF. Let $\overline{\mathrm{B}}_n$ be the Cartan matrix ball $\|T\| \leqslant 1$. Let $\varkappa(P) : \overline{\mathrm{B}}_n \to \mathrm{B}_n$ be the Krein–Shmulian map corresponding to P. By the Brouwer theorem, it has a fixed point T. By (2.3), $b[T]$ is an eigenfunction of $\mathrm{Be}(P)$.

If $\|S\| < 1$, then $\varkappa(P)$ takes $\overline{\mathrm{B}}_n$ to B_n. Therefore a fixed point $\in \mathrm{B}_n$.

See also Theorem 2.7.12. □

3.2. Application to self-adjoint case. Let $P : \mathcal{W}_{2n} \rightrightarrows \mathcal{W}_{2n}$ be a self-adjoint of the category **Sp**. Its Potapov transform satisfies

$$S(P) = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}, \qquad M = \overline{K}, L = L^* \tag{3.3}$$

Theorem 3.2 *Suppose* $\mathrm{Be}(P)$ *has a Gaussian eigenvector* $b[T] \in \mathbf{F}_n$ *(this holds automatically whenever* $\|S(P)\| < 1$*). Denote by* λ *its eigenvalue. Then*

a) $b[T]$ *is a maximizer, i.e.,* $\|\mathrm{Be}(P)\| = \lambda$.

b) $\lambda = \det(1 - MT)^{-1/2}$

c) The spectrum of $\mathrm{Be}(P)$ *has the following form*

$$\Big\{ \lambda \prod_{j=1}^{n} \mu_j^{l_j} \Big\}$$

where μ_j *are fixed,* $-1 \leqslant \mu_j \leqslant 1$ *and* l_j *range in* $\{0, 1, 2, 3, \dots\}$.

PROOF. By Observation 2.4, for $g \in \mathrm{Sp}(2n, \mathbb{R})$ and $T \in \mathrm{B}_n$

$$\mathrm{Be}(g) b[T] = \mathrm{const} \cdot b[T^{[g]}], \qquad \text{where } T^{[g]} := (\Phi + z\overline{\Psi})^{-1}(\Psi + z\overline{\Phi})$$

Since the action $\mathrm{Sp}(2n, \mathbb{R})$ on the matrix ball is transitive, there is (non-unique) $g \in \mathrm{Sp}(2n, \mathbb{R})$ such that $gb[T] = \lambda \cdot b[0]$. Further, $b[0]$ is an eigenvector of $\mathrm{Be}(gPg^{-1})$; indeed

$$\begin{aligned} \mathrm{Be}(gPg^{-1}) b[0] &= \mathrm{const} \cdot \mathrm{Be}^{\circ}(g) \,\mathrm{Be}(P)\, \mathrm{Be}^{\circ}(g)^{-1} b[0] = \\ &= \mathrm{const}' \cdot \mathrm{Be}^{\circ}(g)\, \mathrm{Be}(P) b[T] = \mathrm{const}'' \cdot \mathrm{Be}^{\circ}(g) b[T] = \mathrm{const}''' \cdot b[0] \end{aligned}$$

Let $\widetilde{S} = \begin{pmatrix} \widetilde{K} & \widetilde{L} \\ \widetilde{L}^t & \widetilde{M} \end{pmatrix}$ be the Potapov transform of gPg^{-1}. Since $b[0]$ is an eigenvector, the formula (2.3) implies

$$0 = \widetilde{K} + \widetilde{L} \cdot 0 (1 - \widetilde{M} \cdot 0)^{-1} \widetilde{L}^t$$

i.e., $\widetilde{K} = 0$. Since P is self-adjoint, it follows that $\overline{M} = 0$ and $\widetilde{L} = \widetilde{L}^*$ (see (3.3).

Thus, $\mathrm{Be}(P)$ can be represented in the form

$$\mathrm{Be}(P) = \mathrm{const} \cdot \mathrm{Be}^{\circ}(g)\, \mathrm{B} \begin{bmatrix} 0 & \widetilde{L} \\ \widetilde{L}^t & 0 \end{bmatrix} \mathrm{Be}^{\circ}(g)^{-1}, \tag{3.4}$$

The middle factor is an operator of change of variable (see Subsection 4.1.7)

$$B\begin{bmatrix} 0 & \widetilde{L} \\ \widetilde{L}^t & 0 \end{bmatrix} f(z) = f(zL) \tag{3.5}$$

We can reduce the self-adjoint matrix $\widetilde{L}$ to the diagonal form, $\widetilde{L} = U\Delta U^{-1}$, here the matrix U is unitary and Δ is a diagonal matrix, denote by μ_j be its eigenvalues.

Without loss of generality, we can think that $\widetilde{L} = \Delta$, otherwise we change coordinates in $\mathbb{C}^n$ by the matrix U. Thus, our middle factor has the form

$$B\begin{bmatrix} 0 & \Delta \\ \Delta & 0 \end{bmatrix} f(z_1, \dots, z_n) = f(\mu_1 z_1, \dots, \mu_n z_n)$$

The monomials $\prod z_j^{l_j}$ are the eigenvectors, the eigenvalues are $\prod \mu_j^{l_j}$, and $f(z) = 1$ is a maximizer (since $\|L\| \leqslant 1$).

The initial operator $\mathrm{Be}(P)$ is conjugate to the operator of change of variable (3.5) to within a scalar. Further

$$\mathrm{Be}^\circ(g)^{-1} \cdot b[0] = \mathrm{const} \cdot b[T]$$

Therefore, $b[T]$ is the maximizer for $\mathrm{Be}(P)$. Hence the constant in (3.4) coincide with the eigenvalue of $b[T]$, i.e., $\sigma = \det(1 - MT)^{-1/2}$. □

3.3. An estimate of a norm.

Lemma 3.3 *Let A be a square matrix with $\|A\| < 1$. Then*

a) $|\det(1 - A)| \geqslant \det(1 - |A|)$

b) For each matrix R with $\|R\| < 1$, $|\det(1 - RA)| \geqslant \det(1 - |A|)$.

PROOF. a) Let a_j be the eigenvalues of A, α_i be the singular values. We must verify the inequality

$$\prod\nolimits_j (1 - |a_j|) \geqslant \prod\nolimits_j (1 - \alpha_j),$$

or equivalently, $\sum_j \ln(1 - |a_j|) \geqslant \sum_j \ln(1 - \alpha_j)$, or

$$\sum_{n=1}^{\infty} \frac{1}{n} \Big(\sum_j |a_j|^n\Big) \leqslant \sum_{n=1}^{\infty} \frac{1}{n} \Big(\sum_j |\alpha_j|^n\Big).$$

The last inequality follows from the inequality

$$\sum\nolimits_j |a_j|^n \leqslant \sum\nolimits_j \alpha_j^n,$$

which is a special case of the Weyl inequalities for eigenvalues and singular numbers, see [180].

b) Let β_j be the singular values of RA. By the mini-max characterization of singular values (2.5.3), it follows that $\beta_j \leqslant \alpha_j$. □

Theorem 3.4 *For a self-adjoint Gaussian operator $B[S]$,*

$$\left\| B \begin{bmatrix} K & L \\ L^t & M \end{bmatrix} \right\| \leqslant \det(1-|M|)^{-1/2}$$

PROOF. First, let $\|S\|<1$. In this case, the eigenvector $b[T]$ is contained in $\mathbf{F}_n$, the norm of the operator is $\det(1-MT)^{-1/2} \leqslant \det(1-|M|)^{-1/2}$.

Next, let us apply limit considerations. Consider a matrix $(1-\varepsilon)S$,

$$\|B\big[(1-\varepsilon)S\big]\| \leqslant \det\big(1-(1-\varepsilon)|M|\big)^{-1/2} \leqslant \det(1-|M|)^{-1/2}$$

The kernels of $B\big[(1-\varepsilon)S\big]$ converge pointwise to the kernel of $B[S]$ and norms of $B[(1-\varepsilon)S]$ are uniformly bounded. By Proposition 4.1.13, it follows that the family $B[(1-\varepsilon)S]$ weakly converges to $B[S]$, the operator $B[S]$ is bounded, and its norm is majorized by the same constant. □

This also completes a proof of boundedness of operators $B[S]$ in Hilbert spaces $\mathbf{F}_n$.

3.4. Operators $B[S]$ in $\mathcal{S}\mathbf{F}_n$. It remains to prove the statement b) of Theorem 1.5, i.e., we must show that our operators are well defined in in $\mathcal{S}\mathbf{F}$.

Lemma 3.5 *a) For $g \in \mathrm{Sp}(2n,\mathbb{R})$,*

$$\mathrm{Be}^\circ(g)\widehat{a}(v)\,\mathrm{Be}^\circ(g)^{-1} = \widehat{a}(vg) \tag{3.6}$$

in particular, domains of definiteness of these operators in $\mathbf{F}_n$ coincide.

b) The operator $\mathrm{Be}^\circ(g)$ preserves the space $\mathcal{S}\mathbf{F}_n$.

PROOF. a) By Lemma 2.7, the identity holds on a dense subspace $\mathfrak{F}_n$. The operator $\mathrm{Be}^\circ(g)$ is unitary, therefore the conjugation by $Be^\circ(g)$ takes closed operators to closed operators.

b) We use a characterization of $\mathcal{S}\mathbf{F}_n$ mentioned in proof of Proposition 4.2.5.a. By the statement a), operators $\mathrm{Be}^\circ(g)$ regard this characterization. □

Also operators of change of variable preserve $\mathcal{S}\mathbf{F}$ (this also easily follows from Proposition 4.2.5)). It remains to note that each Gaussian operator decomposes into a product $\mathrm{Be}^\circ(g_1)Q\,\mathrm{Be}^\circ(g_2)$, where Q is an operator of a change of variable and g_1, g_2 are symplectic matrices (here we refer to Potapov–Olshanski decomposition, Theorem 3.5.2).

5.4 Bibliographical remarks

The representation of symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ in the Fock space (mainly for $n=\infty$) was considered by F.A.Berezin in his seminal book 'Method of second quantization' [11]. The boundedness theorem was obtained by G.I.Olshanski near 1983. Gaussian operators in the Fock space are exposed in a wider generality in [118].