# Chapter 6

# Gaussian operators. Details

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To be short, here we consider only operators $B[S] : \mathbf{F}_n \to \mathbf{F}_n$.

## 6.1 Canonical forms and invariants

**1.1. Tensor products.**

**Observation 1.1** *a)* $\mathbf{F}_n \otimes \mathbf{F}_k \simeq \mathbf{F}_{n+k}$

*b) If operators* $A : \mathbf{F}_m \to \mathbf{F}_n$ *and* $A' : \mathbf{F}_{m'} \to \mathbf{F}_{n'}$ *have kernels* $K(z,u)$ *and* $K'(z',u')$ *respectively, then the operator* $A \otimes A'$ *has the kernel* $K(z,u)K(z',u')$

See the definition of products of Hilbert spaces, Subsection 1.1.6. The canonical operator $\mathbf{F}_n \otimes \mathbf{F}_k \to \mathbf{F}_{n+k}$ takes a vector $f \otimes f'$ to $f(z)f'(z')$.

**1.2. Canonical forms of Gaussian operators.**

**Theorem 1.2** *a) Each operator* $B[S]$ *in* $\mathbf{F}_n$ *admits a decomposition*

$$B[S] = \lambda \cdot U_1 B[R] U_2 \tag{1.1}$$

*where* $\lambda \in \mathbb{C}^*$ *is a constant,* $U_1$, $U_2$ *are unitary Gaussian operators (corresponding to elements of* $\mathrm{Sp}(2n,\mathbb{R})$*) and* $B[R]$ *is a tensor product of operators* $\mathbf{F}_1 \to \mathbf{F}_1$ *of two following types*

$$\text{I.} \qquad B\begin{pmatrix} 0 & s_j \\ s_j & 0 \end{pmatrix}, \quad \textit{where } 0 \leqslant s_j \leqslant 1, \tag{1.2}$$

$$\text{II.} \qquad B\begin{pmatrix} \sigma_k & 1-\sigma_k \\ 1-\sigma_k & \sigma_k \end{pmatrix}, \quad \textit{where } 0 < \sigma_k < 1, \tag{1.3}$$

*a) The collection $s_j$ is uniquely determined by the operator $B[S]$. Also the number of blocks of each type is uniquely determined by the operator $B[S]$.*

*b) If $\|S\| < 1$, then there are only blocks of the first type with $0 \leqslant s < 1$.*

REMARK. The numbers $\sigma_j$ are not invariants. Moreover, all the operators in $\mathbf{F}_1$ of the type II are conjugate by elements of $\mathrm{Sp}(2, \mathbb{R})$. Apparently this remark looks like non-obvious, however these operators correspond to the operators

$$A(\sigma)f(x) = e^{-\sigma x^2/2}f(x)$$

in $L^2(\mathbb{R})$. The latter operators are conjugate under the dilatations $H(a)f(x) = a^{1/2}f(ax)$.

Also, passing to limit as $\sigma \to 0$, we get the identity operator (type I with $s = 1$). □

PROOF. This is a rephrasing of canonical forms of symplectic contractions, see Subsections 1.1.10, 3.3.5. □

**1.3. Invariants $s_j$ of Gaussian operators.** We say that the numbers

$$\{s_1, s_2, \dots, s_n\}$$

are the *invariants* of the Gaussian operator $B[S] : \mathbf{F}_n \to \mathbf{F}_n$; to the matrices of the type II we assign $s := 1$.

**Theorem 1.3** *The roots $\lambda$ of the equation*

$$\det\left((1 - S^*S) - \lambda\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - S^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S\right)\right) = 0 \qquad (1.4)$$

*are* $\mathrm{Sp}(2n, \mathbb{R}) \times \mathrm{Sp}(2n, \mathbb{R})$*-invariants of the operator $B[S]$. In addition,*

$$\lambda = \pm(1 - s_j^2)/(1 + s_j^2) \qquad (1.5)$$

PROOF. Let $\mathcal{W}_{2n}$ be the complexification of a real symplectic space, as in Subsection 5.1.5. We have a contractive linear relation $P : \mathcal{W}_{2n} \rightrightarrows \mathcal{W}_{2n}$ corresponding to the matrix $S$ as in (5.1.5).

Define the two following Hermitian forms on $\mathcal{W}_{2n} \oplus \mathcal{W}_{2n}$.

$$M^{\ominus}(v \oplus w, v' \oplus w') := M(v, v') - M(w, w')$$
$$M^{\oplus}(v \oplus w, v' \oplus w') := M(v, v') + M(w, w')$$

The group $\mathrm{Sp}(2n, \mathbb{R}) \times \mathrm{Sp}(2n, \mathbb{R})$ acts in $\mathcal{W}_{2n} \oplus \mathcal{W}_{2n}$ in the natural way and preserves both forms.

Further, restricting these forms to the subspace $P$, we get a pair of Hermitian forms on this subspace. Writing out these forms in the coordinates $(v_-, w_+)$, we obtain the Hermitian matrices

$$(1 - S^*S), \qquad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - S^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S$$

We write invariants of a pair of Hermitian matrices as it was explained in Theorem 2.6.9. This provides the first statement.

Next, we evaluate the invariants $\lambda_j$ for the matrix $S := \begin{pmatrix} 0 & \Sigma \\ \Sigma & 0 \end{pmatrix}$, where $\Sigma$ is the diagonal matrix with entries $s_j$. Comparing the results, we come to (1.5). $\square$

**1.4. Canonical forms of self-adjoint operators.**

**Proposition 1.4** *a) Each self-adjoint operator $B[S]$ in $\mathbf{F}_n$ admits a representation $B[S] = \lambda \cdot UB[R]U^{-1}$ where $\lambda \in \mathbb{C}^*$, $U$ is a unitary Gaussian operator, and $B[R]$ is a tensor product of operators $\mathbf{F}_1 \to \mathbf{F}_1$ of 2 following types*

$$\text{I.} \qquad B\begin{pmatrix} 0 & t_j \\ t_j & 0 \end{pmatrix}, \quad \textit{where } -1 \leqslant t_j \leqslant 1, \tag{1.6}$$

$$\text{II.} \qquad B\begin{pmatrix} \tau_k & \pm(1-\tau_k) \\ \pm(1-\tau_k) & \tau_k \end{pmatrix}, \quad \textit{where } 0 < \tau_k < 1, \tag{1.7}$$

We omit a proof, since the statement is a symplectic variant of Lemma 2.7.9.

We say that the numbers $t_1, t_2, \dots$ are the invariants of $B[S]$. To each matrix of the type II we assign $t = \pm 1$ according the sign in the matrix.

**1.5. Invariants $t_j$ of self-adjoint operators.** Consider a self-adjoint operator $B[S] = B\begin{bmatrix} K & L \\ L^t & M \end{bmatrix} = \mathrm{Be}[P]$. For simplicity, assume $\|S\| < 1$, and $\det L \neq 0$. In other words, $P$ is a graph of a (pseudo)-self-adjoint strictly contractive element $g \in \mathrm{Sp}(2n, \mathbb{C})$.

**Proposition 1.5** *Under these conditions, the roots of the equation*

$$\det\Bigl(L^t + tM(1 - tL)^{-1}K - t\Bigr) = 0 \tag{1.8}$$

*lying in the interval $(-1, 1)$ coincide with the invariants $t_j$.*

PROOF. Consider the corresponding element

$$g = \begin{pmatrix} L^{-1} & -L^{-1}K \\ ML^{-1} & L^t - ML^{-1}K \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{C})$$

(this is formula (8.4) for the inverse Potapov transform). Since $g$ is symplectic, its spectrum is invariant under the transformation $\tau \mapsto \tau^{-1}$. Since $g$ is pseudo-self-adjoint and contractive, its spectrum is real (see Lemma 2.7.9). We write the characteristic equation $\det(g - \tau) = 0$ and apply formula (1.1.11) for a determinant of a block matrix

$$\begin{aligned} 0 = \det &\begin{pmatrix} L^{-1} - \tau & -L^{-1}K \\ ML^{-1} & L^t - ML^{-1}K - \tau \end{pmatrix} = \\ &= \det(L^{-1} - \tau)\det\Bigl(L^t - ML^{-1}K - \tau + ML^{-1}(L^{-1} - \tau)^{-1}L^{-1}K\Bigr) \end{aligned} \tag{1.9}$$

Next, we transform the following subexpression of the last fromula

$$\begin{aligned} -ML^{-1}K + ML^{-1}(L^{-1}-\tau)^{-1}L^{-1}K &= M(-L^{-1}+L^{-1}(L^{-1}-\tau)^{-1}L^{-1})K \\ &= M(-L^{-1}+L^{-1}(1-\tau L)^{-1})K = \tau M(1-\tau L)^{-1}K \end{aligned}$$

Finally, we obtain

$$0 = \det(L^{-1}-\tau)\det\Big(L^t + \tau N(1-\tau L)^{-1}M - \tau\Big) \tag{1.10}$$

By the construction, the roots of this equation are invariants of $B[S]$.

Next, put $S := \begin{pmatrix} 0 & \Xi \\ \Xi & 0 \end{pmatrix}$, where $\Xi$ is a diagonal matrix with entries $t_j$. Then the roots of equation (1.10), i.e.,

$$0 = \det(\Xi^{-1}-\tau)\det(\Xi^t - \tau)$$

are precisely $t_j^{\pm 1}$.

Further, the collection of roots (1.10) is invariant under the transformation $\tau \mapsto \tau^{-1}$. Roots of the first factor satisfy $|\tau_j| > 1$, therefore all the eigenvalues in the interval $(-1,1)$ are roots of the second factor. □

REMARK. Nevertheless, the total collection of roots of (1.8) is not an invariant of $B[S]$. Usually, a root of $\det(L^{-1}-\tau)$ is not a root of the total characteristic equation (1.10), because the second factor has a pole at the same point. However, a pole can be absent (for instance $K = M = 0$); then the roots of $\det(L^{-1}-\tau)$ are essential. □

Our considerations also imply the following statement.

**Lemma 1.6** *Suppose $\|S\| < 1$ (and $B[S]$ is not necessary self-adjoint). Then collection of roots of the equation (1.8) lying inside circle $|\lambda| < 1$ does not change under conjugations of $B[S]$ by unitary Gaussian operators.*

## 6.2 Norms

**2.1. Norms of a self-adjoint operators.**

**Theorem 2.1** *Let $\|S\| < 1$, and let $B[S]$ be self-adjoint, i.e., $S = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ satisfies $K = \overline{M}$, $L = L^*$. Let $t_j$ be the invariants of the operator $B[S]$. Then*

$$\|B[S]\| = \det\left[\begin{pmatrix} 1-L & -K \\ -M & 1-L^t \end{pmatrix}^{-1/2}\right]\prod_{j=1}^{n}(1-t_j) \tag{2.1}$$

$$= \det\left[\begin{pmatrix} 1+L & -K \\ -M & 1+L^t \end{pmatrix}^{-1/2}\right]\prod_{j=1}^{n}(1+t_j) \tag{2.2}$$

PROOF. Note that $B[S]$ is a trace class operator (indeed, we know its spectrum, see Theorem 5.3.2[1]).

Let us calculate its trace in two ways. First, the trace is equal to the integral of the kernel of the operator over the diagonal:

$$\operatorname{tr} B[S] = \int \exp\left\{\frac{1}{2}(z\overline{z})\begin{pmatrix} K & L \\ L^t & M\end{pmatrix}\begin{pmatrix} z \\ \overline{z}\end{pmatrix} \exp\{-|z|^2\}\right\} dz$$
$$= \det\left[\begin{pmatrix} 1-L & -K \\ -M & 1-L^t\end{pmatrix}^{-1/2}\right].$$

Secondly, the trace is the sum of eigenvalues. We know the spectrum of $B[S]$ (see Theorem 5.3.2). Thus,

$$\operatorname{tr} B[S] = \|B[S]\| \sum t_1^{k_1} \cdots t_n^{k_n} = \|B[S]\| \prod_{j=1}^{n} (1-t_j)^{-1}.$$

Comparing the two expressions, we obtain the first formula.

To derive the second formula, consider the operator $B\begin{bmatrix} 0 & -1 \\ -1 & 0\end{bmatrix} f(z) = f(-z)$, it commutes with any $B[S]$:

$$B\begin{bmatrix} 0 & -1 \\ -1 & 0\end{bmatrix} B[S] = B[S]\, B\begin{bmatrix} 0 & -1 \\ -1 & 0\end{bmatrix} = B\left[\begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix} S \begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}\right],$$

Calculating $\operatorname{tr}\left(B\begin{bmatrix} 0 & -E \\ -E & 0\end{bmatrix} B[S]\right)$ in the same two ways, we come to (2.2). □

**2.2. Norms of arbitrary operators** $B[S]$**.** Denote $J = \begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}$.

**Theorem 2.2** *a) Let* $\|S\| < 1$*. Then*

$$\|B[S]\| = \det(1 - S^* S)^{-1/4} \prod_{j=1}^{n} (1 - s_j^2)^{1/2}. \tag{2.3}$$

$$= \det(1 - S^* JSJ)^{-1/4} \prod_{j=1}^{n} (1 + s_j^2)^{1/2}. \tag{2.4}$$

*b) The second formula holds for an arbitrary* $S$*.*

PROOF. We calculate $\operatorname{tr}(B[S]^* B[S])$ in two ways.

First, we reduce the operator to the canonical form

$$B[S] = \lambda \cdot U_1 B \begin{bmatrix} 0 & \Xi \\ \Xi & 0\end{bmatrix} U_2$$

[1]the same arguments shows that for $\|S\| < 1$ the operator $B[S]$ is contained in all the Shatten classes

with unitary $U_1$, $U_2$ and diagonal $\Xi$. Then

$$B[S]^* B[S] = |\lambda|^2 U_1 B \begin{bmatrix} 0 & \Xi \\ \Xi & 0 \end{bmatrix}^2 U_1^{-1}$$

Hence the collection of eigenvalues of $B[S]^* B[S]$ has form $\{|\lambda|^2 s_1^{2k_1} \dots s_n^{2k_n}\}$ where $k_j$ range in $\{0, 1, 2, \dots\}$ (see also Theorem 3.2). The maximal eigenvalue $|\lambda|^2$ coincides with $\|B[S]^* B[S]\| = \|B[S]\|^2$. Thus

$$\operatorname{tr} B[S]^* B[S] = \|B[S]\|^2 \sum s_1^{2k_1} \cdots s_n^{2k_n} = \|B[S]\|^2 \prod_{j=1}^{n} (1 - s_j^2)^{-1}. \qquad (2.5)$$

**Problem 2.1** Let $A : \mathbf{F}_n \to \mathbf{F}_n$ be a Hilbert–Schmidt operator, let $K$ be its kernel. Then

$$\operatorname{tr} A^* A = \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \bigl|K(z,u)\bigr|^2 e^{-|z|^2} e^{-|u|^2}\, d\lambda(z)\, d\lambda(u)$$

In our case,

$$\operatorname{tr}(B[S]^* B[S]) = \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \bigl|\exp\{\frac{1}{2}(z \ \overline{u}) S (z \ \overline{u})^t\}\bigr|^2 e^{-|z|^2} e^{-|u|^2}\, d\lambda(z)\, d\lambda(u)$$

To simplify notation, we write $w := \begin{pmatrix} z & \overline{u} \end{pmatrix}$ and get

$$\int_{\mathbb{C}^{2n}} \exp\Bigl\{\frac{1}{2} \begin{pmatrix} w & \overline{w} \end{pmatrix} \begin{pmatrix} S & 1 \\ 1 & S^* \end{pmatrix} \begin{pmatrix} w \\ \overline{w} \end{pmatrix}\Bigr\}\, d\lambda(z)\, d\lambda(u) = \det \begin{pmatrix} S & 1 \\ 1 & S^* \end{pmatrix}^{-1/2} =$$
$$= \det(1 - S^* S)^{-1/2} \qquad (2.6)$$

Comparing two expressions(2.5) and (2.6), we obtain the first formula for norm.

Calculating $\operatorname{tr} \left( B[S]^* B \begin{bmatrix} 0 & -E \\ -E & 0 \end{bmatrix} B[S] \right)$ in the two ways, we get (2.4).

b) The right hand side in (2.4) has no indeterminacy as a function in $S$. After that, it remains to perform some exercises in functional analysis. We omit details. □

**2.3. One more formula.**

**Theorem 2.3** *Let* $S = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$*, let* $L$ *be invertible, and* $\|S\| < 1$*. Then*

$$\left\| B \begin{bmatrix} K & L \\ L^t & M \end{bmatrix} \right\| = |\det(L)|^{-1/2} \prod_{j=1}^{n} |s_j|^{1/2}. \qquad (2.7)$$

Proof. By virtue of Theorem 1.2, the matrix $S$ can be represented as a $*$-product[2]

$$S = \begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix} * \begin{pmatrix} 0 & \Sigma \\ \Sigma & 0 \end{pmatrix} * \begin{pmatrix} G & H \\ H^t & F \end{pmatrix},$$

[2]See (5.1.10).

where $\Sigma$ is a diagonal matrix with eigenvalues $s_1, \dots, s_n$; $\begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix}$, $\begin{pmatrix} G & H \\ H^t & F \end{pmatrix}$ are unitary matrices (see remarks to Theorem 5.1.5). By Theorem 5.1.5.f), the operators

$$|\det(Q)|^{1/2} B \cdot \begin{bmatrix} P & Q \\ Q^t & R \end{bmatrix}, \qquad |\det(H)|^{1/2} \cdot B \begin{bmatrix} G & H \\ H^t & F \end{bmatrix}$$

are unitary. Taking into account the fact that $\left\| B \begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix} \right\| = 1$, we find that the norm of the operator

$$X = |\det(Q)|^{1/2} |\det(H)|^{1/2} \cdot B \begin{bmatrix} P & Q \\ Q^t & R \end{bmatrix} \cdot B \begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix} \cdot B \begin{bmatrix} G & H \\ H^t & F \end{bmatrix}$$

is equal to 1. Using formula (5.1.7) to calculate the right-hand side, we obtain

$$X = |\det(Q)|^{1/2} |\det(H)|^{1/2} \det(1 - \Sigma R \Sigma G)^{-1/2} \cdot B \begin{bmatrix} K & L \\ L^t & M \end{bmatrix}.$$

However,

$$L = Q\Sigma(1 - \Sigma R \Sigma G)^{-1} H,$$

Since $\|X\| = 1$, comparing the two last expressions, we obtain

$$\left\| B \begin{bmatrix} K & L \\ L^t & M \end{bmatrix} \right\| = |\det(Q)|^{-1/2} \cdot |\det(H)|^{-1/2}| \cdot |\det(1 - \Sigma R \Sigma G)|^{1/2} = \\ |\det(L)|^{-1/2} |\det(\Sigma)|^{1/2} \qquad (2.8)$$

and this implies the desired statement. □

## 6.3 Spectra and eigenvectors

### 3.1. Spectra of compact Gaussian operators.

**Theorem 3.1** *a) Let $\|S\| < 1$. Then the eigenvalues of $B[S]$ have the form*

$$C \cdot \prod_{j=1}^{n} \lambda_j^{m_j} \qquad (3.1)$$

*where $C$ is a constant and $\lambda_1, \dots, \lambda_n$ are fixed complex numbers, $|\lambda_k| < 1$; the exponents $m_j$ range in the set $0, 1, 2, \dots$.*

*b) The numbers $\lambda_j$ are the solutions of the equation*

$$\det\Big(L^t + \lambda M(1 - \lambda L)^{-1} K - \lambda\Big) = 0 \qquad (3.2)$$

*that are contained inside the disk $|\lambda| < 1$.*

*c) There is a unique eigenvector of* $B[S]$ *having a form* $b[T] = \exp\{\frac{1}{2}zTz^t\}$ *with* $\|T\| < 1$. *Moreover, the corresponding eigenvalue coincides with* $C$.

*d) All the eigenvectors of* $B[S]$ *have the form* $p(z)\exp\{\frac{1}{2}zTz^t\}$, *where* $p(z)$ *are polynomials. In the case of coinciding eigenvalues, root subspaces consist of vectors having such form.*

**3.2. Proof.** The existence of a Gaussian eigenvector was proved in Theorem 3.1. The uniqueness can be easily deduced from Compression Theorem 2.9.7.

**Lemma 3.2** *Let* $B[S]$ *be a Gaussian operator, let* $B[S]b[T] = b[R]$. *Then, for each polynomial* $p(z)$, *the* $B[S]$ *takes* $p(z)\exp\{\frac{1}{2}zTz^t\}$ *to a function of form* $q(z)\exp\{\frac{1}{2}zRz^t\}$, *where* $q(z)$ *is a polynomial; moreover the degree of* $q$ *does not exceed the degree of* $p$.

In fact, Lemma was proved in proof of Proposition 5.2.5; we assume $\tau = 0$ in (5.2.5).

Further, without loss of generality, we can assume, that the Gaussian eigenvector is $b[0]$. Otherwise, consider a unitary Gaussian operator $c\cdot B[Q]$ such that $B[Q]b[0] = \sigma b[T]$. Next, let us pass to the new operator $B[S'] = B[Q]^{-1}B[S]B[Q]$, it satisfy $B[S']b[0] = b[0]$. The invariance of roots $|\lambda_j| < 1$ of the equation (3.2) was claimed in Lemma 1.6.

Thus, we assume $B[S]\cdot 1 = 1$. Therefore $S = \begin{pmatrix} 0 & L \\ L^t & M \end{pmatrix}$. This reduces equation (3.2) to the form $\det(L^t - \lambda) = 0$. That is, $\lambda_j$ are the eigenvalues of $L$.

For a better visualization, let us pass to the Wick form,

$$B\begin{pmatrix} 0 & L \\ L^t & M \end{pmatrix} = \exp\Big\{\sum l_{ij}z_i\frac{\partial}{\partial z_j}\Big\}\cdot\exp\Big\{\frac{1}{2}\sum_{ij} m_{ij}\frac{\partial^2}{\partial z_i\partial z_j}\Big\} \tag{3.3}$$

Denote by $\mathrm{Pol}^{\leqslant k}$ the space of polynomials of the degree $\leqslant k$, by $\mathrm{Pol}^{=k}$ the space of homogeneous polynomials of degree $k$. There is an obvious isomorphism

$$\mathrm{Pol}^{\leqslant k}/\mathrm{Pol}^{\leqslant k-1} \simeq \mathrm{Pol}^{=k} \tag{3.4}$$

Further, the operator (3.3) is block-triangle with respect to the filtration

$$\dots \subset \mathrm{Pol}^{\leqslant k} \subset \mathrm{Pol}^{\leqslant k+1} \subset \dots$$

i.e., $B[S]\,\mathrm{Pol}^{\leqslant k} \subset \mathrm{Pol}^{\leqslant k}$.

Let us examine the diagonal blocks which are transformations $\mathrm{Pol}^{=k} \to \mathrm{Pol}^{=k}$. For $f \in \mathrm{Pol}^{\leqslant k}$,

$$\exp\Big\{\frac{1}{2}\sum_{ij} m_{ij}\frac{\partial^2}{\partial z_i\partial z_j}\Big\} f - f \in \mathrm{Pol}^{\leqslant k-2}$$

For arbitrary $g$,

$$\exp\Big\{\sum l_{ij}z_i\frac{\partial}{\partial z_j}\Big\} g(z) = g(zL)$$

Thus, the diagonal blocks of $B[S]$ are transformations $g(z) \mapsto g(zL)$.

Hence the eigenvalues of diagonal blocks of (3.3) are $\prod_{\sum k_j = n} \lambda_j^{k_j}$.

Now all the statements concerning eigenvalues, eigenvectors, and root vectors become obvious. $\square$

REMARK. In fact, the eigenvectors can be written explicitly in the terms of matching functions which are defined below, Section 6.5.

## 6.4 Quadratic operators and exponents

**4.1. A normalization of Gaussian operators.** Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R})$ be a symplectic contraction sufficiently close to unit. Define the operator $\mathrm{Be}^\nabla(g)$ in the Fock space by

$$\mathrm{Be}^\nabla(g) := \det a^{-1/2} \cdot B \begin{bmatrix} -a^{-1}b & a^{-1} \\ a^{t-1} & ca^{-1} \end{bmatrix}$$

Since $g$ is near 1, the square root in this formula is well-defined.

**Lemma 4.1** *For two symplectic contractive matrices $g_1$, $g_2$ sufficiently closed to unit*

$$\mathrm{Be}^\nabla(g_1)\,\mathrm{Be}^\nabla(g_2) = \mathrm{Be}^\nabla(g_1 g_2)$$

PROOF. This is more-or-less calculation (5.2.11). $\square$

**4.2. Quadratic operators.** Let $X := \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^t \end{pmatrix}$ be a dissipative element of the Lie algebra $\mathfrak{sp}(2n, \mathbb{C})$. By definition

$$\exp\left\{\varepsilon \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^t \end{pmatrix}\right\} =: \begin{pmatrix} a(\varepsilon) & b(\varepsilon) \\ c(\varepsilon) & d(\varepsilon) \end{pmatrix}, \qquad \varepsilon > 0$$

is a symplectic contraction. The corresponding operator in the Fock space is

$$\mathrm{Be}^\nabla \begin{pmatrix} a(\varepsilon) & b(\varepsilon) \\ c(\varepsilon) & d(\varepsilon) \end{pmatrix} = \det a(\varepsilon)^{-1/2} \cdot B \begin{bmatrix} -a(\varepsilon)^{-1}b(\varepsilon) & a(\varepsilon)^{-1} \\ a(\varepsilon)^{t-1} & c(\varepsilon)a(\varepsilon)^{-1} \end{bmatrix}$$

Denote by $K_\varepsilon(z, u)$ the kernel of this operator. Expanding formally $K_\varepsilon$ in $\varepsilon$, we get

$$K_\varepsilon(z,u) = -\frac{\varepsilon}{2}\operatorname{tr}\alpha + \exp\{z\overline{u}\}\left\{1 + \frac{\varepsilon}{2}\begin{pmatrix} z & \overline{u} \end{pmatrix} \begin{pmatrix} -\beta & -\alpha \\ -\alpha^t & \gamma \end{pmatrix} \begin{pmatrix} z \\ \overline{u} \end{pmatrix}\right\} + O(\varepsilon^2)$$

Applying the expression for Wick symbols, see Proposition 4.1.11, we come to

$$\mathrm{Be}^\nabla \begin{pmatrix} a(\varepsilon) & b(\varepsilon) \\ c(\varepsilon) & d(\varepsilon) \end{pmatrix} = 1 + \varepsilon \mathfrak{W} \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^t \end{pmatrix} + O(\varepsilon^2)$$

where

$$\mathfrak{W}\begin{pmatrix}\alpha & \beta\\ \gamma & -\alpha^t\end{pmatrix} := -\frac{1}{2}\operatorname{tr}\alpha - \frac{1}{2}\sum_{k,l}\beta_{kl}z_k z_l - \sum_{k,l}\alpha_{kl}z_k\frac{\partial}{\partial z_l} + \frac{1}{2}\sum_{k,l}\gamma_{kl}\frac{\partial^2}{\partial z_k\partial z_l} \quad (4.1)$$

Such operators are called *quadratic operators*.

**4.3. Commutators of quadratic operators.**

**Theorem 4.2** *For any $R$, $Q \in \mathfrak{sp}(2n,\mathbb{C})$ and for any $f \in \mathcal{S}\mathbf{F}_n$,*

$$[\mathfrak{W}(R), \mathfrak{W}(Q)]f = \mathfrak{W}([R,Q])f$$

*i.e., $\mathfrak{W}$ is a representation of the Lie algebra $\mathfrak{sp}(2n,\mathbb{C})$.*

This can be easily verified by a direct calculation

**4.4. Correspondence Lie semigroup – Lie algebra.** The main statement of this section is

**Theorem 4.3** *For a pseudo-dissipative $R$ and $t > 0$,*

$$\exp\{t\,\mathfrak{W}(R)\} = \mathrm{Be}^{\nabla}\big(\exp\{tR\}\big) \quad (4.2)$$

Remark. At a first glance, the theorem is trivial. For a matrix $A$ of finite size,

$$\frac{d}{d\varepsilon}\exp\varepsilon A\Big|_{\varepsilon=0} = A$$

Formally, our our equality is the same. However, the theorem even is yet not formulated, since the operators $\mathfrak{W}(R)$ are unbounded; this gives rise to the usual difficulties. □

**4.5. Self-adjointness.** We suppose that the reader is familiar with unbounded self-adjoint and dissipative operators (for definitions and criterions, see, for instance [145]–[146]).

**Theorem 4.4** *Let an operator (4.1) is formally self-adjoint, i.e., $\alpha = \alpha^*$, $\gamma = \beta^*$. Then it is essentially self-adjoint on the space $\mathrm{Pol}_n \subset \mathbf{F}_n$ of polynomials.*

We omit a proof, it is contained in many texts in a wider generality, see, for instance, [11], [118].

**4.6. Formal dissipativity.** Let $S$ be an operator in a Hilbert space defined on a dense subspace $V$. We say that $S$ is *formally dissipative* if $\mathrm{Re}\langle Sv, v\rangle \leqslant 0$ for each $v \in V$.

*Let us define quadratic operators on the space $\mathcal{S}\mathbf{F}_n$.*

**Lemma 4.5** *For $g \in \mathrm{Sp}(2n,\mathbb{R})$,*

$$\mathrm{Be}^{\nabla}(g)\mathfrak{W}(R)\,\mathrm{Be}^{\nabla}(g)^{-1} = \mathfrak{W}(gRg^{-1})$$

PROOF. In other words,

$$\frac{d}{d\varepsilon}\operatorname{Be}^{\nabla}\left(g\exp\{\varepsilon R\}g^{-1}\right)\Big|_{\varepsilon=0}=\operatorname{Be}^{\nabla}(g)\cdot\frac{d}{d\varepsilon}\operatorname{Be}^{\nabla}(\exp\{\varepsilon R\})\Big|_{\varepsilon=0}\cdot\operatorname{Be}^{\nabla}(g)^{-1}$$

This is obvious. □

**Theorem 4.6** *A quadratic operator $\mathfrak{W}(R)$ is formally dissipative iff the matrix $R\in\mathfrak{sp}(2n,\mathbb{C})$ is pseudo-dissipative.*

PROOF. First of all, expand $R=H+iT$, where $H$, $T\in\mathfrak{sp}(2n,\mathbb{R})$. Since $\mathfrak{W}(H)^*=-\mathfrak{W}(H)$, the dissipativity of $\mathfrak{W}(R)$ is equivalent to the dissipativity of the self-adjoint operator $\mathfrak{W}(iT)$.

Denote by $C$ the cone of all the matrices $R\in\mathfrak{sp}(2n,\mathbb{R})$ such that $iR$ is pseudo-dissipative.

Denote by $C'$ the cone of all the matrices $R\in\mathfrak{sp}(2n,\mathbb{R})$ such that the quadratic operator $\mathfrak{W}(iR)$ is dissipative.

**Lemma 4.7** $C\subset C'$.

PROOF OF THE LEMMA. We must prove the implication

$$\{iR \text{ is pseudo-dissipative}\}\Longrightarrow\{\mathfrak{W}(iR)\text{ is dissipative in }\mathbf{F}_n\}$$

Taking in account Lemma 4.5, we observe that it suffices to prove it for $R$ being in the canonical form, see Theorem 3.5.2. Corresponding quadratic operators in $\mathbf{F}_n$ are

$$-\sum_{j\leqslant p}\lambda_j z_j\frac{\partial}{\partial z_j}-\sum_{j>p}\Big(z_j+\frac{\partial}{\partial z_j}\Big)^2$$

with $\lambda_j\geqslant 0$. Such operators are dissipative. □

However, $C$ is a maximal convex cone in $\mathfrak{sp}(2n,\mathbb{R})$ (see Theorem 3.5.3), Therefore $C=C'$ and this completes the proof of the Theorem. □

**4.7. Dissipativity and exponents.** Let $B(t)$, $t\geqslant 0$ be an one-parameter semigroup of contractive operators in a Hilbert space, i.e., $B(t_1)B(t_2)=B(t_1+t_2)$. Its generator $S$ is the operator $S$ defined by

$$Sv:=\lim\frac{1}{\varepsilon}(B(\varepsilon)-1)v$$

The domain of definiteness $\operatorname{dom}(S)$ consist of all the vectors $v$, for which the limit exists. We also write $B(t)=\exp(tS)$

The operator $S$ is called the *generator of the one-parameter group $B(t)$*; by definition, *dissipative operators* are operators that can be obtained in this way.

There is a following condition of dissipativity, see [56], 9.20:

*—If a formally dissipative operator has a dense set of analytic vectors[3], then its closure is a dissipative operator.*

[3] A vector $v$ is an *analytic vector* of an (unbounded) operator $A$, if the series $\sum t^n\|A^n v\|/n!$ is convergent for some $t>0$.

**Theorem 4.8** *The closure of a formally dissipative quadratic operator is a dissipative operator, and moreover (4.2) holds.*

PROOF. It suffices to prove, that the vectors $\varphi_w(z) = \exp\{z\overline{w}^t\} \in \mathbf{F}_n$ are analytic. Indeed, let us evaluate explicitly

$$\mathrm{Be}^{\nabla}(g)\varphi_w(z) := \det a^{-1/2} \cdot \exp\Bigl\{\frac{1}{2}\begin{pmatrix} z & w\end{pmatrix}\begin{bmatrix} -a^{-1}b & a^{-1}\varphi_a(z) \\ a^{t-1} & ca^{-1}\end{bmatrix}\begin{pmatrix} z^t \\ w^t\end{pmatrix}\Bigr\} =$$
$$= \det a^{-1/2} \exp\Bigl\{\frac{1}{2} wca^{-1}w^t\Bigr\} \cdot b\bigl[-a^{-1}b|wa^{-1}\bigr]$$

where $b[\cdot]$ is a Gaussian vector. If $g$ is in a small neighbourhood of 1, then our vector is in $\mathbf{F}_n$. Thus the map $\Gamma_w : g \mapsto \mathrm{Be}^{\nabla}(g)\varphi_w$ is a well-defined map

$$\bigl\{\text{Neighbourhood of 1 in } \mathrm{Sp}(2n, \mathbb{C})\bigr\} \to \mathbf{F}_n$$

It can be easily shown, that this mapping is weakly analytic, i.e., for each $f \in \mathbf{F}_n$, the function $\langle \Gamma_w(g), f\rangle$ is analytic (we must validate a differentiation of the integral with respect to parameter). However week analyticity implies strong analyticity (see [145]). By definition, every strong holomorphic function admits an expansion into a convergent Taylor series; this implies the required estimate for its coefficients. □

## 6.5 Matching functions and matrix elements of Gaussian operators

**5.1. Configurations.** Let $H$ be a finite set, $h_1, \dots, h_k$ be its elements. A *configuration* over $H$ is a pair $(\omega, \pi)$, where $\omega$ is a finite set and $\pi$ is a map $\omega \to H$. By $\alpha_j$ we denote the number of points in the set $\pi^{-1}(h_j)$.

Two configurations $(\omega, \pi)$, $(\omega', \pi')$ over $H$ are *equivalent* if there is a bijection $\varkappa : \omega \to \omega'$ such that $\pi = \pi' \circ \pi$.

REMARK. See Fig. 6.1. The set $H$ is the horizontal chain of black circles. The set $\omega$ consists of white circles, it is organized as collections of columns over elements of $H$. □

REMARK. In fact, this definition is nothing but a formalization of the following notion: *a configuration is a set with multiplicities; we think that each point $h_j$ has $\alpha_j$ copies.* Two configurations are equivalent iff their multiplicities coincide. □

We denote the set of all the configurations on $A$ by $\Omega(A)$.

**5.2. Matchings.** Let $\omega$ be a configuration. A *matching* of $\omega$ is a partition of $\omega$ into two-point and one-point sets.

If we draw $\omega$ as a collection of points, we can regard a matching as a graph such that each vertex is an end of one or zero edges.

By $\Xi[\omega]$ we denote the set of all the matchings of the set $\omega$.

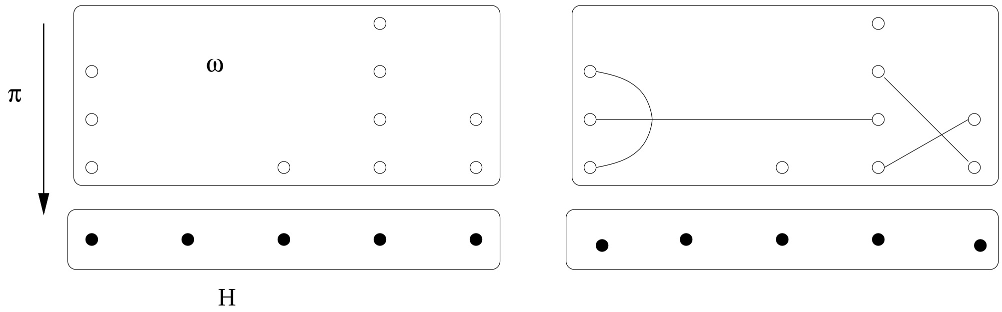


Fig. 6.1: A configuration. A matching on configuration.

| | | | | | | $s_j$ | |
|---|---|---|---|---|---|---|---|
| | 1 | 0 | 0 | 1 | 0 | 0 | Σ= 3 |
| | 0 | 0 | 0 | 0 | 0 | 0 | Σ= 0 |
| | 0 | 0 | 0 | 0 | 0 | 1 | Σ= 1 |
| $t_{ij}$ | 1 | 0 | 0 | 0 | 2 | 1 | Σ= 4 |
| | 0 | 0 | 0 | 2 | 0 | 0 | Σ= 2 |
| $s_j$ | 0 | 0 | 1 | 1 | 0 | | |
| | Σ=3 | Σ= 0 | Σ=1 | Σ=4 | Σ=2 | | |

Fig. 6.2: The numbers $s_{ij}$, $t_j$ corresponding to Figure 6.1.

For a matching $\xi$, we denote by two($\xi$) the set of all two-point subsets of the matching, by one($\xi$) the set of all one-point subsets, see Fig. 6.1.

For a matching $\xi$, we denote by $s_{ij}$ the number of arcs between fibers $\pi^{-1}(h_i)$ and $\pi^{-1}(h_j)$. Denote by $t_j$ the number of one-point subsets in a fiber $\pi^{-1}(h_j)$. Obviously, these numbers satisfy the identities $s_{ij} = s_{ji}$ and

$$\sum_j s_{ij} + t_i = \alpha_i \tag{5.1}$$

see Fig. 6.2

**Lemma 5.1** *Let $\omega$ be a configuration with multiplicities $\alpha_j$. The number of its matchings with given $s_{ij}$, $t_j$ is*

$$\frac{\prod_i \alpha_i!}{\prod_i 2^{s_{ii}} \prod_{i\leqslant j} s_{ij}! \cdot \prod_i t_i!} \tag{5.2}$$

Proof. Denote by $S_n$ the group of permutations of a set with $n$ elements. Denote by $Q = Q(s_{ij}, t_j)$ the set of matching with given $s_{ij}$, $t_j$.

Consider the group $G$ of all the permutations of the set $\omega$ preserving the fibers of the map $\pi$ (see Figure 6.1, we consider permutations inside vertical columns). By definition,

$$G = S_{\alpha_1} \times S_{\alpha_2} \times \dots$$

Obviously, $G$ acts transitively on $Q$. It remains to describe the stabilizer $Z$ of a point of $Q$.

An element of $Z$ can

— rearrange one-point sets inside a fiber;

— permute arcs connecting $i$-th fiber and $j$-th fiber (in particular, for $i = j$);

— change ends of an arc connecting elements of one fiber.

Thus the stabilizer is

$$Z \simeq \prod_j S_{t_j} \times \prod_{i<j} S_{s_{ij}} \times \prod_i (\mathbb{Z}/2\mathbb{Z})^{s_{jj}}$$

This gives the denominator in (5.2). □

**5.3. Matching functions.** Let $(\omega, \pi)$ be a configuration over a set $H$ with multiplicities $\alpha_j$. Let $A(h_i, h_j) = a_{ij}$ be a symmetric function on $H \times H$, $a_{ij} = a_{ji}$; let $b_j = b(h_j)$ be a function on $H$. We define the *matching function* $\mathcal{M}(A, b|\alpha)$ by

$$\mathcal{M}(A, b|\alpha) := \sum_{\xi \in \Xi(\omega)} \prod_{\{u,v\} \in \mathrm{two}(\xi)|} A(\pi(u), \pi(v)) \prod_{\{w\} \in \mathrm{one}(\xi)} b(\pi(w)) \tag{5.3}$$

In other words, for a given matching, we write out a product in the following way. Having an arc connecting $i$-th column and $j$-th column, we write the factor $a_{ij}$. Having an isolated point in $j$-th column, we write the factor $b_j$. Then we write the sum over all the matchings of a given configuration.

**Proposition 5.2**

$$\mathcal{M}(A, b|\alpha) := \sum_{s_{ij}, t_i :\ \sum_j s_{ij} + t_i = \alpha_i} \frac{\prod_i \alpha_i!}{\prod_i 2^{s_{ii}} \cdot \prod_{i \leqslant j} s_{ij}! \cdot \prod_i t_i!} \cdot \prod_{i \leqslant j} a_{ij}^{s_{ij}} \prod_j b_i^{t_i}$$

PROOF. This is an immediate corollary of the definition and the previous lemma. □

**5.4. Generating functions.**

**Theorem 5.3**

$$\exp\Bigl\{\frac{1}{2} \sum_{i,j} a_{ij} z_i z_j + \sum_j b_j z_j\Bigr\} = \sum_{\alpha_1 \geqslant 0, \alpha_2 \geqslant 0, \dots} \mathcal{M}(A, b|\alpha_1, \alpha_2, \dots) \frac{z_1^{\alpha_1}}{\alpha_2!} \frac{z_2^{\alpha_2}}{\alpha_2!} \dots \tag{5.4}$$

PROOF. This is obvious. Indeed,

$$\exp\Big\{\dots\Big\} = \prod_{i\neq j} \exp\{a_{ij}z_iz_j\} \prod_j \exp\{\frac{1}{2}a_{jj}z_j^2\} \prod_k \exp\{c_jz_j\} =$$

$$= \prod_{i\neq j}\Big\{\sum_k \frac{1}{k!}a_{ij}^k z_i^k z_j^k\Big\} \cdot \prod_j \Big\{\sum_l \frac{1}{2^l l!}a_{jj}^l z_j^{2l}\Big\} \cdot \prod_k \Big\{\sum_m \frac{1}{m!}c_j^m z_j^m\Big\}$$

Opening brackets we obtain a desired expression. □

**5.5. Partial differential equations for matching functions.**

**Corollary 5.4** *The functions $F(A,b) = \mathcal{M}(A,b|\alpha)$ satisfy the equations*

$$\Big(\frac{\partial}{\partial a_{ij}} - \frac{\partial^2}{\partial b_i \partial b_j}\Big)F(A,b) = 0, \qquad i \neq j \tag{5.5}$$

$$\Big(\frac{\partial}{\partial a_{jj}} - \frac{1}{2}\frac{\partial^2}{\partial b_j^2}\Big)F(A,b) = 0 \tag{5.6}$$

*Also, consider the matrix (recall that $a_{ij} = a_{ji}$)*

$$\begin{pmatrix} 2\partial/\partial a_{11} & \partial/\partial a_{12} & \partial/\partial a_{13} & \dots \\ \partial/\partial a_{21} & 2\partial/\partial a_{22} & \partial/\partial a_{23} & \dots \\ \partial/\partial a_{31} & \partial/\partial a_{32} & 2\partial/\partial a_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{5.7}$$

*For each $2 \times 2$ minor $\Delta$ of this matrix, our functions $F(A,b) = \mathcal{M}(A,b|\alpha)$ satisfy*

$$\Delta F(A,b) = 0 \tag{5.8}$$

PROOF. Indeed, the left-hand side of (5.4) satisfies the equations (5.5)–(5.6), hence the Taylor coefficients of the right-hand side satisfy the same equations.

The equations (5.8) follow from (5.5)–(5.6). □

**Problem 5.1** a) The functions $F_\alpha(A,b) := \mathcal{M}(A,b|\alpha)$ form a basis in the space of all the polynomial solutions of the system (5.5)–(5.6).

b) A polynomial solution $F(A,b)$ of the system (5.5)–(5.6) is determined by $F(0,b)$.

**5.6. Matrix elements of Gaussian operators.** By Proposition 4.1.6, the matrix elements of an operator in a Fock space are the Taylor coefficient of its kernel.

Thus, by Theorem 5.3, matrix elements of a Gaussian operator are matching functions.

**5.7. Digression. Fock spaces as $l_2$.** Consider the countable space $\Omega_n := \mathbb{Z}_+^n$, we denote its elements by $\alpha = (\alpha_1, \dots, \alpha_n)$. We introduce a measure on the set $\Omega_n$ from the assumption: the measure of a point $\alpha$ is $\alpha! := \prod \alpha_j!$.

We also can regard $\Omega_n$ as the space of all configurations over the set $\{1, 2, \dots, n\}$.

Consider the Hilbert space $\ell_2(\Omega_n)$, the inner product is given by the formula

$$\langle \mathfrak{f}, \mathfrak{g}\rangle = \sum_{\alpha\in\Omega_n} \mathfrak{f}(\alpha)\overline{\mathfrak{g}(\alpha)}\alpha!$$

Define a canonical unitary operator $U : \mathbf{F}_n \to \ell_2(\Omega_n)$. To each function

$$f(z) = \sum_\alpha \mathfrak{f}(\alpha) z^\alpha \in \mathbf{F}_n$$

we assign the sequence $\mathfrak{f}(\alpha)$ of its Taylor coefficients.

By Theorem 5.3, a function

$$\exp\Bigl\{\frac{1}{2}\sum a_{ij} z_i z_j + \sum_j b_j z_j\Bigr\} \in \mathbf{F}_n$$

correspond to the sequence $\mathfrak{b}(\alpha) = \mathcal{M}(A, b|\alpha)/\alpha!$. Taking in account the identity (5.1.3), we get

$$\sum_\alpha \frac{1}{\alpha!}\mathcal{M}(A, b|\alpha)\overline{\mathcal{M}(P, q|\alpha)} = \det(1 - A\overline{P})^{-1/2} \exp\Bigl\{\begin{pmatrix} b & \overline{q}\end{pmatrix} \begin{pmatrix} 1 & -A \\ \overline{P} & 1\end{pmatrix} \begin{pmatrix} b \\ \overline{q}\end{pmatrix}\Bigr\}$$

Certainly, this this looks as a trivial remark. However, the last identity immediately lose its self-evidence if we forgot the method of its derivation.

**5.8. Exercises on determinantal system of differential equations.**

**Problem 5.2** a)Let $\chi$ be tempered distribution on $\mathbb{R}^n$. Show that the function $\ll \chi, \exp\{-\frac{1}{2}\sum a_{kl}x_k x_l\} \gg$ satisfies the system (5.7)–5.8).

b) Evaluate a Gaussian integral over octant,

$$\int_{x_1>0,\dots,x_n>0} \exp\{-\frac{1}{2}\sum a_{ij}x_i x_j\}\,dx = \\ = \int_{x_1>0,\dots,x_n>0} \exp\{-\frac{1}{2}\sum_j a_{jj}x_j^2\}\exp\{-\sum_{i>j} a_{ij}x_i x_j\}\,dx$$

expanding the second factor into the Taylor series. Represent the integral as a series in matching polynomials. Under which conditions on $a_{ij}$ this way is correct?

c) Consider a simplicial cone in $\mathbb{R}^n$. Show that a calculation of its solid angle is equivalent to a calculation of a Gaussian integral over octant.

**Problem 5.3** Let $x_i$ be an infinite collection of formal variables. Denote by $h_k(x)$ the complete symmetric function,

$$h_k(x) = \sum \text{of all monomials of degree } k$$

Expand $h_{\lambda_1}\dots h_{\lambda_N}$ in a sum of monomials. Compare the result with coefficients of the Taylor series of $\exp\{\sum_{kl} z_k u_l\}$.

## 6.6 Bibliographical remarks to Chapter 6

Calculation of norms is based on G.I.Olshanski [135]. All the remaining material of this chapter is semi-well-known and semi-obvious.