Chapter 7

Spaces of holomorphic functions in matrix balls

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Here we discuss a certain natural one-parametric family of representations which includes the Weil representation of $\mathrm{Sp}(2n,\mathbb{R})$.

7.1 Reproducing kernels

Text-books in the functional analysis propose orthogonal bases as a tool for explicit calculations. As Berezin claimed, one can use an arbitrary total system of vectors ξ_x instead of a basis, we need only explicit formula for inner products of ξ_x. In numerous contexts (mainly, for spaces of functions depending on a large or infinite number of variables), such *systems of coherent states* are more convenient and flexible tool than explicit orthogonal bases.

For the following discussion, the Fock space $\mathbf{F}_n$ is a good basic example. Numerous natural examples arise in the next three sections.

1.1. Gram matrix.

Theorem 1.1 *Let R be an $n\times n$ Hermitian matrix. The following conditions are equivalent*

i) R is non-negative definite.

ii) There is a Euclidean space and vectors ξ_1, ..., ξ_n such that $\langle \xi_i,\xi_j\rangle = r_{ij}$.

iii) For each collection $1\leqslant i_1<\dots<i_k\leqslant n$

$$\det_{\alpha,\beta}\{r_{i_\alpha i_\beta}\}\geqslant 0$$

PROOF. ii$\Longrightarrow$i. We must verify the inequality $\sum r_{ij}a_i\overline{a}_j \geqslant 0$. Indeed,

$$\sum r_{ij}a_i\overline{a}_j = \sum a_i\overline{a}_j\langle\xi_i, \xi_j\rangle = \langle\sum a_i\xi_i, \sum a_j\xi_j\rangle \geqslant 0$$

i$\Longrightarrow$ii. Consider a linear space V and a basis e_j. Put $\langle e_i, e_j\rangle := r_{ij}$. We get a positive semi-definite Hermitian form on V. Let H be its kernel. Then the space V/H is Euclidean; we define $\xi_j \in V/H$ being the images of e_j.

i$\Longleftrightarrow$ iii. This is the Sylvester criterion. □

Problem 1.1 (Menger Theorem) Let R be a symmetric $n \times n$ matrix with non-negative elements. The following conditions are equivalent

— there are points $\eta_1, \dots, \eta_n$ in some real Euclidean space such that

$$\|\eta_i - \eta_j\|^2 = r_{ij}$$

— the matrix R is *conditionally positive definite*, i.e..,

$$\sum r_{ij}a_ia_j \geqslant 0 \qquad \text{for all real vectors } a = (a_1, a_2, \dots) \text{ satisfying } \sum a_j = 0$$

1.2. Positive definite kernels. Let $\mathcal{X}$ be a set. Consider a function L on $\mathfrak{X} \times \mathfrak{X}$ such that

$$L(x, y) = \overline{L(y, x)}$$

It is called a *positive-definite kernel* if for each finite collection $x_1, \dots, x_n \in \mathcal{X}$ the matrix $\{L(x_i, x_j)\}$ is positive semi-definite.

EXAMPLE. Let H be a Hilbert space. Let ξ_x be a system of vectors in H indexed by $x \in \mathfrak{X}$. Then $L(x, y) := \langle\xi_x, \xi_y\rangle$ is a positive definite kernel on $\mathfrak{X}$. □

1.3. The construction of Hilbert spaces from positive definite kernels.

Theorem 1.2 *Let $\mathfrak{X}$ be a separable*[1] *metric space. Assume that a positive definite kernel L is continuous as a function on $\mathfrak{X} \times \mathfrak{X}$.*

a) There is a separable Hilbert space H and a total system of vectors ξ_x indexed by elements $x \in \mathfrak{X}$ such that $\langle\xi_x, \xi_y\rangle = L(x, y)$.

b) A space H is unique in the following sense: if H' is an another Hilbert space and $\xi'_a \in \mathsf{H}$ is an another system of vectors having the same properties, then there is a unitary operator $U : \mathsf{H} \to \mathsf{H}'$ such that $U\xi_x = \xi'_x$.

PROOF. a) Consider a countable dense set $y_1, y_2, \dots$ in $\mathfrak{X}$. For each finite set $\{y_1, \dots, y_k\}$, consider the corresponding Euclidean space W_k and the total system of vectors vectors $\eta_1, \dots, \eta_k \in W_k$ such that $\langle\eta_i, \eta_j\rangle = L(y_i, y_j)$ (see Theorem 1.1).

We have canonical isometric embeddings

$$\dots \to W_{k-1} \to W_k \to W_{k+1} \to \dots$$

[1] A metric space is *separable* if it contains a countable dense subset

which take $\eta_k \in W_k$ to $\eta_k \in W_{k+1}$. Consider the pre-Hilbert space $\cup_k W_k$ and the associated Hilbert space H. We claim that H is the desired object.

For each $x \in \mathfrak{X}$, choose a subsequence y_{α_1}, y_{α_2}, ... convergent to x. To simplify notation denote it by w_1, w_2, etc. Consider the corresponding sequence $\eta_{w_m} \in \mathsf{H}$. Then

$$\|\eta_{w_m} - \eta_{w_k}\|^2 = L(w_m, w_m) + L(w_k, w_k) - L(w_m, w_k) - L(w_k, w_m)$$

Since L is continuous, this expression tends to 0 as $m, k \to \infty$. Hence there exists a limit $\xi_x := \lim_{m\to\infty} \xi_{w_m}$. By continuity considerations, $\langle \xi_x, \xi_{x'} \rangle = L(x, x')$.

b) The construction of W_k is canonical. □

Problem 1.2 Extend the Menger Theorem in a similar way.

The following statement is a variant of the previous Theorem 1.2.

Theorem 1.3 *Let $\mathfrak{X}$ be an arbitrary set.*

a) For each positive definite kernel L on $\mathfrak{X}$, there is a Hilbert space H and a total system of vectors ξ_x indexed by elements $x \in \mathfrak{X}$ such that $\langle \xi_x, \xi_x \rangle = L(x, y)$.

b) A space H is unique in the same sense as above.

Proof. a) Consider the linear space W, whose elements are finite sums $\sum_i a_i \xi_{x_i}$, where $a_i \in \mathbb{C}$ and ξ_x are formal symbols enumerated by elements of $\mathfrak{X}$. We define the inner product in W by

$$\langle \sum_i a_i \xi_{x_i}, \sum_j b_j \xi_{y_j} \rangle := \sum_{i,j} L(x_i, y_j) a_i \overline{b}_j$$

In particular,

$$\langle \sum_i a_i \xi_{x_i}, \sum_i a_i \xi_{x_i} \rangle := \sum_{i,j} L(x_i, x_j) a_i \overline{a}_j \geqslant 0$$

because the kernel L is positive-definite. Thus, W is a pre-Hilbert space. Our space H is the associated Hilbert space.

b) The space of linear combinations of vectors $\xi'_x \in \mathsf{H}'$ is canonically isometric to W. □

We denote this Hilbert space by

$$\mathsf{H} = \mathsf{H}(\mathfrak{X}; L)$$

and say that ξ_x is an *overfilled system.* In literature, $\{\xi_x\}$ is also called a *system of coherent states* or a *supercomplete basis.*

1.4. Functional realization of H. Let L be a positive definite kernel. For each $v \in \mathsf{H}(\mathfrak{X}; L)$, define the function

$$f_v(x) := \langle v, \xi_x \rangle$$

on the set $\mathfrak{X}$. Thus we obtain a certain linear space $\mathsf{H}^\circ = \mathsf{H}^\circ(\mathfrak{X}; L)$ consisting of functions f_v on $\mathfrak{X}$. By construction, $\mathsf{H}^\circ(\mathfrak{X}; L)$ is in one-to-one correspondence with $\mathsf{H}(\mathfrak{X}; L)$, therefore $\mathsf{H}^\circ(\mathfrak{X}; L)$ inherits a structure of a Hilbert space.

Theorem 1.4 *a) The function corresponding to a vector* ξ_a *is*

$$\varphi_a(x) := L(a, x)$$

b) For each $f \in \mathsf{H}^\circ(\mathfrak{X}; L)$,

$$\langle f, \varphi_a \rangle_{H^\circ} = f(a) \qquad \textit{(reproducing property)} \tag{1.1}$$

c) If $f^{(j)} \in \mathsf{H}^\circ(\mathfrak{X}; L)$ *weakly converges to* f, *then* $f^{(j)}$ *converges to* f *pointwise.*

d) Linear combinations of functions φ_a *are dense in* H°.

PROOF. a) is obvious.

b) Let $f = f_v$. Then

$$\langle f, \varphi_a \rangle_{\mathsf{H}^\circ} = \langle v, \xi_a \rangle_{\mathsf{H}} = f_v(a)$$

c) Let $f^{(j)} = f_{v_j}$, $f = f_v$. Then $\langle v_j, \xi_a \rangle$ converges to $\langle v, \xi_a \rangle$. However, this is precisely the convergence $f^{(j)}(a)$ to $f(a)$.

d) By definition linear combinations of ξ_a are dense in $\mathsf{H}(\mathfrak{X}; L)$. □

Due to the reproducing property, the kernel L is also called the *reproducing kernel* of the space $\mathsf{H}^\circ(\mathfrak{X}; L)$.

REMARK. The reproducing property is an implicit description of the inner product in $\mathsf{H}^\circ(\mathfrak{X}; L)$. Also the theorem provides us with a (non-perfect) tool for watching a convergence.

EXAMPLE. Let $\mathfrak{X}$ be a circle with the standard coordinate α. Let $L(\alpha, \beta) = \cos\bigl(\alpha - \beta)/2\bigr)$. Then the space H° is two-dimensional. The vectors $\xi_\alpha \in \mathbb{R}^2$ can be chosen as $\xi_\alpha := (\cos\alpha, \sin\alpha)$. This example shows that the space $\mathsf{H}^\circ(\mathfrak{X}; L)$ can drastically differs from the space of 'all functions' on $\mathfrak{X}$; however a kernel L 'knows' not only an inner product but also a functional space. □

Proposition 1.5 *Let* $\mathfrak{X}$ *be a metric space,* L *be continuous on* $\mathfrak{X} \times \mathfrak{X}$.

a) If a sequence $f^{(j)} \in \mathsf{H}^\circ(\mathfrak{X}; L)$ *converges to* f *in the norm topology, then* $f^{(j)}$ *converges to* f *uniformly on each compact subset* $\mathfrak{X}$.

b) For each point $b \in \mathfrak{X}$ *there is a ball* $B(b, r)$ *of center* b *and radius* $r > 0$ *such that every sequence* $f^{(j)} \in \mathsf{H}^\circ$ *that converges in* H° *converges uniformly on* $B(b, r)$.

PROOF. a), b) Let y range in a set C. Let $f^{(j)} = f_{v_j}$, $f = f_v$. Then

$$|f^{(j)}(y) - f(y)| = |\langle v_j, \xi_y \rangle - \langle v, \xi_y \rangle| = |\langle (v_j - v), \xi_y \rangle| \leqslant$$
$$\leqslant \|v_j - v\| \cdot \|\xi_y\| \leqslant \|v_j - v\| \cdot \sup_{y \in C} L(y, y)^{1/2}$$

If the set C is compact, then the supremum is finite. Also, for each point $b \in \mathfrak{X}$ there is neighborhood C, where the function $L(x, x)$ is bounded. Since $\|v_j - v\| \to 0$, in both cases we get the uniform convergence on C. □

Corollary 1.6 *If a kernel L is continuous, then elements of $\mathsf{H}^\circ(\mathfrak{X};L)$ are continuous functions.*

Theorem 1.7 *Let M be a Hilbert space whose elements are functions on a set $\mathfrak{X}$. Assume for each $a \in \mathfrak{X}$ the evaluation linear functional $f \mapsto f(a)$ is continuous on M[2]. Then M is determined by some reproducing kernel.*

PROOF. Since our functional is continuous, for each a there is a function φ_a such that $\langle f, \varphi_a\rangle = f(a)$. We take $L(a,b) := \langle \varphi_a, \varphi_b\rangle$. Then $L(a,b)$ is a positive definite kernel and φ_a is an overfilled system. □

1.5. Dual functional realization. Now let Ω be a smooth manifold, let $\mathcal{E}'(\Omega)$ be the space of compactly supported distributions on Ω. Let $L(x,y)$ be a smooth reproducing kernel in Ω.

Define an Hermitian form in $\mathcal{E}'(\Omega)$ by

$$B(\chi_1,\chi_2)) :=\ll L, \chi_1 \otimes \overline{\chi}_2 \gg$$

where $\chi_1 \otimes \chi_2 =: \chi(x) \cdot \chi(y)$.

Lemma 1.8 *This inner product is positive semi-definite.*

We denote the corresponding Hilbert space by $\mathsf{H}^\star(\mathfrak{X}, L)$ and put of a proof.

Lemma 1.9 *The form $\mathrm{B}^\star$ is jointly continuous as a function on $\mathcal{E}' \times \mathcal{E}'$.*

PROOF OF LEMMA 1.9. Obvious. □

Next, consider the subspace $\mathfrak{D} \subset \mathcal{E}'$ consisting of finite linear combinations of delta-functions δ_x. Endow $\mathfrak{D}$ with the form $B^\star$. Then the map $\delta_x \mapsto \xi_x$ induces an isometric embedding $\mathfrak{D} \to \mathsf{H}(\mathfrak{X}, L)$. Hence $B^\star$ is positive semi-definite on $\mathfrak{D}$.

PROOF OF LEMMA 1.8. The subspace $\mathfrak{D}$ is dense in $\mathcal{E}'$. Since the form $B^\star$ is positive semidefinite on $\mathfrak{D}$ and the jointly continuous, it follows that it is positive semi-definite on $\mathcal{E}'$. □

Moreover, $\mathfrak{D}$ is dense $\mathcal{E}'$ with respect to the inner product $B^\star$. In other words, we obtain the following theorem.

Theorem 1.10 $\mathsf{H}^\star(\Omega; L) \simeq \mathsf{H}(\Omega; L)$.

Next, we combine the last isomorphism with the isomorphpism $\mathsf{H}(\Omega; L) \to \mathsf{H}^\circ(\Omega; L)$. Thus, we obtain the map $\mathcal{E}'(\Omega) \to \mathsf{H}^\circ(\Omega; L)$ given by

$$\chi \mapsto \Phi_\chi(x) := \int_{a\in\Omega} \varphi_a(x)\chi(a)$$

[2]This proviso is necessary de jure. However, existence of a discontinuous linear functional on a Hilbert space is one of the forms of the axiom of choice, see Howevel?, Rubin "Consequences of the axiom of choice".

Corollary 1.11 *a) For every $\chi \in \mathcal{E}'(\Omega)$, the function $\Phi_\chi \in \mathsf{H}^\circ(\Omega; L)$*

$$b) \qquad \langle \Phi_\chi, \Phi_{\chi'} \rangle = \iint_{a\in\Omega, b\in\Omega} L(a,b)\chi(a)\overline{\chi'(b)}$$

An important special case is χ being a derivative of a delta-function.

Corollary 1.12 *For sake of simplicity, assume Ω is an open subset in $\mathbb{R}^N$. Then for each $a \in \Omega$,*

$$\varphi_a^{[\alpha]} := \frac{\partial^\alpha}{\partial z^\alpha}\varphi_z\Big|_{z=a} \in \mathsf{H}^\circ(\Omega,)$$

and

$$\langle \varphi_a^{[\alpha]}, \varphi_b^{[\beta]} \rangle = \frac{\partial^{\alpha+\beta}}{\partial a^\alpha \partial b^\beta} L(a,b)$$

Remarks. 1) Usually, $\mathcal{E}'$ is a *proper* dense subspace in $\mathsf{H}^\star(\mathfrak{X})$. In many interesting cases, one can represent arbitrary elements of $\mathsf{H}^\star(\mathfrak{X})$ by distributions supported by some 'boundary' $\partial\Omega$ of Ω; however I do not know a general theorem of such type.

2) Usually, the map $\mathcal{E}' \to \mathsf{H}^\star$ has a nontrivial kernel. In other words, a vector in $\mathsf{H}^\star(\mathfrak{X})$ can be represented by distributions in numerous ways. □

1.6. Reproducing kernels and orthogonal bases.

Theorem 1.13 *Let e_j be an arbitrary orthonormal basis in $\mathsf{H}^\circ(\mathfrak{X}, L)$. Then*

$$L(x,y) = \sum_j e_j(x)\overline{e_j(y)} \tag{1.2}$$

Proof. Expanding φ_a by the orthogonal basis e_j, we get

$$\varphi_a(y) = \sum\nolimits_j \langle \varphi_a, e_j\rangle e_j(x) = \sum\nolimits_j \overline{e_j(a)} e_j(x) \qquad \square$$

Remark. This statement can be modified as follows. Let p_j, q_j be two (nonorthogonal) dual bases, i.e., $\langle p_i, q_j \rangle = \delta_{ij}$. Then

$$L(x,y) = \sum p_j(x)\overline{q_j(y)} \tag{1.3}$$

Sense of the words 'non-orthogonal basis' is not rather clear and we leave the last identity as an heuristic fact. For some applications of the identities (1.2)–(1.3), see for instance [102].

1.7. Operations with positive definite kernels.

Theorem 1.14 *Let L, M be positive definite kernels on $\mathfrak{X}$. Then $L + M$ and $L \cdot M$ are positive definite kernels.*

PROOF. Let H, Y be Hilbert spaces, $\xi_x \in H$, $\eta_y \in Y$ be overfilled systems

$$\langle \xi_x, \xi_y \rangle_H = L(x,y), \qquad \langle \eta_x, \eta_y \rangle_Y = M(x,y)$$

We consider the spaces $H \oplus Y$ and $H \otimes Y$ equipped with the systems $\xi_x \oplus \eta_x$, $\xi_x \otimes \eta_x$ respectively. Then

$$\langle \xi_x \oplus \eta_x, \xi_y \oplus \eta_y \rangle_{H \oplus Y} = L(x,y) + M(x,y),$$
$$\langle \xi_x \otimes \eta_x, \xi_y \otimes \eta_y \rangle_{H \otimes Y} = L(x,y) \cdot M(x,y)$$

and this implies our statement. □

Theorem 1.15 *Let L_j be a sequence of positive definite kernels on $\mathfrak{X}$, let L be its pointwise limit. Then L is positive definite.*

PROOF. This is a straightforward corollary of the definition. □

Proposition 1.16 *If a kernel $L(x,y)$ is positive definite, then for each function $h(x)$ the kernel $h(x)\overline{h(y)}L(x,y)$ is positive definite.*

PROOF. We consider an overfilled system $h(x)\xi_x$. □

1.8. Example. The kernel $L_0(z,u) = 1$ on $\mathbb{C}$ is positive definite. Therefore, kernels $L_j(z,u) = z^j \overline{u}^j$ are positive definite. Hence, for an arbitrary sequence $\nu_n \geqslant 0$, the kernel $\sum_{j=1}^n \nu_j z^j \overline{u}^j$ is positive definite on $\mathbb{C}$. Denote by r^2 the radius of convergence of the series $\sum \nu_j w^j$. Then the series

$$K(z, \overline{u}) := \sum_{j=0}^{\infty} \nu_j z^j \overline{u}^j$$

is a positive definite kernel on the disk $|z| < r$.

Next, we apply Corollary 1.12. Differentiating K in $\overline{u}$, we get $\nu_j z^j \in \mathsf{H}^\circ$, i.e., $z^j \in \mathsf{H}^\circ$ whenever $\nu_j \neq 0$. Moreover,

$$\langle z^j, z^k \rangle = \nu_j^{-1} \delta_{jk} \tag{1.4}$$

Thus $\{z^j\}$ is an orthogonal system.

It can be readily checked that the series $K(z,a) = \sum \nu_j z^j \overline{a}^j$ converges in a Hilbert space; therefore the system z^j is a basis.

Also we get an illustration to Therem 1.13

1.9. Digression. Kernels of operators. For a bounded operator $A : \mathsf{H}^\circ(\mathfrak{Y}, M) \to \mathsf{H}^\circ(\mathfrak{X}, L)$ define its symbol $\mathcal{K}_A(x,y)$ as above (4.1.8), namely

$$\mathcal{K}_A(x,y) := \langle A\varphi_y, \varphi_x \rangle_{\mathsf{H}^\circ(\mathfrak{X})}$$

Theorem 1.17 *a)*
$$Af(c) = \langle f(y), \mathcal{K}(y,c) \rangle_{\mathsf{H}^\circ(\mathfrak{Y})},$$
where $\mathcal{K}(y,c)$ is regarded as a function of the variable y.

b) Let A, B be bounded operators,

$$B : \mathsf{H}^\circ(\mathfrak{Z}; N) \to \mathsf{H}^\circ(\mathfrak{Y}; M), \qquad A : \mathsf{H}^\circ(\mathfrak{Y}, M) \to \mathsf{H}^\circ(\mathfrak{X}, L)$$

Then the kernel $\mathcal{K}_{AB}$ of the product AB is

$$\mathcal{K}_{AB}(u,v) = \langle \mathcal{K}_A(u,y), \overline{\mathcal{K}_B(y,v)} \rangle_{\mathsf{H}^\circ(\mathfrak{Y})}$$

here $\mathcal{K}_A$, $\mathcal{K}_B$ in the right-hand side are functions of $y \in \mathfrak{Y}$ depending on the parameters u, v.

1.10. Digression. Orthogonal bases again. Let $e_j(x)$ be an orthogonal basis in $H^\circ(\mathfrak{X}, L)$, $\langle e_j, e_j \rangle = \nu_j$. Expand the overfilled system $\varphi_a(z)$ in basis e_j,

$$\varphi_a(z) = \sum_j c_j(a) e_j(z)$$

Let A be a bounded operator $H^\circ(\mathfrak{X}, L) \to H^\circ(\mathfrak{X}, L)$. Then its kernel is

$$\mathcal{K}_A(x,y) = \langle A\varphi_y, \varphi_x \rangle = \langle A \sum c_i(y) e_i, \sum c_j(x) e_j \rangle = \sum_j \nu_j c_j(y) \overline{c_j(x)} \langle A e_i, e_j \rangle$$

This is an extension of resent formula (1.2 and also of (4.1.12).

7.2 Highest weight representations of $\mathrm{SU}(1,1)$

2.1. Example. The Bergman spaces in a disk. The *Bergman space* $\mathcal{H}_2(D)$ is the space of holomorphic functions square integrable in the unit disk D. The inner product is given by

$$\langle f, g \rangle = \frac{1}{\pi} \int_{|z|<1} f(z) \overline{g(z)}\, dz\, d\overline{z} \tag{2.1}$$

where $d\lambda(z)$ denote the Lebesgue measure on the disk.

The vectors z^k form an orthogonal basis in $\mathcal{H}_2$, indeed

$$\langle z^k, z^m \rangle = \frac{1}{\pi} \int z^k \overline{z}^m dz\, d\overline{z} = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 r^{k+m+1} e^{i(k-m)\varphi}\, dr\, d\varphi = \frac{1}{k+1} \delta_{k,m}$$

Proposition 2.1 *The function*

$$\varphi_a(z) = (1 - z\overline{a})^{-2} = \sum_{k=0}^{\infty} (k+1) z^k \overline{a}^k$$

satisfies the reproducing property $\langle f, \varphi_a \rangle = f(a)$. In other words, the Bergman space is determined by the reproducing kernel $(1 - z\overline{u})^{-2}$.

PROOF. For $f = \sum c_k z^k$,

$$\langle f, \varphi_a \rangle = \sum_k c_k a^k (k+1) \langle z^k, z^k \rangle = \sum_k c_k a^k = f(a) \qquad \square$$

2.2. Example. The Hardy space in the disk. The Hardy space[3] $\mathcal{H}_1(D)$ is the space of Taylor series $f(z) = \sum_{k\geqslant 0} c_k z^k$ such that $\sum |c_k|^2 < \infty$. The inner product is given by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\varphi})\overline{g(e^{i\varphi})}\, d\varphi \tag{2.2}$$

Functions $f(e^{i\varphi})$, $g(e^{i\varphi})$ are L^2-convergent Fourier series, therefore the integral makes sense. Also,

$$\langle \sum c_k z^k, \sum b_k z^k\rangle = \sum c_k \overline{b}_k$$

Problem 2.1 The Hardy space is determined by the reproducing kernel $(1 - z\overline{u})^{-1}$

2.3. The Bargmann–Pukansky scale. Fix $s \geqslant 0$. Consider the reproducing kernel on D given by

$$(1 - z\overline{u})^{-s} = 1 + \frac{s}{1!}z\overline{u} + \frac{s(s+1)}{2!}z^2\overline{u}^2 + \dots$$

Since all the Taylor coefficients are positive, the kernel is positive definite (see Subsection 1.8). Denote by $\mathcal{H}_s(D)$ the corresponding Hilbert space.

Theorem 2.2 *a) The space $\mathcal{H}_s(D)$ consists of holomorphic functions.*

b) For $s > 1$, the inner product in $\mathcal{H}_s(D)$ can be written as

$$\langle f, g\rangle_s = \frac{s-1}{\pi}\int_{|z|<1} f(z)\overline{g(z)}(1 - |z|^2)^{s-2}\, d\lambda(z) \tag{2.3}$$

c) The space $\mathcal{H}_s(D)$ consists of Taylor series $f(z) = \sum c_n z^n$ satisfying

$$\sum_{j\geqslant 0} \frac{j!}{s(s+1)\dots(s+j-1)}|c_j|^2 < \infty, \text{ or, equivalently} \sum_{j\geqslant 0} j^s |c_j|^2 < \infty \tag{2.4}$$

and the inner product is

$$\langle \sum c_n z^n, \sum c'_n z^n\rangle = \sum_{j\geqslant 0} \frac{j!}{s(s+1)\dots(s+j-1)} c_j \overline{c}'_j \tag{2.5}$$

Proof. a) Since the system z^n is an orthogonal basis, each element $f \in \mathcal{H}_s(D)$ is a sum of power series convergent in the sense of Hilbert space. Hence it uniformly converges; therefore the sum is a holomorphic function.

b) is a straightforward calculation (as for the Bergman space).

c) is a rephrasing of (1.4). In row (2.4), we apply a standard asymptotic expansion for the Γ-function, $\Gamma(x + a)/\Gamma(x + b) \sim x^{a-b}$. □

2.4. Remark. Boundary values. We emphasis that formula (2.3) breaks down if $s \leqslant 1$.

Recall the following well-known fact (see text-books on distributions)

[3] H^2 in the usual notation

Theorem 2.3 *a) Let c_k, where $k \in \mathbb{Z}$, be a bilateral sequence such that*

$$|c_k| \leqslant C \cdot (|k|+1)^M \qquad \text{for some } M,\, C$$

Then the Fourier series $\sum c_k e^{ik\varphi}$ converges in the sense of distributions. Moreover, for each distribution f on the circle,

$$f = \sum\nolimits_{k=-\infty}^{\infty} \ll f(\varphi), e^{-ik\varphi} \gg \cdot e^{ik\varphi}$$

b) A sum of a Fourier series is a smooth function iff

$$|c_n| \leqslant \mathrm{const} \cdot (1+|n|)^{-M} \qquad \text{for all } M$$

c) A distribution f is contained in the Sobolev space[4] W^α iff

$$\sum |c_n|^2 (|n|+1)^{2\alpha} < \infty$$

For a function $f(z) = \sum c_k z^k$ on the disk, define its boundary value as the sum of Fourier series $f(e^{i\varphi}) = \sum c_k e^{ik\varphi}$ in the sense of distributions[5]. Equivalently,

$$f(e^{i\varphi}) := \lim_{r \to 1-0} f(re^{i\varphi}), \qquad \text{in the sense of distributions}$$

Observation 2.4 *The space $\mathcal{H}_s(D)$ consists of holomorphic functions whose boundary values are contained in the Sobolev space $W^{-s/2}$.*

This is rephrasing of (2.4). □

2.5. Remark. The integral formula in the unallowed domain.

Observation 2.5 *Let f, $g \in \mathfrak{H}_s(D)$ be smooth up to the boundary.*

a) The integral (2.3 converges for $\mathrm{Re}\, s > 1$ *and determines a functions $I(s)$ holomorphic in s.*

b) The series (2.5) converges for $\mathrm{Re}\, s > 0$ *and determines a function $G(s)$ holomorphic in s.*

This in an exercise on differentiation of series and integrals in parameters.□
However, $I(s) = G(s)$ for real $s \geqslant 1$. In other words, the integral (2.3) admits an analytic continuation to the domain $\mathrm{Re}\, s > 0$.

Recall a more standard way of holomorphic continuations of the integral (2.3). Pass to the polar coordinates. Our problem is reduced to continuation of the integral

$$(s-1)\int_0^{2\pi} \left[\int_0^1 F(\varphi, r)(1-r^2)^{s-2} r\, dr\right] d\varphi \tag{2.6}$$

[4]The reader nonfamiliar with Sobolev spaces can consider this statement as a definition
[5]There numerous ways to define boundary values pointwise; however this leads to highly nontrivial phenomena; for instance, see a recent work [88].

We denote $G(\varphi, r) := F(\varphi, r)(1+r)^{s-2}r$ and write the interior integral as

$$\Big[\dots\Big] = \int_0^1 G(\varphi, r)(1-r)^{s-2} dr$$

Next, integrating by parts (for $\operatorname{Re} s > 1$), we get

$$\Big[\dots\Big] = -G(\varphi, 0) - \frac{1}{s-1}\int_0^1 G'_r(\varphi, r)(1-r)^{s-1}\, dr$$

We come to the integral that converges for $\operatorname{Re} s > 1$. The last expression has a pole at $s = 1$, however in the integral formula (2.3) this pole is killed by the factor $(s-1)$. Therefore, the whole expression (2.3) is holomorphic in $\operatorname{Re} s > 0$.

Separately, let us discuss the case $s = 1$.

Observation 2.6 *Let h be a smooth function in the disk $|z| \leqslant 1$. Then*

$$\lim_{s\to 1+0} \frac{s-1}{\pi} \iint_{|z|<1} h(z, \overline{z})(1-|z|^2)^{s-2}\, d\lambda(z) = \frac{1}{2\pi}\int h(e^{i\varphi})\, d\varphi$$

i.e., the family of distributions $(s-1)(1-|z|^2)^{s-2}$ tends to a delta-function supported by the circle.

We leave this statement as an exercise in the theory of distribution (see a similar statement in Subsection 1.3.1).

Thus the limit of our inner products as $s \to 1+0$ is the inner product in the Hardy space. However, our arguments are redundant, since this follows from the explicit formula for reproducing kernels.

2.6. Unitary representations of SU(1, 1)**.** Let $g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha}\end{pmatrix}$, where $|\alpha|^2 - |\beta|^2 = 1$, i.e., $g \in \mathrm{SU}(1,1) \simeq \mathrm{SL}(2,\mathbb{R})$.

For a fixed $s \in \mathbb{R}$, consider transformations

$$T_s\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha}\end{pmatrix} = f\Big(\frac{\beta + z\overline{\alpha}}{\alpha + z\overline{\beta}}\Big)(\alpha + z\overline{\beta})^{-s}$$

Proposition 2.7 *If s is integer, then $T_s(g)$ is a representation of the group* SU(1, 1)*.*

Proof is a straightforward calculation. □

Now let s be arbitrary. We define

$$(\alpha + z\overline{\beta})^{-s} := e^{-s\ln\alpha}(1 + z\overline{\beta}\alpha^{-1})^{-s}, \qquad |z| < 1$$

This function splits into a countable family of branches which differ by scalars $e^{-2\pi i s}$.

There are two way to formulate the previous proposition in this case. First, for a given matrix g we can we can choose an arbitrary branch. *In this case, we get a projective representation of* $\mathrm{SU}(1,1)$.

On another hand, the function $\ln\alpha$ is a single-valued function on the universal covering $\mathrm{SU}(1,1)^\sim$ of $\mathrm{SU}(1,1)$ (see our discussion in Section 3.6). *In this way, we obtain a linear representation of* $\mathrm{SU}(1,1)^\sim$.

Theorem 2.8 *For $s \geqslant 0$, the operators $T_s(g)$ are unitary in the space $\mathcal{H}_s$.*

PROOF. First, let $s > 1$

$$\langle T_s(g)f_1, T_s(g)f_2\rangle_s =$$
$$= \frac{s-1}{\pi}\iint f_1\Bigl(\frac{\beta+z\overline{\alpha}}{\alpha+z\overline{\beta}}\Bigr)(\alpha+z\overline{\beta})^{-s}\overline{f_2\Bigl(\frac{\beta+z\overline{\alpha}}{\alpha+z\overline{\beta}}\Bigr)}\,\overline{(\alpha+z\overline{\beta})}^{-s}(1-|z|^2)^{s-2}\,d\lambda(z)$$

Problem 2.2 Change the variable $u := (\beta + z\overline{\alpha})/(\alpha + z\overline{\beta})$ and come to $\langle f_1, f_2\rangle$. However, this is done in a wider generality in Section 7.4.

Thus, for $s > 1$ the operators are unitary. Taking to account the analytic continuation, we obtain the statement for all $s > 0$. □

The representations $T_s(g)$ are called *unitary highest weight representations* of $\mathrm{SU}(1,2)$.

7.3 Hilbert spaces of holomorphic functions

3.1. Spaces of holomorphic functions.

Theorem 3.1 *Let Ω be a domain in $\mathbb{C}^n$. Let $L(z,\overline{u})$ be a positive definite kernel holomorphic in z and anti-holomorphic in u. Then the Hilbert space $\mathsf{H}^\circ(\Omega; L)$ consists of functions holomorphic on Ω.*

PROOF. By the Hartogs Theorem (see, for instance, [164]), a function that is separately holomorphic in each argument is jointly holomorphic. Hence $K(z,u)$ is continuous on $\Omega\times\Omega$. Therefore the convergence in our space implies the uniform convergence on compact subsets.

Functions $\varphi_a(z) := K(z,a)$ are holomorphic. Their finite linear combinations are dense in H°. But a uniform limit of holomorphic functions is holomorphic.□

3.2. Weighted Bergman spaces. Let $\zeta(z)$ be a nonnegative function on Ω. Denote by $\mathcal{B}(\Omega,\zeta)$ the space of holomorphic functions on Ω satisfying the condition
$$\int_\Omega |f(z)|^2\zeta(z)\,d\lambda(z) < \infty$$
where $d\lambda(z)$ denotes the Lebesgue measure on Ω. Define the L^2-scalar product
$$\langle f, g\rangle = \int_\Omega f(z)\overline{g(z)}\zeta(z)\,d\lambda(z) \tag{3.1}$$

in the space $\mathcal{B}(\Omega, \zeta)$.

Theorem 3.2 *Let ζ be continuous and strictly positive on Ω. Then the space $\mathcal{B}(\Omega, \zeta)$ is closed in $L^2(\Omega, \zeta)$. Moreover, this space is determined by some reproducing kernel $K(z, \overline{u})$ holomorphic in z and anti-holomorphic in $\overline{u}$.*

Lemma 3.3 *The theorem holds for the polydisk $|z_j| < 1$ and the weight $\zeta = 1$.*

Proof of the lemma. See Subsection 2.1. Our space is determined by the reproducing kernel $\prod(1 - z_j\overline{u}_j)^{-2}$. □

Proof of the theorem. Consider an arbitrary point $w \in \Omega$ and a small polydisk $D^n \subset \Omega$ with a center at w. For an arbitrary f,

$$\int_\Omega |f(z)|^2 \zeta(z)\, d\lambda(z) \geqslant \min_{z \in D^n} \zeta(z) \cdot \int_{D^n} |f(z)|^2 d\lambda(z) \tag{3.2}$$

Now, let $f_j \in \mathcal{B}(\Omega, \zeta)$ converges in L^2-sense to f. Then it converges in $L^2(D^n)$. By the lemma, the limit f is holomorphic in D. Hence it is holomorphic in Ω.

By the same lemma, $f_j(w)$ converges to $f(w)$. Therefore, the evaluation functional is continuous and we apply Theorem 1.7. □

3.3. Digression. Discussion of Theorem 3.2. The condition $\zeta(z) > 0$ of this theorem is too restrictive for some applications, we discuss how to weaken it.

A. Denote by D^n the polydisk $|z_1| < 1, \ldots, |z_n| < 1$. Let $\sigma(z)$ be a continuous strictly positive continuous function on the torus $\mathbb{T}^n : |z_1| = \cdots = |z_n| = 1$. Consider a space $\mathcal{A}(D^n)$ of all functions in D admitting a holomorphic extension to a larger polydisk $|z_j| < 1 + \delta$. Equip $\mathcal{A}(D^n)$ with the inner product

$$\langle f, g \rangle = \int_{\mathbb{T}^n} f(z)\overline{g(z)}\sigma(z)\, d\lambda(z)$$

where $d\lambda$ is the Lebesgue measure on torus.

Proposition 3.4 *The closure $\overline{\mathcal{A}}(D^n)$ of $\mathcal{A}(D^n)$ in $L^2(D^n, \sigma)$ consists of functions holomorphic in D^n. Moreover $\overline{\mathcal{A}}(D^n)$ is determined by some reproducing kernel.*

Proof. Let us show that for each $a \in D^n$, the evaluation linear functional $\ell_a(f) = f(a)$ on $\mathcal{A}(D)$ is continuous. Represent $f \in \mathcal{A}(D^n)$ by the Cauchy integral formula

$$f(a) = \frac{1}{(2\pi i)^n} \int_{|z_1| = |z_2| = \cdots = 1} f(z) \prod \frac{dz_j}{z_j - a_j} =$$
$$= \frac{1}{(2\pi)^n} \cdot \int_{|z_1| = |z_2| = \cdots = 1} f(z) \cdot \left\{ \sigma(z)^{-1} \prod (z_j - a_j)^{-1} \right\} \cdot \sigma(z)\, d\lambda(z)$$

where $d\lambda(z)$ is the Lesbegue measure on the torus. Denote by $\gamma_a(z)$ the expression in the curly brackets. The last integral is the $L^2(\mathbb{T}^n, \sigma)$-inner product $\langle f, \gamma_a \rangle$. Therefore, ℓ_a is a bounded linear functional.

Moreover, for a ranging in a compact set $C \subset D^n$, the linear functionals ℓ_a are uniformly bounded.

Let $f_j \in \mathcal{A}(D^n)$ be a sequence convergent to a function f in the $L^2(\mathbb{T}^n, \sigma)$-sense. Due to the continuity of evaluation functionals, this sequence converges pointwise. Moreover, this sequence is L^2-bounded, therefore it is uniformly bounded in each compact subset $C \subset D$. Next, a uniformly bounded pointwise convergent sequence of holomorphic functions converges uniformly on compact sets. Therefore f is holomorphic. □

B. Let $\Sigma \subset \mathbb{C}^n$ be a smooth closed hypersurface. Let $\Omega \subset \mathbb{C}^n$ (respectively $\overline{\Omega}$) be the open (respectively closed) domain bounded by Σ. Let $\tau(z)$ be a strictly positive continuous function on Σ. Consider the space $\mathcal{A}(\Omega)$ of functions admitting a holomorphic continuation to some neighborhood of $\overline{\Omega}$. Equip $\mathcal{A}(\Omega)$ with the inner product

$$\langle f, g\rangle = \int_\Sigma f(z)\overline{g(z)}\tau(z)\, d\lambda(z)$$

where $d\lambda(z)$ is the surface Lebesgue measure on Ξ.

Denote by $\overline{\mathcal{A}}(\Omega)$ the closure of $\mathcal{A}(\Omega)$ in $L^2(\Sigma, \tau)$.

Proposition 3.5 *The space $\overline{\mathcal{A}}(\Omega)$ consists of functions holomorphic in Ω; this space is determined by a reproducing kernel.*

PROOF. We repeat the proof of the previous proposition, we only replace the Cauchy integral formula with the Bochner–Martinelli formula.

Denote by ω the following $(2n-1)$-differential form on $\mathbb{C}^n$

$$\omega(z) := \sum_{j=1}^n \overline{z}_j d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_{j-1} \wedge dz_{j+1} \cdots \wedge d\overline{z}_n \wedge dz_1 \wedge \cdots \wedge dz_n$$

By the Martinelli–Bochner formula (see [164]),

$$f(u) = \frac{(n-1)!}{(2\pi i)^n} \int_{z\in\Sigma} f(z) \frac{\omega(u-z)}{|u-z|^{2n}}$$

and we repeat the arguments given in the previous proof. □

C. COUNTEREXAMPLE. In the three previous statements, we require a strict positivity of a weight function.

Consider the disk $D : |z| < 1$ on $\mathbb{C}$. Denote by $\mathcal{A}(D)$ the space of functions that are holomorphic in a neighbourhood of $|z| \leqslant 1$.

Observation 3.6 *There is a weight function $\sigma(z)$, $z \in D$, that is smooth and strictly positive outside the segment $[-1, 0]$ such that the closure of $\mathcal{A}(D)$ in $L^2(D, \sigma)$ contains a function non-holomorphic in D.*

PROOF. Let $U \subset \mathbb{C}$ be a simply connected domain. Any function f holomorphic in $U \subset \mathbb{C}$ is a limit of a sequence of polynomials that uniformly converges on all compact subsets in U (see, for instance [103]).

We take U being the circle $|z| < 1+\delta$ cut by the segment $[-1-\delta, 0]$. Let $f(z) = \sqrt{z}$ in U. Let $h_j(z)$ be sequence of polynomials convergent to $\sqrt{z}$ uniformly on all the compact subsets in U. It is more-or-less clear that if $\sigma(z)$ sufficiently rapidly decreases near the cut, then this uniform convergence implies the convergence in $L^2(D, \zeta)$. Hence the closure of $\mathcal{B}(D, \sigma)$ in L^2 contains the function $\sqrt{z}$ discontinuous along the cut. □

D. However, the function $\zeta(z)$ in the previous counterexample must be flat along $[-1,0]$ (i.e., its partial derivatives of all the orders are 0 on $[-1,0]$).

Now let $\sigma(z)$ be a smooth nonnegative function on the circle $\mathbb{T} : |z| = 1$ having a unique zero of order $2k$ at $z = 1$. Let $\mathcal{A}(D)$ be the same as above. Define the inner product in $\mathcal{A}(D)$ by

$$\langle f, g\rangle = \int_{|z|=1} f(z)\overline{g(z)}\sigma(z)\, d\lambda(z)$$

Observation 3.7 *The closure of $\mathcal{A}(D)$ in $L^2(\mathbb{T},\sigma)$ contains only functions holomorphic in the disk $|z| < 1$.*

PROOF. Denote by $\mathcal{A}^{\square}(D) \supset \mathcal{A}^{\square}(D)$ the set of functions of the form $f(z) = h(z)(z-1)^{-k}$, where $h(z) \in \mathcal{A}(D)$. Consider the closure of $\mathcal{A}^{\square}(D)$ in $L^2(D,\sigma)$.

However, convergence of f_j in the sense of $L^2(\mathbb{T},\sigma)$ is equivalent to the convergence of h_j in the sense of $L^2(\mathbb{T},\sigma(z)|1-z|^{2k})$. In the latter case, the limit of h_j is holomorphic (because of Proposition 3.4). $\square$

E. Consider the space of entire functions on $\mathbb{C}$ determined by the reproducing kernel $\sin(z-\overline{u})/(z-\overline{u})$, i.e. the Payley–Wiener space PW, which was discussed in Subsection 4.??. The inner product in PW admits the integral representation

$$\langle f, g\rangle_{\mathrm{PW}} = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx$$

(by definition). Also, PW is closed subspace in $L^2(\mathbb{R})$.

By Whittacker–Kotelnikov–Shanon Theorem, the same inner product admits an another 'integral' representation

$$\langle f, g\rangle = \sum_{n=-\infty}^{\infty} f(n)\overline{g(n)}$$

(moreover, there are other integral representations of such type, see [23]).

7.4 Berezin scale

We extend the construction of Section 7.2 to arbitrary symplectic groups $\mathrm{Sp}(2n,\mathbb{R})$.

4.1. Action of $\mathrm{Sp}(2n,\mathbb{R})$ on the Siegel wedge. Consider the linear space $\mathrm{Symm}(n,\mathbb{C})$ of symmetric complex $n\times n$-matrices. Denote $\mathrm{P}_n \subset \mathrm{Symm}(n,\mathbb{C})$ the Siegel wedge, i.e, the set of matrices satisfying $\operatorname{Im} z > 0$, see Subsection 3.3.8. We realize the group $\mathrm{Sp}(2n,\mathbb{R})$ as the group of real symplectic matrices $g = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$. Recall that this group acts on the P_n by the linear fractional transformations:

$$z \mapsto z^{[g]} := (a+zc)^{-1}(b+zd)$$

There is an alternative expression for $z^{[g]}$.

$$z^{[g]} = (d^t z + b^t)(c^t z + a^t)^{-1}$$

To establish the equivalence of these two forms, we must verify the equality

$$(b+zd)(c^t z+a^t) = (a+zc)(d^t z+b^t)$$

We open brackets and refer to the defining relations (3.1.4) for the symplectic group.

Lemma 4.1 *The complex Jacobian of the map $z \mapsto z^{[g]}$ is*

$$J_{\mathbb{C}}(g,z) = \det(a+zc)^{-(n+1)/2}$$

PROOF. By Lemma 2.11.1, the differential of our transformation is

$$dz \mapsto (a+zc)^{-1}\, dz\, (-cz^{[g]}+d) \tag{4.1}$$

Next

$$-cz^{[g]}+d = -c(d^t z+b^t)(c^t z+a^t)^{-1}+d = \big[-c(d^t z+b^t)+d(c^t z+a^t)\big](c^t z+a^t)^{-1} =$$
$$= \big[(-cd^t+dc^t)z+(-cb^t+da^t)\big](c^t z+a^t)^{-1} = (c^t z+a^t)^{-1}$$

Here we again apply the defining relations (3.1.4) for the symplectic group and obtain 1 in the square brackets. However, the last result is predictable, since expression (4.1) has to be symmetric for symmetric dz.

Thus the differential is

$$dz \mapsto (a+zc)^{-1}\, dz\, (c^t z+a^t)^{-1}$$

Now the statement follows from the next lemma.

Lemma 4.2 *Consider the linear operator $x \mapsto \alpha x \alpha^t$ in the space* $\mathrm{Symm}(n,\mathbb{C})$. *Its determinant is* $\det(\alpha)^{n+1}$.

PROOF OF LEMMA 4.1. Denote the determinant of our operator by $\gamma(\alpha)$. Obviously,

$$\gamma(\alpha\alpha') = \gamma(\alpha)\gamma(\alpha')$$

Represent $\alpha = A\Delta B$, where A, $B \in \mathrm{SU}(n)$ and Δ is a diagonal matrix.

First, the determinant of the map $x \mapsto BxB^t$ is 1, because the group $\mathrm{SU}(n)$ has not nontrivial homomorphisms to $\mathbb{C}^*$.

It remains to evaluate the determinant of the map $x \mapsto \Delta x \Delta$. Denote by E_{ij} the matrix units. The matrices E_{kk} and $E_{ij}+E_{ji}$, where $i > j$, form a basis of the space $\mathrm{Symm}(n,\mathbb{C})$. These vectors are the eigenvectors of our linear operator. The eigenvalues are λ_k^2, $\lambda_j\lambda_j$. The product of eigenvalues is

$$\prod \lambda_j^{n+1} = \det \Delta^{n+1} = \det \alpha^{n+1} \qquad \square$$

4.2. A general remark. Let a group G act on a space X. Let $\tau(g,x)$ be a $\mathbb{C}^*$-valued function on $G \times X$. Consider the following operators in the space of functions on X

$$T(g)f(x) = f(xg)\tau(g,x) \tag{4.2}$$

Observation 4.3 *$T(g)$ is a representation of the group G, i.e.*

$$T(g_1)T(g_2) = T(g_1 g_2)$$

iff $\tau(g,x)$ satisfies the multiplier identity

$$\tau(g_1 g_2, x) = \tau(g_1, x g_2)\tau(g_2, x) \tag{4.3}$$

For instance, let X be an open subset in $\mathbb{R}^N$. Then

1) the Jacobian $J_{\mathbb{R}}(g,x)$ satisfies this identity.

2) for each $\alpha \in \mathbb{C}$, the function $\tau(g,x) := J_{\mathbb{R}}(g,x)^\alpha$ also satisfies the identity.

The first remark remains valid for complex domains and complex Jacobians $J_{\mathbb{C}}(g,z)$. Concerning the second remark, we meet a difficulty with a definition of complex powers.

4.3. Action of $\mathrm{Sp}(2n,\mathbb{R})$ in the spaces of holomorphic functions. For each $\alpha \in \mathbb{C}$, define the following operators $T_\alpha(g)$ in the space of holomorphic functions on P_n,

$$T_\alpha(g) f(z) = f\big(z^{[g]}\big) \det(a + zc)^{-\alpha} \tag{4.4}$$

Proposition 4.4 *T_α is a projective representation of $\mathrm{Sp}(2n,\mathbb{R})$. If α is integer, then this representation is linear.*

PROOF. The multiplier identity can be easily verified by a straightforward calculation. However we prefer a longer way.

First of all,

$$\det(a + zc) = J_{\mathbb{C}}(g,z)^{1/n} \neq 0 \qquad \text{for } z \in \mathrm{P}_n$$

because the Jacobian of a biholomorphic map does not vanish. Hence the many-valued function $\det(a+zc)^\alpha$ splits into a countable number of holomorphic branches, these branches differ by scalar factors $e^{2\pi i \alpha}$.

Since the Jacobian satisfies the multiplier identity (4.3), i.e.,

$$J_{\mathbb{C}}(g_1 g_2, z) = J_{\mathbb{C}}(g_2, z) J_{\mathbb{C}}(g_1, z^{[g_2]})$$

it follows that the powers of the Jacobian satisfy the same identity to within scalars (that appear because powers are multi-valued). □

4.4. Next general remark. Invariant kernels. Now let a group G act on a space Ω via transformations

$$T(g) f(x) = f(xg)\tau(g,x)$$

Let $L(x,y)$ be a reproducing kernel on Ω.

Theorem 4.5 *The operators $T(g)$ are unitary in $\mathsf{H}^\circ(\Omega, L)$ iff L is invariant in the following sense:*

$$L(xg, yg) = L(x,y) c(g,x)^{-1} \overline{c(g,y)}^{\,-1}$$

In particular, if Ω is G-homogeneous, then an invariant reproducing kernel L is uniquely determined by the multiplier $c(g,x)$.

PROOF. Let ψ_x be the corresponding overfilled system. The operators $T(g)$ are unitary, hence

$$\langle f, T(g^{-1})\psi_x\rangle_{\mathsf{H}^\circ} = \langle T(g)f, \psi_x\rangle = \{\text{value of } T(g)f \text{ at } x\} = f(xg)\tau(g,x)$$

Thus,

$$T(g^{-1})\psi_x = \tau(g,x)^{-1}\psi_{xg}$$

Next

$$L(xg,yg) = \langle \psi_{xg}, \psi_{yg}\rangle_{\mathsf{H}^\circ} = \langle \tau(g,x)^{-1}T(g^{-1})\psi_x, \tau(g,y)^{-1}T(g^{-1})\psi_y\rangle =$$
$$= \tau(g,x)^{-1}\overline{\tau(g,y)}^{-1}\langle T(g^{-1})\psi_x, T(g^{-1})\psi_y\rangle = \tau(g,x)^{-1}\overline{\tau(g,y)}^{-1}\langle \psi_x,\psi_y\rangle$$

4.5. Berezin kernels on matrix wedges. The statement of main result. Now we going to examine unitarity property of the representation T_α of $\mathrm{Sp}(2n,\mathbb{R})$ defined above, see (4.4).

Theorem 4.6 *A kernel on* P_n *invariant under the operators* $T_\alpha(g)$ *has the form*

$$K_\alpha(z,u) = \mathrm{const}\cdot\det\Bigl(\frac{z-u^*}{2i}\Bigr)^{-\alpha} \tag{4.5}$$

PROOF. An invariant kernel is unique because the action is transitive. The invariance of K_α is equivalent to the identity

$$\det\Bigl(\frac{z^{[g]}-(u^{[g]})^*}{2i}\Bigr) = \det(a+zc)^{-1}\det(c^tu^*+a^t)^{-1}\det\Bigl(\frac{z-u^*}{2i}\Bigr)$$

This follows from[6]

$$z^{[g]} - \bigl(u^{[g]}\bigr)^* = (a+zc)^{-1}\bigl(z-u^*\bigr)(c^tu^*+a^t)^{-1} \tag{4.6}$$

To prove the last identity, we write

$$z^{[g]} - \bigl(u^{[g]}\bigr)^* = (a+zc)^{-1}(b+zd) - (d^tu^*+b^t)(c^tu^*+a^t)^{-1} =$$
$$= (a+zc)^{-1}\Bigl((b+zd)(c^tu^*+a^t) - (a+zc)(d^tu^*+b^t)\Bigr)(c^tu^*+a^t)^{-1}$$

Applying the defining relations (3.1.4) for $\mathrm{Sp}(2n,\mathbb{R})$, we get $z-u^*$ in the big brackets. □

Theorem 4.7 *The Berezin kernel* $K_\alpha(z,u)$ *is positive definite iff*

$$\alpha = 0,\ 1/2,\ 1,\ \dots,\ (n-1)/2,\ \textit{or}\ \alpha > (n-1)/2 \tag{4.7}$$

Proof is given in Subsections 4.8–4.13.

We denote the corresponding Hilbert space by $\mathcal{H}_\alpha$.

[6] In turn, it is a variant of (11.4)

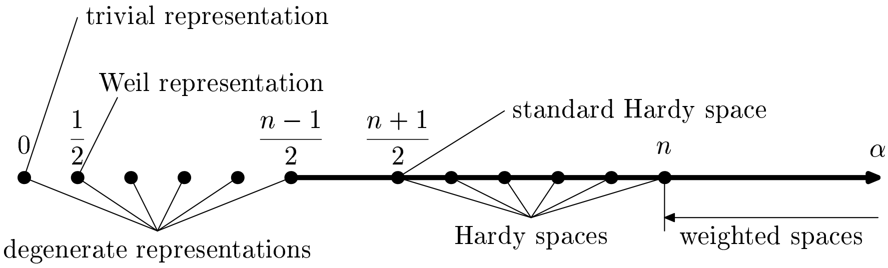

Fig. 7.1: To Theorem 4.7

In the next subsection, we give another description of $\mathcal{H}_\alpha$ for $\alpha > n$ (and in particular, this gives a partial proof of Theorem, however it is not used in the main proof).

4.6. Weighted spaces. Consider the space $\mathcal{H}_\alpha$ of holomorphic functions f on the P_n satisfying

$$\int_{\mathrm{P}_n} |f(z)|^2 \big[\det \operatorname{Im}(z)\big]^{\alpha-n-1}\, d\lambda(z) < \infty$$

where $d\lambda(z)$ is the Lebesgue measure on the space P_n. Equip this space with the inner product

$$\langle f, g\rangle_\alpha = \mathrm{const} \int_{\mathrm{P}_n} f(z)\overline{g(z)} \det \operatorname{Im}(z)^{\alpha-n-1}\, d\lambda(z) \tag{4.8}$$

Theorem 4.8 *For $\alpha > n$ the Hilbert space $\mathcal{H}_\alpha$ is nonempty and the representation $T_\alpha(g)$ of* $\mathrm{Sp}(2n,\mathbb{R})$ *is unitary in $\mathcal{H}_\alpha$.*

Since an invariant reproducing kernel is unique, this space is determined by the kernel K_α.

Incomplete proof. Substitute $z = u^{[g]}$ into the integral. The real Jacobian is

$$\frac{d\lambda(z^{[g]})}{d\lambda(z)} = J_{\mathbb{R}}(g,u) = |J_{\mathbb{C}}(g,u)|^2 = |\det(a+zc)|^{-2(n+1)}$$

Next, we use formula (4.6)

$$\det(\operatorname{Im} z^{[g]}) = \det\Big(\frac{z^{[g]} - \overline{z^{[g]}}}{2i}\Big) = \det(\operatorname{Im} z)|\det(a+zc)|^{-2}$$

Hence, the right-hand part of (4.8) is

$$\langle f_1, f_2\rangle_\alpha = \int_{\mathrm{P}_n} f_1(u^{[g]})\overline{f_2(u^{[g]})} \det|(a+uc)|^{-2\alpha} \det \operatorname{Im}(u)^{\alpha-n-1}\, d\lambda(u) =$$

$$= \int_{\mathrm{P}_n} f_1(u^{[g]}) \det(a+uc)^{-\alpha}\, \overline{f_2(u^{[g]}) \det(a+uc)^{-\alpha}} \det \operatorname{Im}(u)^{\alpha-n-1}\, d\lambda(u) =$$

$$= \langle T_\alpha(g) f_1, T_\alpha(g) f_2\rangle_\alpha$$

Thus the operators $T_\alpha(g)$ are unitary.

For $\alpha < n$, the space $\mathcal{H}_\alpha$ is empty. Indeed, the boundary $\partial \mathrm{P}_n$ of P_n consists of matrices

$$x + iy, \qquad \text{where } x \in \mathrm{Symm}(n, \mathbb{R}),\ y \in \mathrm{Symm}(n, \mathbb{R}),\ y \geqslant 0,\ \det y = 0.$$

It is a singular real manifold, a point $x + iy$ is nonsingular iff $\mathrm{rk}\, y = n - 1$. Near a nonsingular point the weight function $\det \mathrm{Im}(z)^{\alpha-n-1} = \det y^{\alpha-n-1}$ has the usual power singularity. It is locally integrable iff the power is $\geqslant -1$.

If $\alpha > n+1$, then $\mathcal{H}_\alpha$ is nonempty. Indeed, in this case function $\mathrm{Im}(z)^{\alpha-n-1}$ is continuous on the wedge; it remains to find a function that sufficiently rapidly decreases in P_n. We can put $f(z) := \det\big((z+i)/2i\big)^{-N}$ for large N.

The case $n < \alpha < n + 1$ is not completely obvious (because the function $\det y^{\alpha-n-1}$ has rather complicated singularities at the manifold $\mathrm{rk}\, y \leqslant n - 2$.)

Lemma 4.9 *The function $\det(i + z)^{-\alpha}$ is contained in $\mathcal{H}_\alpha$ for all $\alpha > n$.*

The L^2-norm of $\det(i+z)^{-\alpha}$ can be evaluated explicitly, we can propose this as an exercise; however a similar (and more complicated) calculation is given below in Subsections 5.3–5.6.

4.7. Next general comment. The Bochner Theorem.

Theorem 4.10 (Bochner Theorem) *A function $\Psi(x - y)$ is a positive definite kernel on $\mathbb{R}^n$ iff Ψ is a Fourier transform of a finite positive Borel measure μ on $\mathbb{R}^m$, i.e.,*

$$\Psi(x) = \int_{\mathbb{R}^m} e^{itx}\, d\mu(t)$$

Proof. The additive group $\mathbb{R}^n$ acts on the space $\mathsf{H}^\star(\mathbb{R}^n, \Psi)$ by unitary operators

$$T(a)f(x) = f(x + a)$$

Further,

$$\langle T(a)\delta(x), \delta(x)\rangle_{\mathsf{H}^\star} = \Psi(a) \tag{4.9}$$

By definition of $\mathsf{H}^\star$, the functions $\delta(x - a)$ form a total family. In other words, the group $\mathbb{R}^n$ has a cyclic vector, namely $\Delta(x)$.

We apply a multi-parametric variant of the Stone Theorem. Consider a unitary representation T of the additive group $\mathbb{R}^n$; assume that T has a cyclic vector v. Then T can be realized in L^2 on $\mathbb{R}^n$ with respect to a finite measure μ on $\mathbb{R}^n$ by the operators

$$\widetilde{T}(a)f(t) = e^{ita} f(t)$$

the cyclic vector v corresponds to $f = 1$. In our case, v is $\delta(x)$ and

$$\langle \widetilde{T}(a) \cdot 1, 1\rangle_{L^2} = \int e^{ita} \cdot 1 \cdot 1\, d\mu(t) \tag{4.10}$$

Comparing (4.9) and (4.10), we observe, that μ is the desired measure. □

Let Γ be a nonempty open sharp[7] convex cone in $\mathbb{R}^m$. Let $\Gamma^\circ \subset \mathbb{R}^m$ be the *dual cone*, i.e., Γ° is the set of all vectors, such that $\langle x, y\rangle > 0$. Denote by $\overline{\Gamma}^\circ$ the closure of the cone Γ°.

A *tube domain* in $\mathbb{C}^m$ is a domain having the form $\Omega = \mathbb{R}^m \oplus i \cdot \Gamma$, where Γ is an open convex cone.

We say that a holomorphic function f in a tube domain has a *polynomial growth* if

$$|f(z)| \leqslant C(1+|z|)^N, \qquad \text{for some } N \text{ and } C.$$

Theorem 4.11 *A holomorphic function f in the tube domain $\mathbb{R}^m \oplus i\Gamma$ has a polynomial growth iff f is a Fourier (Laplace) transform[8] of a tempered distribution supported by the closed cone $\overline{\Gamma}^\circ$*

This is a variant of the Paley–Wiener–Schwartz Theorem, see, for instance, [69], v.1, Theorem 7.4.3.

Theorem 4.12 *Let Ψ be a holomorphic function of polynomial growth in the tube domain $\Omega = \mathbb{R}^m \oplus i\Gamma$. The function $\Psi(z-\overline{u})$ on $\Omega\times\Omega$ is a positive definite kernel iff Ψ is a Fourier transform of a positive measure supported by the closed cone $\overline{\Gamma}^\circ$*

PROOF. By Theorem 4.11,

$$\Psi(z) = \int e^{itz} R(t) \tag{4.11}$$

where R is a tempered distribution supported by $\overline{\Gamma}^\circ$. Put $\psi_c(x) := \Psi(x+ic)$ Let

$$\psi_c(x) = \Psi(x+ic) = \int_{\mathbb{R}^n} e^{it(x+ic)} R(t), \qquad \text{where } c \in \Gamma \text{ is fixed}$$

The function $\psi_c(x-y)$ is a positive definite kernel on $\mathbb{R}^n$ and hence ψ_c is a Fourier transform of a positive measure

$$\Psi(x+ic) = \int e^{itx} d\mu(t)$$

Comparing the last expression and (4.11), we obtain $d\mu(t) = e^{-cx}R(t)$, therefore the distribution $R(t)$ is a positive measure. □

4.8. The Laplace transform and the Berezin kernels. Consider the space $\mathrm{Symm}(n,\mathbb{R}) \simeq \mathbb{R}^{n(n+1)/2}$ of real symmetric $n\times n$-matrices. Endow it with the inner product

$$\langle x, y\rangle_{\mathrm{Symm}(n,\mathbb{R})} = \operatorname{tr} xy$$

Let $\Gamma \subset \mathrm{Symm}(n,\mathbb{R})$ be the cone Pos_n of positive definite matrices.

[7] A convex cone is sharp if it does not contain a line

[8] The discussed situation is a multivariate generalization of the Laplace transform

Problem 4.1 Show that the cone dual to Pos_n is Pos_n.

The Berezin kernel K_α is of the form $\Psi_\alpha(z-\overline{u})$ with

$$\Phi_\alpha(z) := \det(z/i)^{-\alpha}$$

It is a holomorphic function of polynomial growth in the tube

$$\mathrm{P}_n = \mathrm{Symm}(n,\mathbb{R}) + i\mathrm{Pos}_n \subset \mathrm{Symm}(n,\mathbb{C})$$

Therefore is a Fourier transform of a certain positive measure ν_α supported by the closed cone $\overline{\mathrm{Pos}_n}$.

Next, $\Phi_\alpha(z)$ satisfies the following homogeneity conditions

$$\Phi_\alpha(gzg^t) = |\det(g)|^{-2\alpha}\Phi_\alpha(z), \text{where } g \in \mathrm{GL}(n,\mathbb{R})$$

Hence the distribution ν_α on $\overline{\mathrm{Pos}_n}$ satisfies the homogeneity condition

$$\nu_\alpha(gxg^t) = \det(g)^{-2\alpha}\nu_\alpha(x) \tag{4.12}$$

Now we can find ν_α from this homogeneity. The immediate reaction is to write

$$\nu_\alpha(x) = \det(x)^{\alpha-(n+1)/2}\, dx \tag{4.13}$$

This is almost truth, however the situation is slightly mode delicate.

4.9. Description of the measures ν_α. Regular case.

Theorem 4.13 *a) For $\alpha > (n-1)/2$, $z \in \mathrm{P}_n$*

$$\int_{x\in\mathrm{Pos}_n} \det(x)^{\alpha-(n+1)/2} \exp\{i \operatorname{tr} xz\}\, dx = \Gamma[n;\alpha]\det(z/i)^{-\alpha} \tag{4.14}$$

where

$$dx := \prod_{k\leqslant l} dx_{kl}$$

and the constant $\Gamma[n;\alpha]$ is the Whishart–Siegel integral

$$\Gamma[n;\alpha] := \int_{x\in\mathrm{Pos}_n} \det x^{\alpha-(n+1)/2} \exp\{-\operatorname{tr} x\}\, dx \tag{4.15}$$

b) The Whishart–Siegel integral converges iff $\alpha > (n-1)/2$ and equals

$$\Gamma[n;\alpha] : \pi^{n(n-1)/4} \prod_{k=1}^{n} \Gamma(\alpha-(k-1)/2) \tag{4.16}$$

Recall that the Γ*-function* is defined by

$$\Gamma(\beta) := \int_0^\infty x^{\beta-1}e^{-x} dx \tag{4.17}$$

It is a meromorphic function having poles at $\beta = 0,\ -1,\ -2, \ldots$.

Corollary 4.14 *For $\alpha > (n-1)/2$, the Berezin kernel K_α is positive definite.*

PROOF OF COROLLARY. Indeed, in this case $\nu_\alpha := \det(x)^{\alpha-(n+1)/2}\,dx$ is a positive measure on the cone Pos_n. For $\alpha > (n+1) > 2$ the density of ν_α is continuous, therefore ν_α is a tempered distribution.

The boundary of the cone Pos_n is a singular manifold $\det(x) = 0$, $x \geqslant 0$. Nonsingular points satisfy $\operatorname{rk} x = n-1$. For $\alpha \leqslant (n-1)/2$, the density of the measure is not locally integrable near a nonsingular point; therefore generally speaking the integrals $\int \varphi(x)\,d\nu_\alpha(x)$ are divergent.

For $(n-1)/2 < \alpha < (n+1)/2$ the local integrability of the density in not self-evident. However, $\exp\{-\operatorname{tr} x\}$ is a strictly positive function with convergent integral with respect to ν_α, and this implies the local integrability; thus the measure ν is a tempered distributin.

DERIVATION OF FORMULA (4.14). Firstly, assume $z = ir$, where $r \in \mathrm{Pos}_n$. Then

$$i\operatorname{tr} zx = -\operatorname{tr} rx = -\operatorname{tr}\sqrt{r}x\sqrt{r}$$

We write the integral as

$$\int \det(r)^{-\alpha+n+1}\det(\sqrt{r}x\sqrt{r})^{\alpha-n-1}\exp\{-\operatorname{tr}\sqrt{r}x\sqrt{r}\}\det(r)^{-n-1}d(\sqrt{r}x\sqrt{r})$$

After the substitution $x' = \sqrt{r}x\sqrt{r}$, we come to (4.14).

Next, both sides of (4.14) are holomorphic in $z \in \mathrm{P}_n$, hence the identity holds for all $z \in \mathrm{P}_n$ by the analytic continuation.

This is almost sufficient for a proof Corollary 4.14, however it remains the difficulty with the interval $\alpha \in \big((n-1)/2, (n+1)/2\big)$ mentioned above. $\square$

4.10. The evaluation of the Whishart–Siegel integral. Represent the matrix x as a block $(1+n)\times(1+n)$ matrix and write out its determinant by formula (1.1.11)

$$x = \begin{pmatrix} p & q \\ q^t & r \end{pmatrix}, \qquad \det x = \det(r)\cdot(p - qr^{-1}q^t)$$

Then

$$\Gamma[n;\alpha] := \int_{x>0} \det x^{\alpha-(n+1)/2}\exp\{-\operatorname{tr} x\}\,dx =$$
$$= \int_{r>0,\, p-qr^{-1}q^t>0} \det(r)^{\alpha-(n+1)/2}(p-qr^{-1}q^t)^{\alpha-(n+1)/2}\exp\{-p-\operatorname{tr} r\}\,dp\,dq\,dr$$

Firstly, we integrate $dp\,dq$ with a fixed $r>0$,

$$\int_{p>qrq^t}(p-qrq^t)^{\alpha-(n+1)/2}e^{-p}dp\;dq =$$
$$= \int_{p>qrq^t}(p-qr^{-1/2}r^{-1/2}q^t)^{\alpha-(n+1)/2}e^{-p}\det r^{1/2}dp\;d(qr^{-1/2})$$

We pass to the new variable $h := qr^{1/2} \in \mathbb{R}^{n-1}$ and come to

$$\det r^{1/2} \int_{p>hh^t} (p-hh^t)^{\alpha-(n+1)/2} e^{-p}\, dp\, dh$$

Next, we introduce a new variable $y := p - hh^t$ instead of p and obtain

$$\det r^{1/2} \int_{y>0,\, h\in\mathbb{R}^{n-1}} y^{\alpha-(n+1)/2} e^{-y-hh^t}\, dy\, dh =$$
$$= \det r^{1/2} \int_{y>0} y^{\alpha-(n+1)/2} e^{-y}\, dy \cdot \int_{\mathbb{R}^{n-1}} e^{-hh^t}\, dh = \det r^{1/2} \Gamma(\alpha - \tfrac{n-1}{2})\, \pi^{(n-1)/2}$$

Thus

$$\Gamma[n,\alpha] = \Gamma(\alpha - \tfrac{n-1}{2})\, \pi^{(n-1)/2} \int_{r>0} \det(r)^{\alpha-n/2} e^{-\operatorname{tr} r} dr =$$
$$= \Gamma[n-1;\alpha]\Gamma(\alpha - \tfrac{n-1}{2})\, \pi^{(n-1)/2}$$

and this implies (4.16). □

4.11. Constructions of measures ν_α. The singular case. Denote by $\mathcal{O}_j \subset \mathrm{Pos}_n$ the set of all positive semi-definite matrices of rank j. Denote by $\overline{\mathcal{O}}_j$ the closure of $\mathcal{O}_j$, i.e., $\overline{\mathcal{O}}_j = \cup_{k\leqslant j} \mathcal{O}_k$.

The sets $\mathcal{O}_j$ are precisely $\mathrm{GL}(n,\mathbb{R})$-orbits on $\overline{\mathrm{Pos}}_n$; the orbit $\mathcal{O}_n$ is the open cone Pos_n. Al the orbits $\mathcal{O}_j$ except $\mathcal{O}_n$ are also $\mathrm{SL}(n,\mathbb{R})$-homogeneous.

Theorem 4.15 *a) For each $j = 0,1,\dots,n-1$ there is a unique to within scalar $\mathrm{SL}(n,\mathbb{R})$-invariant measure $d\nu_j$ on the orbit $\mathcal{O}_j$. For $g \in \mathrm{GL}(n,\mathbb{R})$,*

$$d\nu(gxg^t) = |\det(g)|^j d\nu(x) \tag{4.18}$$

b) This measure represents a tempered distribution on $\mathbb{R}^{n(n+1)/2}$.

c) The Fourier transform of this measure is $\mathrm{const}\cdot \det(z/2i)^{-j/2}$

Proof of the statement a. We intend to apply Theorem C.10.

Consider the symmetric matrix $I_j = \begin{pmatrix} 1_j & 0 \\ 0 & 0_{n-j} \end{pmatrix} \in \mathcal{O}_j$ or, equivalently, consider the degenerate quadratic form $\xi_1^2 + \dots + \xi_j^2$ on $\mathbb{R}^n$. A linear transformation $g \in \mathrm{GL}(n,\mathbb{R})$ preserving this form must preserve its kernel, i.e., the subspace L generated by the last $(n-j)$ basic vectors; g must also induce an orthogonal transformation on $\mathbb{R}^n/L$. Therefore, the stabilizer $H \subset \mathrm{GL}(n,\mathbb{R})$ of this quadratic form consists of triangular matrices

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}, \qquad a \in \mathrm{O}(j),\, c \in \mathrm{GL}(n-j,\mathbb{R}) \tag{4.19}$$

The stabilizer $H' \subset H$ of I_j in $\mathrm{SL}^{\pm}(n,\mathbb{R})$ consists of matrices having the same form with $\det(c) = \pm 1$.

The tangent space to $\mathcal{O}_j = \mathrm{GL}(n,\mathbb{R})/H$ at the point I_j is the space of $j \times (n-j)$-matrices. The group H acts on this space by the transformations $\xi \mapsto a\xi c^t$. The determinant of this map is $\det(a)^{n-j}\det(c)^j$ (see Lemma 2.11.3). For elements of H' we get ± 1. We apply Theorem C.10 and obtain the existence and uniqueness of $\mathrm{SL}^{\pm}(n,\mathbb{R})$-invariant measure ν_α.

Let us prove (4.18). Consider a scalar matrix $\Lambda := \lambda \cdot 1_n \in \mathrm{GL}(n,\mathbb{R})$. It can be readily checked that the measure $\nu_\alpha(\Lambda x \Lambda)$ is $\mathrm{SL}(n,\mathbb{R})$-invariant. Therefore

$$\nu_\alpha(\Lambda x \Lambda) = \varkappa_\lambda \cdot \nu_\alpha(x),$$

Obviously, the constant $\varkappa_\lambda$ is multiplicative with respect to λ, i.e.,

$$\varkappa_{\lambda\lambda'} = \varkappa_\lambda \varkappa_{\lambda'}$$

Hence $\varkappa_\lambda = |\lambda|^a$ for some a. Since each $g \in \mathrm{GL}(n,\mathbb{R})$ admits a representation $g = \Lambda \cdot h$ with $h \in \mathrm{SL}^{\pm}(n,\mathbb{R})$, we have

$$\nu(gxg^t) = C \cdot \nu(x), \qquad C = |\det(g)|^b$$

with constant $b := a/n$. To find the exponent b, we consider g having the form (4.19). Then the factor C coincides with absolute value of the determinant of the differential and we know this determinant. $\square$

Statement c) follows from a) and b). A proof of b) is contained in the next subsection.

4.12. Convolutions and a more explicit description of measures ν_j. Firstly, consider $\mathcal{O}_1$. Each $x \in \mathcal{O}_1$ has a form $x = v^t v$ where $v \in \mathbb{R}^n$ is a matrix-row. This representation of x is unique to within the change of sign, $v \mapsto -v$. The action of $\mathrm{GL}(n,\mathbb{R})$ in these coordinates is the usual linear action $v \mapsto vg$. The $\mathrm{SL}(n,\mathbb{R})$-invariant measure is the Lebesgue measure dv on $\mathbb{R}^n$.

Let φ be in the Schwartz space $\mathcal{S}\big(\mathrm{Symm}(n,\mathbb{R})\big)$. Then

$$\int_{\mathcal{O}_1} \varphi(x)\, d\nu_1(x) = \mathrm{const} \int_{\mathbb{R}^n} \varphi(v^t v)\, dx$$

and the convergence of the integral in the right-hand side is obvious. Thus ν_1 determines a tempered distribution.

Now we construct all the other measures ν_j as convolutions

$$\nu_2 = \nu_1 * \nu_1, \qquad \nu_3 = \nu_1 * \nu_1 * \nu_1, \ldots$$

Since all the ν_j are supported by the convex cone Pos_n, the convolutions are well-defined and are contained in $\mathcal{S}'(\mathrm{Symm}(n))$ (see Subsection 4.4.1). Next, ν_1 is $\mathrm{SL}(n,\mathbb{R})$-invariant, therefore all the ν_j are $\mathrm{SL}(n,\mathbb{R})$-invariant. For this reason ν_j are actually the measures ν_j from the previous subsection.

Problem 4.2 The measures ν_j can be described as follows. Equip the space $\mathrm{Mat}(j,n;\mathbb{R})$ with the Lebesgue measure. Consider the map $\mathrm{Mat}(j,n;\mathbb{R}) \to \mathrm{Symm}(n,\mathbb{R})$ given by

$u \mapsto uu^t$. The measure ν_j coincides (to within a scalar factor) with the pushforward of the Lebesgue measure under this map. In other words,

$$\int_{\mathcal{O}_j} \varphi(x)\, d\nu_j(x) = \text{const} \int_{\mathrm{Mat}(j,n;\mathbb{R})} \varphi(uu^t)\, du$$

4.13. End of Proof of Theorem 4.7. Thus, we proved the affirmative part of the theorem.

Let us establish the absence of positivity for all the remaining α. To apply Theorem 4.12, we must describe all the positive measures ν on the cone $\overline{\mathrm{Pos}_n}$ homogeneous in the sense (4.12).

Firstly, any such μ determines a homogeneous measure μ' on the open cone $\mathrm{Pos}_n \simeq \mathrm{GL}(n,\mathbb{R})/O(n,\mathbb{R})$. The usual arguments (see Subsections C.10–C.11) show that μ' has the form described above.

Consider the measure $\mu-\mu'$, which is zero on Pos_n. It determines a $\mathrm{SL}(n,\mathbb{R})$-invariant measure μ'' on $\mathcal{O}_{n-1}$. Such measure is unique up to a factor, therefore it coincides with ν_{n-1} to within a factor. Next we consider the measure $\mu-\mu'-\mu''$. This measure is supported by the closure of $\mathcal{O}_{n-2}$ and we repeat the same argument. Etc.

Thus the only positive homogeneous measures on Pos_n are the measures constructed above. □

4.14. Irreducibility.

Proposition 4.16 *The representations T_α of the group $\mathrm{Sp}(2n,\mathbb{R})$ in $\mathcal{H}_\alpha$ are irreducible.*

Proof. Let W be an invariant subspace. It is determined by a reproducing kernel. However, an invariant reproducing kernel is unique. □

4.15. Weighted Laplace transforms. Let ν_α be the positive measure on $\overline{\mathrm{Pos}_n}$ whose Fourier–Laplace transform is $\det(z/2i)^{-\alpha}$.

Consider the *weighted Laplace transform*

$$\mathcal{L}_\alpha f(z) = \int_{\mathrm{Pos}_n} f(x) e^{i\,\mathrm{tr}\,xz}\, d\nu_\alpha(x)$$

Theorem 4.17 *The operator $\mathcal{L}_\alpha$ is a unitary operator from $L^2(\overline{\mathrm{Pos}_n},\nu_\alpha)$ to the space $\mathcal{H}_\alpha(\mathrm{P}_n)$ of holomorphic functions.*

Proof. Consider the function

$$\varphi_z(x) := e^{i\,\mathrm{tr}\,xz}$$

on the cone Pos_n,

$$\langle \varphi_z, \varphi_u \rangle_{L^2} = \int e^{\mathrm{tr}\,x(iz-i\overline{u})} d\mu_\alpha(x) = \det\big((z-u)/2i\big)^{-\alpha} = K_\alpha(z,\overline{u})$$

Thus $\mathcal{H}_\alpha(\mathrm{P}_n) = \mathsf{H}^\circ(\mathrm{P}_n, K_\alpha)$ and the weighted Laplace transform is the standard map $\mathsf{H}(\mathrm{P}_n,K_\alpha) \to \mathsf{H}^\circ(\mathrm{P}_n,K_\alpha)$. □

4.16. Rephrasing of Theorem 4.7. Let B_n be the matrix ball, see Subsection 3.3.5. We realize the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ as in Subsection 3.2.3.

Fix $\alpha \in \mathbb{C}$. The group $\mathrm{Sp}(2n,\mathbb{R})$ acts in the space of holomorphic functions on B_n by the formula

$$T_\alpha \begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix} f(z) = f\big((\Phi + z\overline{\Psi})^{-1}(\Psi + z\overline{\Phi})\big) \det(\Phi + z\overline{\Psi})^{-\alpha} \tag{4.20}$$

Theorem 4.18 *a) The kernel*

$$L_\alpha(z,u) = \det(1 - zu^*)^{-\alpha}$$

on B_n *is positive definite iff* α *satisfies the conditions (4.7).*

b) Under the conditions of positivity, the representation T_α *is unitary in the Hilbert space* $\mathcal{H}_\alpha(\mathrm{B}_n)$ *determined by the kernel* $L_\alpha(z,\overline{u})$.

This is the same Theorem 4.7, we simply identify the wedge P_n and B_n by the Cayley transform. For a holomorphic function g on B_n we define the holomorphic function $\mathfrak{C}_\alpha g(z)$ on P_n as

$$\mathfrak{C}_\alpha f(z) = f\left(\frac{i-z}{i+z}\right) \det(i+z)^{-\alpha} \tag{4.21}$$

4.17. Digression. Stein–Sahi kernels. In what follows we outline a landscape without proofs.

We consider the space $\mathcal{L}_n$ of unitary symmetric matrices z,

$$zz^* = 1, \qquad z = z^t$$

Recall that the group $\mathrm{U}(n)$ acts on this space as $z \mapsto hzh^t$, the action is transitive, $\mathcal{L}_n \simeq \mathrm{U}(n)/\mathrm{O}(n)$. Our space lyes on the boundary of the matrix ball B_n, the group $\mathrm{Sp}(2n,\mathbb{R})$ acts on $\mathcal{L}_n$ by the same transformations (4.20).

Let σ, $\tau \in \mathbb{C}$, $\operatorname{Re}\sigma \leqslant 0$, $\operatorname{Re}\tau \leqslant 0$. Consider the function ℓ_σ on $\mathcal{L}_n$,

$$\ell_\sigma(z) = \lim_{\varepsilon\to+0} \det\Big[\big(1-(1-\varepsilon)z\big)^{-\sigma}\Big]$$

We define the *Stein–Sahi kernel*

$$L_{\sigma,\tau}(z,u) := \ell_\sigma(zu^*)\ell_\tau(uz^*) =: \det(1-zu^*)^{-\alpha}\det(1-uz^*)^{-\tau}$$

where z, $u \in \mathcal{L}_n$.

Next, we define a sesquilinear form on $C^\infty(\mathcal{L}_n)$ by

$$\langle f, g\rangle_{\sigma,\tau} := \iint_{\mathrm{U}(n)\times\mathrm{U}(n)} L_{\sigma,\tau}(z,u) f_1(z)\overline{f_2(z)}\, d\mu(z)\, d\mu(u) \tag{4.22}$$

where μ denotes the $\mathrm{U}(n)$-invariant measure on $\mathcal{L}_n$. If $\sigma, \tau \in \mathbb{R}$, then this form is Hermitian.

For $g = \begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{R})$, define the operators $T_{\sigma,\tau}(g)$ in $C^\infty(\mathrm{U}(n))$ by

$$T_{\sigma,\tau}(g)f(z) = f\big((\Phi+z\overline{\Psi})^{-1}(\Psi+z\overline{\Phi})\big)\det(\Phi+z\overline{\Psi})^{-(n+1)/2-\sigma}\,\overline{\det(\Phi+z\overline{\Psi})}^{\,-(n+1)/2-\tau}$$

Proposition 4.19 *The sesquilinear form $\langle\cdot,\cdot\rangle_{\sigma,\tau}$ is invariant with respect to the operators $T_{\sigma,\tau}$*

(This is an exercise on Jacobians and formula (2.11.4)).

Observation 4.20 *Representations $T_{\sigma,\tau}$ and $T_{\sigma+1,\tau-1}$ are equivalent.*

The intertwining operator is $Af(z) = \det(z)f(z)$.

Proposition 4.21 *For given f_1, $f_2 \in C^\infty(\mathcal{L}_n)$, the expression $\langle f_1, f_2\rangle_{\sigma,\tau}$ admits a meromorphic continuation to the whole complex plane $(\sigma,\tau)\in\mathbb{C}^2$, possible poles are located on the hyperplanes $\operatorname{Re}\sigma$, $\operatorname{Re}\tau = 1,\ 3/2,\ 2,\ \ldots$*

(The existence of meromorphic continuation is a general phenomenon of the distribution theory; however, the precise position of the poles is not self-evident).

Observation 4.22 *The representations T_α discussed above coincide with the representations $T_{\alpha,0}$ defined now*

More precisely, for α satisfying the conditions (4.7) of positivity, the form $\langle\cdot,\cdot\rangle_{\alpha,0}$ is positive semidefinite (it is degenerate); therefore we get a Hilbert space $H_{\alpha,0}$ and a unitary representation of $\mathrm{Sp}(2n,\mathbb{R})$. The operator

$$Jf(z) = \mathrm{const}\cdot\int_{\mathcal{L}} L_{\sigma,\tau}(z,u)f(u)\,d\mu(u), \qquad z\in \mathrm{B}_n,\, u\in\mathcal{L}_n$$

is a unitary intertwining operator $H_{\alpha,0}\to\mathcal{H}_\alpha$.[9]

Theorem 4.23 *Let $n>1$. a) Let σ, τ be real $\notin \frac{1}{2}\mathbb{Z}$. The form $\langle\cdot,\cdot\rangle_{\sigma,\tau}$ is positive definite iff (σ,τ) is contained in the chain of squares on Fig.7.2.*

b) For $\sigma+\tau=(n+1)/2$ our inner product coincides with the L^2 inner product (i.e., the distribution $\ell_\sigma(z)\ell_{(n+1)/2-\sigma}(z^*)$ is the delta-function supported by $z=1$).

The proof is based on explicit expansion of the distribution $\ell_\sigma(z)\ell_\tau(z^*)$ in $\mathrm{U}(n)$-harmonics on $\mathrm{U}(n)/\mathrm{O}(n)$. However, this is the basic non-trivial point in our considerations.

Remark. Now we can restrict the representations $T_{\sigma,\tau}$ to symmetric subgroups; this leads to the Stein–Sahi kernels on symmetric spaces mentioned in Section 2.12.

4.18. Unipotent representations. The analogs of discrete representations T_α also exist.

Theorem 4.24 *Let $n>1$. Fix σ, $\tau = 0, 1/2, \ldots, n/2$, $\sigma+\tau=\nu/2\leqslant n/2$. Choose t, $s\in\mathbb{R}$ satisfying $t+s\neq 0$. Then*

$$U_{s,t}(\cdot,\cdot) := \lim_{\varepsilon\to 0}\langle\cdot,\cdot\rangle_{\sigma+\varepsilon s,\tau+\varepsilon t}$$

is finite; $U_{s,t}$ is a $T_{\sigma,\tau}$-invariant Hermitian form on $C^\infty(\mathcal{L}_n)$.

b) Denote by $\mathcal{K}$ the common kernel of all the forms $U_{t,s}$. Then the representation $T_{\sigma,\tau}$ in $C^\infty(\mathcal{L})/\mathcal{K}$ is a direct sum of ν unitary representations of $\mathrm{Sp}(2n,\mathbb{R})$. The inner products can be explicitly expressed as linear combinations of $U_{s,t}$.

[9] This is nothing but the construction of Subsection 1.5 ('dual functional model'); however now distributions are supported by the boundary $\mathcal{L}_n$ of the domain B_n and some additional substantiation is necessary.

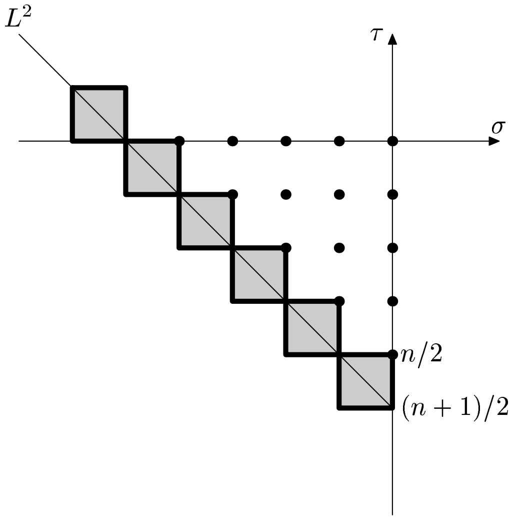

Fig. 7.2: Conditions of positivity of Hermitian forms $\langle\cdot,\cdot\rangle_{\sigma,\tau}$.

7.5 Hardy spaces

5.1. The standard Hardy space.

Theorem 5.1 *The space $\mathcal{H}_{(n+1)/2}$ is a Hardy space, i.e., the inner product in $\mathcal{H}_{(n+1)/2}$ is given by the formula*

$$\langle f,g\rangle_{\mathcal{H}_{(n+1)/2}} = \int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx$$

PROOF. This is a corollary of Theorem 4.17. The measure $d\nu_{(n+1)/2}$ is the Lebesgue measure. Hence we have the usual Laplace transform and apply the usual Plancherel formula. □

Since f is the usual Laplace transform of L^2-function, it follows that $f(x)$ is well defined as an L^2-function on $L^2(\mathbb{R}^n)$.

5.2. Exotic Hardy spaces. Put

$$j = 0,1,\dots,n-1, \qquad \alpha := (n+1+j)/2$$

Represent $z \in \overline{\mathrm{P}}_n$ as $z = x+iy$, where $x \in \mathrm{Symm}(n,\mathbb{R})$ and $y \in \overline{\mathrm{Pos}}_n$. Denote by μ_α the measure on P_n given by

$$d\mu_\alpha(z) = dx\,d\nu_j(y)$$

where ν_j is the measure on the boundary of the cone $\overline{\mathrm{Pos}}_n$ discussed in the previous section.

Our next aim is the following statement.

— *For $j = 0, 1, \ldots, n-1$, the space $\mathcal{H}_{(n+j+1)/2}(\mathrm{P}_n)$ is the spaces of holomorphic functions on P_n with the inner product*

$$\langle f, g\rangle = \int_{\overline{\mathrm{P}}_n} f(z)\overline{g(z)}\, d\mu_\alpha(z) = \int_{\mathrm{Symm}(n,\mathbb{R})} \int_{\mathcal{O}_j} f(x+iy)\overline{g(x+iy)}\, d\nu_j(y)\, dx$$

Let us formulate the statement more carefully. Consider the space of polynomials on the matrix ball B_n and consider the image $\mathsf{W}_\alpha(\mathrm{P}_n)$ of this space under the transform $\mathfrak{C}_\alpha$ given by (4.21).

Lemma 5.2 $\mathsf{W}_\alpha \subset L^2(\mathrm{P}_n; d\mu_\alpha)$

Theorem 5.3 *The closure of* W_α *in* $L^2(\mathrm{P}_n; d\mu_j)$ *consists of function holomorphic in* P_n *and coincides with* $\mathcal{H}_\alpha(\mathrm{P}_n)$. *Moreover, the* $L^2(\mathrm{P}_n, \mu_\alpha)$*-inner product in* $\mathcal{H}_\alpha(\mathrm{P}_n)$ *coincides with the standard inner product.*

Proof occupies the rest of this section.

5.3. Reduction of Lemma 5.2 to Hua integrals. Consider the function $G(z) = \det(i+z)^{-\alpha}$ which is the image of the function $g(z) = 1$ under the map $\mathfrak{C}_\alpha$(see 4.21). Since a polynomial f is bounded on B_n, we have $|\mathfrak{C}f(z)| \leqslant \mathrm{const}(f)\cdot|G(z)|$. Hence, it suffices to show that $G \in L^2(\mathrm{P}_n, \mu_\alpha)$.

$$|G(z)|^2 = |\det(i+z)|^{-2\alpha} = |\det(x+i(1+y)|^{-2\alpha} = \\ = \det(1+y)^{-2\alpha}\det\big(i+(1+y)^{-1/2}x(1+y)^{-1/2}\big)^{-2\alpha}$$

We must integrate this expression with respect to $d\mu_\alpha$. First, we integrate with respect to x with fixed y; we keep in mind Lemma 4.1,

$$\int_{\mathrm{Symm}(n,\mathbb{R})} \big|\det\big(i+(1+y)^{-1/2}x(1+y)^{-1/2}\big)\big|^{-2\alpha} dx = \\ = \det(1+y)^{(n+1)/2}\int \big|\det\big(i+(1+y)^{-1/2}x(1+y)^{-1/2}\big)\big|^{-2\alpha} d(1+y)^{-1/2}x(1+y)^{-1/2} = \\ = \det(1+y)^{(n+1)/2}\int \big|\det(i+\widetilde{x})\big|^{-2\alpha} d\widetilde{x} = \det(1+y)^{(n+1)/2}\int \det(1+\widetilde{x}^2)^{-\alpha} d\widetilde{x} \tag{5.1}$$

We split the integral

$$\int_{\mathrm{Symm}(n,\mathbb{R})} \det(1+x^2)^{-\alpha} d\widetilde{x} \tag{5.2}$$

as a factor (independent on y) and integrate the rest with respect to y. We get

$$\int_{\mathcal{O}_j} \det(1+y)^{-(n+1)/2-j} d\nu_j(y)$$

By Problem 4.2, this integral can be transformed to the form

$$\mathrm{const}\cdot\int_{\mathrm{Mat}(j,n,\mathbb{R})} \det(1+vv^t)^{-(n+1)/2-j}\, dv \tag{5.3}$$

Lemma 5.4 *The integrals (5.2) and (5.3) are convergent.*

These integrals were evaluated by Hua Loo Keng [73], we present an another calculation, however it uses two common-mathematical facts that are formulated (without proofs) in the two following subsections.

5.4. Integration with respect to eigenvalues. Denote by Δ_n the wedge in $\mathbb{R}^n$ determined by the inequalities

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$$

Consider the map $\mathcal{L} : \mathrm{Symm}(n, \mathbb{R}) \to \Delta_n$ that takes a matrix x to collection of its eigenvalues.

Theorem 5.5 *The image of the Lebesgue measure under $\mathcal{L}$ is*

$$\mathrm{const} \cdot \prod_{k<l} (\lambda_k - \lambda_l) \prod_m d\lambda_m$$

Next, denote by Ξ_j the cone in $\mathbb{R}^j$ defined by the inequalities

$$\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_j \geqslant 0$$

Let $j \leqslant n$. Consider the map $\mathcal{M} : \mathrm{Mat}(j, n; \mathbb{R}) \to \Xi$ that takes $v \in \mathrm{Mat}(j, n; \mathbb{R})$ to its collection of singular values.

Theorem 5.6 *The push-forward of the Lebesgue measure under $\mathcal{M}$ is*

$$\mathrm{const} \cdot \prod_{k<l} (\mu_k^2 - \mu_l^2) \prod_k \mu_k^{n-j} \prod_k d\mu_k$$

These theorems are proved in numerous books, for instance, [73].

5.5. The Selberg integral.

Theorem 5.7 *a) Let* $\mathrm{Re}\, c > 0$, $\mathrm{Re}\, a > 0$, $\mathrm{Re}\, b > 0$. *Then*[10]

$$\int_0^\infty \ldots \int_0^\infty \prod_{k=1}^p t_j^{a-1} (1 + t_k)^{-a-b-2c(p-1)-1} \prod_{1 \leqslant k < l \leqslant p} |t_k - t_l|^{2c} \prod_{k=1}^p dt_k =$$

$$= \prod_{k=1}^p \frac{\Gamma(a + (k-1)c)\Gamma(b + (k-1)c)\Gamma(1 + kc)}{\Gamma(a + b + (p + k - 2)c)\Gamma(1 + c)} \qquad (5.4)$$

b) Let $\mathrm{Re}\, c > 0$, $\mathrm{Re}\, a + \mathrm{Re}\, b > (2p - 2)\,\mathrm{Re}\, c + 1$. *Then*

$$\frac{1}{(2\pi)^n} \int_{-\infty}^\infty \ldots \int_{-\infty}^\infty \prod_{k=1}^p (1 + it_k)^{-a} (1 - it_k)^{-b} \prod_{1 \leqslant k < l \leqslant p} |t_k - t_l|^{2c} \prod_{j=1}^p dt_j =$$

$$= 2^{-(a+b)p + c(n-1)p} \prod_{k=1}^p \frac{\Gamma(a + b - (p + k - 2)c - 1)\Gamma(1 + kc)}{\Gamma(a - (k-1)c)\Gamma(b - (k-1)c)\Gamma(1 + c)} \qquad (5.5)$$

[10] these conditions and conditions below are sufficient but not necessary for the convergence

It is a highly nontrivial theorem, for various proofs, see [3], [127].

5.6. Proof of Lemma 5.4. We apply an integration in eigenvalues. In both cases we get the Selberg integral with $c = 1$. For (5.2) we obtain

$$\int_{\mathbb{R}^n} \prod_{k=1}^{n} (1 + i\lambda_k)^{-\alpha} (1 - i\lambda_k)^{-\alpha} \prod_{1 \leqslant k < l \leqslant n} |\lambda_k - \lambda_n| \prod_k d\lambda_k$$

We have $\alpha \geqslant (n + 1)/2$ and the integral converges.

In the second case (5.3), we obtain

$$\int_0^\infty \dots \int_0^\infty \prod_{k=1}^{j} (1 + \mu_j^2)^{-(n+1)/2-j} \prod_{k<l} |\mu_k^2 - \mu_l^2| \prod_{k=1}^{j} \mu_k^{n-j} \prod_{k=1}^{j} d\mu_k =$$

$$= 2^{-n} \int_0^\infty \dots \int_0^\infty \prod t_k^{(n-j-1)/2} (1 + t_k)^{-(n+1)/2-j} \prod_{k<l} |t_k - t_l| \prod_{k=1}^{j} dt_k$$

We have $p = j$, $a = (n + 1 - j)/2$, $b = j/2$ and the integral converges.

REMARK. Another way of evaluation of Hua-type integrals is proposed in Theorem 2.10.9.

5.7. Proof of Theorem 5.3. 1) Consider the closure $\overline{\mathrm{B}}_n$ of the matrix ball and the subsets $\mathfrak{O}_j$ in its boundary consisting of matrices u satisfying the condition $\mathrm{rk}(1 - uu^*) = j$. Equivalently, $u \in \mathfrak{O}_j$ iff its Cayley transform has the form $z = x + iy$ with $y \in \mathcal{O}_j$.

Define the measure $\varkappa_j^\circ$ on $\mathfrak{O}_j$ as

$$\varkappa_j \text{ is } |\det(i + u)|^{2\alpha} \cdot \Big\{ \text{ pushforward of } \mu_\alpha \text{ under the Cayley transform } \Big\}$$

Then the operator

$$\widetilde{\mathfrak{C}}_\alpha f(z) = f\left(\frac{i - z}{i + z}\right) \det(i + z)^{-\alpha}$$

is unitary as an operator

$$\mathfrak{C}_\alpha : L^2(\mathcal{S}_j, \varkappa_j) \to L^2(\mathrm{P}_n, \mu_\alpha)$$

recall, that $\alpha = (n + j + 1)/2$. The operator $\mathfrak{C}_\alpha$ discussed above (4.21) is given by the same formula; however formally it is another operator.

Lemma 5.8 *The measure $\varkappa_j$ is finite.*

PROOF OF THE LEMMA. The image of the function $f = 1$ under $\overline{\mathfrak{C}}_\alpha$ is the function $\det(i + z)^{-\alpha}$. The latter function is contained L^2, therefore the former also is in L^2. □

Lemma 5.9 *The closure of the space of polynomials in $L^2(\mathfrak{O}_j, \varkappa_j)$ consists of functions holomorphic in $\mathcal{B}_n$.*

PROOF OF THE LEMMA. First, we wish to construct a compactly supported smooth function τ on $\mathfrak{O}_j$ such that

$$f(0) = \langle f, \tau \rangle_{L^2(\mathfrak{O}, \varkappa_j)}$$

for all f holomorphic in a neighborhood of $\overline{B}_n$. Consider a point $\zeta \in \mathfrak{O}_j$ and a complex manifold $\mathbb{D}_\zeta$ consisting of matrices of form $\lambda\zeta$, where $\lambda \in C$ satisfy $|\lambda| \leqslant 1$. Denote by $\mathbb{T}_\zeta$ the boundary of $\mathbb{D}_\zeta$. We write the Cauchy integral formula in $\mathbb{D}_\zeta$, i.e.,

$$f(0) = \frac{1}{2\pi} \int_{\mathbb{T}_\zeta} f(\zeta e^{i\varphi})\, d\varphi \tag{5.6}$$

Consider a hypersurface M intersecting transversally a circle $\mathbb{T}_\xi$ at ξ and a small disk Q in M surrounding ξ. The circles $\mathbb{T}_\zeta$ with $\zeta \in Q$ sweep a tube neighborhood of the initial circle $\mathbb{T}_\xi$. Denote by π an arbitrary smooth probabilistic measure on Q. Compound the Cauchy integral representations we get

$$f(0) = \frac{1}{2\pi} \int_{\zeta \in Q} \int_{\mathbb{T}_\zeta} f(\zeta e^{i\varphi})\, d\varphi\, d\pi(\zeta)$$

However, this has the form (5.6).

Since B_n is homogeneous, each evaluation linear functional $f \to f(a)$ can be written as an L^2-inner product. It remains to apply the argument with the uniform boundedness, see proof of Proposition 3.4. □

END OF THE PROOF. Return to P_n. Thus, $\mathsf{W}_\alpha \subset L^2(\mathrm{P}_n, d\mu_\alpha$ and its closure consists of holomorphic functions. It remains to evaluate the reproducing kernel. Consider the subgroup $P \subset \mathrm{Sp}(2n, \mathbb{R})$ consisting of matrices $g = \begin{pmatrix} h^{t-1} & b \\ 0 & h \end{pmatrix}$. This group acts on P_n by affine transformations $z \mapsto h^t b + h^t z h$. It can be easily checked that this action is transitive. The measure μ_α is invariant with respect to shifts along the edge of wedge. Also

$$d\mu_\alpha(h^t z h) = d(h^t x h)\, d\nu(h^t y h^t) = |\det h|^{n+1+j} d\mu_\alpha(z)$$

Thus operators

$$T_\alpha \begin{pmatrix} h^{t-1} & b \\ 0 & h \end{pmatrix} f(z) = f(h^t z h + h^t b) |\det h|^{(n+1+j)/2}$$

are unitary in $L^2(\overline{\mathrm{P}}_n, \mu_\alpha)$. Next we evaluate the reproducing kernel from the invariance by Theorem 4.5 and obtain the required result. □

7.6 Realization of the Weil representation in holomorphic functions on matrix ball.

Taking different overfilled systems in the Fock space $\mathbf{F}_n$, we get different functional models of the Weil representation.

6.1. Subrepresentations of Weil representation.

Proposition 6.1 *The Weil representation has only two subrepresentations, that are realized respectively in the subspace* $\mathbf{F}_n^{even}$ *of even functions* ($f(z) = f(-z)$) *and the subspace* $\mathbf{F}_n^{odd}$ *of odd* ($f(z) = -f(-z)$) *functions in* $\mathbf{F}_n$.

Obviously, each Gaussian operator takes even functions to even functions and odd functions to odd functions. Thus, we must show that there are no other subrepresentations. From the point of view of the general representation theory, this is a completely trivial fact. However the representation theory is not developed in this book.

PROOF. Consider the subgroup $\mathrm{SU}(n) \subset \mathrm{Sp}(2n, \mathbb{R})$. It acts in $\mathbf{F}_n$ by changes of variables

$$\rho(g)f(z) = f(zg)$$

as in 4.1.7. The subspaces S_n^m of homogeneous polynomials are invariant with respect to $\mathrm{SU}(n)$.

Lemma 6.2 *The representation of* $\mathrm{SU}(n)$ *in* S_n^m *is irreducible.*

PROOF OF THE LEMMA. Consider the subgroup $D \subset \mathrm{U}(n)$ consisting of diagonal matrices. Monomials $z^\alpha \in \mathbf{F}_n$ are the eigenvectors. Moreover, the corresponding eigenvalues are pairwise distinct. Let Y be a D-invariant subspace. Then the projection P on Y commutes with D. Then it must be diagonal in the monomial basis z^α, therefore Y is a 'coordinate subspace', i.e., Y is spanned by some set of basic elements. Let $z_1^{\alpha_1} \dots z_n^{\alpha_n} \in Y$. Applying $g \in \mathrm{SU}(n)$ to this monomial, we get

$$\prod_j (a_{j1}z_1 + \dots + a_{jn})^{\alpha_j}$$

Clearly, for a generic g all the coefficients of this polynomial are nonzero. Therefore, Y is the whole space. □

Lemma 6.3 *Any* $\mathrm{SU}(n)$*-invariant subspace* H *in* $\mathbf{F}_n$ *has the form*

$$H = \oplus \mathsf{S}_n^{m_j} \tag{6.1}$$

PROOF OF THE LEMMA. Let P be the orthogonal projector on invariant subspace H. Then it commutes with the operators of representation

$$\rho(g)P = P\rho(g), \qquad g \in \mathrm{SU}(n)$$

Represent P as a block matrix $P = P_{ml}$ according to the decomposition $\mathbf{F}_n = \oplus \mathsf{S}_n^m$. Then each block $P_{ml} : \mathsf{S}_n^m \to \mathsf{S}_n^m$ commutes with $\mathrm{Be}(g)$, i.e., it is an intertwining operator (see Addendum B). By the Schur Lemma, $P_{kl} = 0$ for $k \neq l$ and P_{kk} are scalar operators. Indeed, representations of $\mathrm{SU}(n)$ in different spaces S_n^m are non-equivalent (for instance if $n > 1$ then their dimensions are pairwise distinct).

Since P is a projector, P_{mm} is 0 or 1, therefore H has the desired form. □

Let us return to the proof of the proposition.

Decompose $\mathbf{F}_n$ into a direct sum of irreducible representations, all the summands have the form (6.1). The vector $1 \in \mathbf{F}_n$ must be contained in some summand H (by Lemma 6.3), $\mathrm{Be}(g)$ is a Gaussian vector $\mathrm{const}\, b[T]$. Expansion of $b[T]$ into a Taylor series contains nonzero terms in all the even degrees. By Lemma 6.3, W consists of all the even functions.

Applying the same arguments to the function $f(z) = z_1$, we get that $\mathbf{F}_n^{odd}$ is irreducible.

6.2. Integral transform: even functions. First, we consider all the Gaussian vectors $b[A]$ as an overfilled system in $\mathbf{F}_n^{even}$.

For $f \in \mathbf{F}_n^{even}$, define a function on the matrix ball B_n by the rule

$$\mathcal{R}_+ f(A) = \int_{\mathbb{C}^n} f(z) \exp\{\frac{1}{2}\overline{z}A\overline{z}^t\} \exp(-|z|^2)\, d\lambda(z) = \langle f, b[\overline{A}]\rangle \tag{6.2}$$

Obviously, a $\mathcal{R}_+ f(A)$ is holomorphic function of the variable $A \in \mathrm{B}_n$. Therefore we get a certain space of holomorphic functions on B_n.

Theorem 6.4 *a) The reproducing kernel of this space is*

$$L(a,b) = \det(1 - ab^*)^{-1/2}$$

i.e., our space is the space $\mathcal{H}_{1/2}(\mathrm{B}_n)$ of the Berezin scale.

b) The Gaussian operators[11] $\mathrm{Be}(g) : \mathbf{F}_n \to \mathbf{F}_n$, *where* $\mathrm{Sp}(2n,\mathbb{R})$, *correspond to*

$$T\begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix} f(A) = f\big((\Phi + A\overline{\Psi})^{-1}(\Psi + A\overline{\Phi})\big) \det(1 + A\overline{\Psi}\Phi^{-1})^{-1/2}$$

The operators Be° *correspond to* $T_{1/2}(g)$.

Proof. The statement a) is a rephrasing of formula (5.1.1) for inner products of Gaussian vectors.

The statement b) follows from a formula (5.2.2) for action of Gaussian operators on Gaussian vectors. □

Theorem 6.5 *Functions $F = \mathcal{R}_+ f$ satisfy the system (6.5.7)–(6.5.8) of partial differential equations.*

To prove theorem, we must differentiate the integral (6.2) with respect to a parameter. To avoid a ground of the differentiation, we refer to the following Theorem 6.6.

6.3. Partial differential equations for $f \in \mathsf{H}^\circ(\Omega, L)$.

Theorem 6.6 *Let Ω be a domain in $\mathbb{C}^n$, $L(z,\overline{u})$ be a reproducing kernel, $\mathsf{H}^\circ(\Omega;L)$ be the corresponding space of holomorphic functions. Let $D = D_z$ be a partial differential operator on Ω. Let*

$$D_z L(z,\overline{u}) = 0 \qquad \text{for all } u \in \Omega.$$

Then $D_z f(z) = 0$ for all $f \in \mathsf{H}^\circ(\Omega;L)$.

[11] see Theorem 5.1.5.

PROOF. Each element of $\mathsf{H}^\circ(\Omega; L)$ is a uniform limit of a sequence of elements having a form $\sum a_j L(z, \overline{u}_j)$ (sums are finite). All such sums satisfy to the equation $Df = 0$, hence their uniform limits satisfy to the same equation.□

PROOF OF THEOREM 6.5. We write

$$\int \exp\bigl\{\frac{1}{2}\overline{z}A\overline{z}^t\bigr\}\bigl\{\frac{1}{2}\overline{z}B\overline{z}^t\bigr\}e^{-|z|^2}\,d\lambda(z) = \det(1 - A\overline{B})^{-1/2}$$

In this case, we can differentiate the left-hand side in parameters a_{kl}. For instance if j, k, l, m, are pairwise distinct, then

$$\frac{\partial^2}{\partial a_{jk}\partial a_{lm}}\exp\bigl\{\frac{1}{2}\overline{z}A\overline{z}^t\bigr\} = \overline{z}_j\overline{z}_k\overline{z}_l\overline{z}_m \exp\bigl\{\frac{1}{2}\overline{z}A\overline{z}^t\bigr\} = \frac{\partial^2}{\partial a_{jl}\partial a_{km}}\exp\bigl\{\frac{1}{2}\overline{z}A\overline{z}^t\bigr\}$$

Therefore, the left-hand side satisfies the required partial differential equation. □

Problem 6.1 Show directly that the function $\det(1 - A\overline{B})^{-1/2}$ for a fixed B satisfies our system of partial differential equations (this is not obvious).

6.4. A basis in $\mathcal{H}_{1/2}$. In this subsection, we use notation $\mathcal{M}(A, b; \alpha)$ for matching functions, see Section 6.5

Proposition 6.7 *Let $\alpha = (\alpha_1, \dots, \alpha_n)$, where α_j are non-negative integers and $\sum \alpha_j$ is even. Then the system of functions*

$$e_\alpha(A) = \mathcal{M}(A, 0; \alpha)$$

forms an orthogonal basis in $\mathcal{H}_{1/2}(\mathrm{B}_n)$

PROOF. We push-forward the standard basis $z_1^{\alpha_1} \dots z_n^{\alpha_n} \in \mathbf{F}_n$. □

6.5. Integral transforms. Odd functions. Next, consider the space $\mathbf{F}_n^{even}$ consisting of odd functions and the overfilled system $(\sum \overline{h}_j z_j)\exp\{\frac{1}{2}z\overline{A}z^t\}$. Consider the integral transform

$$\mathcal{R}_- f(A, h) = \int_{\mathbb{C}^n} f(z)\,(\sum h_j\overline{z}_j)\exp\bigl\{\frac{1}{2}\overline{z}A\overline{z}^t\bigr\}e^{-|z|^2}\,dz\,d\overline{z}$$

We obtain a Hilbert space of holomorphic functions on $\mathrm{B}_n \times \mathbb{C}^n$, that are linear with respect to the variable h and satisfy the partial differential equations

$$\Bigl(\frac{\partial^2}{\partial h_k \partial a_{lm}} - \frac{\partial^2}{\partial h_l \partial a_{km}}\Bigr) f(A, h) = 0$$
$$\Bigl(\frac{\partial^2}{\partial h_k \partial a_{jj}} - \frac{1}{2}\frac{\partial^2}{\partial h_j \partial a_{jk}}\Bigr) f(A, h) = 0$$

and the equations (6.5.7)–(6.5.8).

Since our functions are linear with respect to the variable h, it is reasonable to regard them as $\mathbb{C}^n$-valued function on B_n.

The reader (if necessary) can easily write the reproducing kernel and the action of $\mathrm{Sp}(2n,\mathbb{R})$.

6.6. A realization on an extended matrix ball. Now we consider the whole Fock space $\mathbf{F}_n$ and the overfilled system

$$\exp\{\frac{1}{2}\overline{z}\overline{A}z^t + \overline{b}\overline{z}^t\}$$

Then we obtain a Hilbert space of functions on $\mathrm{B}_n \times \mathbb{C}^n$ satisfying the partial differential equations (6.5.5)–(6.5.8).

The reproducing kernel and the action of $\mathrm{Sp}(2n,\mathbb{R})$ in this model can be easily written starting formula (5.1.3)

7.7 Determinantal systems of differential equations

7.1. Determinantal systems. Let z_{kl} be complex variables, $z_{kl} = z_{lk}$. Compose the $n \times n$-matrix of the corresponding partial derivatives

$$\begin{pmatrix} 2\partial/\partial z_{11} & \partial/\partial z_{12} & \partial/\partial z_{13} & \dots \\ \partial/\partial z_{21} & 2\partial/\partial z_{22} & \partial/\partial z_{23} & \dots \\ \partial/\partial z_{31} & \partial/\partial z_{32} & 2\partial/\partial z_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{7.1}$$

Let f be a holomorphic function on $\mathrm{Symm}(n,\mathbb{C})$. We say, that $f(z)$ satisfies the determinantal system $\mathfrak{D}_k$ if $f(z)$ is annulated by all the $(k+1)\times(k+1)$-minors of this matrix.

Lemma 7.1 *If f satisfies the system $\mathfrak{D}_k$ and g satisfies the system $\mathfrak{D}_l$, then fg satisfies the system $\mathfrak{D}_{k+l}$*

PROOF. Let I, J,... denote subsets in $\{1,2,\dots,n\}$, let $|I|$, ... denote their cardinality. By $\Delta(I,J)$ we denote the minor of (7.1) corresponding to rows with numbers $\in I$ and columns with numbers $\in J$. Let $|I| = |J| = k+l+1$. Then

$$\Delta(I,J)f(z)g(z) = \sum_{I'\subset I, J'\subset J, |I'|=|J'|} \Delta(I',J')f(z)\cdot\Delta(I\setminus I', J\setminus J')g(z)$$

If $|I'| \geqslant k+1$, then $\Delta(I',J')f(z) = 0$. Otherwise, $\Delta(I\setminus I', J\setminus J')g(z) = 0$. $\square$

Observation 7.2 *If $f(z)$ satisfies $\mathfrak{D}_k$, and $c = c^t$, then $f(z+c)$ and $f(aza^t)$ satisfy the same system*

7.2. Hilbert spaces $\mathcal{H}_{k/2}$.

Theorem 7.3 *Let $k \leqslant n-1$.*

a) Each $f \in \mathcal{H}_{k/2}(\mathrm{B}_n)$ satisfies the determinantal system $\mathfrak{D}_k$.

b) Each $f \in \mathcal{H}_{k/2}(\mathrm{P}_n)$ satisfies the determinantal system $\mathfrak{D}_k$.

PROOF. a) As we seen above in 6.2, a function $\det(1-bz)^{-1/2}$ satisfies the system $\mathfrak{D}_1$. By Lemma 7.1, $\det(1-bz)^{-k/2}$ satisfies $\mathfrak{D}_k$. It remains to apply Theorem 6.6.

b) Again, $\det(1-z)^{-1/2}$ satisfies $\mathfrak{D}_1$. Hence $\det(c-z)^{-1/2}$ for $c=c^t$ satisfies the same system. It remains to repeat the proof of a). □

ANOTHER PROOF. By Theorem 4.17, each element $f \in \mathcal{H}_{k/2}(\mathrm{P}_n)$ is a Fourier–Laplace transform of a distribution $\varkappa$ having the form

$$\ll h, \varkappa \gg = \int_{x \in \mathcal{O}_k} h(x) r(x)\, d\nu_{k/2}(x)$$

where $r(x)$ is a L^2-function on $\mathcal{O}_k$. A point $x \in \mathcal{O}_k$ satisfies the equations

$$M[I,J;x] := \det \begin{pmatrix} x_{i_1 j_1} & \dots & x_{i_k j_1} \\ \vdots & \ddots & \vdots \\ x_{i_1 j_k} & \dots & x_{i_k j_k} \end{pmatrix} = 0$$

Hence

$$M[I,J;x]\varkappa(x) = 0$$

After the Fourier–Laplace transform, we get the determinantal system for f.□

7.3. Concluding remarks. We formulate without proofs several statements about determinantal systems. Proofs require some references to the representation theory or some redundant efforts for avoiding such references.

A. For $\alpha > (n-1)/2$ the space $\mathcal{H}_\alpha(\mathrm{B}_n)$ contains all polynomials. If $k = 0, 1, \dots,$ $n-1$, then the space $\mathcal{H}_{k/2}(\mathrm{B}_n)$ contains all the polynomials satisfying the system $\mathfrak{D}_k$.

B. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{C})$. Let f be a function holomorphic in an open set in $\mathrm{Symm}(n, \mathbb{C})$ and satisfying the system $\mathfrak{D}_k$. Then

$$f\big((a+zc)^{-1}(b+zd)\big) \det(a+zc)^{-k/2}$$

also is a solution of $\mathfrak{D}_k$

C. Denote by $\Delta_j(z)$ the determinant of the left upper $j \times j$ corner of z. For non-negative integers $l_1 \geqslant l_2 \geqslant \dots \geqslant l_n$ consider the function[12]

$$\Xi(\mathbf{l}; z) = \Delta_1^{l_1 - l_2} \Delta_2^{l_2 - l_3} \dots \Delta_{n-1}^{l_{n-1} - l_n} \Delta_n^{l_n}$$

Denote by $H(\mathbf{l})$ the $\mathrm{U}(n)$-cyclic span of $\Xi(\mathbf{l})$. Then the space of polynomials on $\mathrm{Symm}(n)$ is $\oplus_{\mathbf{l}} H(\mathbf{l})$ and

$$\mathcal{H}_{k/2}(\mathrm{B}_n) = \bigoplus_{\mathbf{l}:\, l_{k+1} = \dots = l_n = 0} H(\mathbf{l})$$

[12] it is a $\mathrm{U}(n)$-highest vector

7.8 The symplectic category and the spaces $\mathcal{H}_\alpha$

In what follows, we extend the action of symplectic groups in the spaces $\mathcal{H}_\alpha$ to an action of the symplectic category $\mathbf{Sp}$.

8.1. Formulation of the statement. Here we prefer to explore the matrix ball model. Recall the construction of Subsection 3.7.3. Let $P : \mathcal{W}_{2n} \rightrightarrows \mathcal{W}_{2m}$ be a morphism of the symplectic category $\mathbf{Sp}$. Let $\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ be its Potapov transform.

Theorem 8.1 *Let α range in the same set as above (see Theorem 4.7).*

a) The operator $\mathfrak{T}_\alpha(P)$ given by the formula

$$\mathfrak{T}_\alpha(P)f(z) = f\big(K + Lz(1-Mz)^{-1}L^t\big)\det(1-Mz)^{-\alpha}$$

is a bounded operator

$$\mathfrak{T}_\alpha(P) : \mathcal{H}_\alpha(\mathrm{B}_m) \to \mathcal{H}_\alpha(B_n)$$

b) For $P : \mathcal{W}_{2n} \rightrightarrows \mathcal{W}_{2m}$, $P' : \mathcal{W}_{2k} \rightrightarrows \mathcal{W}_{2n}$

$$\mathfrak{T}_\alpha(P)\mathfrak{T}_\alpha(P') = \det(1-MK')^{-\alpha}\mathfrak{T}_\alpha(PP') \tag{8.1}$$

Remark. If $g \in \mathrm{Sp}(2n, \mathbb{R})$, then the corresponding operator $\mathfrak{T}_\alpha(g)$ differs from the operator $T_\alpha(g)$ by a constant factor. □

The last statement is a special case of Theorem 3.7.2, so we must only to proove the boundedness. This fact extends Theorem on boundedness of Gaussian operators. Fortunately, the proof of the latter theorem given in the Chapter 5 survives in our case.

8.2. Action on overfilled basis. Consider the following functions on B_n

$$\Phi_\alpha[T;z] := \det(1-Tz)^{-\alpha} \in \mathcal{H}_\alpha, \qquad z \in \mathrm{B}_n$$

Lemma 8.2

$$\mathfrak{T}_\alpha(P)\Phi_\alpha[T;z] = \det(1-KT)^{-\alpha}\Phi_\alpha[M + L^tT(1-KT)^{-1}L;\, z]$$

Proof. Our statement is equivalent to the identity

$$\det\Big(1 - T[K + L(1-zM)^{-1}zL^t]\Big)^{-\alpha}\det(1-Mz)^{-\alpha} = \det(1-KT)^{-\alpha}\det\Big(1 - z[M + L^tT(1-KT)^{-1}L]\Big)^{-\alpha}$$

This can be verified in a staightforward way; however, this is the cocycle identity (3.7.4)=(3.7.5). □

Corollary 8.3 *Let $k = 0, 1/2, \ldots, (n-1)/2$. Let $f \in \mathcal{H}_{k/2}$. Then $\mathfrak{T}_\alpha(P)f$ satisfies the determinant system $\mathfrak{D}_k$.*

PROOF. Denote by $\mathrm{Hol}(\mathrm{B}_n)$ the space of functions holomorphic on B_n endowed with the topology of uniform convergence on compact sets. Evidently, the operators $\mathfrak{T}_\alpha(P)$ are continuous as operators $\mathrm{Hol}(\mathrm{B}_n) \to \mathrm{Hol}(\mathrm{B}_m)$. Any $f \in \mathcal{H}_{k/2}$ is a uniform limit of finite linear combinations of functions $\Phi_{k/2}[T;z]$. Hence $\mathfrak{T}_{k/2}f$ is also a uniform limit of finite linear combinations of functions $\Phi_{k/2}[S;z]$. Such functions satisfy the system $\mathfrak{D}_k$, therefore their uniform limits satisfy the same system. □

8.3. Norms and maximazers.

Proposition 8.4 *Let $P : \mathcal{W}_{2n} \rightrightarrows \mathcal{W}_{2n}$ be a self-adjoint strictly contractive morphism.*

a) The operator $\mathfrak{T}_\alpha(P)$ has a (unique) eigenvector of the form $\Phi_\alpha[T;z]$

b) The eigenvector $\Phi_\alpha[T;z]$ is a maximizer and $\|\mathfrak{T}_\alpha(P)\|$ coincides with the corresponding eigenvalue.

c) $\|\mathfrak{T}_\alpha(P)\| \leqslant \det(1-|M|)^{-\alpha}$

d) The last estimate holds for an arbitrary self-adjoint morphisms of **Sp**.

PROOF imitates considerations of Section 5.3. We look to Lemma 8.2, apply the fixed point principle and come to a) (compare with Theorem 5.3.1). Conjugating $\mathfrak{T}_\alpha(P)$ by an element of symplectic group, we can assume that the fixed point is 0. Hence the Potapov transform of P is of the form $\begin{pmatrix} 0 & L \\ L^t & 0 \end{pmatrix}$ (see Theorem 5.3.2). Thus, without loss of generality, we can assume that

$$\mathfrak{T}_\alpha(P)f(z) = f(LzL^t), \qquad \text{where } L = L^*,\ \|L\| < 1 \tag{8.2}$$

Lemma 8.5 *For the last operator $\|\mathfrak{T}_\alpha(P)\| = 1$, the norm is achieved on the vector $f(z) = 1$.*

PROOF OF LEMMA. Without loss of generality, we can assume that the matrix L is diagonal. Consider all the operators

$$A_\Lambda f(z) = f(\Lambda z \Lambda), \qquad \text{where} \qquad \Lambda = \begin{pmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{8.3}$$

If $|\lambda_j| = 1$, then A_Λ is unitary. Consider a common eigenbasis for such operators. Evidently it consists of polynomials. Evidently, all the eigenvalues have form $\prod \lambda_j^{a_j}$. The maximal eigenvalue is 1 and the corresponding eigenvector is $f(z) = 1$. □

END OF PROOF OF PROPOSITION. Repeating arguments of Theorem 5.3.2, we obtain the statement b). Proof of Theorem 5.3.4 survives literally, and we obtain c) and d). □

Theorem 8.1 is trivial modulo the last Proposition. □

7.9 Bibliographical remarks to Chapter 7

9.1. Positive definite kernels. For references on positive definite kernels, see I.Schoenberg [158]. Hilbert spaces of holomorphic functions were introduced by S.Bergman. Kernels of operators were introduced by F.A.Berezin.

9.2. Berezin scale. Highest weight representations of $\mathrm{SL}(2,\mathbb{R})$ were constructed by V.Bargmann in 1948 and by Pukansky, (1960, for the universal cover). Representations of semisimple groups in spaces of holomorphic functions were introduced by Harish-Chandra.

Theorem 4.7 was firstly obtained by F.A.Berezin [14], and also S.G.Gindikin [54], M.Vergne–H.Rossi [173], and N.Wallach [177], Gelbart. The set (4.7) usually is called *Wallach set*. The exotic Hardy spaces were introduced by J.Arazy and H.Upmeier [6]. Similar results exist for matrix balls $\mathrm{B}_{p,q}$ discussed above in Chapter 2, for the matrix balls of complex skew-symmetric matrices, and also for tube related to the cone

$$x_0^2 > x_1^2 + \cdots + x_n^2, \qquad x_0 > 0$$

Our approach is based on [173]. Berezin's approach was based on expansions of reproducing kernels in characters, see exposition of this way [39].

For unitary representations in spaces of vector-valued holomorphic functions, see [37].

9.3. Stein–Sahi representations. This a general construction which was discovered by E.Stein **??** for $\mathrm{SL}(2n,\mathbb{C})$, extended to $\mathrm{SL}(2n,\mathbb{R})$, $\mathrm{SL}(2n,\mathbb{H})$ by D.Vogan, and to other series by Sahi, see **??**, **??**. The model discussed above is proposed in [129].

9.4. Whishart–Siegel integral. It is a special case of Gindikin matrix Gamma-function, see [53], [39]. Also some variants of Gamma-function (with integration over non-convex cones) were considered by E.Stein–F.Ricci [148] and M.Sato–T.Shintani [156].

9.5. Determinantal systems. They appeared independently in numerous works by different reasons. I even afraid to give references.