Chapter 8

Cartier model

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8.1 Zak transform

1.1. The Zak transform. For a function f on $\mathbb{R}^n$ we define the function $\mathcal{Z}f$ on $\mathbb{R}^{2n}$ (*Zak transform*) by the formula

$$\mathcal{Z}f(x,\xi) = \sum_{k\in\mathbb{Z}^n} f(x+k)e^{2\pi i k\xi} \tag{1.1}$$

Here x, $\xi \in \mathbb{R}^n$, and $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$.

Proposition 1.1 *A function $g = \mathcal{Z}f$ satisfies the following quasiperiodicity conditions*

$$g(x, \xi + m) = g(x, \xi), \tag{1.2}$$

$$g(x + l, \xi) = g(x, \xi)e^{-2\pi i l\xi} \tag{1.3}$$

where m, $l \in \mathbb{Z}^n$.

Proof. To verify the second condition, we write

$$\mathcal{Z}(x + l, \xi) = \sum_{k\in\mathbb{Z}^n} f(x + k + l)e^{2\pi i k\xi}$$

and change the summation index $k' = k + l$. $\square$

Thus a function $\mathcal{Z}f(x,\xi)$ is completely determined by its values on the $2n$-dimensional cube $[0,1]^n \times [0,1]^n$,

$$x_j \in [0,1], \qquad \xi_j \in [0,1]$$

Now we have 3 possibilities to treat function $g = \mathcal{Z}f$:

1. g is a quasiperiodic function on $\mathbb{R}^{2n}$;
2. g is a function on the cube $[0,1]^{2n}$;
3. g is a section of a line bundle on the torus $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$,

We prefer the first two languages.

1.2. Inversion. Consider the space $L^2([0,1]^{2n})$ endowed with the inner product

$$\langle g_1, g_2 \rangle = \int_{[0,1]^{2n}} g_1(x,\xi)\overline{g_2(x,\xi)}dx\, d\xi \tag{1.4}$$

REMARK. By the quasiperiodicity, we can write the inner product as

$$\langle g_1, g_2 \rangle = \int_{\xi\in\alpha+[0,1]^n} \int_{x\in a+[0,1]^n} g_1(x,\xi)\overline{g_2(x,\xi)}dx\, d\xi$$

where $a \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^n$ are arbitrary vectors. The result does not depend on a choice of a and α. □

Theorem 1.2 *a) The Zak transform is a unitary operator*

$$\mathcal{Z} : L^2(\mathbb{R}^n) \to L^2([0,1]^{2n})$$

b) The inversion formula is

$$f(x) = \int_{[0,1]^n} g(x,\xi)\, d\xi \tag{1.5}$$

(here we regard g as a function on the whole $\mathbb{R}^{2n}$).

PROOF. Let $n = 1$. Let us regard $\mathbb{R}$ as a union of segments $[p, p+1]$, or equivalently as

$$\mathbb{R} \simeq [0,1] \times \mathbb{Z},$$

namely, to a point $(x,p) \in [0,1]\times\mathbb{Z}$ we assign the point $x + p \in \mathbb{R}$.

Write out a function $f(x)$ as a function $F(x,p)$ in 'new coordinates'. On this language, the Zak transform is the map $L^2\big([0,1]\times\mathbb{Z}\big) \to L^2\big([0,1]^2\big)$ given by

$$F(x,p) \mapsto g(x,\xi) = \sum F(x,p)e^{2\pi i p\xi}$$

i.e., it is a summation of Fourier series in p. Thus, for each $x \in [0,1]$, we have a unitary map $\ell_2(\mathbb{Z}) \to L^2[0,1]$. Uniting all the x, we come to a).

The inverse map $L^2[0,1] \to \ell_2(\mathbb{Z})$ is the evalution of the Fourier coeffitients

$$F(x,p) = \int_0^{2\pi} g(x,\xi) e^{-2\pi i p \xi}\, d\xi \tag{1.6}$$

Assuming $p = 0$, we obtain the inversion formula for $x \in [0,1]$.

For an arbitrary $p \in \mathbb{Z}$, we transform (1.6) by the quasiperiodicity and get

$$F(x,p) = \int_0^{2\pi} g(x+p,\xi)\, d\xi$$

This is the desired inversion formula. □

1.3. The image of Schwartz space. Denote by $\mathcal{Q}(\mathbb{R}^{2n})$ the space of all the smooth functions satisfying the quasiperiodicity conditions (1.2), (1.3).

Theorem 1.3 *The Zak transform is the one-to-one map of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{Q}(\mathbb{R}^{2n})$.*

Remark. We emphasis that the Zak transform indentify a space of functions in n variables and a space of functions in $2n$ variables (i.e., it changes the number of the variables).

Proof. Let $f \in \mathcal{S}(\mathbb{R})$. Let $g = \mathcal{Z} f$. Then

$$\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial \xi^\beta} g(x,\xi) = \sum_k \frac{\partial^\alpha f(x+k)}{\partial x^\alpha} (2\pi i k)^\beta e^{i\xi k}$$

For $f \in \mathcal{S}(\mathbb{R})$ all such series uniformly converge, hence $g \in C^\infty(\mathbb{R}^{2n})$.

Conversely, let $g \in \mathcal{Q}(\mathbb{R}^2)$. Evaluate formally f by the inversion formula (1.5). Differentiating it with respect to the parameter, we get

$$\frac{\partial}{\partial x^\beta} f(x) = \int_0^1 \frac{\partial}{\partial x^\beta} g(x,\xi)\, d\xi$$

We must show that $\varphi(x)$ rapidly decreases in x.

$$\frac{\partial}{\partial x^\beta} f(x+N) = \int_0^1 \frac{\partial}{\partial x^\beta} g(x+N,\xi)\, d\xi = \int_0^1 \frac{\partial}{\partial x^\beta} g(x,\xi) e^{-2\pi N x \xi}\, d\xi$$

Since $\frac{\partial}{\partial x^\beta} f(x,\xi)$ is periodic in ξ, its Fourier coefficients rapidly decrease. □

Therefore we also have a bijection between the dual spaces $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{Q}'(\mathbb{R}^{2n})$.

In particular, the image of the function $\delta(x-a) \in \mathcal{S}'(\mathbb{R}^n)$ is the linear functional $\chi(g)$ on $\mathcal{Q}(\mathbb{R}^2)$ given by

$$\chi(g) = \int_0^{2\pi} f(a,\xi)\, d\xi$$

The delta function $\delta(x-a, \xi-\alpha) \in \mathcal{Q}'(\mathbb{R}^2)$ corresponds to

$$\sum \delta(x-a+k) e^{2\pi i \alpha k}$$

Of course, our words is nothing but the formulae (1.1), (1.5).

1.4. Images of differential operators.

Theorem 1.4 *The Zak transform takes the operator $\partial/\partial x_j$ to the operator $\partial/\partial x_j$ and the operator $f \mapsto x_j f$ to the operator*

$$g(x,\xi) \mapsto \Bigl(x_j + \frac{1}{2\pi i}\frac{\partial}{\partial \xi_j}\Bigr) g(x,\xi)$$

PROOF. The first statement is obvious. The $\mathcal{Z}$-image of $xf(x)$ is

$$\sum_k (x+k) f(x+k) e^{2\pi i k\xi} = x \sum_k f(x+k) e^{2\pi i k \xi} + \frac{1}{2\pi i}\frac{\partial}{\partial \xi} \sum_k f(x+k) e^{2\pi i k\xi}$$

EXAMPLE. The pushforward of the heat equation

$$\Bigl(\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\Bigr) f(x,t) = 0$$

under the Zak transform is

$$\frac{\partial}{\partial t} F(x,\xi,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2} F(x,\xi,t)$$

At a first glance, ξ is absent in this equation. More precisely, the equation fibers to equations defined on lines $\xi = \text{const}$. Formal differential operator are independent on ξ. However, their boundary conditions are nothing but the quasiperiodicity (1.3), and these conditions are different on different lines.

1.5. Heisenberg group. The Zak transform provides the following correspondence of operators

$$T_a f(x) = f(x+a) \qquad \longleftrightarrow \qquad \widetilde{T}_a g(x,\xi) = g(x+a,\xi); \tag{1.7}$$

$$S_b f(x) = f(x) e^{2\pi i b x} \qquad \longleftrightarrow \qquad \widetilde{S}_b g(x,\xi) = g(x,\xi+b) e^{2\pi i b x} \tag{1.8}$$

1.6. Digression. Convolutions.

Problem 1.1 Let f_1, $f_2 \in \mathcal{S}(\mathbb{R}^n)$, let $g_1 = \mathcal{Z} f_1$, $g_2 = \mathcal{Z} f_2$.

a) The $\mathcal{Z}$-image of the convolution

$$r(x) = \int_{\mathbb{R}^n} f_1(y)\, f_2(x-y)\, dy$$

is

$$R(x,\xi) = \int_{[0,1]^n} g_1(y,\xi)\, g_2(x-y,\xi)\, dy$$

b) The $\mathcal{Z}$-image of the function

$$t(x) := \sum_{p \in \mathbb{Z}^n} f_1(x+p) f_2(x-p)$$

is $g_1(x,\xi) g_2(x,\xi)$.

c) The $\mathcal{Z}$-image of $f_1(x) f_2(x)$ is

$$H(x,\xi) = \int_{[0,1]^n} g_1(x,\eta)\, g_2(x,\xi-\eta)\, d\eta$$

1.7. Digression. Zeros of the Zak transform. First, let $n = 1$.

Proposition 1.5 *A function $f \in \mathcal{Q}(\mathbb{R}^2)$ has a zero.*

A function $g \in \mathcal{Q}(\mathbb{R}^2)$ is a map $\mathbb{R}^2 \to \mathbb{C}$. Denote by $J_g(x,\xi)$ its Jacobian. We say that g is in a general position if $J_f \neq 0$ at all the zeros of g.

If (a,α) is a zero of g, then all the points $(a+n, \alpha+m)$ are zeros. Hence it is natural to consider a zero of g as a point of the torus $\mathbb{R}^2/\mathbb{Z}^2$.

Proposition 1.6 *For g in a general position, denote by $(a_j, \alpha_j) \in \mathbb{R}^2/\mathbb{Z}^2$ all its zeros. Then*

$$\sum_j \operatorname{sgn} J_g(a_j, \alpha_j) = +1 \tag{1.9}$$

PROOF OF PROPOSITIONS 1.5, 1.6. Both statements are corollaries of the following elementary topological fact.

— *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain, let $\sigma(t)$ be a counter-clockwise parametrization of the boundary of Ω. Let $g : \Omega \to \mathbb{R}^2$ be a smooth map in a general position. Then*

$$\sum_{u \in \mathbb{R}^2:\ g(u)=0} J_g(u)$$

equals the winding number $\mathbf{w}\big[g(\sigma(t))\big]$, i.e., the number of (counter-clockwise) turns of the curve $g(\sigma(t))$ about 0.

First, assume g does nt vanish on the boundary $ABCDA$ of the square $[0,1]^2 \subset \mathbb{R}^2$, see Fig.8.1a). Consider the curve $g(ABCDA) \subset \mathbb{C}$, see Fig.8.1b). By quasiperiodicity conditions (1.2)–(1.3),

$$g(A) = g(B) = g(C) = g(D)$$

our curve consists of 4 closed loops; moreover, the its winding number is

$$\mathbf{w}\big[g(ABCDA)\big] = \mathbf{w}\big[g(AB)\big] + \mathbf{w}\big[g(BC)\big] + \mathbf{w}\big[g(CD)\big] + \mathbf{w}\big[g(DA)\big]$$

By (1.2), the curves $g(AB)$ and $g(CD)$ coinside, but they are passed in opposite directions. Hence $\mathbf{w}\big[g(AB)\big] + \mathbf{w}\big[g(CD)\big] = 0$. Next,

$$g(1,\xi) = g(0,\xi) e^{2\pi i \xi}$$

Hence $\mathbf{w}\big[g(BC)\big] = 1 + \mathbf{w}\big[g(AD)\big] = 1 - \mathbf{w}\big[g(DA)\big]$. Thus, $\mathbf{w}\big[g(ABCDA)\big] = 1$.

If $g \in \mathcal{Q}(\mathbb{R}^2)$ has a zero on the contour $ABCDA$, then we replace it by an appropriate function $g(x+a, \xi+\alpha) e^{2\pi i \alpha x} \in \mathcal{Q}(\mathbb{R}^2)$ nonvanishing on the contour $ABCD$. $\square$

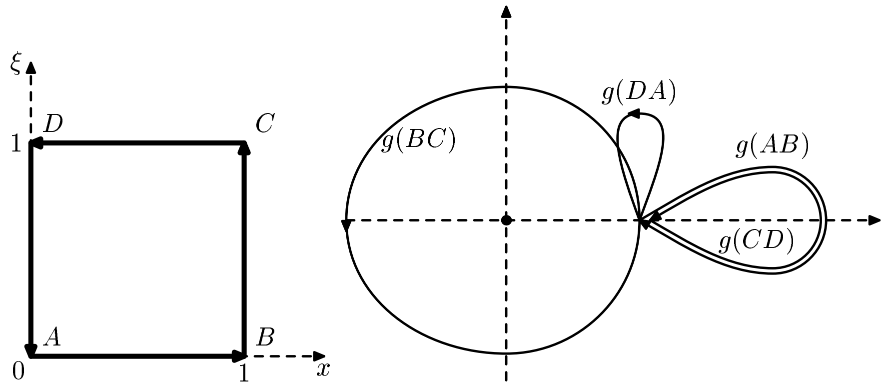

Fig. 8.1: Zeros of a function g.

Now let $n \geqslant 2$, let $f \in \mathcal{Q}(\mathbb{R}^{2n})$. Assume that the set S_f defined by $f(x,\xi) = 0$ is a surface of codimension 2 having singularities at a surface of codimension $\geqslant 3$. The surface S_f has a canonical orientation at nonsingular points (because $\mathbb{R}^{2n}$ and the target space $\mathbb{C}$ are oriented).

Proposition 1.7 *a) The $\mathbb{Z}$-homological class of S_f in the torus $\mathbb{T}^{2n}$ does not depend on a choice of f*

b) It coincides with the sum $\sum_{j=1}^{n} \varkappa_j$, where $\varkappa_j$ is a $(2n-2)$-dimensional subtorus defined by the equations $x_j = \text{const}$, $\xi_j = \text{const}$.

We leave this statement as an exercise in topology.

8.2 The push-forward of the Fourier transform

2.1. The Poisson summation formula. Denote by Δ the distribution on $\mathbb{R}^n$ given by

$$\Delta(x) = \sum_{(k_1,\dots,k_n)\in\mathbb{Z}^n} \prod_j \delta(x_j - k_j)$$

Normalize the usual Fourier transform[1] by

$$\widehat{f}(x) = \mathcal{F}f(x) = \int_{\mathbb{R}^n} f(y) e^{2\pi i x y^t}\, dy$$

Theorem 2.1 $\mathcal{F}\Delta = \Delta$. *In other words, for each $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$\sum_{k\in\mathbb{Z}^n} f(k) = \sum_{l\in\mathbb{Z}^n} \widehat{f}(l) \qquad \textit{(The Poisson summation formula)} \qquad (2.1)$$

[1] This normalization differs from (1.1.8) used above

It suffices to prove the statement for $n = 1$.

FIRST PROOF. Consider the operators T, S in $\mathcal{S}'(\mathbb{R})$ given by

$$Tf(x) = f(x+1), \qquad Sf(x) = e^{2\pi i x} f(x)$$

Obviously,

$$\mathcal{F}T = S^{-1}\mathcal{F}, \qquad \mathcal{F}S = T\mathcal{F} \tag{2.2}$$

Lemma 2.2 *The Δ is a unique to within scalar distribution $\chi \in \mathcal{S}'(\mathbb{R}^n)$ invariant with respect to the transformations S, T.*

PROOF OF LEMMA 2.2. The S-invariance is equivalent to

$$(e^{2\pi i x} - 1)\chi(x) = 0,$$

Hence the distribution χ is supported by zeros of $(e^{2\pi i x} - 1)$ i.e., by $\mathbb{Z}$. The derivative $(e^{2\pi i x} - 1)' = 2\pi i e^{2\pi i x}$ does not vanish at integer points, therefore χ is a sum of δ-functions supported by $x \in \mathbb{Z}$.

By T-invariance, χ is periodic, $\chi(x+1) = \chi(x)$. Hence $\chi = c\Delta$. □

By (2.2), $\mathcal{F}\Delta$ is invariant with respect to S, T. Therefore $\mathcal{F}\Delta = c \cdot \Delta$. Hence

$$\mathcal{F}^2\Delta = c \cdot \mathcal{F}\Delta = c^2\Delta$$

On the other hand, $\mathcal{F}^2 g(x) = g(-x)$, therefore $(\mathcal{F}^2)\Delta(x) = \Delta(-x) = \Delta(x)$. Hence $c^2 = 1$.

For $\psi(x) := \exp(-\pi x^2)$ we have $\mathcal{F}\psi = \psi$. Hence

$$\ll \mathcal{F}\Delta, \psi \gg = \ll \Delta, \mathcal{F}\psi \gg = \ll \Delta, \psi \gg$$

Thus, $c = +1$. □

THE SECOND PROOF. We verify (2.1). For $f \in \mathcal{S}(\mathbb{R})$, consider the periodic function $\varphi(x)$ given by

$$\varphi(x) = \sum_{m=-\infty}^{\infty} f(x+m)$$

Expand it into a Fourier series:

$$\varphi(x) = \sum_{k=-\infty}^{\infty} c_k e^{-2\pi i k x}$$

where

$$\begin{aligned} c_k = \int_0^1 \varphi(x) e^{2\pi k i x}\, dx = \sum_m \int_0^1 f(x+m) e^{2\pi k i x}\, dx = \\ = \sum_m \int_{-m}^{-m+1} f(x) e^{2\pi k i x}\, dx = \int_{-\infty}^{\infty} f(x) e^{2\pi k i x}\, dx = \widehat{f}(k) \end{aligned}$$

Hence

$$\sum_k \widehat{f}(k) = \sum c_k = \varphi(0) = \sum_m f(m) \qquad \square$$

Corollary 2.3 *Denote*

$$\Delta[a,\xi|x] := \sum_{k=-\infty}^{\infty} \delta(x-a-k)e^{2\pi k i\xi}$$

Then

$$\mathcal{F}\Delta[a,\xi] = e^{-2\pi i a\xi}\Delta[-\xi,a]$$

PROOF. We use the standard properties of the Fourier transform (1.1.9),

$$\Delta[0,\xi;x] = \sum \delta(x-k)e^{2\pi i k\xi} = e^{2\pi i x\xi}\sum \delta(x-k) = e^{2\pi i x\xi}\Delta[0,0;x]$$

Therefore the Fourier transform of $\Delta[0,\xi;x]$ is $\Delta[-\xi,0;x]$. Then we shift $x \to x-a$ and come to the required formula. □

2.2. The pushforward of the Fourier transform.

Theorem 2.4

$$\mathcal{Z}\mathcal{F}\mathcal{Z}^{-1}g(x,\xi) = e^{-2\pi i x\xi}g(-\xi,x) \tag{2.3}$$

PROOF. This is a rephrasing of the previous corollary because $\mathcal{Z}f(a) = \ll \Delta[\xi,a;x], f \gg$. □

ANOTHER PROOF. Let $f \in \mathcal{S}(\mathbb{R})$, $g := \mathcal{Z}f$,

$$\mathcal{F}f(\eta) = \int_{-\infty}^{\infty} f(y)e^{2\pi i y\eta}dy = \int_0^1 \Big[\sum_{k=-\infty}^{\infty} f(y+k)e^{2\pi i\eta(y+k)}\Big]\,dy =$$
$$= \int_0^1 e^{2\pi i y\eta}\Big[\sum_{k=-\infty}^{\infty} f(y+k)e^{2\pi i k\eta}\Big]\,dy = \int_0^1 e^{2\pi i y\eta}g(y,\eta)\,dy =$$
$$= \int_0^1 e^{-2\pi i y\eta}g(-y,\eta)\,dy \tag{2.4}$$

However, the last expression look as the inverse Zak transform of a function $h(x,\xi) := e^{-2\pi i x\xi}g(-\xi,x)$. It is important that $h \in \mathcal{Q}$ (to achieve this, we do the last substitution). □

REMARK. The two-step evaluation of the Fourier transform in (2.4) is an important trick for numerical computations, see Section 9.4. □

2.3. Another normalization of the Zak transform. Formula (2.3) looks as a rotation by 90°. We want to renormalize the Zak transform and to remove the exponential factor.

Namely, for a function f on $\mathbb{R}^n$ define the function $g = \mathcal{Z}^\circ f$ on $\mathbb{R}^{2n}$ by

$$\mathcal{Z}^\circ f(x,\xi) := e^{\pi i x\xi}\mathcal{Z}f(x,\xi)$$

Then

$$\mathcal{Z}^\circ \cdot \mathcal{F} \cdot (\mathcal{Z}^\circ)^{-1}g(x,\xi) = g(-\xi,x)$$

The quasiperiodicity conditions also change, a function $g = \mathcal{Z}^\circ f$ now saisfies

$$g(x+p, \xi+q) = g(x,\xi)\exp\{\pi i(xq^t - \xi p^t)\}\cdot \exp\{\pi i pq^t\} \tag{2.5}$$

where x, $\xi \in \mathbb{R}^n$, p, $q \in \mathbb{Z}^n$. Notice that these conditions are symmetric under the transposition $x \leftrightarrow \xi$. Also,

$$\exp\{\pi i pq^t\} = \pm 1 \qquad \text{for all } p,\, q \in \mathbb{Z}^n$$

Denote by $\mathcal{Q}^\circ(\mathbb{R}^{2n})$ the space of C^∞-functions g on $\mathbb{R}^{2n}$ satisfying these quasiperiodicity conditions; this space is the image of $\mathcal{S}(\mathbb{R}^{2n})$ under $\mathcal{Z}^\circ$.

Since $|e^{\pi i x\xi}| = 1$, it follows that the modified transform $\mathcal{Z}^\circ$ is a unitary operator $L^2(\mathbb{R}^n) \to L^2[0,1]^{2n}$.

2.4. Two Heisenberg groups. The duplicated Heisenberg group Heis_{2n} acts in the space $C^\infty(\mathbb{R}^{2n})$ in the usual way, see Section 1.2, i.e., via operators

$$U\big(a\oplus\alpha\oplus b\oplus\beta\big)g(x,\xi) = g\big((x+a)\oplus(\xi+\alpha)\big)\exp\big\{\pi i(x\beta+\xi b)\big\}\exp\big\{\frac{\pi i}{2}(a\beta+b\alpha)\big\}$$

These operators satisfy the usual relations

$$\begin{aligned} &U\big(a\oplus\alpha\oplus b\oplus\beta\big)\cdot U\big(a'\oplus\alpha'\oplus b'\oplus\beta'\big) = \\ &= \exp\Big\{\frac{\pi i}{2}(a\beta' - a'\beta + b'\alpha - b\alpha')\Big\}\cdot U\Big((a+a')\oplus(\alpha+\alpha')\oplus(b+b')\oplus(\beta+\beta')\Big)\end{aligned}$$

This group of operators does not appear in our considerations. However it contains two subgroups Heis_n^{aux} and Heis_n^{bas} which are an important tool in the next section. These subgroups are distinguished respectively by the equations

$$\begin{cases} a = b \\ \alpha = -\beta \end{cases} \qquad \text{and} \qquad \begin{cases} a = -b \\ \alpha = \beta \end{cases}$$

Both subgroups are isomorphic to Heis_n.

For further references, we present explicit formulae for actions of both groups. The *basic* Heisenberg group Heis_n^{bas} acts by[2]

$$T(a\oplus\alpha)g(x,\xi) = g(x+a, \xi+\alpha)\exp\{\pi i(x\alpha^t - \xi a^t)\} \tag{2.6}$$

where a, $\alpha \in \mathbb{R}^n$. These operators satisfy

$$T(a\oplus\alpha)T(b\oplus\beta) = e^{\pi i(a\beta^t - b\alpha^t)}T\big((a+b)\oplus(\alpha+\beta)\big) \tag{2.7}$$

The *auxiliary* Heisenberg group Heis_n^{aux} acts by

$$\mathfrak{T}(u\oplus v)g(x,\xi) = g(x+u, \xi+v)e^{-i\pi(xv^t - \xi u^t)}e^{-\pi i uv^t} \tag{2.8}$$

Then

$$\mathfrak{T}(a\oplus b)\mathfrak{T}(u\oplus v) = e^{-2\pi i bu^t}\mathfrak{T}\big((a+u)\oplus(b+v)\big) \tag{2.9}$$

In formula (2.8), we modify a normalization of operators; the purpose of this modification will be clear immediately (Observation 2.6).

[2] The notation $a\oplus\alpha$ is equivalent to (a,α).

Observation 2.5 *The groups* Heis_n^{aux} *and* Heis_n^{bas} *commute with each other.*

This follows from the product formula in Heis_{2n}. In other words, these subgroups correspond to a pair of complementary orthogonal subspaces in the symplectic space $\mathbb{R}^n$.

Observation 2.6 *The subspace* $\mathcal{Q}^\circ(\mathbb{R}^{2n})$ *is precisely the space of functions fixed by all the operators* $\mathfrak{T}(p,q)$, *where* p, q *range in* $\mathbb{Z}^n$.

This is a reformulation of quasiperiodicity.

These two statements imply

Corollary 2.7 *The space* $\mathcal{Q}^\circ(\mathbb{R}^{2n})$ *is invariant under the group* Heis_n^{bas}.

Observation 2.8 *The transform* $\mathcal{Z}^\circ$ *intertwines the standard representation of* Heis_n *in* $\mathcal{S}(\mathbb{R}^n)$ *and the representation* $T(a \oplus \alpha)$.

This is a rephrasing of (1.7), (1.8)

8.3 Action of $\mathrm{Sp}(2n,\mathbb{Z})$

Our next purpose is to push forward the action of the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ and the symplectic category to the spaces $\mathcal{Q}^\circ(\mathbb{R}^{2n})$ (see Chapter 1). This will be done in the full generality in the Section 8.4. Here we transport the action of the subgroup in $\mathrm{Sp}(2n,\mathbb{R})$ consisting of matrices

$$g = \begin{pmatrix} A & 2\pi B \\ (2\pi)^{-1}C & D \end{pmatrix} = \begin{pmatrix} (2\pi) & 0 \\ 0 & (2\pi)^{-1} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} (2\pi) & 0 \\ 0 & (2\pi)^{-1} \end{pmatrix}^{-1}$$

where the matrices A, B, C, D have integer entries.

3.1. The group $\mathrm{Sp}(2n,\mathbb{Z})$ and its Igusa subgroup $\Gamma_{1,2}$. Denote by $\mathrm{Sp}(2n,\mathbb{Z})$ the subgroup of $\mathrm{Sp}(2n,\mathbb{Z})$ consisting of matrices with integer entries.

Lemma 3.1 $\mathrm{Sp}(2n,\mathbb{Z})$ *is a group*

PROOF. Indeed, for a symplectic matrix g,

$$g^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Therefore $g \in \mathrm{Sp}(2n,\mathbb{Z})$ implies $g^{-1} \in \mathrm{Sp}(2n,\mathbb{Z})$. □

Fix an integer positive N. Define the *principal congruence subgroup* $\Gamma_N \subset \mathrm{Sp}(2n,\mathbb{Z})$ consisting of matrices g such that N divides $g_{ij} - \delta_{ij}$.

Problem 3.1 Denote by $\mathbb{Z}_N$ the quotient-ring $\mathbb{Z}/N\mathbb{Z}$. To a matrix $g \in \mathrm{Sp}(2n,\mathbb{Z})$ we assign the matrix $[g]_N$ over $\mathbb{Z}_N$, whose matrix elements are $g_{ij} \mod (N)$. Then the map $g \mapsto [g]_N$ is a homorphism $\mathrm{Sp}(2n,\mathbb{Z}) \to \mathrm{Sp}(2n,\mathbb{Z}_N)$. Its kernel is Γ_N.

General values of N appear in our considerations below, in Section 11.6. Now consider the case $N = 2$. We are going to define a certain intermideate $\Gamma_{1,2}$, such that

$$\Gamma_2 \subset \Gamma_{1,2} \subset \mathrm{Sp}(2n,\mathbb{Z})$$

Denote by $\mathbb{F}_2$ the field with two elements. Define the group $\mathrm{Sp}(2n,\mathbb{F}_2)$ as the group of $(n+n)\times(n+n)$ matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ over the field $\mathbb{F}_2$ preserving the bilinear form

$$B(v_+ \oplus v_-, w_+ \oplus w_-) = v_+(w_-)^t - v_-(w_+)^t = v_+(w_-)^t + v_-(w_+)^t$$

This form is skew-symmetric, but it is also symmetric, since $-1 = +1$ in $\mathbb{F}_2$. The matrix g satisfies the conditions (see (3.1.4))

$$ad^t + bc^t = 1, \quad ab^t,\ cd^t \text{ are symmetric} \tag{3.1}$$

Consider the quadratic form on $\mathbb{F}_2^{2n}$ given by

$$Q(v_+ \oplus v_-) = v_+ \cdot v_-^t$$

Evidentely,

$$B(v,w) = Q(v+w) - Q(v) - Q(w)$$

Denote by $\mathrm{O}(2n,\mathbb{F}_2)$ the group of matrices over $\mathbb{F}_2$ preserving the quadratic form Q. By the last identity,

$$\mathrm{O}(2n,\mathbb{F}_2) \subset \mathrm{Sp}(2n,\mathbb{F}_2)$$

Lemma 3.2 *A matrix* $g \in \mathrm{Sp}(2n,\mathbb{F}_2)$ *is contained in* $\mathrm{O}(2n,\mathbb{F}_2)$ *iff the matrices* ab^t, cd^t *have zeros on the diagonal.*

PROOF. A matrix g preserves Q if

$$Q(v_+ \oplus v_-) = Q\left((v_+ \oplus v_-)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \quad \Longleftrightarrow$$
$$\Longleftrightarrow \quad v_+(v_-)^t = (v_+a + v_-c)\cdot(v_+b + v_-d)^t \quad \Longleftrightarrow$$
$$\Longleftrightarrow \quad \left\{v_+(v_-)^t = v_+(ad^t + bc^t)v_-,\ 0 = v_+(ab^t)v_+^t,\ 0 = v_-(cd^t)v_-^t\right\}$$

Compare the last row with conditions (3.1) for $g \in \mathrm{Sp}(2n,\mathbb{F}_2)$. The first identity holds automatically. Next, a priory, ab^t and cd^t are symmetric.

Now let R be an arbitrary symmetric matrix over $\mathbb{F}_2$. Consider the corresponding quadratic form

$$xRx^t = \sum r_{ij}x_ix_j = \sum_i r_{ii}x_i^2 + \sum_{i>j}(r_{ij}+r_{ji})x_ix_j = \sum_i r_{ii}x_i^2$$

This expression is zero, iff $r_{ii} = 0$. This remark implies our statement. $\square$

We denote by $\Gamma_{1,2} \supset \Gamma_2$ the *Igusa subgroup* of $\mathrm{Sp}(2n,\mathbb{Z})$ consisting of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that the matrices $a^t c$ and $b^t d$ have even elements on the diagonals. In other words, $\Gamma_{1,2}$ is the pull-back of $\mathrm{O}(2n,\mathbb{F}_2) \subset \mathrm{Sp}(2n,\mathbb{F}_2)$ under the homomorphism $\mathrm{Sp}(2n,\mathbb{Z}) \to \mathrm{Sp}(2n,\mathbb{F}_2)$.

3.2. The affine action of $\mathrm{Sp}(2n,\mathbb{F}_2)$**.** Recall that an affine transformation of a linear space V is a map of the form $h \mapsto hA + v$, where h ranges in V, $v \in V$, and $A : V \mapsto V$ is a linear transformation. In other words, an affine transformation is a composition of a linear transformation and a shift.

Consider the linear space $V := (\mathbb{F}_2)^{2n}$ endowed with a bilinear form B and quadratic form Q as above. Denote by V' the space of linear functionals on V.

Theorem 3.3 *The group* $\mathrm{Sp}(2n,\mathbb{F}_2)$ *acts on* $V' = \mathbb{F}_2^{2n}$ *by the affine transformations*

$$U(g)\ell(v) = \ell(vg) + Q(vg) - Q(v), \qquad \ell \in V',\, v \in V \tag{3.2}$$

Lemma 3.4 *For* $g \in \mathrm{Sp}(2n,\mathbb{F}_2)$*, the function* $\varphi(v) := Q(vg) - Q(v)$ *is a linear on* V*.*

Proof of the lemma. We must watch the proof of Lemma 3.2.

$$Q(vg) - Q(v) = \sum \alpha_j (v_j^+)^2 + \sum \beta_j (v_j^-)^2$$

where α_j, β_j are the diagonal values of matrices ab^t, cd^t.

Since $y^2 = y$ for all $y \in \mathbb{F}_2$, the last expression is a linear functional. □

Proof of the theorem. Consider the space W consisting of $\mathbb{F}_2$-valued functions τ on V having the form

$$\tau(v) = Q(v) + \ell(v), \qquad \text{where } \ell \text{ is a linear functional on } V$$

By the lemma, the formula

$$Z(g)\tau(v) = \tau(vg)$$

defines an action Z of $\mathrm{Sp}(2n,\mathbb{F}_2)$ on the set W.

We identify $V' \leftrightarrow \mathsf{W}$ by $\ell \mapsto \ell + Q$. The induced action of $\mathrm{Sp}(2n,\mathbb{F}_2)$ on V' is (3.2). □

Remark. An affine transformation $U(g)$ is linear iff $g \in \mathrm{O}(2n,\mathbb{F}_2)$. □

3.3. Action of $\Gamma_{1,2}$ in $\mathcal{Q}(\mathbb{R}^{2n})$. Recall that in previous section we introduced the group Heis_n^{bas} acting in $\mathcal{Q}^\circ$ by operators T given by (2.6).

Theorem 3.5 *a) For* $h = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{1,2}$ *and* $f \in \mathcal{Q}^\circ(\mathbb{R}^{2n})$*, the function*

$$W(h)f(x,\xi) := f\left(\begin{pmatrix} x & \xi \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = g(xA + \xi C, xB + \xi D) \tag{3.3}$$

is contained in $\mathcal{Q}^\circ(\mathbb{R}^{2n})$

$$b) \qquad W(h_1)W(h_2) = W(h_1h_2)$$

$$c) \qquad W(h)T(a\oplus\alpha) = T\big((a\oplus\alpha)h\big)W(h)$$

Define two bilinear forms. The first one is the symplectic form on $\mathbb{R}^{2n}$,

$$\Lambda(x\oplus\xi, y\oplus\eta) = x\eta^t - \xi y^t$$

The second form is the $\mathbb{F}_2$-valued quadratic form Ξ on $\mathbb{Z}^{2n}$ given by

$$\Xi(p\oplus q) = pq^t \mod 2$$

In these notation, the quasiperiodicity condition (2.5) is

$$g(x+p,\xi+q) = g(x,\xi)\exp\big\{\pi i\Lambda\big(x\oplus\xi, p\oplus q\big)\big\}\exp\big\{\pi i\Xi(p\oplus q)\big\}$$

PROOF. a) Recall that the matrix $s\in\Gamma_{1,2}$ preserves the integer lattice $\mathbb{Z}^{2n}$ and both forms Λ, Ξ. Therefore it regards the quasiperiodicity.

b) is obvious, c) follows from a straightforward calculation. □

As a corollary, we obtain the following theorem.

Theorem 3.6 *Let* $g = \begin{pmatrix} A & B\\ C & D\end{pmatrix} \in \Gamma_{1,2}$. *Then the operators of the Weil representation*

$$\mathrm{We}\left[\begin{pmatrix}2\pi & 0\\ 0 & (2\pi)^{-1}\end{pmatrix} g \begin{pmatrix}2\pi & 0\\ 0 & (2\pi)^{-1}\end{pmatrix}^{-1}\right]$$

defined by (1.2.9) correspond to the operators $W(g)$ *under the Zak transform.*

PROOF. Both operators are determined by the commutation relations with the Heisenberg group, see the last Theorem 3.5 and Theorem 1.2.5. The Zak transform intertwines the actions of the Heisenberg group in $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{Q}^\circ(\mathbb{R}^{2n})$. The factor 2π appear due to different normalizations in formulae (1.2.7), (2.7). □

Corollary 3.7 *The restriction of the Weil representation to* $\Gamma_{1,2}$ *is equivalent to a linear representation.*

PROOF. Indeed, we have no scalar factor in Theorem 3.5.b). □

3.4. An example: matrix $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$**.** Our next purpose is to write out the action of the whole subgroup $\mathrm{Sp}(2n,\mathbb{Z})$ in $\mathcal{Q}^\circ$. First, we discuss an example.

Consider the operator $R:\mathcal{S}(\mathbb{R})\to\mathcal{S}(\mathbb{R})$ given by

$$Rf(x) := \mathrm{We}\begin{pmatrix}1 & 2\pi\\ 0 & 1\end{pmatrix} f(x) = \exp\{\pi i x^2\} f(x)$$

Proposition 3.8 *The corresponding operator in $\mathcal{Q}^\circ(\mathbb{R}^1)$ is*

$$\widetilde{R}g(x,\xi) = e^{-i\pi x/2} g(x, \xi + x + 1/2)$$

REMARK. Comparatively the previous Theorem 3.5, we have here an additional shift of the argument. □

PROOF. Consider the distribution $\Delta^\circ[a,\xi;x]$ of the variable x given by

$$\Delta^\circ[a,\xi;x] = e^{\pi i a\xi} \sum_k \delta(x-a-k) e^{i\xi k}$$

Let $g := \mathcal{Z}^\circ f$. Then

$$g(a,\xi) = \ll f(x), \Delta^\circ[a,\xi;x] \gg, \qquad \widetilde{R}g(a,\xi) = \ll f(x), \exp\{\pi i x^2\}\Delta^\circ[a,\xi;x] \gg$$

We evaluate

$$\exp\{\pi i x^2\}\Delta^\circ(a,\xi;x) = \exp\{\pi i x^2\} \cdot e^{\pi i a\xi} \sum_k \delta(x-a-k) e^{2\pi k\xi} =$$

$$= e^{\pi i a\xi} \sum_k \delta(x-a-k) e^{2\pi k\xi}\, e^{\pi i (a+k)^2} = e^{\pi i (a\xi + a^2)} \sum_k \delta(x-a-k) e^{2\pi i k(a+\xi)} e^{\pi i k^2}$$

Since k is integer, $e^{\pi i k^2} = e^{\pi i k}$. We obtain

$$e^{-i\pi a/2} e^{\pi i a(\xi + a + 1/2)} \sum_k \delta(x-a-k) e^{2\pi i k(a+\xi+1/2)} = e^{-i\pi a/2}\Delta^\circ[a, \xi + a + 1/2; x]$$

This is equivalent to the desired formula. □

3.5. Action of $\mathrm{Sp}(2n,\mathbb{Z})$. We let us try to define the action of $\mathrm{Sp}(2n,\mathbb{Z})$ by the same formula (3.3) as for $\Gamma_{2,1}$.

Lemma 3.9 *For $f \in \mathcal{Q}^\circ(\mathbb{R}^{2n})$, $h = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{Z})$, the function*

$$g(x,\xi) := \widetilde{W}(h) f(x,\xi) := f\big((x \oplus \xi)h\big) \tag{3.4}$$

satisfies the quasiperiodicity conditions

$$g(x+p, \xi+q) = g(x,\xi)\exp\big\{\pi i(xq^t - \xi p^t + pq^t)\big\} \times \tag{3.5}$$

$$\times \exp\big\{\pi i(\Xi((p\oplus q)h) - \Xi(p \oplus q))\big\} \tag{3.6}$$

for p, $q \in \mathbb{Z}^n$.

PROOF. See proof of Theorem 3.5. □

In the last formulae, the first row (3.5) corresponds to the quasiperiodicity conditions for $\mathcal{Q}^\circ$, Nevertheless we also have an additional factor ± 1 in the second row (3.6). The previous subsection suggests that our formulae must be corrected by half-integer shifts. In fact, these correcting operators $\mathfrak{T}(\cdot)$ are contained in the Heisenberg group Heis_n^{aux} defined in the previous section.

Now let u, $v \in \mathbb{Z}^n$. Consider the operators $\mathfrak{T}(u/2 \oplus v/2) : C^\infty(\mathbb{R}^{2n}) \to C^\infty(\mathbb{R}^{2n})$, see (2.8). Denote by

$$\mathfrak{T}_\downarrow(u/2 \oplus v/2) : \mathcal{Q}^\circ(\mathbb{R}^{2n}) \to C^\infty(\mathbb{R}^{2n})$$

the same operator restricted to the subspace $\mathcal{Q}^\circ(\mathbb{R}^{2n})$.

Lemma 3.10 *Let p, $q \in \mathbb{Z}^n$. Then*

$$\mathfrak{T}_\downarrow\big((u/2+p) \oplus (v/2+q)\big) = \pm\mathfrak{T}_\downarrow\big(u/2 \oplus v/2\big)$$

Consider the standard map $w \mapsto \bar{w}$ from $\mathbb{Z}^{2n}$ to $\mathbb{F}_2^{2n}$. The lemma claims that the operator $\mathfrak{T}_\downarrow$ is determined by $\bar{u} \oplus \bar{v} \in \mathbb{F}_2^{2n}$ canonically to within a factor ± 1.

Proof of the lemma. By formula (2.8),

$$\mathfrak{T}(u \oplus v)\mathfrak{T}(p \oplus q) = \pm\mathfrak{T}\big((p+u) \oplus (q+v)\big)$$

Next, we apply Observation 2.6. For $g \in \mathcal{Q}^\circ(\mathbb{R}^{2n})$,

$$\mathfrak{T}\big((p+u) \oplus (q+v)\big)g = \pm\mathfrak{T}(u \oplus v)\mathfrak{T}(p \oplus q)g = \pm\mathfrak{T}(u \oplus v)g \qquad \square$$

Lemma 3.11 *The image of $\mathfrak{T}_\downarrow(u \oplus v)$ consists of functions satisfying*

$$h(x+p, \xi+q) = e^{2\pi i(vp^t - uq^t)}\Big(h(x,\xi)e^{\pi i(xq^t - \xi p^t) + \pi i p q^t}\Big) \tag{3.7}$$

or equivalently,

$$\mathfrak{T}(p \oplus q)h(x,\xi) = e^{2\pi i(vp^t - uq^t)}h(x,\xi) \tag{3.8}$$

Proof. Straightforward calculation. $\square$

We emphasis that the correcting factors in (3.8), (3.6) are ± 1.

Now we are ready to define operators $W(h)$ for $h \in \mathrm{Sp}(2n,\mathbb{Z})$. In Subsection 3.1 we defined the 'symplectic' form Λ and the quadratic form Q on the space $\mathbb{F}_2^{2n}$. Referring to Theorem 3.3, for $h \in \mathrm{Sp}(2n,\mathbb{Z})$, we consider an element $\bar{u} \oplus \bar{v} \in \mathbb{F}_2^{2n}$ such that

$$Q\big((\bar{p} \oplus \bar{q})h\big) - Q\big(\bar{p} \oplus \bar{q}\big) = \Lambda(\bar{u} \oplus \bar{v}, \bar{p} \oplus \bar{q})$$

for all $\bar{p}$, $\bar{q} \in \mathbb{F}_2^n$. Put

$$W(h)g := \pm\mathfrak{T}_\downarrow(u \oplus v)^{-1}\widetilde{W}(h)$$

where the operator $\widetilde{W}$ is given by (3.4).

Theorem 3.12 *a) Operators $W(h)g$ preserve the space $\mathcal{Q}^\circ(\mathbb{R}^{2n})$.*

b) The operator $W(h)$ satisfies the commutation relation

$$W(h)T\big(a \oplus \alpha\big) = T\big((a \oplus \alpha)s\big)W(h)$$

where T are the operators of Heisenberg group.

c) $W(h_1)W(h_2) = \pm W(h_1 h_2)$.

d) Operators $W(s)$ correspond to operators $\mathrm{We}(s)$ *under the Zak transform.*

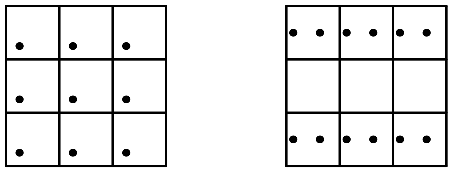

Fig. 8.2: To Proposition 3.13, $r = 2$. To evaluate a value of $U_2 f(x,\xi)$ we need in values $f(x/2, 2\xi)$, $f((x+1)/2, 2\xi)$.

PROOF. a) This follows from Lemmas 3.11 and 3.9. The operator $\widetilde{W}$ breaks quasiperiodicity conditions and $\mathfrak{T}_\downarrow$ return them away.

b) The operators $\widetilde{W}$ satisfy the condition

$$\widetilde{W}(h) T\big(a \oplus \alpha\big) = T\big((a \oplus \alpha)h\big)\widetilde{(h)}$$

(see Theorem 3.5). By Observation 2.5, operators $\mathfrak{T}(\cdot)$ and $T(\cdot)$ commute and this implies the desired relations.

c), d) By b), the operators $W(s)$ are operators of Weil representation. $\square$

3.6. Digression. Rational dilatations.

Proposition 3.13 *a) Let r be a positive integer. Consider the operator U_r in $\mathcal{S}(\mathbb{R})$ given by*

$$U_r f(x) = f(x/r)$$

Then the corresponding operator $\widetilde{U}_r = \mathcal{Z}^\circ U_r \mathcal{Z}^\circ$ in $\mathcal{Q}^\circ(\mathbb{R}^2)$ is given by

$$\widetilde{U}_r g(x,\xi) = \sum_{\sigma=0}^{r-1} e^{\pi i \sigma \xi} g\big((x+\sigma)/r, r\xi\big) \tag{3.9}$$

b) The operator $\widetilde{U}_r$ satisfies the condition

$$T(a \oplus \alpha)\widetilde{U}_r = \widetilde{U}_r T(a/r \oplus \alpha r)$$

c) The operator $\widetilde{U}_r^{-1}$ is given by

$$\widetilde{U}_r^{-1} g(x,\xi) = \frac{1}{r}\sum_{\sigma=0}^{r-1} e^{-\pi i \sigma x} g\left(rx, \frac{\xi+\sigma}{r}\right)$$

PROOF. a) For

$$g(x,\xi) = e^{\pi i x\xi}\sum_k f(x+k) e^{2\pi i k\xi}$$

we have

$$\widetilde{U}_r g(x,\xi) = e^{\pi i x\xi}\sum_k f\left(\frac{x+k}{r}\right) e^{2\pi i k\xi}$$

Representing $k = lr + \sigma$, we obtain

$$e^{\pi ix\xi}\sum_{\sigma=0}^{r-1}\sum_{l=-\infty}^{\infty} f\left(\frac{x+\sigma}{r}+l\right)e^{2\pi i(lr+\sigma)\xi} =$$

$$= \sum_{\sigma=0}^{r-1} e^{\pi i\sigma\xi}e^{\pi i\cdot\frac{x+\sigma}{r}\cdot r\xi} f\left(\frac{x+\sigma}{r}+l\right)e^{2\pi il(r\xi)} = \sum_{\sigma=0}^{r-1} e^{\pi i\sigma\xi} g\Big(\frac{x+\xi}{r}+\sigma, r\xi\Big)$$

and this gives (3.9).

The statement b) follows from a) or can be verified by a trivial calculation. Let us prove c).

$$\widetilde{U}_r^{-1}g(x,\xi) = e^{\pi ix\xi}\sum_k f(r(x+k))e^{2\pi ik\xi} = \frac{1}{r}e^{\pi ix\xi}\sum_{l=-\infty}^{\infty}\sum_{\sigma=0}^{r-1} f(rx+l)e^{2\pi il(\xi+\sigma)/r}$$

Here we apply the standard property of roots of unity,

$$\sum_{\sigma=0}^{r-1} e^{2\pi il\sigma/r} = \begin{cases} r, & \text{if } l/r \text{ is integer} \\ 0, & \text{otherwise} \end{cases}$$

Next, we transform our expression to the form

$$\frac{1}{r}\sum_{\sigma=0}^{r-1} e^{-\pi i\sigma x}e^{\pi irx(\xi+\sigma)/r}\sum_{l=-\infty}^{\infty} f(rx+l)e^{2\pi il(\xi+\sigma)/r} = \frac{1}{r}\sum_{\sigma=0}^{r-1} e^{-\pi i\sigma x} g\Big(rx, \frac{\xi+\sigma}{r}\Big)$$

and this proves c). □

3.7. Action of $\mathrm{Sp}(2n,\mathbb{Q})$**.**

Problem 3.2 Show, that each matrix $g \in \mathrm{GL}(n,\mathbb{Q})$ can be represented in the form $g = \alpha D\beta$, where α, $\beta \in \mathrm{GL}(n,\mathbb{Z})$, and D is a diagonal matrix.

Problem 3.3 Each matrix $g \in \mathrm{Sp}(2n,\mathbb{Q})$ can be represented in the form $g = \alpha D\beta$, where α, $\beta \in \mathrm{Sp}(2n,\mathbb{Z})$, and D is a diagonal matrix.

We give a proof below, ???

Now we can write

$$W(g) := W(\alpha)W(D)W(\beta)$$

for an arbitrary matrix $g \in \mathrm{Sp}(2n,\mathbb{Q})$, this operator corresponds to the operator $\mathrm{We}(g)$ in $\mathcal{S}(\mathbb{R}^n)$. We know $W(\alpha)$, $W(\beta)$ by Theorem 3.12 and we know $W(D)$ due to Proposition 3.13. Therefore we know (at least, potentially) $W(g)$.

In the next section we present a more simple approach to this problem. However, now we describe approximate structure of the final answer.

We consider the graph of an operator $g : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and its shifts by elements of the lattice $\mathbb{Z}^{4n}$. In this way, we get a countable locally finite collection of $2n$-dimensional planes. Then we shift this picture by an appropriate rational vector. The kernel of $W(g)$ is a delta-distribution supported by the intersection of this picture with the cube $[0,1]^{4n}$.

8.4 Theta-functions and theta-kernels

In this section, we evaluate images of the Gaussian integral operators under the Zak transform.

4.1. Theta-functions. Let T be an $n \times n$ matrix with positive real part. We define the *theta-function*[3]

$$\theta[T; x, \xi] = \sum_{k \in \mathbb{Z}^n} \exp\{-\frac{1}{2}(x+k)T(x^t + k^t)^t\} e^{2\pi i k \xi}$$

being the image of the Gaussian $\exp\{-\frac{1}{2}xTx^t\}$ under the Zak transform.

Theorem 4.1

$$\langle \theta[T; x, \xi], \theta[S; x, \xi] \rangle_{L^2[0,2\pi]^{2n}} = (2\pi)^{n/2} \det(S+T)^{-1/2} \tag{4.1}$$

PROOF. Since the Zak transform is unitary, the left-hand side of (4.1) equals

$$\langle \exp\{-\tfrac{1}{2}xTx^t\}, \exp\{-\tfrac{1}{2}xSx^t\} \rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \exp\Big\{-\frac{1}{2}x(T+S)x^t\Big\}\, dx$$

and we apply(1.12). □

4.2. Theta-distributions. Denote by $\mathcal{Q}'(\mathbb{R}^{2n})$ the space dual to $\mathcal{Q}(\mathbb{R}^{2n})$. Recall that a Gaussian distribution on $\mathbb{R}^n$ is a distribution having the form

$$\mathfrak{t} = \lambda \cdot \exp\{\frac{1}{2}xTx^t\} \cdot \delta_L \tag{4.2}$$

where L is a linear subspace in $\mathbb{R}^n$ (in particular, we admit $L = 0$ and $L = \mathbb{R}^n$), δ_L is its delta-function, and T is a matrix with a *non-negative* definite real part.

We define *θ-distributions* as images of Gaussian distributions under the Zak transform. There are several more direct ways to define theta-distributions.

First, the Zak transform is a continuous bijection $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{Q}'(\mathbb{R}^{2n})$. In particular, $\mathcal{Z}$ establish a continuous one-to-one correspondence between the cone of Gaussian distributions $\subset \mathcal{S}'(\mathbb{R}^n)$ and the cone of theta-distributions $\subset \mathcal{Q}'(\mathbb{R}^{2n})$.

For instance, let T be a matrix with a positive semi-definite real part. Then we define the corresponding θ-distribution as

$$\ll \theta[T], g(x,\xi) \gg = \lim_{\varepsilon \to 0+} \iint_{[0,1]^{2n}} \theta[T+\varepsilon; x, \xi] g(x,\xi)\, dx\, d\xi$$

Another way to describe θ-distributions is following.

[3]In the standard language, they are called *theta-functions with chracteristics.*

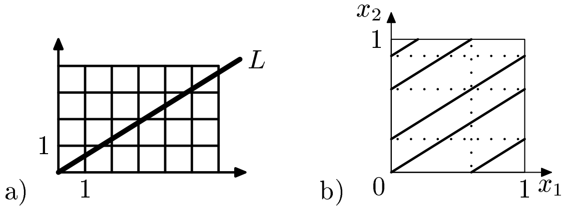

Fig. 8.3: To Proposition 4.2, $n = 2$.
a) To formula [4.4)
b) To formula (4.3). The coordinates ξ_1, ξ_2 are absent on this figure. We show domain of integration with respect to x_1, x_2. If the angular coefficient of L is rational, then we obtain an integration (in x) with respect to a finite family of segments. Otherwise, we get an integration with respect to a dense winding.

Proposition 4.2 *Let a Gaussian distribution $\mathfrak{t}$ on $\mathbb{R}^n$ be given by (4.2). Assume that L is not contained in coordinate hyperplanes in $\mathbb{R}^n$. Then the corresponding linear functional on $\mathcal{Q}(\mathbb{R}^{2n})$ is*

$$\ll \Theta[\mathfrak{t}], g \gg = \\ = \sum_{k\in\mathbb{Z}^n} \int_{x\in[0,1]^n} \Big\{ \delta_L(x-k) \exp\{-\frac{1}{2}(x-k)T(x-k)^t\} \int_{\xi\in[0,1]^n} e^{-2\pi i k\xi} g(x,\xi)\, d\xi \Big\} \tag{4.3}$$

The iterated integral and the series are absolutely convergent.[4]

Proof. By definition, If L is not contained in coordinate hyperplanes, then

$$\ll \mathfrak{t}, f \gg_{\mathbb{R}^n} = \int_{\mathbb{R}^n} f(x) \exp\{-\frac{1}{2} xTx^t\} \delta_L(x)$$

Since $f \in \mathcal{S}(\mathbb{R}^n)$, the integral absolutely converges. We write it as

$$\sum_k \int_{x\in k+[0,1]^n} f(x) \exp\{-\frac{1}{2} xTx^t\} \delta_L(x) \tag{4.4}$$

(if L is contained in a face of the cube, then a sense of integrals is not clear; however this is forbidden). The series remains to be absolutely convergent. Applying the inversion formula (1.5), we come to

$$\sum_k \int_{x\in k+[0,1]^n} \exp\{-\frac{1}{2} xTx^t\} \delta_L(x) \Big[\int_{\xi\in[0,1]^n} g(x,\xi)\, d\xi \Big] \tag{4.5}$$

Next, we shift the argument $x \mapsto x + k$ and apply the quasiperiodicity (1.3). $\square$

[4] But $\sum \int |\cdot|$ diverges.

4.3. Example. Rational matrices. For simplicity, assume L is the whole $\mathbb{R}^n$. Let $T = 2\pi i R$, where R has rational coefficients. By the last proposition, the corresponding θ-distribution, is a sum of the series

$$\sum_{k\in\mathbb{Z}^n} \exp\bigl\{-\pi i(x-k)R(x-k)^t - 2\pi i\xi k^t\bigr\} =$$

$$= \exp\bigl\{-\pi i x R x^t\bigr\} \sum_{k\in\mathbb{Z}^n} \exp\bigl\{-2\pi i(xR+\xi)k^t - \pi i k R k^t\bigr\}$$

The last sum admits an explicit summation. Indeed, the factor $\exp\{\pi i k R k^t\}$ is periodic and has only finite number of values. Precisely, there is a lattice $M \subset \mathbb{Z}^n$ such that

$$\exp\{\pi i(k+m)R(k+m)^t\} = \exp\{\pi i k R k^t\}, \qquad m \in M$$

Thus we write out the sum as

$$\sum = \sum_{j\in\mathbb{Z}^n/M} \Bigl[\exp\{-\pi i[jRj^t + 2(xR+\xi)j^t]\} \cdot \sum_{m\in M} \exp\bigl\{-2\pi i(xR+\xi)m^t\bigr\}\Bigr]$$

Denote by $M^\diamond$ the dual lattice, i.e., the set of all points $l \in \mathbb{R}^n$ such that $lm^t \in \mathbb{Z}$ for all $m \in M^t$. Then

$$\sum_{m\in M} e^{-2\pi i m^t} = \sum_{l\in M^\diamond} \delta(x-l)$$

Finally, we obtain that our initial sum has a form

$$\Phi(x,\xi) \sum_{\ell\in M^\diamond} \delta(xR+\xi-l)$$

where Φ is an explicit linear combination of Gaussians.

REMARK. Recall, that we must consider the restriction of this distribution to the cub $[0,1]^n$.

Problem 4.1 Let L be a subspace defined by rational equations, $T = iR$, where R is a matrix with rational entries.

4.4. Partial differential equations for θ-functions. A Gaussian vector $\exp\bigl\{-\frac{1}{2}xSx^t\bigr\}$ satisfies the system of partial differential equations

$$\Bigl(\frac{\partial}{\partial x_j} + \sum_k s_{kj}x_k\Bigr)\exp\bigl\{-\frac{1}{2}xSx^t\bigr\} = 0$$

By (1.4), θ-functions satisfy the partial differential equations

$$\Bigl[\frac{\partial}{\partial x_j} + \sum_k s_{kj}\Bigl(x_k + \frac{1}{2\pi i}\frac{\partial}{\partial \xi_k}\Bigr]\Bigr)\theta[S;x,\xi] = 0$$

Observation 4.3 *The differential equations and the quasiperiodicity conditions determine θ-function uniquely to within a scalar.*

PROOF. Otherwise, the differential equations for Gaussian vector have an additional solution. $\square$

Similar systems can be easily written for θ-distributions (see Theorem 6.2).

4.5. Theta-kernels. Formulation of results. For a symmetric $(m+n)\times(m+n)$ matrix S with positive definite real part denote, define the theta-kernel $\mathcal{K}[S](x,\xi;y,\eta)$ as a function on $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n$ given by

$$\mathcal{K}[S](x,\xi;y,\eta) := \theta\big[S;(x,y),(\xi,-\eta)\big] = \\ = \sum_{k\in\mathbb{Z}^m}\sum_{l\in\mathbb{Z}^n} \exp\left\{-\frac{1}{2}\begin{pmatrix} x+k & y+l\end{pmatrix} S \begin{pmatrix} x^t+k^t \\ y^t+l^t\end{pmatrix}\right\} e^{2\pi i k\cdot\xi} e^{-2\pi i l\cdot\eta}$$

where $(x,\xi)\in\mathbb{R}^{2n}$, $(y,\xi)\in\mathbb{R}^{2m}$.

Define the corresponding integral operator

$$\mathfrak{W}_{n,m}[S]:\ \mathcal{Q}(\mathbb{R}^{2n}) \to \mathcal{Q}(\mathbb{R}^{2m})$$

by

$$(\mathfrak{W}_{n,m}[S]g)(x,\xi) = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^{2n}} \mathcal{K}[S](x,\xi;y,\eta)g(y,\eta)dy\,d\eta. \tag{4.6}$$

where $g\in Q(\mathbb{R}^{2n})$.

Observation 4.4 *$\mathcal{K}$ satisfies the quasiperiodicity conditions*

$$\mathcal{K}[S](x+k,\xi;y+l,\eta) = e^{-2\pi i(\xi\cdot k+i\eta\cdot l)}\mathcal{K}[S](x,\xi;y,\eta), \tag{4.7}$$

$$\mathcal{K}[S](x,\xi+k;y,\eta+l) = \mathcal{K}[S](x,\xi;y,\eta). \tag{4.8}$$

Indeed, these formulae are nothing but the quasiperiodicity conditions for θ-functions. $\square$

Observation 4.5 *$\mathcal{K}$ is a kernel of an operator $\mathcal{Q}(\mathbb{R}^n)\to\mathcal{Q}(\mathbb{R}^m)$.*

More precisely, the conditions (4.7 –4.7) allow to write our integral (4.6) as an integral over an arbitrary shifted cub $(x_0,\xi_0;y_0,\eta_0)+[0,1]^{4n}$, the result does not depend on a choice of (a,α). In the next subsection, this allows to integrate (4.6) by part without appearance of boundary summands. Also these conditions provide the quasiperiodicity of the left-hand side of (4.6).

Theorem 4.6 *Let $P_1:\mathcal{V}_k\rightrightarrows\mathcal{V}_m$ $P_2:\mathcal{V}_m\rightrightarrows\mathcal{V}_n$ be morphisms of the category* Sp *(see Section 1.8) Let*

$$S(P_1) = i\begin{pmatrix} A & B\\ B^t & C\end{pmatrix},\qquad S(P_2) = i\begin{pmatrix} K & L\\ L^t & M\end{pmatrix} \tag{4.9}$$

be their Potapov transforms in the sense of Subsection 1.9.6. Then

$$\mathfrak{W}_{q,m}[S(P_1)]\mathfrak{W}_{n,q}[S(P_2)] = (2\pi)^{m/2}\det(C+K)^{-1/2}\mathfrak{W}_{n,m}[S(P_1P_2)] \tag{4.10}$$

In particular, we obtain a projective representation of the category **Sp**.

Recall that the Gaussian operators $B[S]$, where $S = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$, were defined in Section 1.3; namely $B[S]$ is determined by the kernel

$$K(x,y) := \exp\Bigl\{ \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \begin{pmatrix} x^t \\ y^t \end{pmatrix} \Bigr\}$$

Theorem 4.7 $\mathcal{Z}_m\mathcal{B}[S]\mathcal{Z}_n^{-1} = \mathfrak{W}_{n,m}[S]$

Consider a general Gauss operator $\mathcal{B} : S(\mathbb{R}^{2n}) \to S'(\mathbb{R}^{2m})$ having the kernel

$$\exp\Bigl\{ -\frac{1}{2} \begin{pmatrix} x^t & y^t \end{pmatrix} S \begin{pmatrix} x \\ y \end{pmatrix} \Bigr\} \cdot \delta_L(x,y)$$

Let $\Theta_{L,S}(x,\xi,y,\eta)$ be the corresponding θ-distribution in $\mathcal{Q}'(\mathbb{R}^{2n+2m})$.

Theorem 4.8 *The operator*

$$\mathcal{Z}_m\,\mathcal{B}\mathcal{Z}_n^{-1} :\ \mathcal{Q}(\mathbb{R}^{2n}) \to \mathcal{Q}(\mathbb{R}^{2m})$$

has the kernel $(2\pi)^{-n}\Theta_{L,S}(x,\xi,y,-\eta]$.

This follows from the previous theorem by the continuity.

In particular, this statement gives an explicit formula for the Weil representation of the group $\mathrm{Sp}(2n,\mathbb{R})$ in $\mathcal{Q}(\mathbb{R}^{2n})$.

Remark. For $g \in \mathrm{Sp}(2n,\mathbb{Q})$, we get an elementary expression for the kernel, see Subsection 4.3.

The rest of this section is a proof of Theorems 4.6–4.8.

4.6. Almost proof of Theorems 4.6–4.7. We are going to prove that the operator

$$\mathfrak{W} := \mathcal{Z}_m\mathrm{B}[S]\mathcal{Z}_n^{-1} : \mathcal{Q}(\mathbb{R}^{2n}) \to \mathcal{Q}(\mathbb{R}^{2m})$$

is equal to the operator with the kernel $(2\pi)^{-n}\mathcal{K}[S]$ *to within a scalar factor* (Theorem 4.7 claims that this factor is 1). Since we know the rules of multipication of Gaussian operators, this statement also implies product formula (4.10) to within a scalar.

We repeat the usual arguments with commutation relations. For α^+, $\alpha^- \in \mathbb{C}^n$, β^+, $\beta^- \in \mathbb{C}^m$ satisfying

$$\begin{pmatrix} \alpha^+ & \beta^+ \end{pmatrix} = \begin{pmatrix} \alpha^- & \beta^- \end{pmatrix} \begin{pmatrix} A & -B \\ B^t & -C \end{pmatrix}$$

we have

$$\sum_{j=1}^{m}\Bigl(\alpha_j^+ x_j + \alpha_j^- \frac{d}{dx}\Bigr)K(x,y) = \sum_{j=1}^{n}\Bigl(\beta_j^+ y_j - \beta_j^- \frac{d}{dy_j}\Bigr)K(x,y)$$

or equivalenly

$$\sum_{j=1}^{m}\Bigl(\alpha_j^+ x_j + \alpha_j^- \frac{d}{dx}\Bigr)B[S] = B[S]\sum_{j=1}^{n}\Bigl(\beta_j^+ y_j + \beta_j^- \frac{d}{dy_j}\Bigr) \tag{4.11}$$

Hence, by Theorem 1.4

$$\sum_{j=1}^{m}\Bigl(\alpha_j^+ \Bigl[x_j + \frac{1}{2\pi i}\frac{\partial}{\partial \xi_j}\Bigr] + \alpha_j^- \frac{\partial}{\partial x_j}\Bigr)\mathcal{Z}_n B[S]\mathcal{Z}_m^{-1} =$$
$$= \mathcal{Z}_n B[S]\mathcal{Z}_m^{-1}\sum_{j=1}^{n}\Bigl(\beta_j^+ \Bigl[y_j + \frac{1}{2\pi i}\frac{\partial}{\partial \eta_j}\Bigr] + \beta_j^- \frac{\partial}{\partial y_j}\Bigr)$$

We rewrite these equations as equations for the kernel of the operator $\mathcal{Z}_n B[S]\mathcal{Z}_m^{-1}$ and obtain the partial differential system for a θ-kernel which has an unique solution to within a constant factor, see Observation 4.3 □

4.7. Composition formula. Proof of Theorem 4.6. Let $\mathfrak{W}_1 := \mathfrak{W}_{q,m}[S_1]$ and $\mathfrak{W}_2 := \mathfrak{W}_{n,q}[S_2]$ be our operators. Denote their kernels by $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively.

We must evaluate the convolution of kernels, i.e.

$$\int_{[0,2\pi]^{2q?}} \mathcal{K}_1(x,\xi;s,\zeta)\mathcal{K}_2(s,\zeta;y,\eta)g(y,\eta)\,ds\,d\zeta,$$

Expanding the theta-functions into series, we obtain

$$\int\limits_{[0,2\pi]^{2q}}\sum_k\sum_p \exp\Bigl\{-\frac{1}{2}\begin{pmatrix}x+k & s+p\end{pmatrix}\begin{pmatrix}A & B\\ B^t & C\end{pmatrix}\begin{pmatrix}x^t+k^t\\ s^t+p^t\end{pmatrix}\Bigr\}e^{2\pi i k\cdot\xi}e^{-2\pi i p\cdot\zeta}\times$$
$$\times\sum_r\sum_l \exp\Bigl\{-\frac{1}{2}\begin{pmatrix}s+r & y+l\end{pmatrix}\begin{pmatrix}K & L\\ L^t & M\end{pmatrix}\begin{pmatrix}s^t+r^t\\ y^t+l^t\end{pmatrix}\Bigr\}e^{2\pi i r\cdot\zeta}e^{-2\pi i l\cdot\eta}dsd\zeta =$$

$$\sum_k\sum_l e^{ik\cdot\xi}e^{-il\cdot\eta}\int\limits_{[0,2\pi]^{2q}}\sum_p \exp\Bigl\{-\frac{1}{2}\begin{pmatrix}x+k & s+p\end{pmatrix}\begin{pmatrix}A & B\\ B^t & C\end{pmatrix}\begin{pmatrix}x^t+k^t\\ s^t+p^t\end{pmatrix}\Bigr\}e^{-2\pi i p\cdot\zeta}\times$$
$$\times\sum_r \exp\Bigl\{-\frac{1}{2}\begin{pmatrix}s+r & y+l\end{pmatrix}\begin{pmatrix}K & L\\ L^t & M\end{pmatrix}\begin{pmatrix}s^t+r^t\\ y^t+l^t\end{pmatrix}\Bigr\}e^{2\pi i r\cdot\zeta}dsd\zeta.$$

For fixed x, y, ξ, η the integral is the $L^2[0,1]^{2m}$-inner product of two theta-functions. These theta-function are of the Zak transforms of two Gaussian functions, namely of

$$\mathbf{b}_1(s) := \exp\Big\{-\frac{1}{2}\begin{pmatrix} x+k & s\end{pmatrix}\begin{pmatrix} A & B \\ B^t & C\end{pmatrix}\begin{pmatrix} x^t+k^t \\ s^t\end{pmatrix}\Big\}$$
$$\mathbf{b}_2(s) := \exp\Big\{-\frac{1}{2}\begin{pmatrix} s & y+l\end{pmatrix}\begin{pmatrix} \overline{K} & \overline{L} \\ \overline{L}^t & \overline{M}\end{pmatrix}\begin{pmatrix} s^t \\ y^t+l^t\end{pmatrix}\Big\}$$

Since the Zak transform is a unitary operator, we can write the last integral as an inner product in $L^2(\mathbb{R}^m)$, i.e.,

$$\int_{\mathbb{R}^m} \mathbf{b}_1(s)\overline{\mathbf{b}_2(s)}\,ds$$

The last expression is a Gaussian integral. We easily evaluate it and obtain a Gaussian in x, y. We omit a calculation (actually, it was done in the proof of Theorem 1.4.2). After summation in k and l, we get a θ-kernel. □

4.8. End of proof of Theorem 4.6. In Section 4.6 we evaluated the kernel of the operator $\mathcal{Z}_m\mathrm{B}[S]\mathcal{Z}_n^{-1}$ to within in a scalar, i.e.,

$$\mathcal{Z}_m\mathrm{B}[S]\mathcal{Z}_n^{-1} = \sigma(m,n;S)\mathfrak{W}_{n,m}[S] \tag{4.12}$$

where $\sigma(m,n;S)\in\mathbb{C}$. We want to prove that

$$\sigma(m,n;S) = 1 \qquad \text{for all}\quad m,\ n,\ S. \tag{4.13}$$

A priori we know the following facts about the function σ.

Lemma 4.9 a) *$\sigma(m,n;S)$ is a holomorphic function in the variable S.*

b) $\sigma(0,m;S) = 1$

c) $\sigma\big(m,n;S(P_1)\big)\sigma\big(k,m;S(P_2)\big) = \sigma(k,n;S(P_1P_2))$.

Proof. a) Both sides of (4.12) are holomorphic in S.

b) Indeed, the Gaussian operators corresponding to elements $\mathrm{Mor}_{\mathbf{Sp}}(V_0,V_m)$ are Gaussian vectors, see Subsection3.11. By definition, images of Gaussians under the Zak transform are theta-functions, see Subsection 4.1.

c) We observe (see (1.4.1), (4.10)) that in the identities

$$\mathcal{B}[S(P_1)]\mathcal{B}[S(P_2)] = \lambda(P_1,P_2)\,\mathcal{B}[S(P_1P_2)]$$

$$\mathfrak{W}_{m,n}[S(P_1)]\mathfrak{W}_{k,m}(S(P_2)) = \lambda(P_1,P_2)\,\mathfrak{W}_{k,n}(S(P_1P_2))$$

the scalar factors $\lambda(P_1,P_2)$ coincide. This implies c). □

Corollary 4.10 $\sigma(n,n;P) = 1$.

Proof. Let P ranges in the semigroup $\Gamma\mathrm{Sp}(2n,\mathbb{R})$ of strict symplectic contractions of $\mathbb{C}^{2n}$. This semigroup is open in $\mathrm{Sp}(2n,\mathbb{C})$. Evidently, each homomorphism $\sigma : \Gamma\mathrm{Sp}(2n,\mathbb{R}) \to \mathbb{C}^*$ admits a unique extension to a homomorphism $\mathrm{Sp}(2n,\mathbb{R}) \to \mathbb{C}^*$ and hence $\sigma = 1$.

The semigroup $\Gamma\mathrm{Sp}(2n,\mathbb{R})$ is dense in $\mathrm{Mor}(\mathcal{V}_{2n},\mathcal{V}_{2n})$ and this implies the statement. □

Corollary 4.11 *$\sigma(n,m;P)$ does not depend on P.*

Proof. Let R range in $\Gamma\mathrm{Sp}(2n,\mathbb{R})$ and Q range in $\Gamma\mathrm{Sp}(2m,\mathbb{R})$. By the previous corollary, $\sigma(n,m;QPR) = \sigma(n,m;P)$. □

Proof of Theorem 4.7. By the previous corollary, $\sigma(n,m;P) := \sigma(n,m) =$ const. On the other hand, $\sigma(0,m) = \sigma(0,n)\sigma(m,n)$ and $\sigma(0,k) = 1$. □

8.5 Bibliographical remarks

4.9. The Cartier model for representation of the Heisenberg group was proposed by P.Cartier [28]. May be the 'Zak transform' (J.Zak, [182]) also must be attributed to him. Another term for the Zak transform is *A.Weil–J.Brezin transform*, see [24]. See also the book of G.Lion and M.Vergne [96] and the paper of A.Janssen [76].

4.10. Theta-kernels. We follow [42], the statement was earlier conjectured by R.Howe.