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AN ESTIMATE OF THE NUMBER OF PARAMETERS DEFINING AN n-DIMENSIONAL ALGEBRA

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ABSTRACT. Consider an arbitrary family of nonisomorphic n-dimensional complex Lie algebras (respectively, associative algebras, commutative algebras) that depends continuously on a certain set of parameters $t_1, \dots, t_N \in \mathbf{C}$. The asymptotics is obtained for the largest number N of parameters possible when n is fixed:

$$\frac{2}{27}n^3 + O(n^{8/3}), \quad \frac{4}{27}n^3 + O(n^{8/3}), \quad \frac{2}{27}n^3 + O(n^{8/3})$$

respectively. A decomposition into irreducible components is also studied for the algebraic variety Lie_n of all possible Lie algebra structures on the linear space $\mathbf{C}^n$.

Bibliography: 19 titles.

We consider an n-dimensional complex Lie algebra with a basis $e_1, \dots, e_n$ and defining relations $[e_i, e_j] = \sum c_{ij}^k e_k$. The structure constants c_{ij}^k must satisfy the following two conditions: $c_{ij}^k = -c_{ij}^k$ (anticommutativity) and

$$\sum_\alpha (c_{ij}^\alpha c_{\alpha k}^\beta + c_{jk}^\alpha c_{\alpha i}^\beta + c_{ki}^\alpha c_{\alpha j}^\beta) = 0$$

(Jacobi identity). These equations determine an algebraic subvariety in n^3-dimensional space with coordinates c_{ij}^k. This variety is usually called the variety of structure constants of Lie algebras; we denote it by Lie_n. In a similar manner we define the variety Assoc_n of structure constants of n-dimensional associative algebras with unity and the variety Comm_n of structure constants of n-dimensional commutative-and-associative algebras with unity. In this paper we prove that the dimensions of these varieties do not exceed

$$\frac{2}{27}n^3 + O(n^{8/3}), \quad \frac{4}{27}n^3 + O(n^{8/3}), \quad \frac{2}{27}n^3 + O(n^{8/3})$$

respectively. Simple examples (metabelian algebras, see [17], as well as §1.6.F and §3.2) show that the leading terms of the above asymptotics are precise (the genuine order of the remainder seems to be equal to $O(n^2)$). Consider, in particular, an arbitrary family of nonisomorphic Lie algebras analytically depending on a certain set of parameters. Then the number of parameters is at most $(\frac{2}{27})n^3 + O(n^{8/3})$, and it can reach a value of order $(\frac{2}{27})n^3 + O(n^2)$ (in fact, any change of coordinates can reduce the number of parameters by at most $\dim(\mathrm{GL}(n)) = n^2$).

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After the paper had been submitted for publication, I. R. Shafarevich informed the author about the paper [28], where it had been proved that the number of nonisomorphic n-dimensional associative algebras over a field of q elements does not exceed $q^{4n^3/27+O(n^{8/3})}$. Even earlier Sims [16] had proved that the number of groups of order p^n is at most $p^{2n^3/27+O(n^{8/3})}$. The proofs of these results follow essentially the same pattern.

In §1 we discuss various properties of the variety Lie_n: possible ways of separating components, classification of the components for small n, and structure of Lie_n in the neighborhood of a given point (in other words, deformations of Lie algebras). In §2 we prove the estimates for the number of parameters in the case of Lie_n. In §3 we discuss Assoc_n and Comm_n.

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§1. The varieties Lie_n

1.0. Notation. Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{r}(\mathfrak{g})$ its radical, $\mathfrak{p}(\mathfrak{g})$ its maximal semisimple subalgebra, $\mathfrak{n}(\mathfrak{g})$ its nilpotent radical, and $Z(\mathfrak{g})$ its center. Let $\leftthreetimes$ designate the semidirect product ($\mathfrak{g} = \mathfrak{p}(\mathfrak{g}) \leftthreetimes \mathfrak{r}(\mathfrak{g})$).

We will use the words "semisimple" and "nilpotent" only in the case where the corresponding element x belongs to an algebraic Lie algebra $\mathfrak{g}$ (if the one-parameter subgroup determined by x consists of semisimple or unipotent elements respectively). A subalgebra in an algebraic algebra $\mathfrak{g}$ is called *toroidal* if its associated subgroup in the algebraic group is a maximal torus. We recall that all toroidal sublagebras are conjugate. The dimension of a toroidal subalgebra is called the *rank* $\mathrm{rk}(\mathfrak{g})$ of an algebraic algebra $\mathfrak{g}$.

Given a Lie algebra $\mathfrak{g}$, let $\mathrm{der}(\mathfrak{g})$, $\mathrm{int}(\mathfrak{g})$, and $D(\mathfrak{g}) = \mathrm{der}(\mathfrak{g})/\mathrm{int}(\mathfrak{g})$ denote the algebras of all derivations of $\mathfrak{g}$, of inner derivations, and of outer derivations, respectively, and $\mathrm{Aut}(\mathfrak{g})$ the automorphism group of $\mathfrak{g}$. For a nilpotent algebra $\mathfrak{n}$ we set $\mathrm{drk}(\mathfrak{n}) = \mathrm{rk}(D(\mathfrak{n})) = \mathrm{rk}(\mathrm{der}(\mathfrak{n})) \leq \dim \mathfrak{n}$.

We recall that by a theorem of Morozov [2] there exist only finitely many nilpotent Lie algebras of dimension less than or equal to 6. We introduce notation for some of these: $\mathbf{C}^k$ for an abelian algebra, Γ_4 for an indecomposable four-dimensional nilpotent algebra ($[x_1, x_2] = x_3$, $[x_1, x_3] = x_4$), and $H_{2\alpha+1}$ for an algebra with basis $x_1, \dots, x_\alpha, y_1, \dots, y_\alpha, z$ and relations $[x_j, y_j] = z$ (Heisenberg algebra). Let $\mathrm{Aff}(n)$ denote the algebra of affine transformations of an n-dimensional space, and V_α the irreducible $\mathrm{sl}(2)$-module of dimension α (for instance, $\mathrm{Aff}(2) = (\mathrm{sl}(2) \oplus \mathbf{C}) \leftthreetimes V_2$).

The words "dense", "open", and "closed" can be understood both in the usual sense and in the sense of the Zariski topology. The abbreviation IAF (irreducible algebraic family) will stand for an arbitrary irreducible algebraic subset in Lie_n which is not necessary closed. Now let Ω be an IAF. We say that an A is a set of generic points in Ω if there exists an open dense subset $B \subset \Omega$ such that $B \subset A$.

1.1. Changes of bases define the action of $\mathrm{GL}(n)$ on Lie_n. The classification of its orbits is equivalent to the classification of Lie algebras up to isomorphism.

1.2. Let $\mathfrak{g}(\varepsilon)$ be a family of Lie algebras analytically depending on a small parameter ε, $\mathfrak{g}(0) = \mathfrak{g}$. Then all the $\mathfrak{g}(\varepsilon)$ are called *deformations* of $\mathfrak{g}$. Often two approaches are distinguished in studying deformations: the "global" one based on the study of the structure of Lie_n in the nieghborhood of $\mathfrak{g}$, (see [10], [8], [14], and [3]) and the "cohomological" one based on the decomposition of the commutator in $\mathfrak{g}(\varepsilon)$ into a series in ε (see [5], [10], and [17]). Let $[x, y](\varepsilon)$ be the commutator in $\mathfrak{g}(\varepsilon)$,

$$[x, y](\varepsilon) = [x, y](0) + \varepsilon q_1(x, y) + \varepsilon^2 q_2(x, y) + \dots,$$

where $q_j(x,y) = -q_j(y,x)$. Then the Jacobi identity imposes a linear condition on $q_1(x,y)$ which is equivalent to $q_1 \in Z^2(\mathfrak{g},\mathfrak{g})$. (In what follows we use standard notation for Lie algebra cohomology: $H^q(\mathfrak{g},M) = Z^q(\mathfrak{g},M)/B^q(\mathfrak{g},M)$. See [5] and [19].) $B^2(\mathfrak{g},\mathfrak{g})$ consists of trivial deformations resulting from changes of bases in $\mathfrak{g}$. Thus the essentially different infinitesimal deformations of $\mathfrak{g}$ are parametrized by $H^2(\mathfrak{g},\mathfrak{g})$.

However, at singular points of Lie_n the group $H^2(\mathfrak{g},\mathfrak{g})$ is enormous and, moreover, Lie_n has multiple components (see §1.6.E, and also [14], [8], and [5]).

We say that an algebra $\mathfrak{a}$ is a *degeneration* of $\mathfrak{g}$ if its $\mathrm{GL}(n)$-orbit is contained in the $\mathrm{GL}(n)$-orbit of $\mathfrak{g}$.

1.3. Definition. An algebra $\mathfrak{g}$ is called *rigid* if it has no nontrivial deformations.

In orther words, the $\mathrm{GL}(n)$-orbit of $\mathfrak{g}$ is an open dense subset in a certain component $A(\mathfrak{g})$ of Lie_n. Clearly $\dim A(\mathfrak{g}) = n^2 - \dim \mathrm{Aut}(\mathfrak{g})$. It is also obvious that the condition $H^2(\mathfrak{g},\mathfrak{g}) = 0$ implies rigidity whereas the converse is not true (see §1.6.E, and also [14], [8], and [5]).

PROPOSITION 1. *Any semisimple Lie algebra $\mathfrak{g}$ is rigid, and* $\dim A(\mathfrak{g}) = n^2 - n$.

This proposition being both simple and important, we give three proofs as follows: 1. $H^2(\mathfrak{g},\mathfrak{g}) = 0$ for $\mathfrak{g}$ semisimple (see [5] and [19]). 2. Killing form being nondegenerate is an open condition. 3. Absence of abelian ideals is an open condition.

1.4. Let $\mathfrak{g}$ be an insoluble Lie algebra, and let $\Pi(\mathfrak{g})$ denote the pair $(\mathfrak{p}(\mathfrak{g}), M)$, M being the adjoint representation of $\mathfrak{p}(\mathfrak{g})$ on $\mathfrak{g}$. Suppose A is a component of Lie_n containing at least one insoluble algebra. Then, at almost all points $\mathfrak{g} \in A$, $\Pi(\mathfrak{g})$ is a constant function; for these points we set $\Pi(\mathfrak{g}) = (\mathfrak{p}_0, M_0)$.

PROPOSITION 2. *When $\mathfrak{g}$ runs through A, $\Pi(\mathfrak{g})$ runs through the set of all pairs of the form $(\mathfrak{p}_j, M_j)$, where $\mathfrak{p}_j \subset \mathfrak{p}_0$ is a semisimple subalgebra and M_j is the restriction of M_0 to $\mathfrak{p}_j$.*

PROOF. Existence. Let $\mathfrak{p}_j \subset \mathfrak{p}_0$ be a semisimple subalgebra, $x_1, \dots, x_\alpha$ its basis, and $y_1, \dots, y_\beta$ a basis of a $\mathfrak{p}_j$-invariant complement for $\mathfrak{p}_j$ in $\mathfrak{p}_0$; let $y_{\beta+1}, \dots, y_\gamma$ be a basis for the radical. Let $c_{ij}^k(\varepsilon)$ denote the structure constants of $\mathfrak{g}$ with respect to $x_1, \dots, x_\alpha, \varepsilon y_1, \dots, \varepsilon y_\gamma$. Then $c_{ij}^k(0)$ are the structure constants of an algebra $\mathfrak{a}$ such that $\Pi(\mathfrak{a}) = (\mathfrak{p}_j, M_0|_{\mathfrak{p}_j})$ and $\mathfrak{a}$ is a degeneration of $\mathfrak{g}$.

The remaining part of the proposition is implied by the following.

LEMMA. *Let $\mathfrak{g}(\varepsilon)$ be an analytic family of Lie algebras with $\mathfrak{g}(0)$ insoluble and let $\mathfrak{p}$ be a maximal semisimple subalgebra of $\mathfrak{g}(0)$. Then any deformation of $\mathfrak{g}(\varepsilon)$ may be reduced by an analytic change of basis to the form $\mathfrak{g}'(\varepsilon)$ such that $[x,a](\varepsilon) = [x,a](0)$ for any $x \in \mathfrak{p}$ and any $a \in \mathfrak{g}$.*

REMARK. Modulo $O(\varepsilon^2)$ our lemma is equivalent to the equality $H^2(\mathfrak{a},\mathfrak{a}) = H^2(\mathfrak{r}(\mathfrak{a}),\mathfrak{a})^{\mathfrak{p}}$ (see [19], Theorem 13).

We proceed by induction. Choose $x_j \in \mathfrak{p}$ and a_l in the radical, and write

$$[x_j, x_k](\varepsilon) = [x_j, x_k] + \varepsilon^m q(x_j, x_k) + O(\varepsilon^{m+1}),$$
$$[x_j, a_l](\varepsilon) = [x_j, a_l] + \varepsilon^m p(x_j, a_l) + O(\varepsilon^{m+1}).$$

Using the Jacobi identity for x_i, x_j, and x_k, we get $q \in Z^2(\mathfrak{p},\mathfrak{g})$. But $H^2(\mathfrak{p},\mathfrak{g}) = 0$ (see [5]); hence by a transformation of the form $x \to x + \varepsilon^m u(x)$ we can arrive at $q = 0$. But then the Jacobi identity for x_i, x_j, and a_k implies $p \in Z^1(\mathfrak{p}, \mathrm{Hom}(\mathfrak{r}(\mathfrak{g}),\mathfrak{g}))$. However $H^1(\mathfrak{p}, \mathrm{Hom}(\mathfrak{r}(\mathfrak{g}),\mathfrak{g})) = 0$ and therefore by a transformation of the form $a \to a + \varepsilon^m v(a)$ we can attain $p = 0$. Repeating the procedure, we derive the existence of the required transformation in the class of formal series. But this transformation satisfies a finite set of analytic equations; hence, by a lemma of M. Artin [7], the required transformation exists in the class of convergent series.

1.5. Root systems. Let $I \subset \mathfrak{g}$ be an ideal of codimension 1. A linear deformation of $\mathfrak{g}$ associated with I is any Lie algebra in which I is an ideal of codimension 1. It is obvious that the set of linear deformations is indexed by the elements of the linear space $\operatorname{der}(I)$.

PROPOSITION 3. *Let Q be a component of* Lie_n. *Then the spectrum of the operator* $\operatorname{Ad} x$ *on Q ($x \in \mathfrak{g}$, $\mathfrak{g} \in Q$) runs (up to a permutation) through the set of all collections of numbers satisfying a certain system of linear equations with integer coefficients.*

PROOF. a) Suppose the generic algebra $\mathfrak{g}$ of Q does not contain any ideal of codimension 1. Then the spectrum of the generic $\operatorname{Ad} x$, $x \in \mathfrak{g}$, has the same structure as the spectrum of the generic element of $\mathfrak{p}(\mathfrak{g})$ in the adjoint representation of $\mathfrak{p}(\mathfrak{g})$ on $\mathfrak{g}$, proving our claim. For in this case we have $\mathfrak{r}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{r}(\mathfrak{g})]$. But for any $\mathfrak{g}$ one has $[\mathfrak{g}, \mathfrak{r}(\mathfrak{g})] \subset \mathfrak{n}(\mathfrak{g})$ since $\mathfrak{g}/\mathfrak{n}(\mathfrak{g})$ is reductive. Therefore $\mathfrak{r}(\mathfrak{g})$ is nilpotent, and hence the spectrum of $\operatorname{Ad} x$, $x \in \mathfrak{g}$, is completely determined by the projection of x to $\mathfrak{p}(\mathfrak{g}) = \mathfrak{g}/\mathfrak{r}(\mathfrak{g})$.

b) Let the generic algebra $\mathfrak{g}$ of the component Q contain an ideal of codimension 1. Denote by K one of these ideals. Let $\mathscr{A}(\mathfrak{g}, K)$ denote the set of all linear deformations of $\mathfrak{g}$ associated with K. Then, obviously, for almost all $\mathfrak{g} \in Q$ one has $\mathscr{A}(\mathfrak{g}, K) \subset Q$.

The set of all spectra of $\operatorname{Ad} x|_K$ with $x \in a$, $a \in \mathscr{A}(\mathfrak{g}, K)$, coincides with that for all operators $y \in \operatorname{der}(K)$ acting on K. But $\operatorname{der}(K)$ is an algebraic algebra (see [1]) and hence the spectrum of y runs through the set of all solutions of a system of linear equations with integer coefficients (see [1], 7.7.3 and 7.7.9). We denote this system by $\Xi(\mathfrak{g}, K)$. Now let $\mathfrak{g}$ be fixed and let Ξ_0 be a system of linear equations with integer coefficients. Then the set of all $I \subset \mathfrak{g}$ for which Ξ_0 is a consequence of $\Xi(\mathfrak{g}, I)$ is the union of a finite number of IAF. Hence there exists a system $\Xi(\mathfrak{g})$ such that $\Xi(\mathfrak{g})$ is equivalent to $\Xi(\mathfrak{g}, I)$ for the I running through a set of full measure. Arguing similarly, we prove the existence of a system $\Xi(Q)$ such that, on a set of full measure in Q, $\Xi(\mathfrak{g})$ is equivalent to $\Xi(Q)$. The proof is complete.

REMARK 1. It is easy to see the existence of the unique component in Lie_n with the spectrum of Ad running through the set of all collections of numbers of the form $(0, \lambda_1, \dots, \lambda_{n-1})$, where the λ_j are arbitrary (see 1.6.B). Apparently there also exists a component with the spectrum of Ad of the form $(0, \dots, 0)$; that is, with all algebras in the component nilpotent.

REMARK 2. Let $x \in \mathfrak{g}$ be a generic element, and $\mathfrak{g} = \bigoplus \mathfrak{g}_\lambda$ the decomposition of $\mathfrak{g}$ into the direct sum of root spaces with respect to $\operatorname{Ad} x$. Then, obviously,

$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}. \tag{1}$$

For simplicity, let $\mathfrak{g}_\lambda$ and $\mathfrak{g}_\mu$ be one-dimensional and $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \neq 0$; also let $\mathfrak{g}(\varepsilon)$ be a small perturbation of $\mathfrak{g}$, and $\lambda(\varepsilon)$ and $\mu(\varepsilon)$ two eigenvalues of $\operatorname{Ad} x$ in $\mathfrak{g}(\varepsilon)$, with $\lambda(0) = \lambda$ and $\mu(0) = \mu$. Then the spectrum of $\operatorname{Ad} x$ in $\mathfrak{g}(\varepsilon)$ contains the eigenvalue $\lambda(\varepsilon) + \mu(\varepsilon)$.

1.6. The evaluation of deformations for particular $\mathfrak{g}$'s is usually greatly facilitated by the use of root systems and Proposition 2. It is also useful to involve various semicontinuous characteristics: dimensions of the center, of the nilradical, of the largest abelian ideal, and of the terms of various central series. (It should be noted that the property of being decomposable into the direct sum is neither open nor closed.) From time to time it is necessary to use some finer invariants such as the Jordan form for the generic $\operatorname{Ad} x$ or the dimension of the subspace spanned by $[x_1, [x_1, x_2]]$ and $[x_2, [x_1, x_2]]$ with x_1 and x_2 generic.

A.α. All algebras of the form $\mathrm{sl}(2) \ltimes (V_{2\alpha_1} \oplus \cdots \oplus V_{2\alpha_j})$ are rigid. Indeed, the summands in the decomposition of $V_{2k} \otimes V_{2l}$ have the form V_β with all β odd; hence the deformation of the radical cannot be endowed with a nontrivial multiplication. On the other hand the multiplicity of 0 in the spectrum of generic $\mathrm{Ad}\, x$ is equal to 1, and so the semisimple part cannot get larger in the process of deformation (see [3] and [9]).

β. Let $c(n)$ be the number of the components in Lie_n; then $c(2) = 1$, $c(3) = 2$, $c(4) = 4$, $c(5) = 7$, $c(6) = 17$, and $c(7) = 49$ (see [9] and [3]); it follows from Bézout's theorem that $c(n) < 2^{n^4}$ (see [3]). A certain complication of the argument in A.α shows that the direct sums of algebras in A.α are rigid. Let $q(n)$ denote the number of algebras arising in this way and, as usual, let $p(n)$ be the partition number. Then

$$\sum q(n)t^n = \prod_{k\geq 0} (1 - t^{2k+3})^{-p(k)}.$$

PROPOSITION. *$c(n)$ increases more rapidly than any function of the form* $\exp(n^\alpha)$ *with* $\alpha < 1$.

PROOF. A theorem of Meinardus ([6], 6.1) gives a lower bound (apparently a rough one) for $q(n)$.

γ. Any algebra of the form $\mathrm{sl}(2) \ltimes V_{2\alpha+1}$ is rigid except for $2\alpha + 1 = 3, 5, 7, 11$. An algebra of such a form can be obtained by degeneration only from a semisimple algebra of rank 2, since the multiplicity of zero in the spectrum of the generic $\mathrm{Ad}\, x$ is equal to 2. For exceptional $2\alpha + 1$ this possibility occurs (it suffices to choose the principal $\mathrm{sl}(2)$-triple in $\mathrm{sl}(2) \oplus \mathrm{sl}(2)$, $\mathrm{sl}(3)$, $\mathrm{o}(5)$, and g_2 respectively; see [14]).

δ. $\mathrm{sl}(2) \ltimes (V_2 \oplus V_1)$ can be obtained by degeneration from two rigid algebras: $\mathrm{sl}(2) \ltimes H_3$ and $\mathrm{Aff}(2)$.

B. EXAMPLE. Algebras with bases of the form $x_1, \ldots, x_k, y_1, \ldots, y_l$, $k \leq l$, and relations $[x_j, y_k] = \lambda_{jk} y_k$ form a set of generic points on a component of Lie_{k+l}.

PROOF. The argument of Remark 2 in 1.5 shows that the spectrum of perturbed $\mathrm{Ad}\, x$ has the form $(0, \ldots, 0, \mu_1, \ldots, \mu_l)$, where the μ_j satisfy no relations of the form $\mu_\alpha = \mu_\beta + \mu_\gamma$ or $\mu_\alpha = -\mu_\beta$. But then the $\mathrm{Ad}\, x$-invariant subspace I corresponding to the eigenvalues $\mu_1, \ldots, \mu_l$ by (1) is an abelian ideal. The subspace V in $\mathfrak{g}(\varepsilon)$ corresponding to the zeros of the spectrum is a subalgebra (by (1)), and its adjoint representation on I is faithful and semisimple (since $k \leq l$). Thus V is an abelian subalgebra, proving the claim. We remark that for $k = l$ the generic point of the component is $\mathrm{Aff}(1) \oplus \cdots \oplus \mathrm{Aff}(1)$.

C. An algebra of the form $H_3 \oplus \mathbf{C}^{k-3}$ belongs to the closure of any $\mathrm{GL}(k)$-orbit on Lie_k (the proof follows by a sequence of degenerations using linear deformations and the construction of Proposition 2).

D. $\mathrm{Aff}(2)$ and $\mathrm{sl}(2) \ltimes V_2$ are rigid and indecomposable algebras. Their direct sum is not rigid.

E. A standard way of constructing rigid soluble algebras is by "truncating" infinite-dimensional algebras. In particular, under minimal conditions of nondegeneracy algebras of the form $[x_i, x_j] = \alpha_{ij} x_{i+j}$, $0 \leq i, j \leq n$, have a habit of turning out to be rigid for suffciently large n (see [5] and [8]).

EXAMPLE. "*Truncated*" *Virasoro algebra* V_n: $[x_i, x_j] = (j - i)x_{i+j}$, with $0 \leq i \leq j \leq n$, $n \geq 13$. It is known that $H^2(V_n, V_n) = \mathbf{C}$ (see [5] and [8]). An argument similar to that used in 1.6.B shows that the direct sum of algebras of the form V_n is rigid.

COROLLARY. *The dimension of $H^2(\mathfrak{g}, \mathfrak{g})$ with $\mathfrak{g}$ rigid can be arbitrarily high.*

F. We consider the family Ω_α of Lie algebras with basis $x_1, \ldots, x_{2\alpha}, y_1, \ldots, y_\alpha$ and relations $[x_i, x_j] = \sum c_{ij}^k y_k$, where $c_{ij}^k = -c_{ji}^k$. Then Ω_α is an IAF of dimension

$2\alpha^3+O(\alpha^2)$. Hence Lie_n has a component of dimension at least $\frac{2}{27}n^3+O(n^2)$ (see [17]). Applying linear deformations to Ω_α, we obtain an IAF $\Lambda_\alpha \supset \Omega_\alpha$ whose points are Lie algebras which can be reduced to the form $[z,x_j]=x_j$, $[z,y_k]=2y_k$, $[x_i,x_j]=\sum c_{ij}^k y_k$ with $0<i<j\le 2\alpha-1$, $0<k\le\alpha$.

PROPOSITION. *Λ_α is a component of* $\mathrm{Lie}_{3\alpha}$ *for sufficiently large α.*

The proof is based on estimating the dimension of the set $Q\subset\Lambda_\alpha$ of algebras admitting deformations which take them out from Λ_α. The author knows of no particular examples of algebras from $\Lambda_\alpha\backslash Q$.

G. We know very little about the variety N_n of n-dimensional nilpotent algebras (see [17] and [3]). There is an essential misprint in [3]: the generic algebra of N_6 is an algebra with basis x_j, $j=1,2,3,4,7$, and relations $[x_i,x_j]=x_{i+j}$, where $i<\alpha$ (No. 21 from Morozov's list).

1.7. Sets of commuting matrices. Let $M_k(n)$ denote the variety of k-tuples of commuting operators in $\mathbf{C}^n$. The k-tuples of simultaneously diagonalizable matrices, obviously, form the set of generic points in a certain components of $M_k(n)$. A problem of the components of $M_k(n)$ arose in [11]. We will also encounter the varieties $M_k(\mathfrak{g})$ of k-tuples of commuting elements in a Lie algebra $\mathfrak{g}$ and, in the case of algebraic algebras, the varieties $M_k^0(\mathfrak{g})\subset M_k(\mathfrak{g})$ whose elements are k-tuples with no nilpotent linear combinations of components. All these varieties naturally arise in the following setting. We consider the space $\mathbf{C}^{n+k}$ with basis $x_1,\dots,x_k,y_1,\dots,y_n$, and we assume that the subspace spanned by $y_1,\dots,y_n$ is endowed with the structure of a nilpotent Lie algebra $\mathfrak{n}$. We denote by $\Phi R(k,\mathfrak{n})$ the variety of those Lie algebra structures $\mathfrak{g}$ in $\mathbf{C}^{n+k}$ for which there exists an exact sequence $0\to\mathfrak{n}\to\mathfrak{g}\to\mathbf{C}^k\to 0$.

Let $\Phi R^0(k,\mathfrak{n})\subset\Phi R(k,\mathfrak{n})$ be the variety of Lie algebras determined by the additional condition $\mathfrak{n}=\mathfrak{n}(\mathfrak{g})$. With each x_i one can associate an element $a_i'\in\mathrm{der}(\mathfrak{n})$ and, hence, an element $a_i\in D(\mathfrak{n})=\det(\mathfrak{n})/\mathrm{int}(\mathfrak{n})$. The following assertions are obvious.

LEMMA 1. a) *If* $\mathfrak{g}\in\Phi R(k,\mathfrak{n})$ *then* $(a_1,\dots,a_k)\in M_k(D(\mathfrak{n}))$.
b) *If* $\mathfrak{g}\in\Phi R^0(k,\mathfrak{n})$ *then* $(a_1,\dots,a_k)\in M_k^0(D(\mathfrak{n}))$.
c) $\Phi R(1,\mathfrak{n})$ *coincides with* $\det(\mathfrak{n})$.
d) $\Phi R(2,\mathfrak{n})$ (*respectively* $\Phi R^0(2,\mathfrak{n})$) *is a fiber bundle over* $M_2(D(\mathfrak{n}))$ (*respectively, over* $M_2^0(D(\mathfrak{n}))$ *with fiber* $\mathrm{int}(\mathfrak{n})\times\mathrm{int}(\mathfrak{n})\times Z(\mathfrak{n})$.
e) *If* $\mathrm{drk}(\mathfrak{n})<k$, *then* $M_k^0(D(\mathfrak{n}))$ (*hence* $\Phi R_k^0(k,\mathfrak{n})$) *is empty.*

PROPOSITION 4. a) *$M_2(n)$ is irreducible* [11].
b) *$M_k(2)$ and $M_k(3)$ are irreducible.*
c) *$M_k(n)$ with $k\ge 4$ and $n\ge 4$ are reducible.*
d) *$M_3^0(4)$ is irreducible.*

PROOF. a) Let $(A,B)\in M_2(n)$. We need to show that A and B can be simultaneously brought to diagonal form after a suitable small deformation. Let $W_{\lambda,\mu}=\mathrm{Ker}(A-\lambda E)^n\cap\mathrm{Ker}(B-\mu E)^n$ be joint root spaces for A and B. It is clear that C^n is the direct sum of all nonzero spaces of the form $W_{\lambda,\mu}$ and that it suffices to construct the desired deformation in each $W_{\lambda,\mu}$ separately. Now without any loss of generality we may assume A and B nilpotent. Then there exist p and q such that $R=pA+qB$ is decomposable. Let C be a semisimple operator commuting with R, $C\ne\gamma E$. We consider a small deformation of the form $(A,B)\to(A-\varepsilon qC,B+\varepsilon pC)\in M_2(k)$. Now $A-\varepsilon qC$ has at least two distinct eigenvalues and it suffices to construct the desired deformation in the joint root spaces for $A-\varepsilon qC$ and $B+\varepsilon pC$, whose dimension is strictly less than k.

b) and c). See [3].

d) We choose $(A,B,C) \in M_3^0(4)$. Then (A,B,C) has at least 3 nonzero joint root subspaces and the problem reduces to $M_3(2)$.

PROPOSITION 5. a) *$M_2(\mathfrak{p})$, with $\mathfrak{p}$ semisimple, is irreducible* [15].
b) *$M_2(\mathrm{Aff}(k))$ is irreducible.*

The proof is exactly the same as in the case of $M_2(n)$. In part b) it is sufficient to observe that $\mathrm{Aff}(k)$ is the annihilator of a vector in $\mathrm{gl}(k+1)$ and that the deformation in the proof of Proposition 4a) can be chosen in the class of operators annihilating the same vector. (This can be derived from the following elementary assertion. Let A be a nilpotent operator in a finite-dimensional space W, $\dim \mathrm{Ker}\, A > 1$, and $v \in \mathrm{Ker}\, A$. Then there exist nonzero invariant subspaces W_1 and W_2 such that $W = W_1 \oplus W_2$ and $v \in W_1$.)

Let $\mathfrak{n}$ be a nilpotent algebra.

LEMMA 2. a) *If $\dim \mathfrak{n} \leq 5$, then there exists a derivation of $\mathfrak{n}$ all of whose eigenvalues are distinct and nonzero.*
b) *Let $\dim \mathfrak{n} \leq 5$, and let $T \subset \mathrm{Aut}(\mathfrak{n})$ be a maximal torus. Then $\mathfrak{n}$ is rigid in the class of nilpotent algebras where T acts by automorphisms.*
c) *If $\dim \mathfrak{n} = 6$, then there exists a derivation of $\mathfrak{n}$ with all eigenvalues different from zero.*

The proof is a trivial case-by-case analysis (see also [3]).

LEMMA 3. a) *If $\dim \mathfrak{n} \leq 5$, then $M_2^0(D(\mathfrak{n}))$ is either irreducible or empty.*
b) *If $\dim \mathfrak{n} \leq 4$, then $M_3^0(D(\mathfrak{n}))$ is either irreducible or empty.*
The generic point of these varieties is the set of elements belonging to a single toroidal subalgebra.

PROOF. First we remark that the cases $\mathfrak{n} = \mathbf{C}^p$ and $\mathfrak{n} = H_5$ immediately follow from Propositions 4 and 5 ($D(H_5) = \mathrm{sp}(4)$). Let $(x^{(1)}, \dots, x^{(k)}) \in M_k^0(D(\mathfrak{n}))$. Let $x^{(j)} = x_s^{(j)} + x_n^{(j)}$ be the Jordan decomposition, $x_s^{(j)}$ being semisimple and $x_n^{(j)}$ being nilpotent. Then the set of $x_s^{(j)}$ generates a k-dimensional subalgebra X_s in a toroidal subalgebra $A \subset D(\mathfrak{n})$, $x^{(j)}$ lying in the centralizer of X_s. It turns out that in almost all cases one has $\mathrm{drk}(\mathfrak{n}) < k$ or $\mathrm{drk}(\mathfrak{n}) = k$. Referring to Lemma 1e) or Lemma 2a) respectively completes the proof.

The only cases remaining are those of $k = 2$ and $\mathfrak{n} = H_3 \oplus \mathbf{C}$, $H_3 \oplus \mathbf{C}^2$, $\Gamma_4 + \mathbf{C}$ and $g_{5,2}$ ($[x_1, x_2] = x_3$, $[x_1, x_4] = x_5$). We fix a toroidal subalgebra A. It turns out that every centralizer $C(X_s)$ of a two-dimensional subalgebra $X_s \subset A$ is equal to the direct sum of some copies of $\mathrm{sl}(2)$, $\mathbf{C}$, $\mathrm{Aff}(1)$, and $\mathrm{Aff}(2)$. For each of these latter algebras M_2 is an irreducible variety. Transforming $(x_1, x_2) \in M_2(C(X_s))$ into general position, we complete the proof.

1.8. Classification of the components of Lie_n **for** $n \leq 7$. In 1.6.A we have actually classified the components with at least one insoluble algebra. Now let R_n be the variety of structure constants of all n-dimensional soluble Lie algebras. Let $\mathfrak{n}$ be nilpotent and $\mathrm{drk}(\mathfrak{n}) \geq k \geq 1$. We denote by $R^0(k, \mathfrak{n})$ the set of all Lie algebras $\mathfrak{g}$ such that $\mathfrak{n} = \mathfrak{n}(\mathfrak{g})$ and $\mathfrak{g}/\mathfrak{n} = \mathbf{C}^k$ (note that automatically $k \leq \dim \mathfrak{n}$).

THEOREM 1. *The set of closures of $R^0(n - \dim \mathfrak{n}, \mathfrak{n})$ coincides with the set of all irreducible components of R_n for $n \leq 7$.*

REMARK. When $n \leq 7$ all the components of R_n are also the components of Lie_n.

This theorem was discovered independently by Roger Carles (1979) and the author (1981); for $n \leq 5$ it was proved by S. E. Belkin even earlier (1976). Since no complete proof has bee published so far, we give it in what follows. For detailed tables see [8] and [3].

A. The main difficulty is proving the irreducibility of $R^0(k,\mathfrak{n})$. Let $\dim \mathfrak{n} = t$ (then $k+t=n$). Obviously $R^0(k,\mathfrak{n})$ is the fiber bundle over $\mathrm{GL}(\mathfrak{n})/\mathrm{Aut}(\mathfrak{n})$ with fiber $\Phi R^0(k,\mathfrak{n})$ (see 1.7). The base of the bundle is always irreducible. If $k=1$ or 2, then the irreducibility of $\Phi R^0(k,\mathfrak{n})$ follows from Lemmas 1c), 1d), and 3a). If $k=3$, then the canonical fibering $\nu : \Phi R^0(3,\mathfrak{n}) \to M_3^0(D(\mathfrak{n}))$ may have fibers of different dimension, and this might give an "extra" component. This difficulty arises only if $n=7$ and $\mathfrak{n} = \mathbf{C}^4$ or $\mathfrak{n} = H_3 \oplus \mathbf{C}$. But all the algebras in $\Phi R^0(3, H_3 \oplus \mathbf{C})$ are pairwise isomorphic; hence $\Phi R^0(3, H_3\oplus\mathbf{C})$ is irreducible. If $\mathfrak{n} = \mathbf{C}^4$, then the dimension of a fiber can "jump" only on that algebra $\mathfrak{g}$ for which $\nu(\mathfrak{g})$ consists of triples of operators with common kernel (consequently since $\nu(\mathfrak{g}) \in M_3^0(4)$, these operators can be simultaneously brought to the diagonal form). Any such $\mathfrak{g}$ can be brought to the form (see 1.6.B)

$$[x_j, y_j] = y_j, \quad [x_i, x_j] = c_{ij} y_4, \qquad i,j = 1,2,3.$$

But by applying to $\mathfrak{g}$ deformations of the form $[x_j, y_4] = \varepsilon_j y_4$ we can bring $\nu(\mathfrak{g})$ into general position $(c_{23}\varepsilon_1 + c_{31}\varepsilon_2 + c_{12}\varepsilon_3 = 0)$.

B. Suppose $\overline{R^0(k_1,\mathfrak{n}_1)} \supset R^0(k_2,\mathfrak{n}_2)$. At the generic point of $R^0(k,\mathfrak{n})$ we have $[\mathfrak{g},\mathfrak{g}] = \mathfrak{n}$ (Lemma 2c)). Since the dimension of $\mathfrak{n}(\mathfrak{g})$ under small deformations can only get smaller while that of $[\mathfrak{g},\mathfrak{g}]$ can only get larger, we see that $k_1 = k_2$, say $k_1 = k_2 = k$. Now the dimension of $R(1,\mathfrak{n})$ is easy to compute, and it always equals $n^2 - n$ [3]; hence $k=1$ is impossible. As for $k=2$ and $k=3$, these are in contradiction with Lemma 2b).

C. $R_n = \bigcup \overline{R^0(k,\mathfrak{n})}$. Indeed, every algebra outside $\bigcup R^0(k,\mathfrak{n})$ is nilpotent. Now let I be an ideal of codimension 1 in a nilpotent algebra $\mathfrak{g}$. Then $\mathfrak{g} \in R^0(1,I)$ follows by Lemma 2c).

§2. Estimating the number of parameters

In 2.1 we establish the lower bound for the dimension of components in Lie_n, the upper bound being established in 2.2–2.6.

2.1. THEOREM 2. *The dimension of any component of* Lie_n *is at least* $\frac{3}{4}n^2 + \frac{1}{2}n - \frac{9}{4}$. *If n is odd then this value is achieved on the component containing the rigid algebra $\Gamma = \mathrm{sl}(2) \ltimes W$, where W is the direct sum of $(n-3)/2$ copies of V_2.*

The rigidity of Γ has been proved in 1.4.α. On the other hand,

$$\dim \mathrm{Aut}(\Gamma) = \tfrac{1}{4}n^2 + O(n);$$

since with every module automorphism of W one can associate an automorphism of Γ.

LEMMA. *If $\mathfrak{g}$ contains an ideal I of codimension 1, then $\mathfrak{g}$ belongs to an IAF of dimension greater or equal to $(n-1)^2$.*

The proof follows by applying changes of bases in I and linear deformations associated with I.

Now the theorem is a consequence of the following.

LEMMA. *The maximum value of* $\dim \mathrm{Aut}(\mathfrak{g})$ *for $\mathfrak{g}$ without ideals of codimension 1 is attained on Γ.*

PROOF. To start, we remark that $\mathfrak{g}$ is insoluble, $\mathfrak{r}(\mathfrak{g})$ is nilpotent, and $\mathfrak{r}(\mathfrak{g})/[\mathfrak{r}(\mathfrak{g}), \mathfrak{r}(\mathfrak{g})]$ has no $\mathfrak{p}(\mathfrak{g})$-invariants.

We consider all algebras with fixed semisimple part $\mathfrak{p}$, $\mathfrak{g} = \mathfrak{p} \ltimes \mathfrak{r}$, where $\mathfrak{r} = \mathfrak{r}(\mathfrak{g})$. It follows from the Levi-Mal'tsev theorem that, as a linear space, $\mathfrak{g}$ is the direct sum of two subalgebras: 1. Inner derivations. 2. Derivations annihilated on $\mathfrak{p}$. The latter subalgebra corresponds to a subgroup N in $\mathrm{Aut}(\mathfrak{g})$ leaving every element of $\mathfrak{p}$ invariant. In particular, the elements of N are the automorphisms of $\mathfrak{r}$ as a $\mathfrak{p}$-module. Let $\mathfrak{r} = \bigoplus_{i\geq 0} n_i\sigma_i$ be the isotypic decomposition of $\mathfrak{r}$ in which each σ_i is an irreducible $\mathfrak{p}$-module and σ_0 is a one-dimensional $\mathfrak{p}$-module. Let V be a $\mathfrak{p}$-invariant complement for $[\mathfrak{r},\mathfrak{r}]$ in $\mathfrak{r}$. Then any $\rho \in N$ is uniquely determined by its values on V (see 2.4) and ρ preserves the isotypic components. Hence $\dim N \leq \sum_{i>0} n_i^2$. Now the claim becomes obvious.

2.2. We have seen in 1.4 F that there exists a component of Lie_n whose dimension is of the form $\frac{2}{27}n^3 + O(n^2)$.

THEOREM 3. *The dimension of any component of* Lie_n *is at most* $\frac{2}{27}n^3 + O(n^{8/3})$.

COROLLARY. *Suppose we are given a family of nonisomorphic Lie algebras analytically depending on a certain set of parameters. Then the numbers of parameters does not exceed* $\frac{2}{27}n^3 + O(n^{8/3})$.

Now let A be an IAF. Our aim is to determine A by as few parameters as possible. We will be parametrizing an open dense subset A' in A whose points enjoy all properties of generic points which are needed in what follows (formally speaking we will be constructing an injection from A' into $\mathbf{C}^N$). Since $\dim \mathrm{GL}(n) = O(n^2)$, we will usually be assuming that all algebras in A have the same flag of ideals (for instance, that the subspaces $[\mathfrak{g},\mathfrak{g}]$ and $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]$ for all $\mathfrak{g} \in A$ are the same).

2.3. Reductions.

LEMMA 1. *Let A be an IAF whose points are insoluble. Then there exists an IAF of soluble algebras* $B \subset \overline{A}$ *such that* $\dim B = \dim A - O(n^2)$.

For the proof it suffices to consider the family of radicals in algebras of A.

LEMMA 2. *Let $\mathfrak{g}$ be an algebraic Lie algebra. Then the variety of nilpotent elements of $\mathfrak{g}$ is irreducible and its codimension is equal to* $\mathrm{rk}(\mathfrak{g})$.

PROOF. $\mathfrak{g}$ is the semidirect product of its reductive part by its nilradical ([1], 10.6). But for semisimple algebras our lemma is well known ([1], 11.3).

Let Ω be an IAF of soluble algebras with basis $x_1,\dots,x_n$, the nilradical of every $\mathfrak{g}\in\Omega$ being the linear span of $x_{r+1},\dots,x_n$. We extend Ω to a maximal IAF Ω' with the same property. Then Ω' contains all linear deformations of generic algebras $\mathfrak{g}\in\Omega'$ associated with the ideal $I = \{x_1,\dots,x_{r-1},x_{r+1},\dots,x_n\}$. These deformations are indexed by the space $\mathrm{der}(I)$ and, by Lemma 2, Ω' contains an IAF Ω'' of codimension $O(n)$ with nilradical spanned by $x_r,\dots,x_n$. Repeating this procedure, we arrive at the following.

LEMMA 3. *If there exists an IAF of soluble algebras whose dimension is N, then there exists an IAF of nilpotent algebras whose dimension is* $N - O(n^2)$.

2.4. In what follows $\mathfrak{n}$ stands for a nilpotent algebra, $\mathfrak{n}^{(2)} = [\mathfrak{n},\mathfrak{n}]$, and $\mathfrak{n}^{(j+1)} = [\mathfrak{n}^{(j)},\mathfrak{n}]$. We recall that any minimal generating set of $\mathfrak{n}$ is a basis of a complement for $\mathfrak{n}^{(2)}$ in $\mathfrak{n}$.

DEFINITION. Let $x_1,\dots,x_k$ be a generating set for $\mathfrak{n}$. We say that the generators $x_{s+1},\dots,x_n$ are *superfluous* if any element of $[\mathfrak{n},\mathfrak{n}]$ can be expressed in terms of "essential" generators $x_1,\dots,x_s$.

LEMMA 4 (on superfluous generators). *Generators $x_{s+1},\dots,x_n$ are superfluous in $\mathfrak{n}$ if and only if they are superfluous in* $\mathfrak{n}/\mathfrak{n}^{(3)}$.

PROOF. Let S be a commutator of length $k > 1$ in $x_1,\dots,x_n$. Any occurrence of $[x_i,x_j]$ in S can be replaced (modulo $\mathfrak{n}^{(3)}$) by a linear combination of commutators in $x_1, \dots, x_s$. Using the Jacobi identity, we conclude that (modulo $\mathfrak{n}^{(k+1)}$) S can be expressed in terms of $x_1, \dots, x_s$.

Let V and W be linear spaces, $\dim V = k$, and $\dim W = 1$. Then any linear map $\Lambda^2 V \to W$ gives $V \oplus W$ the structure of a metabelian algebra (see 1.6.F). We will always assume that $[V, V] = W$. Let $x_1, \dots, x_k$ be a generic basis in V, V_t the span of $x_1, \dots, x_t$, and $W_t = [V_t, V_t]$. It is obvious that $W_t \subset W_{t+1}$; let s be the first index such that $W_s = W$.

LEMMA 5. *$\dim W_j$ is strictly increasing when $1 \le j \le s$.*

PROOF. Let $\langle a_1, \dots, a_k \rangle$ denote the linear span of all commutators $[a_i, a_j]$, $i, j \le k$, $a_i \in V$. Suppose that $\langle x_1, \dots, x_u \rangle = \langle x_1, \dots, x_{u+1} \rangle = W_u$. Since $x_1, \dots, x_k$ is a generic basis, we have

$$\begin{aligned} W_u &= \langle x_1, \dots, x_{u+1} \rangle = \langle x_1, \dots, x_u, x_{u+2} \rangle \\ &= \langle x_2, \dots, x_{u+2} \rangle = \langle x_1, \dots, x_{u+2} \rangle, \end{aligned}$$

and hence $W_u = W_{u+1} = W_{u+2} = \cdots$.

It is obvious that for generic metabelian algebras one has $s \approx \sqrt{2l}$. A p-group analog of the next lemma was proved in [16].

SIM'S LEMMA. *Let L_s be the variety of structures of metabelian algebras of the form mentioned above ($[V, V] = W$) with exactly s essential generators. Then the dimension of any component of L_s is at most $\frac{1}{2}k^2(l - s) + O((k + l)^{8/3})$.*

PROOF. Let $V = \bigoplus V_\alpha$ be the direct sum decomposition with $k^{1/3} + O(1)$ summands of dimension $k^{2/3} + O(1)$ in general position. By Lemma 5 one has

$$\dim[V_\alpha \oplus V_\beta, V_\alpha \oplus V_\beta] \le l - s + 2k^{2/3} + O(1).$$

For each pair α, β we choose a subspace $[V_\alpha \oplus V_\beta, V_\alpha \oplus V_\beta] = W_{\alpha,\beta}$. Any of these subspaces can be given by $k^{2/3} O(l^2)$ parameters. If x_i and x_j are basic elements of V, then the choice of $[x_i, x_j] \in W_{\alpha,\beta}$ is determined by $l - s + O(k^{2/3})$ parameters. But the total number of these commutators is $k(k-1)/2$. Thus the proof is complete.

2.5. The procedure of choosing parameters. Let Ω be an IAF of nilpotent algebras. Let all the algebras of this IAF have the same flag of ideals including all the terms of the lower central series $\mathfrak{n}^{(k)}$. Fix $\mathfrak{g} \in \Omega$ with $\dim \mathfrak{g} = n$, $\dim \mathfrak{g}/\mathfrak{g}^{(2)} = m$, and $\dim \mathfrak{g}^{(2)}/\mathfrak{g}^{(3)} = k$. We also fix a generating set $x_1, \dots, x_m$ such that $x_1, \dots, x_s$ is a minimal set of essential generators.

α. We choose commutators $[x_i, x_j] \in \mathfrak{g}^{(2)}$ modulo $\mathfrak{g}^{(3)}$. By Sims's Lemma this requires at most $\frac{1}{2}m^2(k - s) + O(n^{8/3})$ parameters.

β. Since $[x_i, x_j]$ are known modulo $\mathfrak{g}^{(3)}$, to define them completely we need another portion of $\frac{1}{2}m^2(n - m - k)$ parameters (we need to specify a linear map $\Lambda^2(\mathfrak{g}/\mathfrak{g}^{(2)}) \to \mathfrak{g}^{(3)}$).

γ. LEMMA 6. *Suppose $[x_i, x_j]$ are known. To recover the Lie algebra structure on $\mathfrak{g}$ it is sufficient to give the operators $\operatorname{Ad} x_1, \dots, \operatorname{Ad} x_s$ on $\mathfrak{g}^{(2)}$.*

This is implied by the following.

LEMMA 7. *If $\operatorname{Ad} x_1, \dots, \operatorname{Ad} x_u$ are known, then $\operatorname{Ad} x_{u+1}$ is uniquely defined on the subalgebra $V_u \subset \mathfrak{g}$ generated by $x_1, \dots, x_u$.*

PROOF. If $w \in V_u$, then $(\mathrm{Ad}\, x_{u+1})w$ can be represented in the form

$$\sum_j c_j \prod_i (\mathrm{Ad}\, x_{\alpha_{ij}}) x_{u+1}, \qquad \alpha_{ij} \le u,$$

proving our lemma.

Since $\mathrm{Ad}\, x_1$ is triangular on $\mathfrak{g}^{(3)}$ and $(\mathrm{Ad}\, x_1)\mathfrak{g}^{(2)} \subset \mathfrak{g}^{(3)}$, it can be determined by at most $\frac{1}{2}((n-m)^2 - k^2)$ parameters. By Lemma 4 the commutators $[x_i, x_j]$, $i, j \le u$, as elements of $\mathfrak{g}^{(2)}/\mathfrak{g}^{(3)}$ generate a subspace of dimension at least $u-1$. Thus if we know $\mathrm{Ad}\, x_1, \dots, \mathrm{Ad}\, x_u$, then we already know $\mathrm{Ad}\, x_{u+1}$ on $u-1$ vectors, so that to determine it completely we need at most $\frac{1}{2}((n-m)^2 - k^2) - (u-1)(n-m-k)$ parameters. Finally, all of $\mathrm{Ad}\, x_1, \dots, \mathrm{Ad}\, x_s$ are determined by at most

$$(s/2)((n-m)^2 - k^2) - (s^2/2)(n-m-k) + O(n^2)$$

parameters.

2.6. Let $x = m/n$, $y = k/n$, and $p = s/n$. We need to estimate the maximal value of the cubic form

$$R = \frac{x^2}{2}(1-x-p) + \frac{p}{2}((1-x)^2 - y^2) + \frac{p^2}{2}(1-x-y)$$

on the simplex $0 \le p \le x$, y, $x + y \le 1$ ($s \le m$ follows from Lemma 4). By rather cumbersome calculations we determine that $R \le 2/27$ and that the maximal value is achieved on the line $x = 2/3$, $p = 0$ ($\partial R/\partial y < 0$; hence $y = p$, and then we determine the maximum varying p with x fixed).

REMARK 1. Apparently, the following is true: if Ω is an IAF of dimension $\frac{2}{27}n^3 + o(n^3)$, then for almost all $\mathfrak{g} \in \Omega$ one has $\dim \mathfrak{n}(\mathfrak{g}) = n - o(n)$, $\dim \mathfrak{n}(\mathfrak{g})^{(2)} = \frac{2}{3}n + o(n)$, and $\dim \mathfrak{n}(\mathfrak{g})^{(3)} = o(n)$.

REMARK 2. We say that a generating set $x_1, \dots, x_\alpha$ of an algebra $\mathfrak{g}$ is *regular* if $x_3, \dots, x_\alpha \in \mathfrak{r}(\mathfrak{g})$ and $x_1, x_2 \in \mathfrak{p}(\mathfrak{g})$ (if $\mathfrak{p}(\mathfrak{g})$ is nonempty). The dimension of the variety of structure constants of Lie algebras with α regular generators does not exceed $(\alpha+1)n^2/2$.

§3. The varieties Assoc_n and Comm_n

3.1. Let A be an associative algebra with unity. Then by the Wedderburn-Mal′tsev theorem $A = M \leftthreetimes N$, where M is a semisimple algebra (hence $M = \bigoplus \mathrm{Mat}(k)$, where $\mathrm{Mat}(k)$ is the full matrix algebra in $\mathbf{C}^k$) and N the nilpotent radical. Clearly, each N is the direct sum of irreducible bimodules (each of them being, in fact, the set of all matrices of order $r \times l$) on which $\mathrm{Mat}(r)$ acts by multiplications on the left and $\mathrm{Mat}(l)$ on the right.

Let $\Pi(A)$ denote the pair (M, Q), where $Q = A$ as an M-bimodule. Suppose at a generic point of a component Ω one has $\Pi(A) = (M_0, N_0)$.

PROPOSITION 2′. *The set of all values of* $\Pi(A)$ *with* $A \in \Omega$ *is equal to the set of all pairs* (M_j, Q_j) *such that* M_j *is a semisimple subalgebra of* M_0 *and* $Q_j = N_0|_{M_j}$.

REMARK. The role of the root system here is played by the joint spectrum of the operators $L(x)$ and $R(x)$ of respectively left and right multiplication by the generic element $x \in A$. However (in contrast to Lie_n) this joint spectrum is completely recovered from the bimodule structure on N.

3.2. Theorem 3′. *The dimension of any component of* Assoc_n *never exceeds* $\frac{4}{27}n^3 + O(n^{8/3})$.

THEOREM 3″. *The dimension of any component of* Comm_n *never exceeds* $\frac{2}{27}n^3 + O(n^{8/3})$.

The proof is along the lines of Theorem 3, with no use of Lemmas 2 and 3.

3.3. Let A be an algebra with basis e, f, $x_1, \dots, x_{n-2}$, where e and f are orthogonal idempotents. Also let

$$ex_j = x_j f = x_j, \qquad x_j e = f x_j = 0.$$

It is obvious then that A is rigid and that the dimension of the corresponding component has the form $4n + O(1)$.

3.4. The classification of components in Assoc_n is known up to $n = 5$ [12]. The structure of Comm_n is virtually unknown; see [13]. (Similar questions were considered in [4].) Proposition $2'$ in this case is still true, but trivial.

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