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ALMOST INVARIANT STRUCTURES AND CONSTRUCTIONS OF UNITARY REPRESENTATIONS OF THE GROUP OF DIFFEOMORPHISMS OF THE CIRCLE

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YU. A. NERETIN

In this note we give constructions of a number of series of unitary (projective) representations of the group Diff of orientation-preserving diffeomorphisms of the circle S^1. For exceptional values of the parameters we will obtain highest weight representations. The remaining values of the parameters give unknown representations (some special cases were studied previously in [5]–[7]).

For each unitary (perhaps in an indefinite metric) representation ρ of the group $\mathrm{SL}_2(\mathbf{R})$ (or its universal covering $\mathrm{SL}_2^{\sim}(\mathbf{R})$) we construct an embedding of the group Diff (or its universal covering $\mathrm{Diff}^{\sim}$) into some (G, K)-pair (see §1). The choice of (G, K)-pair depends on whether ρ is of real, complex or quaternionic type (see [12]). We shall also use the constructions of representations of (G, K)-pairs from [2].

1. (G, K)-pairs. Let $\mathrm{GL}(\infty), \mathrm{U}(\infty)$, and $\mathrm{O}(\infty)$ respectively denote the general linear group, the (full) unitary group, and the (full) orthogonal group of Hilbert space, and let $\mathrm{Sp}(\infty)$ denote the (full) unitary group of quaternionic Hilbert space, etc.

DEFINITION. Let $G(\infty) \supset K(\infty)$ be groups of this form. Then $(G(\infty), K(\infty))$ is the subgroup of $G(\infty)$ consisting of all operators of the form $A(1+T)$, where $A \in K(\infty)$ and T is a Hilbert-Schmidt operator.

The classical examples are

$$(\mathrm{O}(2\infty), \mathrm{U}(\infty)), \quad (\mathrm{Sp}(2\infty, \mathbf{R}), \mathrm{U}(\infty)), \quad \text{and} \quad (\mathrm{GL}(\infty, \mathbf{R}), \mathrm{O}(\infty));$$

see [1], and also [5]–[11]. In [2] a large collection of representations was constructed for a number of other "natural" (G, K)-pairs (approximately the same result in another language we obtained by Vershik and Kerov in [11]).

2. Example: almost invariant quaternionic structure. Let $\mathrm{Diff}^{(2)}$ be the double covering of Diff, realized as the group of diffeomorphisms of $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ that satisfy the conditions $q(\varphi+\pi) = q(\varphi)+\pi$. Let L_2^- be the subspace of $L_2(S^1)$ consisting of the odd functions $(f(\varphi+\pi) = -f(\varphi))$. Assume that $\mathrm{Diff}^{(2)}$ acts in L_2^- by the formula

$$T(q)f(\varphi) = f(q(\varphi))q'(\varphi)^{(1+is)/2}, \tag{1}$$

where $s \in \mathbf{R}$. It is not complicated to check that the operator

$$Kf(\varphi) = \frac{sB((is+2)/2, is/2)}{4\pi} \mathrm{p.v.} \int_0^{2\pi} \frac{\mathrm{sgn}(\sin(\varphi-\psi))\overline{f(\psi)}\,d\psi}{|\sin(\varphi-\psi)|^{1+is}}$$

gives a quaternionic structure ($K^2 = -1$, $iK = -Ki$) in L_2^-.

THEOREM. *Formula* (1) *gives an embedding of* $\mathrm{Diff}^{(2)}$ *into the group* $(\mathrm{U}(2\infty), \mathrm{Sp}(\infty))$.

The next step: imbed $(\mathrm{U}(2\infty), \mathrm{Sp}(\infty))$ in $(\mathrm{O}(4\infty), \mathrm{U}(2\infty))$ and restrict the spinor representation of $(\mathrm{O}(4\infty), \mathrm{U}(2\infty))$ (see [1], §5) to $\mathrm{Diff}^{(2)}$. For this we need to "cancel"

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inary units (they are parametrized by the sphere S^3, but in fact only one parameter is essential). It is less obvious that there exists a one-parameter series of imbeddings of $(\mathrm{U}(2\infty), \mathrm{Sp}(\infty))$ into $(\mathrm{Sp}(2\infty, \mathbf{R}), \mathrm{U}(\infty))$—see [2] (and then we can apply the Weyl representation; see [1], §4).

3. Notations. A. We realize $\mathrm{Diff}^{\sim}$ as the group of diffeomorphisms of $\mathbf{R}$ satisfying the condition $q(\varphi + 2\pi) = q(\varphi) + 2\pi$. Assume that $-1 < \alpha \leq 1$. Then H_α is the space of C^∞-functions on $\mathbf{R}$ satisfying the condition $f(\varphi + 2\pi) = \exp(i\alpha\pi)f(\varphi)$. In H_α we introduce a basis of functions $v_{k,\alpha} = \exp(i(k + \alpha/2)\varphi)$. Let the group $\mathrm{Diff}^{\sim}$ act in H_α by the formula

$$T_{\alpha,s}(q)f(\varphi) = f(q(\varphi))q'(\varphi)^{(1+s)/2}, \tag{2}$$

where $s \in \mathbf{C}$. We introduce an operator $A_{\alpha,s}\colon H_\alpha \to H_\alpha$:

$$A_{\alpha,s}f(\varphi) = C(\alpha, s) \int_0^{2\pi} \frac{\exp(2\pi i[(\varphi - \psi)/2\pi])f(\psi)\, d\psi}{|\sin((\varphi - \psi)/2)|^{1-s}} \tag{3}$$

where $C(\alpha, s)$ is determined from the condition $A_{\alpha,s}v_\alpha = v_\alpha$. Then

$$A_{\alpha,s}v_k = \frac{2^{2-s}\pi \exp(\alpha\pi/2)C(\alpha, s)}{sB(k + (s + \alpha - 1)/2, -k + (s - \alpha + 1)/2)} v_k. \tag{4}$$

From this we see that $A_{\alpha,s}$ is a well-defined operator-valued function meromorphic with respect to (α, s) with poles lying on the lines $\frac{1}{2}(\alpha \pm s + 1) \in \mathbf{Z}$, and $A_{\alpha,s} = A^{-1}_{\alpha,-s}$. On H_α we introduce a family of Hermitian forms ($t \in \mathbf{R}$):

$$\langle f, g\rangle_0 = \int_0^{2\pi} fg\, d\varphi, \qquad \langle f, g\rangle_t = \langle A_{\alpha,t}f, g\rangle.$$

B. *Motivation for these formulas.* Let the group $\mathrm{PSL}_2(\mathbf{R}) = \mathrm{SL}_2(\mathbf{R})/\mathbf{Z}_2$ act on S^1 as on the projective line, and let $\mathfrak{G}$ be the inverse image of $\mathrm{PSL}_2(\mathbf{R})$ under the homomorphism $\mathrm{Diff}^{\sim} \to \mathrm{Diff}$. It is obvious that $\mathfrak{G} \simeq \mathrm{SL}_2^{\sim}(\mathbf{R})$.

The restriction $P_{\alpha,s}$ of the representation $T_{\alpha,s}$ to the subgroup $\mathfrak{G}$ is a representation of $\mathrm{SL}_2^{\sim}(\mathbf{R})$ of the principal (nonunitary) series. If $is \in \mathbf{R}$, then $P_{\alpha,s}$ is unitary in the metric $\langle\cdot,\cdot\rangle_0$ (the principal unitary series of $\mathrm{SL}_2^{\sim}(\mathbf{R})$). If $s \in \mathbf{R}$, then $P_{\alpha,s}$ is unitary in the metric (generally indefinite) $\langle\cdot,\cdot\rangle_s$. If, in addition, $-1 < s \pm \alpha < 1$, then we obtain the complementary unitary series of $\mathrm{SL}_2^{\sim}(\mathbf{R})$. The poles of formula (3) correspond to highest (lowest) weight representations. In all the remaining cases $A_{\alpha,s}$ is an operator which intertwines $T_{\alpha,s}$ and $T_{\alpha,-s}$ (see [3] and [4]).

C. The space H_0 is realized in a natural way as $C^\infty(S^1)$, and H_1 as the space of odd functions in $C^\infty(S^1)$. Let $H_0^{\mathbf{R}}$ and $H_1^{\mathbf{R}}$ be the spaces of real functions from H_0. and H_1. We also introduce the space $\overline{H}_0^{\mathbf{R}}$ of real functions from H_0 with zero mean ($\int f\, d\varphi = 0$) and the space $\hat{H}_0^{\mathbf{R}}$ which is the quotient of $H_0^{\mathbf{R}}$ by the subspace of constants. In all these spaces we introduce the Hilbert transform

$$If(\varphi) = \frac{1}{\pi}\int_0^{2\pi} \cot\left(\frac{\varphi - \psi}{2}\right) f(\psi)\, d\psi.$$

The operator I introduces a complex structure in the spaces $H_1^{\mathbf{R}}$, $\overline{H}_0^{\mathbf{R}}$, and $\hat{H}_0^{\mathbf{R}}$: $I^2 = -1$ (in fact, $I\exp(in\varphi) = i\,\mathrm{sgn}(n)\exp(in\varphi)$).

4. Almost invariant structures. Below, for each pair (α, s), where $s \in \mathbf{R}$ or $is \in \mathbf{R}$, we construct an imbedding of $\mathrm{Diff}^{\sim}$ into some (G, K)-pair. In each case we indicate the (G, K)-pair, the values of the parameters (α, s), the representation space $\mathcal{H}$, the action D of $\mathrm{Diff}^{\sim}$ on $\mathcal{H}$, and the inner product $(\cdot\,\cdot)$ in which $\mathcal{H}$ should be completed.

A. *Principal series.* Let $is \in \mathbf{R}$ and introduce the inner product $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_0$ in the space H_α.

A1. $(\mathrm{U}(\infty)\times \mathrm{U}(\infty), \mathrm{U}(\infty)) = (G, K)$. The subgroup $K = \mathrm{U}(\infty)$ is embedded in G as the diagonal, $\mathcal{H} = H_\alpha \oplus H_\alpha$, $\mathcal{D} = T_{\alpha,s} \oplus T_{\alpha,-s}$, and the two copies of H_α are identified via the ("almost intertwining") operator $A_{\alpha,s}$.

A2. $(\mathrm{U}(2\infty), \mathrm{Sp}(\infty))$, $\alpha = 1$, $\mathcal{H} = H_1$, and $\mathcal{D} = T_{1,s}$. The structure of a quaternionic Hilbert space in H_1 is introduced using the operator $K_s f(\varphi) = A_{1,s}\overline{f(\varphi)}$. The group $\mathrm{U}(2\infty)$ is the usual unitary group of the space H_1.

A3. $(\mathrm{U}(\infty), \mathrm{O}(\infty))$, $\alpha = 0$, $\mathcal{H} = H_0$, and $\mathcal{D} = T_{0,s}$. The operator $Lf(\varphi) = A_{0,s}\overline{f(\varphi)}$ is antilinear and satisfies the condition $L^2 = E$. Consequently, $V_\pm = \mathrm{Ker}(L \pm E)$ are real orthogonal subspaces of H_0. The $\mathrm{U}(\infty)$ is the usual unitary group, and $\mathrm{O}(\infty)$ is the subgroup of $\mathrm{U}(\infty)$ consisting of the operators that leave $V_\pm$ invariant.

A4. $(\mathrm{O}(2\infty), \mathrm{U}(\infty))$, $\alpha = 1$, $s = 0$, $\mathcal{H} = H_0^{\mathbf{R}}$, and $\mathcal{D} = T_{1,0}$. The complex structure is introduced via the operator I (it is also $A_{1,0}$).

B. *Complementary series.* Let $s \in \mathbf{R}$, $s \neq 0$, $-1 < s \pm \alpha < 1$, $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_s$, and $\mathcal{D} = T_{\alpha,s}$.

B1. $(\mathrm{GL}(\infty,\mathbf{C}), \mathrm{U}(\infty))$ and $\mathcal{H} = H_\alpha$;

B2. $(\mathrm{GL}(\infty,\mathbf{R}), \mathrm{O}(\infty))$, $\alpha = 0$, and $\mathcal{H} = H_0^{\mathbf{R}}$,

B3. $(\mathrm{Sp}(2\infty,\mathbf{R}), \mathrm{U}(\infty))$, $\alpha = 0$, $s = 1$, and $\mathcal{H} = \overline{H}_0^{\mathbf{R}}$; the form $\langle\cdot,\cdot\rangle_1$ is regularized as

$$\langle f, g\rangle = \int_0^{2\pi}\int_0^{2\pi} \ln\left|\sin\left(\frac{\varphi_1-\varphi_2}{2}\right)\right| f(\varphi_1) f(\varphi_2)\, d\varphi_1\, d\varphi_2$$

The complex structure is introduced by the operator I, the invariant symplectic form is $\{f, g\} = \langle If, g\rangle_1$.

B3′. $(\mathrm{Sp}(2\infty,\mathbf{R}), \mathrm{U}(\infty))$, $\alpha = 0$, $s = -1$, $\mathcal{H} = \hat{H}_0^{\mathbf{R}}$, and $\langle f, g\rangle_{-1} = \langle f, Ig'\rangle_0$. The rest is analogous; see [9].

C. *Indefinite series.* Let $s \in \mathbf{R}$, let (for simplicity) $s > 0$ and $\frac{1}{2}(s \pm \alpha + 1) \notin \mathbf{Z}$, and let $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_s$ and $\mathcal{D} = T_{\alpha,s}$.

C1. $(\mathrm{GL}(\infty,\mathbf{C}), \mathrm{U}(p,\infty))$, $-1 < s \pm \alpha - 2p < 1$, and $\mathcal{H} = H_\alpha$;

C2. $(\mathrm{GL}(2\infty,\mathbf{C}), \mathrm{U}(\infty,\infty))$, remaining (α, s), and $\mathcal{H} = H_\alpha$;

C3. $(\mathrm{GL}(\infty,\mathbf{R}), \mathrm{O}(p,\infty))$, $\alpha = 0$, $\mathcal{H} = H_0^{\mathbf{R}}$, and $-1 < s - 2p < 1$.

C4. $(\mathrm{GL}(2\infty,\mathbf{R}), \mathrm{O}(\infty,\infty))$, $\alpha = 1$, and $\mathcal{H} = H_1^{\mathbf{R}}$.

D. *One more series.* Let $is \in \mathbf{R}$, and $\mathcal{D} = T_{\alpha,s}$, $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_0$.

D1. $(\mathrm{U}(2\infty), \mathrm{U}(\infty)\times\mathrm{U}(\infty))$ and $\mathcal{H} = H_\alpha$; let $H_\alpha = H_\alpha^+ \oplus H_\alpha^-$, where H_α^+ (respectively H_α^-) is spanned by the $v_{k,\alpha}$ with $k > 0$ (respectively, with $k < 0$). The group $\mathrm{U}(\infty)\times \mathrm{U}(\infty)$ is the group of unitary operators that preserve the decomposition $H_\alpha = H_\alpha^+ \oplus H_\alpha^-$.

D2. $(\mathrm{O}(2\infty), \mathrm{U}(\infty))$, $\alpha = 1$, and $s = 0$. See A4.

D3. $(\mathrm{O}(2\infty+1), \mathrm{U}(\infty))$, $\alpha = 0$, $s = 0$, and $\mathcal{H} = H_0^{\mathbf{R}}$. The complex structure on the subspace $\overline{H}_0^{\mathbf{R}} \subset H_0^{\mathbf{R}}$ is introduced via the operator I. The group $\mathrm{U}(\infty)$ consists of the operators orthogonal in $H_0^{\mathbf{R}}$ that preserve the subspace $\overline{H}_0^{\mathbf{R}}$ and are unitary on $H_0^{\mathbf{R}}$ (communicated to the author by R. S. Ismagilov).

5. Construction of the unitary representations of Diff$^\sim$. In the case of series A, B, and D it is sufficient to restrict to the Ol′shanskiĭ representations of (G, K)-pairs to Diff$^\sim$. In case D3 we first need to imbed $(\mathrm{O}(2\infty+1), \mathrm{U}(\infty))$ into $(\mathrm{O}(2\infty+2), \mathrm{U}(\infty+1))$. In the case of series C we obtain (G, K)-pairs that are not natural in the sense of [2]. For C1 and C3 it should be noted that

$$(\mathrm{GL}(\infty,\mathbf{C}), \mathrm{U}(p,\infty)) = (\mathrm{GL}(\infty,\mathbf{C}), \mathrm{U}(\infty)),$$
$$(\mathrm{GL}(\infty,\mathbf{R}), \mathrm{O}(p,\infty)) = (\mathrm{GL}(\infty,\mathbf{R}), \mathrm{O}(\infty)),$$

and in cases C2 and C4 the imbedding in fact goes into the smaller groups

$$(\mathrm{GL}(2\infty, \mathbf{C}), \mathrm{U}(\infty) \times \mathrm{U}(\infty)) \quad \text{and} \quad (\mathrm{GL}(2\infty, \mathbf{R}), \mathrm{O}(\infty) \times \mathrm{O}(\infty)).$$

Then we apply [2].

6. Restrictions of the representations of §5 to $\mathfrak{G} = \mathrm{SL}_2^{\sim}(\mathbf{R})$. In the case of series A and B it has the form $T = R \oplus Q$, where R is an infinite-fold direct integral over the principal and discrete series, and Q is a finite direct sum which can contain a one-dimensional representation, representations of the principal or complementary series (and sometimes representations of the analytic continuation of the "discrete" series). In the case of embeddings of types A and B the subgroup $\mathfrak{G} \subset \mathrm{Diff}^{\sim}$ is imbedded in the subgroup $K \subset (G, K)$. Then the problem is reduced to the problem of the decomposition of tensors over unitary representations of $\mathrm{SL}_2^{\sim}(\mathbf{R})$.

7. Proof of almost invariance. For series A it suffices to verify that the integral operator $T_{\alpha,s}(q)A_{\alpha,s} - A_{\alpha,s}T_{\alpha,-s}(q)$ has a bounded kernel. In the case of series D it is necessary to check the same thing for the operators $T_{\alpha,s}(q)P_{\pm} - P_{\pm}T_{\alpha,s}(q)$, where $P_{\pm}$ are the projection operators onto $H_{\alpha}^{\pm}$. In the case of series B, we calculate the form $\langle Kf, g\rangle = \langle (T_{\alpha,s}^{*}(q)T_{\alpha,s}(q) - E)f, g\rangle_s$. Further, using (4) we estimate the Fourier coefficients of K and show that it is a Hilbert-Schmidt operator in the metric $\langle\cdot,\cdot\rangle_s$. For the series C the proof is obtained by combining the arguments for B and D.

8. A. For some values of (α, s) different constructions of the list of §4 can be used. For example, for $\alpha = 1$ and $s = 0$ the group $\mathrm{Diff}^{\sim}$ can be imbedded in each of the groups

$$(\mathrm{U}(2\infty) \times \mathrm{U}(2\infty), \mathrm{U}(2\infty)) \supset (\mathrm{U}(2\infty), \mathrm{Sp}(\infty)) \supset (\mathrm{O}(2\infty), (\infty)).$$

On the level of the representations of §5 reduction of the pair (G, K) leads to the extraction of the tensor square root of the representation (in the sense of $\rho = \mu \otimes \mu$ or $\rho = \mu \otimes \mu^{*}$, see also [2], [6], and [13]).

B. The highest weight representations of $\mathrm{Diff}^{\sim}$ can be obtained from the imbeddings A4, B3, B3′, and D1–D3. These are discussed in [7]; see also [9] and [10]; a p-adic version of the present paper is contained in [8], and some nonunitary analogues can be found in [15].

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Moscow Institute of Electronic Machine Construction

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