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ON SPINOR REPRESENTATION OF $O(\infty, \mathbf{C})$

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Let G be a Lie group and $G_{\mathbf{C}}$ its complexification. It is known that an infinite-dimensional unitary representation of G can be continued holomorphically only to a neighborhood of the identity in $G_{\mathbf{C}}$. However, the representation operators are unbounded, and their common domain of definition is not invariant. It turns out (see Theorem 2) that if G is an infinite-dimensional orthogonal group, then its spinor representation can be continued to a global holomorphic representation of $G_{\mathbf{C}}$ by unbounded operators in a Hilbert space. This assertion extends the range of applicability of "imbedding techniques" in the theory of representations of infinite-dimensional groups (see [2]–[6]). In particular, it allows one to construct an analytic continuation with respect to a parameter for many known series of representations of infinite-dimensional classical groups and also to obtain somewhat unexpected theorems on integrability for representations of the Virasoro algebra and affine algebras.

1. Notation. The notation $A \in \mathcal{L}_1$ (resp. $A \in \mathcal{L}_2$) will mean that A is a trace class operator (resp. a Hilbert-Schmidt operator). Let $U(\infty)$, $GL(\infty, \mathbf{C})$, $O(\infty, \mathbf{R})$, and $O(\infty, \mathbf{C})$ be, respectively, the full unitary, full linear, and full orthogonal groups of a Hilbert space. Let $G(\infty) \supset K(\infty)$ be groups of the indicated form. Then $(G(\infty), K(\infty))$ is the subgroup of $G(\infty)$ consisting of operators of the form $A(1+T)$, where $A \in K(\infty)$ and $T \in \mathcal{L}_2$.

Let H be a Hilbert space, and let the $\Lambda^k H$ be its exterior powers. Then $\bigoplus_0^\infty \Lambda^k H$ is called *Fock's fermion space* $\Lambda(H)$. Let $\xi_1, \xi_2, \ldots$ be "holomorphic" anticommuting variables:

$$\xi_k \xi_l = -\xi_l \xi_k, \quad \xi_k \overline{\xi}_l = -\overline{\xi}_l \xi_k, \quad \overline{\xi_k \xi_l} = -\overline{\xi}_l \overline{\xi}_k.$$

Then it is convenient to realize $\Lambda(H)$ as the space of polynomials in $\xi_1, \xi_2, \ldots$ completed with respect to the inner product

$$f, g\rangle = \int f(\xi)\overline{g(\xi)}\, d\mu,$$

where

$$d\mu = \left[\exp\left(-\sum_k \frac{\partial^2}{\partial \xi_k \partial \overline{\xi}_k}\right) \prod_k \xi_k \overline{\xi}_k\right] \prod_k d\xi_k\, d\overline{\xi}_k,$$

and the Berezin integral of $\prod_k \xi_k \overline{\xi}_k$ is 1. But if at least one factor in this product is omitted, then the integral equals 0. The monomials in the variables ξ_k form an orthonormal basis in $\Lambda(H)$. The creation and annihilation operators in $\Lambda(H)$ are introduced, respectively, by the formulas $A_k f = \xi_k f$ and $B_k f = (\partial/\partial \xi_k) f$ ($A_j^* = B_j$, $A_k B_l + B_l A_k = \delta_{k,l}$, $A_k A_l = -A_l A_k$, and $B_k B_l = -B_l B_k$).

Let V be the space of linear operators in $\Lambda(H)$ of the form $\sum p_k A_k + \sum q_k B_k$, where $\sum(|p_k|^2 + |q_k|^2) < \infty$. Then V is a closed subspace with respect to the uniform topology in the algebra of all bounded operators. A symmetric bilinear form (P, Q) on V is defined from the condition $PQ + QP = (P, Q) \cdot 1$. The form (P, Q) is positive definite

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in the subspace V' of selfadjoint elements of V. We define $\mathrm{O}(2^\infty, \mathbf{C})$ as the group of all bounded operators in V that preserve the form (P, Q). Its subgroup $\mathrm{O}(2^\infty, \mathbf{R})$ consists of the operators in V that preserve the subspace V'. We specify the elements of $\mathrm{O}(2^\infty, \mathbf{C})$ by matrices of the form $S = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in the basis $A_1, A_2, \ldots, B_1, B_2, \ldots$. The subgroup of block diagonal matrices is isomorphic to $\mathrm{GL}(\infty, \mathbf{C})$. It is easy to see that

$$(\mathrm{O}(2^\infty, \mathbf{C}), \mathrm{GL}(\infty, \mathbf{C})) \cap \mathrm{O}(2^\infty, \mathbf{R}) = (\mathrm{O}(2^\infty, \mathbf{R}), \mathrm{U}(\infty)).$$

Let G be a subgroup of $\mathrm{O}(2^\infty, \mathbf{C})$. Its projective representation ρ in $\Lambda(H)$ is called *spinor* if for any $A \in G$ and $T \in V$

$$\rho(A) T \rho(A)^{-1} = A(T).$$

A spinor representation $(\mathrm{O}(2^\infty, \mathbf{R}),\ \mathrm{U}(\infty))$ was constructed in [1] (see also [9] and [10]). We shall attempt to define $\rho(M)$, $M \in \mathrm{O}(2^\infty, \mathbf{C})$, by means of the formula (this is a holomorphic continuation of (5.15) in [1])

$$(1) \quad \rho\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] f(\eta) = \theta \int \exp\left[-\frac{1}{2}(\eta \bar{\xi}) \begin{pmatrix} CA^{-1} & A^{-1} \\ A^{t-1} & A^{-1}B \end{pmatrix} \begin{pmatrix} \eta \\ \bar{\xi} \end{pmatrix}\right] f(\xi)\, d\mu,$$

where $\theta \in \mathbf{C}$.

2. A bounded spinor representation. We consider the group $\mathrm{O}_r(2^\infty) \subset \mathrm{O}(2^\infty, \mathbf{C})$ consisting of all operators that are representable in the form $U(1+T)(1+S)$, where $U \in \mathrm{U}(\infty)$, $1 + T \in \mathrm{O}(2^\infty, \mathbf{R})$, $1 + S \in \mathrm{O}(2^\infty, \mathbf{C})$, $T \in \mathcal{L}_2$, and $S \in \mathcal{L}_1$.

THEOREM 1. *There exists a spinor representation of the group $\mathrm{O}_r(2^\infty)$ by bounded operators in $\Delta(H)$. The representation is defined by* (1) *on the matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ for which A^{-1} exists.*

3. A large spinor representation. Let $f \in \Lambda(H)$, $f = (f_0, f_1, \ldots)$, where $f_k \in \Lambda^k H$. We consider in $\Lambda(H)$ the subspace Λ of all elements for which $\|f_j\|$ decreases faster than any function of the form e^{-C_j}. A collection of seminorms on Λ is introduced in the obvious way, with respect to which it is a Fréchet space.

THEOREM 2. *Formula* (1) *above gives a well-defined representation of* $(\mathrm{O}(2^\infty, \mathbf{C}),\ \mathrm{GL}(\infty, \mathbf{C}))$ *by continuous operators in the space* Λ.

PROOF. a) The space Λ is invariant under any operator T of a linear change of variables and also operators of the form

$$Pf = \exp\left(\sum p_{ij} \xi_i \xi_j\right) f, \qquad Qf = \exp\left(\sum q_{ij} \frac{\partial^2}{\partial \xi_i \partial \xi_j}\right) f,$$

where $\sum |p_{ij}|^2 < \infty$ and $\sum |q_{ij}|^2 < \infty$.

b) We apply a local Iwasawa decomposition in $(\mathrm{O}(2^\infty, \mathbf{C}),\ \mathrm{GL}(\infty, \mathbf{C}))$ with respect to $\mathrm{GL}(\infty, \mathbf{C})$ to an operator of the form $1 + T$, where $T \in \mathcal{L}_2$ and $\|T\|$ is small. Then $\rho(1+T)$ can be represented as a product of operators of the form Q, T, and P.

c) $\mathrm{GL}(\infty)$ acts on Λ by linear changes of variables.

d) Let $H = \mathbf{C}^n \oplus L$. Then $\Lambda(H) = \Lambda(\mathbf{C}^n) \otimes \Lambda(L)$. This defines an action of $\mathrm{O}(n, \mathbf{C})$ in Λ.

e) The compatibility of b), c), and d) and the fact that the representation on the matrices for which A^{-1} does not exist is well-defined can be verified by the induced action on the creation and annihilation operators.

PROOF OF THEOREM 1. It is necessary to use a polar decomposition for operators of the form $1 + S$, $S \in \mathcal{L}_1$. Then the question of the boundedness of $\rho(1+S)$ reduces to normal operators, for which the spectrum can be explicitly computed.

4. The Virasoro algebra. Let Diff be the group of diffeomorphisms of a circle and V the Virasoro algebra (see, for example, [5], [7], or [8]).

THEOREM 3. *Any irreducible representation of V with highest weight (not necessarily unitarizable) can be integrated to a continuous projective representation of* Diff *in some locally convex space.*

For the proof it is enough to verify the following assertions:

a) Suppose that the universal covering $\mathrm{Diff}^{\sim}$ of the group Diff is realized as the group of diffeomorphisms of $\mathbf{R}$ that satisfy the condition $q(x+2\pi) = q(x) + 2\pi$. Let H_α ($\alpha \in \mathbf{C}$, $0 \le \operatorname{Re}\alpha \le 1$) be the space of functions on $\mathbf{R}$ that satisfy the condition $q(x+2\pi) = e^{2\pi i\alpha} q(x)$. An inner product in H_α is introduced by the formula $\int_0^{2\pi} f\bar{g}\,dx$. Suppose that $\mathrm{Diff}^{\sim}$ acts in H_α by the formula ($\omega \in \mathbf{C}$)

$$T(q)f(x) = f(q(x))q'(x)^{1/2+\omega}.$$

Suppose that $H^+_{\alpha,n} \subset H_\alpha$ is spanned by the vectors $e^{i(k+\alpha)\varphi}$, $k \ge n$, and $H^-_{\alpha,n}$ by the vectors $e^{i(k+\alpha)\varphi}$, $k < n$. Then

$$T(q) \in (\mathrm{GL}(2^\infty, \mathbf{C}),\ \mathrm{GL}(\infty, \mathbf{C}) \times \mathrm{GL}(\infty, \mathbf{C}))$$

(the subgroup $\mathrm{GL}(\infty, \mathbf{C}) \times \mathrm{GL}(\infty, \mathbf{C})$ consists of matrices that preserve the subspaces $H^{\pm}_{\alpha,n}$).

b) We denote by $L^+_{\alpha,n}$ the space $H^+_{\alpha,n}$ with a complex conjugate structure. Then the identity mapping $H_\alpha \to H^-_{\alpha,n} \oplus L^+_{\alpha,n}$ defines an imbedding of $(\mathrm{GL}(2^\infty, \mathbf{C}), \mathrm{GL}(\infty, \mathbf{C}) \times \mathrm{GL}(\infty, \mathbf{C}))$ into $(\mathrm{O}(2^\infty, \mathbf{C}), \mathrm{GL}(\infty, \mathbf{C}))$ (see also [2]).

c) Restricting the spinor representation to $\mathrm{Diff}^{\sim}$, we obtain a series $A(\alpha, \omega, n)$ of representations of V. All the irreducible representations of V with highest weight are contained among its subfactors.

d) Each Diff-subrepresentation in $A(\alpha, \omega, n)$ is the closure of the set of its finitary vectors.

5. Affine algebras. Let G be a complex Lie group, $\mathfrak{G}$ its Lie algebra, and K a maximal compact subgroup. The standard construction for imbedding $C^\infty(S^1, \mathrm{SO}(n))$ into $(\mathrm{O}(2^\infty, \mathbf{R}), \mathrm{U}(\infty))$ (Vershik, Frenkel, and Ismagilov; see, for example, [4]) can be continued to an imbedding of $C^\infty(S^1, \mathrm{SO}(n, \mathbf{C}))$ into $(\mathrm{O}(2^\infty, \mathbf{C}), \mathrm{GL}(\infty))$. The next theorem follows from this.

THEOREM 4. *A basis representation of the affine algebra $C^\infty(S^1, \mathfrak{G})$ can be integrated to a projective representation of $C^\infty(S^1, G)$ that is unitary on $C^\infty(S^1, K)$.*

In the case of the series $C^\infty(S^1, \mathrm{sl}(n, \mathbf{C}))$ integrability follows for all representations with highest weight.

6. REMARK 1. For a Weyl representation of the group $\mathrm{Sp}_0 = (\mathrm{Sp}(2^\infty, \mathbf{R}), \mathrm{U}(\infty))$ the analogs of Theorems 1 and 2 are false. However, the standard representation of the group $\mathrm{Sp}_0 \ltimes H$, where H is an infinite-dimensional Heisenberg group, can be continued holomorphically to a representation of $\mathrm{Sp}_0 \ltimes H_{\mathbf{C}}$ by unbounded operators in Fock's space. We consider the standard holomorphic realization of Fock's boson space (see [1]). Then $\mathrm{Sp}_0 \ltimes H_{\mathbf{C}}$, an invariant dense subset of Ω, forms finitary linear combinations of functions of the form

$$P(z_1, z_2, \ldots) \exp\left(\sum \alpha_i z_i\right) \exp(\langle Az, z\rangle),$$

where P is a homogeneous form in $z_1, z_2, \ldots$, $\sum |\alpha_i|^2 < \infty$, and A is an antilinear Hilbert-Schmidt operator. $\|A\| < \frac{1}{2}$. The author is not aware of any natural topology on Ω. The restriction of our representation to the subgroup $\mathrm{U}(\infty) \ltimes H_{\mathbf{C}}$ is continuous in the topology of uniform convergence on balls.

REMARK 2. For infinite-dimensional classical groups Theorem 2 has consequences of two types. First, Ol′shanskiĭ's fermion representations (see [2]) of real groups can be continued holomorphically to representations of the corresponding complex groups. Second, any series of fermion representations [2] (for certain groups the construction in [2] is equivalent to the Thoma-Voiculescu-Vershik-Kerov construction of quotient representations; see [3] or [2]) that depends on real parameters can be continued holomorphically to a series of representations (already nonunitary) that depends on a complex parameter. It follows from Remark 1 that the same holomorphic continuations also exist for the "intermediate" representations in [2]. It would be interesting to construct analogs of the limit theorems [3] for these series.

REMARK 3. The subgroups of $O_r(2^\infty)$ and $(O(2^\infty, C), GL(\infty, C))$ on which the spinor representation can be chosen to be two-valued are distinguished by the condition that $A - E \in \mathcal{L}_1$ and $D - E \in \mathcal{L}_1$ ($\theta = \pm\sqrt{\det A}$ in (1)).*

REMARK 4. The representations constructed in Theorems 3 and 4 cannot be realized in a Banach space. For this reason it was assumed that integrability can hold only in the unitary case (on unitary integrability see [5]–[8]).

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* *Editor's note.* The Russian cites preprints of these items.