

Foundations of quantum physics

II. The thermal interpretation

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Abstract. This paper presents the thermal interpretation of quantum physics. The insight from Part I of this series that Born's rule has its limitations – hence cannot be the foundation of quantum physics – opens the way for an alternative interpretation: the thermal interpretation of quantum physics. It gives new foundations that connect quantum physics (including quantum mechanics, statistical mechanics, quantum field theory and their applications) to experiment.

The thermal interpretation resolves the problems of the foundations of quantum physics revealed in the critique from Part I. It improves the traditional foundations in several respects:

- The thermal interpretation reflects the actual practice of quantum physics, especially regarding its macroscopic implications.
- The thermal interpretation gives a fair account of the interpretational differences between quantum mechanics and quantum field theory.
- The thermal interpretation gives a natural, realistic meaning to the standard formalism of quantum mechanics and quantum field theory in a single world, without introducing additional hidden variables.
- The thermal interpretation is independent of the measurement problem. The latter becomes a precise problem in statistical mechanics rather than a fuzzy and problematic notion in the foundations. Details will be discussed in Part III.

For the discussion of questions related to this paper, please use the discussion forum <https://www.physicsoverflow.org>.

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1 Introduction

In a statistical description of nature only expectation values or correlations are observable.

Christof Wetterich, 1997 [57]

One is almost tempted to assert that the usual interpretation in terms of sharp eigenvalues is 'wrong', because it cannot be consistently maintained, while the interpretation in terms of expectation values is 'right', because it can be consistently maintained.

John Klauder, 1997 [26, p.6]

This paper presents the thermal interpretation of quantum physics. The insight from Part I [38] of this series was that Born's rule has its limitations and hence cannot be the foundation of quantum physics. Indeed, a foundation that starts with idealized concepts of limited validity is not a safe ground for interpreting reality.

The analysis of Part I opens the way for an alternative interpretation – the thermal interpretation of quantum physics. It gives new foundations that connect all of quantum physics (including quantum mechanics, statistical mechanics, quantum field theory and their applications) to experiment.

Quantum physics, as it is used in practice, does much more than predicting probabilities for the possible results of microscopic experiments. This introductory textbook scope is only the tip of an iceberg. Quantum physics is used to determine the behavior of materials made of specific molecules under changes of pressure or temperature, their response to external electromagnetic fields (e.g., their color), the production of energy from nuclear reactions, the behavior of transistors in the microchips on which modern computers run, and a lot more. Indeed, it appears to predict the whole of macroscopic, phenomenological equilibrium and nonequilibrium thermodynamics in a quantitatively correct way.

Motivated by this fact, this paper defines and discusses a new interpretation of quantum physics, called the **thermal interpretation**. It is based on the lack of a definite boundary between the macroscopic and the microscopic regime, and an application of OCKHAM's razor [44, 24], *frustra fit per plura quod potest fieri per pauciora* – that we should opt for the most economic model explaining a regularity.

Essential use is made of the fact that everything physicists measure is measured in a thermal environment for which statistical thermodynamics is relevant. This is reflected in the characterizing adjective 'thermal' for the interpretation. The thermal interpretation agrees with how one interprets measurements in thermodynamics, the macroscopic part of quantum physics, derived via statistical mechanics. Extrapolating from the macroscopic case, the thermal interpretation considers the functions of the state (or of the parameters characterizing a state from a particular family of states) as the **beables**, the conceptual equivalent of objective properties of what really exists. Some of these are accessible to experiment – namely the expectation values of quantities that have a small uncertainty and vary sufficiently slowly in time and space. Because of the law of large numbers, all thermodynamic

variables are in this category. By its very construction, the thermal interpretation naturally matches the classical properties of our quantum world.

Section 2 gives a detailed motivation of the thermal interpretation and a precise definition of its basic credo. We introduce the Ehrenfest picture of quantum mechanics, the abstract mathematical framework used throughout. It describes a closed, deterministic dynamics for q -expectations (expectation values of Hermitian operators). We discuss the ontological status of the thermal interpretation, making precise the concept of properties of a quantum system, the concept of uncertainty, and the notion of an ensemble. Based on this, we give a formal definition of the thermal interpretation,

In Section 3 we consider the way the thermal interpretation represents statistical and probabilistic aspects of quantum theory. We begin with a discussion of two formal notions of classical probability, their relation to the probability concept used in applied statistics, and their dependence on the description used. We then show how the statistical aspects of the quantum formalism naturally follow from the weak law of large numbers.

In Section 4, we show that the fact that in relativistic quantum field theory, position is a classical parameter while in quantum mechanics it is an uncertain quantity strongly affects the relation between quantum field theory and reality. Among the beables of quantum field theory are smeared field expectations and pair correlation functions, which encode most of what is of experimental relevance in quantum field theory. We discuss notions of causality and nonlocality and their relation to the thermal interpretation.

We also discuss relativistic quantum field theory at finite times, a usually much neglected topic essential for a realistic interpretation of the universe in terms of quantum field theory. According to the thermal interpretation, quantum physics is the basic framework for the description of objective reality (including everything reproducible studied in experimental physics), from the smallest to the largest scales, including the universe as a whole (cf. Subsection 4.3). Classical descriptions are just regarded as limiting cases where Planck's constant \hbar can be set to zero without significant loss of quality of the resulting models.

Except for a brief discussion of the measurement of probabilities in Subsection 3.5, everything related to the thermal interpretation of measurement is postponed to Part III [39] of this series of papers. There it is shown that the thermal interpretation satisfactorily resolves the main stumbling blocks in a clear description of the relation between the quantum formalism and experimental reality.

Hints at a possible thermal interpretation of quantum physics go back at least to 1997; see the above quotes by Wetterich and Klauder. A recent view closely related to the thermal interpretation is the 2017 work by ALLAHVERDYAN et al. [1]. The thermal interpretation of quantum physics itself emerged from my foundational 2003 paper NEUMAIER [32]. It was developed by me in discussions on the newsgroups de.sci.physik, starting in Spring 2004; for the beginnings see NEUMAIER [34]. A first version of it was fully formalized (without naming the interpretation) in Sections 5.1, 5.4 and Chapter 7 of the 2008 edition of the online book by NEUMAIER & WESTRA [41]; see also Sections 8.1, 8.4 and Chapter 10 of the 2011 edition. The name 'thermal interpretation' appeared first in a 2010 lecture (NEUMAIER [35]). Later I created a dedicated website on the topic (NEUMAIER [37]).

The bulk of this paper is intended to be nontechnical and understandable for a wide audience being familiar with some traditional quantum mechanics. For the understanding of the main issues, the knowledge of some basic terms from functional analysis is assumed; these are precisely defined in many mathematics books. However, quite a number of remarks are addressed to experts and then refer to technical aspects explained in the references given.

In the bibliography, the number(s) after each reference give the page number(s) where it is cited.

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2 The thermal interpretation of quantum mechanics

In this section we give a precise definition of the basic credo of the thermal interpretation. Subsection 2.1 introduces the abstract mathematical framework in terms of a Lie product structure on the set of all q-expectations. This induces a Lie–Poisson bracket in terms of which q-expectations behave dynamically as a classical Hamiltonian system, with a dynamics given by the Ehrenfest theorem. This Ehrenfest picture of quantum mechanics is used throughout. Subsection 2.2 discusses the ontological status of the thermal interpretation, making precise the concept of properties of a quantum system. The next two subsections clarify the concepts of uncertainty and the notion of an ensemble. The final Subsection 2.5 then gives a formal definition of the thermal interpretation,

2.1 The Ehrenfest picture of quantum mechanics

As first observed in 1925 by DIRAC [11], classical mechanics and quantum mechanics look very similar when written in terms of the Poisson bracket.

Quantities are represented in classical mechanics by functions from a space of suitable smooth phase space functions $A(p, q)$, and in quantum mechanics by linear operators A on a suitable Euclidean space. We define the **classical Lie product**

$$A \angle B := \{B, A\} = \partial_p A \partial_q B - \partial_p B \partial_q A \quad (1)$$

(read \angle as 'Lie') of classical quantities A, B , and the **quantum Lie bracket**

$$A \angle B := \frac{i}{\hbar} [A, B] = \frac{i}{\hbar} (AB - BA) \quad (2)$$

of quantum mechanical quantities A, B . This infix notation is much more comfortable than the customary bracket notation. In both cases, it is easy to verify anticommutativity,

$$A \angle B = -B \angle A,$$

the product rule

$$A \angle BC = (A \angle B)C + B(A \angle C),$$

and the Jacobi relation

$$A \angle (B \angle C) = (A \angle B) \angle C + B \angle (A \angle C).$$

This shows that \angle turns the space of quantities into a Lie algebra. It also shows that the application of $a \angle$ to a quantity behaves like differentiation.

We also write \int both for the Liouville integral

$$\int A := \int A(p, q) dp dq \quad (3)$$

of a classical quantity A and for the trace

$$\int A := \text{Tr } A \quad (4)$$

of a quantum mechanical quantity A . With this notation, it is easy to verify the invariance under infinitesimal canonical transformations,

$$\int A \angle B = 0,$$

from which one finds the integration by parts formula

$$\int (A \angle B)C = \int A(B \angle C).$$

In the general theory, for which we refer to NEUMAIER [32, 33], these rules are part of a system of axioms for **Euclidean Poisson algebras**, which allows one to develop everything without reference to either the classical or the quantum case.

Quantities and linear functionals are, in general, time-dependent; so we write $\langle f \rangle_t$ for the q -expectation of $f(t)$ at time t . In maximal generality, a **q-expectation** is written in the form

$$\langle A \rangle_t := \int \rho(t) A(t), \quad (5)$$

where $A(t)$ is an arbitrary time-dependent quantity and $\rho(t)$ a time-dependent density operator, a nonnegative Hermitian operator normalized by

$$\int \rho = 1.$$

Their dynamics is given by

$$\dot{A}(t) = H_1(t) \angle A(t) \quad (6)$$

for quantities A ,¹

$$\dot{\rho}(t) = \rho(t) \angle H_2(t) \quad (7)$$

¹More generally, if z is a vector of quantities satisfying (6), quantities given by expressions $A(t) = A(z(t), t)$ with an explicit time dependence satisfy instead of (6) a differential equation of the form

$$\dot{A}(t) = i[H_1(t), A(t)] + \partial_t A(z(t), t).$$

This follows easily from (6) and the chain rule. The generality gained is only apparent since the numbers

for the density operator ρ ; note the different treatment of quantities and the density operator! Here $H_1(t)$ and $H_2(t)$ are arbitrary time-dependent expressions without independent physical meaning; they need not satisfy the differential equations (6) or (7). Integrating (7) shows that $\int \rho$ is time independent, so that the dynamics is consistent with the normalization of ρ .

As a consequence of the dynamical assumptions (6)–(7), the q-expectations (5) have a deterministic dynamics, given by

$$\frac{d}{dt}\langle A \rangle_t = \langle H \angle A \rangle_t, \quad (8)$$

We call (8) the **Ehrenfest equation** since the special case of this equation where A is a position or momentum variable and $H = \frac{p^2}{2m} + V(q)$ is the sum of kinetic and potential energy was found in 1927 by EHRENFEST [14]. Due to the canonical commutation rules, we have

$$\frac{d}{dt}\langle q \rangle_t = \langle H \angle q \rangle_t = \frac{\langle p \rangle_t}{m}, \quad \frac{d}{dt}\langle p \rangle_t = \langle H \angle p \rangle_t = \langle -\nabla V(q) \rangle_t. \quad (9)$$

Note that the Ehrenfest equation does not involve notions of reality or measurement, hence belongs to the formal core of quantum mechanics and is valid independent of interpretation issues.

The product rule implies that $\frac{d}{dt}\langle A \rangle_t$ only depends on the sum $H = H_1 + H_2$, not on H_1 and H_2 separately. Thus there is a kind of gauge freedom in specifying the dynamics, which can be fixed by choosing either H_1 or H_2 arbitrarily. Fixing $H_1 = 0$ (so that $H_2 = H$) makes all quantities A time-independent and defines the **Schrödinger picture**. Fixing H_1 as a reference Hamiltonian without interactions (so that $H_2 = V := H - H_1$ is the **interaction**) defines the **interaction picture**. Fixing $H_2 = 0$ (so that $H_1 = H$) makes the density operator ρ time-independent and defines the **Heisenberg picture**. In the Heisenberg picture, one finds that

$$\langle \phi(u) \rangle_s = \langle \phi(u + s - t) \rangle_t \quad (10)$$

for arbitrary times s, t, u, w, \dots

The Schrödinger picture is fully compatible with the formal core of quantum physics, comprising the postulates (A1)–(A6) discussed in Subsection 2.1 of Part I [38]. In particular,

$\langle A(z(t), t) \rangle_t$ are expressible in terms of canonical ones: In terms of a Fourier expansion

$$A(z(t), t) = \int d\omega e^{i\omega t} A_\omega(z(t)),$$

we see that

$$\langle A(z(t), t) \rangle_t = \int d\omega e^{i\omega t} \langle A_\omega(z(t)) \rangle_t,$$

and the $A_\omega(z(t))$ are canonical quantities respecting (6). This allows us to limit the main text to the case where A has no explicit t -dependence.

the von Neumann equation

$$\frac{d}{dt}\rho(t) = \frac{i}{\hbar}[\rho(t), H] \quad \text{for } t \in [t_1, t_2], \quad (11)$$

holds for closed systems, giving a deterministic dynamics for the density operator. As discussed in Subsection 2.3 of Part I [38], one obtains the Schrödinger equation in the limit of pure states.

In place of the traditional Heisenberg, Schrödinger, and interaction pictures, one can also consider another equivalent picture, in which only q-expectations figure as dynamical variables. The name **Ehrenfest picture** is suggestive since, for the standard multiparticle Hamiltonian and $f = p, q$, this reduces to the Ehrenfest equation. In terms of the Lie bracket on q-expectations defined by the formula

$$\langle A \rangle \angle \langle B \rangle := \langle A \angle B \rangle, \quad (12)$$

the family of q-expectations becomes a Lie algebra \mathbb{L} , and the Ehrenfest equation (8) becomes

$$\frac{d}{dt}\langle A \rangle = \langle H \rangle \angle \langle A \rangle. \quad (13)$$

Equation (13) is quite remarkable as it is manifestly independent of how H is split and how the time-dependent expectation is expressed as (5).

The Ehrenfest picture gives a complete picture of the (classical or) quantum kinematics and a deterministic dynamics for the q-expectations that is equivalent to the Schrödinger picture, the Heisenberg picture, and the interaction pictures.

It is interesting to interpret the above in terms of Hamiltonian dynamics on Poisson manifolds. An extensive discussion of classical Hamiltonian dynamics on Poisson manifolds, in particular using Lie–Poisson brackets, and its application to rigid rotors and fluid dynamical systems is given in MARS DEN & RATIU [31]. A **Poisson manifold** is a smooth manifold together with a Lie product on $\mathbb{E} = C^\infty(M)$ that turns \mathbb{E} into a commutative Poisson algebra. Associated with the Lie algebra \mathbb{L} of q-expectations is the manifold \mathbb{L}^* of continuous linear functionals on \mathbb{L} . On $\mathbb{E} = C^\infty(\mathbb{L}^*)$, a Lie product is given by the classical Lie–Poisson bracket which canonically extends the formula (12) to smooth functions of q-expectations. This turns \mathbb{E} into a commutative Poisson algebra, hence \mathbb{L}^* into a Poisson manifold.² In these terms, the Ehrenfest picture of *quantum* mechanics is just *classical* (but nonsymplectic) Hamiltonian dynamics in the Poisson manifold \mathbb{L}^* , with expected energy $\langle H \rangle$ as the *classical* Hamiltonian. In particular, as can also be seen directly, expected energy is conserved.

2.2 Properties

The **beables** of a physical system in the sense of BELL [4, p.41] are the linear functionals of its density operator ρ , called the **state** of the system, and any complex-valued func-

²In terms of the symplectic Poisson algebra considered by STROCCHI [53] to express the quantum mechanical dynamics of pure states ψ , the present Poisson algebra corresponds to the Poisson subalgebra of even functions of ψ .

tion constructed from these. The linear functionals are in 1-to-1 correspondence with **q-expectations**, i.e., expressions of the form³

$$\overline{A} = \langle A \rangle := \text{Tr } \rho A, \quad (14)$$

where A is a linear operator (or more generally a sesquilinear form) on the Euclidean space \mathbb{H} . By definition, all functions of q-expectations are beables, and all beables arise in this way. This gives a clear formal meaning to the notion of existence, an **ontology**: In the thermal interpretation, something is said to **exist**, to be **real**, and to be **objective** – three ways of expressing the same –, if and only if it can be expressed in terms of beables only.

As observed in 1927 by VON NEUMANN [42, p.255] – who was the first to base quantum mechanics upon expectations rather than probabilities –, the specification of all expectations determines the complete state (density operator) ρ . Thus every function of the state ρ can be rewritten (in any fixed picture) as a function of q-expectations, hence is a beable.

Only a small set of beables are practically (approximately) observable. Clearly, anything computable in quantum physics belongs there. Whenever we are able to compute something from raw measurements according to the rules of some meaningful protocol, and it adequately agrees with something derivable from quantum physics, we call the result of that computation a **measurement** of the latter. This correctly describes the practice of measurement in its most general form. Formal details will be given in Part III [39].

We recall the rules (S1)–(S3) from Subsection 3.4 and (R) from Subsection 4.1 of Part I [38] that we found necessary for a good interpretation:

(S1) The state of a system (at a given time) encodes everything that can be said about the system, and nothing else.

(S2) Every property of a subsystem is also a property of the whole system.

(S3) The state of a system determines the state of all its subsystems.

(R) Something in real life 'is' an instance of the theoretical concept if it matches the theoretical description sufficiently well.

The problems that traditional interpretations have with the relation between the state of a system and that of a subsystem were discussed in Subsection 3.4 of Part I [38]. They are resolved in the thermal interpretation. Indeed, the use of density operators as states implies that the complete state of a system completely and deterministically specifies the complete state of every subsystem.

Rule (S1) holds in the thermal interpretation because everything that can be computed from the state is a beable.

The other rules are satisfied in the thermal interpretation by making precise the notions of 'statement about a system' and 'property of a system'.

³Note that the density operator, viewed as a time-dependent object, is picture-dependent, but with the corresponding time-dependence of the linear operator A as discussed in Subsection 2.1, the q-expectations are picture-independent.

A **statement** is a $\{\text{true}, \text{false}\}$ -valued function of the state. A **property** of a system at time t is a statement P such that $P(\rho(t))$ is true, where $\rho(t)$ is the state of the system at time t .

A **subsystem** of a system is specified by a choice declaring some of the quantities (q-observables) of the system to be the distinguished quantities of the subsystem. This includes a choice for the Hamiltonian of the subsystem. The dynamics of the subsystem is generally not closed, hence not given by the Ehrenfest equation (13). However, in many cases, an approximate closed dynamical description is possible; this will be discussed in more detail in Part III [39].

Note that unlike in traditional interpretations, no tensor product structure is assumed. However, suppose that the latter is present, $\mathbb{H} = \mathbb{H}_S \otimes \mathbb{H}_{\text{env}}$, and the quantities of the subsystem are the linear operators of $\mathbb{H}_S \otimes 1$. Then, without changing any of the predictions for the subsystem, the Hilbert space of the subsystem may be taken to be the smaller Hilbert space \mathbb{H}_S , and the quantities of the subsystem are the linear operators of \mathbb{H}_S . Then the density operator of the subsystem is the **reduced state** obtained as the partial trace over the environment Hilbert space \mathbb{H}_{env} . In this sense, (S3) holds.

Rule (R) just amounts to a definition of what it means of something in real life to 'be an X ', where X is defined as a theoretical concept.

2.3 Uncertainty

A quantity in the general sense is a property ascribed to phenomena, bodies, or substances that can be quantified for, or assigned to, a particular phenomenon, body, or substance. [...] The value of a physical quantity is the quantitative expression of a particular physical quantity as the product of a number and a unit, the number being its numerical value.

Guide for the Use of the International System of Units (TAYLOR [54])

The uncertainty in the result of a measurement generally consists of several components which may be grouped into two categories according to the way in which their numerical value is estimated.

Type A. Those which are evaluated by statistical methods

Type B. Those which are evaluated by other means

[...] The quantities u_j^2 may be treated like variances and the quantities u_j like standard deviations.

NIST Reference on Constants, Units, and Uncertainty [43]

Uncertainty permeates all of human culture, not only science. Everything quantified by real numbers (as opposed to counting) is intrinsically uncertain because we cannot determine a real number with arbitrary accuracy. Even counting objects or events is uncertain in as much the criteria that determine the conditions under which something is counted are ambiguous. (When does the number of people in a room change by one while someone enters the door?)

The thermal interpretation of quantum physics takes the virtually universal presence of uncertainty as the most basic fact of science and gives it a quantitative expression. Some of this uncertainty can be captured by probabilities and statistics, but the nature of much of this uncertainty is conceptual. Thus uncertainty is a far more basic phenomenon than statistics. It is an uncertainty in the notion of measurability itself. What does it mean to have measured something?

To be able to answer this we first need clarity in the terminology. To eliminate any trace of observer issues⁴ from the terminology, we use the word **quantity** (as recommended in the above quote from the Guide to the International System of Units) or – in a more technical context – **q-observable**⁵ whenever quantum tradition uses the word observable. Similarly, to eliminate any trace of a priori statistics from the terminology, we frequently use the terminology **uncertain value** (in [41] simply called value) instead of q-expectation value, and **uncertainty** instead of q-standard deviation.

For the sake of definiteness consider the notion of uncertain position. This may mean two things.

1. It may mean that the position could be certain, as in classical Newtonian physics, except that we do not know the precise value. However, measurements of arbitrary accuracy are at least conceivable.
2. It may mean that the position belongs to an extended object, such as a neutron star, the sun, a city, a house, a tire, or a water wavelet. In this case, there is a clear approximate notion of position, but it does not make sense to specify this position to within millimeter accuracy.

It seems to be impossible to interpret the second case in terms of the first in a natural way. The only physically distinguished point-like position of an extended object is its center of mass. Classically, one could therefore think of defining the exact position of an extended object to be the position of its center of mass. But the sun, a city, a house, or a water wavelet do not even have a well-defined boundary, so even the definition of their center of mass, which depends on what precisely belongs to the object, is ambiguous. And is a tire really located at its center of mass – which is well outside the material the tire is made of? Things get worse in the microscopic realm, where the center of mass of a system of quantum particles has not even an exactly numerically definable meaning.

On closer inspection it seems that the situation of case 2 is very frequent in practice. Indeed, it is the *typical* situation in the macroscopic, classical world. Case 1 appears to be simply a convenient but unrealistic idealization.

⁴Except when relating to tradition, we deliberately avoid the notion of observables, since it is not clear on a fundamental level what it means to ‘observe’ something, and since many things (such as the fine structure constant, neutrino masses, decay rates, scattering cross sections) observable in nature are only indirectly related to what is traditionally called an ‘observable’ in quantum physics.

⁵Note that renaming notions has no observable consequences, but strongly affects the interpretation. To avoid confusion, I follow here as in Part I [38] the convention of ALLAHVERDYAN et al. [1] and add the prefix ‘q-’ to all traditional quantum notions that get here a new interpretation and hence a new terminology. In particular, we use the terms q-observable, q-expectation, q-variance, q-standard deviation, q-probability, q-ensemble for the conventional terms observable, expectation, variance, standard deviation, probability, and ensemble.

The uncertainty in the position of macroscopic objects such as the sun, a city, a house, a tire, or a water wavelet is therefore a conceptual uncertainty impossible to resolve by measurement. The thermal interpretation asserts that quantum uncertainty is an uncertainty of the same conceptual kind.

Thus uncertainty is only partially captured through statistical techniques. The latter apply only in case of highly repetitive uncertain situations, leading to a particular kind of uncertainty called aleatoric uncertainty (see, e.g., [10, 45]). More general kinds of uncertainty are discussed in the NIST Reference on Constants, Units, and Uncertainty [43], which may be regarded as the de facto scientific standard for representing uncertainty. This source explicitly distinguishes between uncertainties "which are evaluated by statistical methods" and those "which are evaluated by other means". For the second category, it is recognized that the uncertainties are not statistical but should be treated "like standard deviations".

The thermal interpretation follows this pattern by explicitly recognizing that not all uncertainty can be expressed statistically, though it is expressed in terms of uncertainty formulas that behave like the corresponding statistical concepts.

2.4 What is an ensemble?

We may imagine a great number of systems of the same nature, but differing in the configurations and velocities which they have at a given instant, and differing not merely infinitesimally, but it may be so as to embrace every conceivable combination of configuration and velocities. [...] The first inquiries in this field were indeed somewhat narrower in their scope than that which has been mentioned, being applied to the particles of a system, rather than to independent systems.

Josiah Willard Gibbs, 1902 [16, pp. vii–viii]

So aufgefaßt, scheint die Gibbssche Definition geradezu widersinnig. Wie soll eine dem Körper wirklich eignende Größe abhängen nicht von dem Zustand, den er hat, sondern den er möglicherweise haben könnte? [...] Es wird eine Gesamtheit mathematisch fingiert [...] erscheint es schwierig, wenn nicht ausgeschlossen, dem Begriffe der kanonischen Gesamtheit eine physikalische Bedeutung abzugewinnen.

Paul Hertz, 1910 [23, p.226f]

The thermal interpretation of quantum physics says that, consistent with statistical thermodynamics, a q-expectation (q-ensemble mean) is interpreted as an in principle approximately measurable quantity. Except when the statistical context is immediate (such as in computer simulations), the q-expectation should not be interpreted as a statistical average over a population⁶ of many realizations.

⁶Physicists usually speak of an ensemble in place of a population. In this paper, the statistical term **population** is used instead, to keep the discussion unambiguous, since in connection with the microcanonical, canonical, or grand canonical ensemble the term **ensemble** is essentially synonymous with a density operator of a particularly simple form.

The q-expectation, conventionally called the ensemble expectation, becomes in the thermal interpretation simply the uncertain value.

Therefore, the notion of q-ensemble is to be understood not as an actual repetition by repeated preparation. It should be understood instead in the sense of a fictitious collection of imagined copies of which *only one* is actually realized –, giving an intuitive excuse for using the statistical formalism for a single system.

The association of a fictitious ensemble to *single* thermal systems goes back to Gibbs, the founder of the ensemble approach to classical statistical mechanics. He was very aware that thermodynamics and hence statistical mechanics applies to single physical systems. His arguments are today as cogent as when he introduced them.

In classical statistical mechanics, the distinction between the deterministic and stochastic description becomes blurred, as *each single* macroscopic system is already described by a phase space density (multiparticle distribution function), although the latter behaves mathematically in every respect like a probability density that expresses the properties of a population of identical systems.

This tension in the terminology is already visible in the famous statistical mechanics textbook by GIBBS [16] in 1902, where he introduced in the preface (from which the above quote is taken) fictitious ensembles to bridge the conceptual gap.

Thus to deduce properties of macroscopic materials, Gibbs uses an ensemble of macroscopic systems – in contrast to Boltzmann, who introduced statistical mechanics for gases by using ensembles of microscopic atoms. Treating a collection of particles in a gas as an ensemble (as Boltzmann did) makes statistical sense as there are a huge number of them. But the Gibbs formalism is applied to single macroscopic systems such as a brick of iron rather than to its many constituents. Treating a single system as part of a fictitious ensemble was a very bold step taken by Gibbs [16, p.5]: *“Let us imagine a great number of independent systems, identical in nature, but differing in phase, that is, in their condition with respect to configuration and velocity.”* This allowed him to extend Boltzmann’s work from ideal gases to arbitrary chemical systems, in a very robust way. His statistical mechanics formalism, as encoded in the textbook (GIBBS [16]), survived the quantum revolution almost without change – the book reads almost like a modern book on statistical mechanics!

Though exceedingly successful, Gibbs’ fictitious ensemble raised in his time severe objections in the physics community, such as the response by Hertz quoted above, who complained that an ensemble is feigned mathematically. Of course Gibbs was aware that imagined systems have no physical implications, but these were needed at a time where mathematics had not yet the abstract character that it has today.

Today, mathematical theories are simply formal systems used without hesitation in applications in which the terms may mean something completely different from their meaning in the uses that gave the names to the terms. For example, the mathematical notion of a vector is today an abstract tool routinely used in contexts where the original geometric notion of a vector is meaningless: No physicist thinks of a quantum mechanical state vector in terms of a little arrow depicting a translational motion.

In the same spirit, mathematical statistics (and hence statistical mechanics) may be used as a tool in which expectation values figure as abstract notions without the need to imagine an ensemble of copies of the single system under study over which the expectation would be an imagined average. Thus we are liberated from having to think of the mathematical expectation values manipulated in statistical mechanics as being true averages over fictitious copies without a physical meaning. Instead, the statistical terminology is simply a reminder of which laws (originally stemming from statistical data analysis) are applicable to these values, in the same way as the geometric terminology of a vector indicates the laws valid for manipulating objects behaving algebraically like vectors.

2.5 Formal definition of the thermal interpretation

Wenn wir aus jenem mathematischen Schema physikalische Resultate ableiten wollen, so müssen wir den quantentheoretischen Größen, also den Matrizen (oder 'Tensoren' im mehrdimensionalen Raum) Zahlen zuordnen. [...] Man kann also sagen: Jeder quantentheoretischen Größe oder Matrix läßt sich eine Zahl, die ihren 'Wert' angibt, mit einem bestimmten wahrscheinlichen Fehler zuordnen; der wahrscheinliche Fehler hängt vom Koordinatensystem ab; für jede quantentheoretische Größe gibt es je ein Koordinatensystem, in dem der wahrscheinliche Fehler für diese Größe verschwindet. Ein bestimmtes Experiment kann also niemals für alle quantentheoretischen Größen genaue Auskunft geben

Werner Heisenberg, 1927 [21, p.181f]

Wir hatten ja immer leichthin gesagt: die Bahn des Elektrons in der Nebelkammer kann man beobachten. Aber vielleicht war das, was man wirklich beobachtet, weniger. Vielleicht konnte man nur eine diskrete Folge von ungenau bestimmten Orten des Elektrons wahrnehmen. Tatsächlich sieht man ja nur einzelne Wassertröpfchen in der Kammer, die sicher sehr viel ausgedehnter sind als ein Elektron. Die richtige Frage mußte also lauten: Kann man in der Quantenmechanik eine Situation darstellen, in der sich ein Elektron ungefähr – das heißt mit einer gewissen Ungenauigkeit – an einem gegebenen Ort befindet und dabei ungefähr – das heißt wieder mit einer gewissen Ungenauigkeit – eine vorgegebene Geschwindigkeit besitzt, und kann man diese Ungenauigkeiten so gering machen, daß man nicht in Schwierigkeiten mit dem Experiment gerät?

Werner Heisenberg, 1972 [22, p.77f]

The thermal interpretation of quantum physics uses for the description of quantum physics a formal framework consisting of

- a Euclidean space \mathbb{H} on which **states** are encoded by density operators, positive semidefinite linear trace 1 operator ρ from \mathbb{H} to itself (in the most regular case; to its completion in the general case);
- a representation of the standard model of the electromagnetic, weak, and strong interactions plus some form of gravity (not yet fully known) to describe the fundamental field content;

- a unitary representation of the Heisenberg, Galilei or Poincaré group⁷ to account for conservative dynamics and the principle of relativity in its nonrelativistic or special relativistic situation, respectively;
- and the Ehrenfest picture for the dynamics of q-expectations.

The representation defines the basic physical quantities (linear operators on \mathbb{H}) characterizing the system – energy, momentum, angular momentum. \mathbb{H} itself is a dense subspace of the physical Hilbert space and forms a common domain for the basic physical quantities.

As described in Subsection 2.2 above, the state defines the properties a particular instance of the system has, and hence what exists in the system.

The thermal interpretation avoids both the philosophically problematic notion of probability, and the anthropomorphic notions of knowledge and measurement. (We shall reconsider the notion of probability in Section 3, that of knowledge in Subsection 3.2, and that of measurement in Part III [39].) Instead, it gives the q-expectation value an interpretation in terms of a fundamental – not further explained but intuitive – notion of uncertainty, generalizing statistical practice:

(GUP) General uncertainty principle: *A Hermitian⁸ quantity A has the **uncertain value** $\bar{A} = \langle A \rangle$ with an **uncertainty** of⁹*

$$\sigma_A := \sqrt{\langle (A - \bar{A})^2 \rangle} = \sqrt{\langle A^2 \rangle - \bar{A}^2}. \quad (15)$$

In particular, the uncertain value \bar{A} is informative whenever its uncertainty σ_A is much less than $|\bar{A}|$

This contrasts with the traditional interpretations, which give the q-expectation value an interpretation in terms of a not further explained, fundamental notion of random measurement or a not further explained, fundamental notion of knowledge (or information), with all their associated foundational difficulties that were explored since Born proposed his interpretation.

Much of quantum physics can be developed without using the notion of probability at all. The notion of uncertain values $\langle A \rangle$ suffices for almost all of quantum chemistry and

⁷Except for a few passing remarks concerning gravity, we assume in this paper a flat spacetime. In a quantum field theory of gravity, we'd also need something like a unitary representation of the diffeomorphism group of the spacetime manifold, which has its own foundational problems.

⁸The uncertain value and its uncertainty makes also sense in the nonhermitian case, but the uncertainty must be defined in this case as

$$\sigma_A := \sqrt{\langle (A - \bar{A})^*(A - \bar{A}) \rangle} = \sqrt{\langle A^*A \rangle - |\bar{A}|^2}.$$

⁹The equivalence of both expressions defining σ_A follows from $\bar{A} = \langle A \rangle$ and

$$\langle (A - \bar{A})^2 \rangle = \langle A^2 - A\bar{A} - \bar{A}A + \bar{A}^2 \rangle = \langle A^2 \rangle - \langle A \rangle \bar{A} - \bar{A} \langle A \rangle + \bar{A}^2 = \langle A^2 \rangle - \bar{A}^2.$$

quantum statistical mechanics. For equilibrium statistical mechanics, this can be seen from the treatment in NEUMAIER & WESTRA [41].

As discovered by HEISENBERG [21], quantum physics predicts its own uncertainty. The q-variances of corresponding components of position q and momentum p cannot be both arbitrarily small since

$$\sigma_{p_j} \sigma_{q_j} \geq \frac{1}{2} \hbar.$$

This inequality is the famous **Heisenberg uncertainty relation** – not to be confused with the general uncertainty principle (GUP) defined above. The Heisenberg uncertainty relation is a special case of the more general statement (due to ROBERTSON [50]) that for non-commuting Hermitian operators A, B ,

$$\sigma_A \sigma_B \geq \frac{1}{2} |\langle [A, B] \rangle|, \quad (16)$$

which follows from the definitions.¹⁰ Like Ehrenfest's equation, the Heisenberg uncertainty relation does not involve notions of reality or measurement, hence belongs to the formal core of quantum mechanics. Both are valid independent of the interpretation of the q-expectations and the q-variances, respectively.

As discussed in Part I [38] (and exemplified in more detail in Part III [39] and Part IV [40]), the **interpretation**, i.e., the identification of formal properties given by uncertain values with real life properties of a physical system, is done by means of the generalized uncertainty principle (GUP), interpreting the q-expectations as beables and the q-variances as their uncertainty, together with

(CC) Callen's criterion (CALLEN [7, p.15]): *Operationally, a system is in a given state if its properties are consistently described by the theory for this state.*

This is enough to find out in each single case how to approximately measure the uncertain value of a quantity of interest, though it may require considerable experimental ingenuity to do so with low uncertainty. The uncertain value \bar{X} is considered informative only when its uncertainty σ_X is much less than $|\bar{X}|$.

This interpretation gives both Ehrenfest's equation and the Heisenberg uncertainty relation their practical importance: Ehrenfest's equation leads to the emergence of classical mechanics as approximate dynamics in cases where the variances are tiny (see Part IV [40] for details). Heisenberg's uncertainty relation enforces an unavoidable limit on the accuracy with which position and momentum can be specified simultaneously.

In the thermal interpretation, even pointlike quantum objects are extended: Every pointlike quantum object has a 3-component position vector \mathbf{q} , hence is extended to the extent

¹⁰Indeed, the relation remains unchanged when subtracting from A and B its q-expectation, hence it suffices to prove it for the case where both q-expectations vanish. In this case, $\langle A^2 \rangle = \sigma_A^2$ and $\langle B^2 \rangle = \sigma_B^2$, and the Cauchy-Schwarz inequality gives $|\langle AB \rangle|^2 \leq \langle A^2 \rangle \langle B^2 \rangle = \sigma_A^2 \sigma_B^2$, hence $|\langle AB \rangle| \leq \sigma_A \sigma_B$. On the other hand, one easily checks that $i \operatorname{Im} \langle AB \rangle = \frac{1}{2} \langle [A, B] \rangle$, so that $\frac{1}{2} |\langle [A, B] \rangle| = |\operatorname{Im} \langle AB \rangle| \leq |\langle AB \rangle|$. Combining both inequalities gives the assertion.

determined by the computable position uncertainty $\sigma_{\mathbf{q}} = \sqrt{\sigma_{\mathbf{q}_1}^2 + \sigma_{\mathbf{q}_2}^2 + \sigma_{\mathbf{q}_3}^2}$, where \mathbf{q} is the position vector of the object. In spacetime, the uncertain positions $\langle \mathbf{q} \rangle_t$ traced out an uncertain world line, and the quantum object can be visualized as moving along a nested family of fuzzy **world tubes**, the union of the intervals $[\langle \mathbf{q} \rangle_t - \kappa \sigma_{\mathbf{q}}(t), \langle \mathbf{q} \rangle_t + \kappa \sigma_{\mathbf{q}}(t)]$, for reasonable κ of order one.

Thus the thermal interpretation answers Heisenberg's question from the early days of quantum mechanics quoted above, "*Can quantum mechanics represent the fact that an electron finds itself approximately in a given place and that it moves approximately with a given velocity, and can we make these approximations so close that they do not cause experimental difficulties?*" It gives the electrons real but uncertain paths, making them move along an extended world tube rather than an infinitesimally thin world line.

3 Thermal interpretation of statistics and probability

In this section we consider the way the thermal interpretation represents statistical and probabilistic aspects of quantum theory. We begin in Subsections 3.1 and 3.2 with a discussion of two formal notions of classical probability, their relation to the probability concept used in applied statistics, and their dependence on the description used.

Historically, the concept of classical probability (including its use in stochastic processes) was given an undisputed formal mathematical foundation in 1933 in terms of the measure-theoretic setting of KOLMOGOROV [27]. Apart from this traditional axiomatic foundation of classical probability theory there is a less well-known equivalent axiomatic treatment by WHITTLE [59] in terms of expectations. Here probabilities appear as the expectations of statements, $\{0, 1\}$ -valued random variables. VON PLATO [49] discusses the history of the concept of probability. KRÜGER et al. [28] discuss the history of probability in the various fields of application. SKLAR [52] discusses the philosophical problems of the probability concept, with an emphasis on statistical mechanics.

Subsection 3.3 then shows how the statistical aspects of the quantum formalism naturally follow from the weak law of large numbers. The notions of c-probability in the classical case and of q-probability in the quantum case are formally defined in Subsection 3.4; they are related to experiment in Subsection 3.5. The final subsection briefly discusses various examples where probabilistic features emerge from fully deterministic situations – a theme to be taken up in much more detail in Part III [39].

3.1 Classical probability via expectation

This subsection gives an elementary introduction to classical probability theory along the lines of WHITTLE [59], similar in spirit to the formal core of quantum mechanics.

Let Ω denote a finite or infinite set of conceivable **experiments** ω .¹¹ Let \mathbb{E} be a vector

¹¹In probability theory Ω is called the **sample space**. In statistics, the $\omega \in \Omega$ may be identified with

space of real valued functions on Ω containing the constant functions. We identify constant functions with their function values. The elements $A \in \mathbb{E}$ are called (real) **random variables**, and the value $A(\omega)$ is called the **realization** of A in experiment ω . The procedure that defines how to obtain the realization $f(\omega)$ for any experiment ω is called the **protocol** defining the random variable f . Functions, order relations, unary and binary operations, and limits on random variables are defined pointwise; note, however, that a pointwise function $f(A)$ of a random variable $A \in \mathbb{E}$ does not necessarily lie in \mathbb{E} .

A **sample** is a finite set S of $|S| > 0$ experiments; the associated **sample mean** of a random variable A is defined by

$$\langle A \rangle_S := \frac{1}{|S|} \sum_{\omega \in S} A(\omega). \quad (17)$$

It is easily checked that any sample mean $\langle \cdot \rangle = \langle \cdot \rangle_S$ satisfies the following rules:

- (E1) $\langle 1 \rangle = 1$.
- (E2) $\langle \alpha A + \beta B \rangle = \alpha \langle A \rangle + \beta \langle B \rangle$ for $\alpha, \beta \in \mathbb{R}$.
- (E3) $A \geq 0$ implies $\langle A \rangle \geq 0$.
- (E4) $A \geq 0$, $\langle A \rangle = 0$ implies $A = 0$.
- (E5) $A_k \downarrow 0$ implies $\langle A_k \rangle \downarrow 0$.

Here \downarrow denotes pointwise convergence from above. To abstract from a particular sample we define a **stochastic model** as an arbitrary mapping that assigns to each random variable A a real number $\langle A \rangle$, called the **expectation** (or **expected value** or **mean**) of A such that the axioms (E1)–(E5) hold whenever the expectations in question exist. All samples, arbitrary convex combinations of samples, and their limits, define a stochastic model. Given real statistical data from real experiments, the quality of a stochastic model is assessed by how well the expectations of key random variables match corresponding sample expectations for samples drawn at random in some informal sense.

As a simple consequence of the axioms, we note:

- (E6) $\langle A^2 \rangle = 0$ implies $A = 0$,
- (E7) $A \leq B$ implies $\langle A \rangle \leq \langle B \rangle$.

Indeed, (E6) follows directly from (E4). For (E7), the assumption gives $B - A \geq 0$, hence $\langle B - A \rangle \geq 0$ by (E3), hence $\langle B \rangle - \langle A \rangle \geq 0$ by (E2), giving $\langle A \rangle \leq \langle B \rangle$.

An example of a random variable is the number n of eyes on the top side of a die. Here $n(\omega) \in \{1, \dots, 6\}$ is the number of eyes on the die visible in experiment ω . We may thus consider \mathbb{E} to be the algebra generated by a single random variable $n = n(\omega)$ taking the values 1, 2, 3, 4, 5, 6. Thus the relevant random variables are the functions $A = A(n)$ defined by

$$A(n)(\omega) := A(n(\omega)).$$

actual experiments carried out in the past or in the future. In classical statistical mechanics, the ω are only hypothetical experiments from a fictitious ensemble in the sense of Gibbs, as discussed in Subsection 2.4. In general, Ω is just a set, and calling the $\omega \in \Omega$ experiments is just a convenient intuition without formal meaning (comparable to calling elements of a vector space vectors even though they may be functions or matrices). For examples, see Subsection 3.6.

A is determined by the vector of the six values $A_1 = A(1), \dots, A_6 = A(6)$. Therefore we may identify \mathbb{E} with the vector space \mathbb{R}^6 with componentwise operations. The stochastic model defined by

$$\langle A \rangle := \frac{1}{6}(A_1 + \dots + A_6)$$

models an ideal, permutation symmetric die.

In practice the set Ω may be different depending on the imagination of people and the intended use. The expectation value is independent of $\omega \in \Omega$ and depends – as in the example just given – on the algebra \mathbb{E} of relevant random variables only. This algebra is always commutative and associative.

A **statement** is a $\{0, 1\}$ -valued random variable A . The statement is **true** (**false**) in an experiment ω if $A(\omega) = 1$ (resp. $A(\omega) = 0$). The **probability** of a statement A , defined as

$$\Pr(A) := \langle A \rangle$$

is a number between 0 and 1. Indeed, we have $0 \leq A \leq 1$. Thus by (E3), $\langle A \rangle \geq 0$. By (E6) and (E1), $\langle A \rangle \leq \langle 1 \rangle = 1$.

Proposition. If A_1, \dots, A_n are alternative statements of which exactly one is true in each experiment then the probabilities $p_{A_i} := \Pr(A_i)$ sum up to 1, and $\Pr(A_i \wedge A_j) = 0$ for $i \neq j$.

Indeed, the random variable $E := \sum_{i=1}^n A_i$ satisfies

$$\chi(\omega) = \sum_{i=1}^n A_i(\omega) = 1 \quad \text{for all } \omega \in \Omega,$$

because by definition exactly one A_i occurs in each experiment ω . Therefore $E = 1$, and

$$1 = \langle 1 \rangle = \langle E \rangle = \left\langle \sum_{i=1}^n A_i \right\rangle = \sum_{i=1}^n \langle A_i \rangle = \sum_{i=1}^n p_{A_i}.$$

Similarly, $(A_i \wedge A_j)(\omega) = A_i(\omega) \wedge A_j(\omega) = 0$ for $i \neq j$ since at most one of $A_i(\omega)$ and $A_j(\omega)$ can be true, hence $A_i \wedge A_j = 0$ and $\Pr(A_i \wedge A_j) = \Pr(0) = \langle 0 \rangle = 0$.

The **cumulative distribution function (CDF)** of a random variable A is the function $\text{cdf} : \mathbb{R} \rightarrow [0, 1]$ defined by as

$$\text{cdf}(x) := \Pr(A \leq x) = \langle [A \leq x] \rangle$$

for all x . Here $[..]$ denotes the statement ($\{0, 1\}$ -valued function) defined by the formula inside the square brackets.

Proposition. Every CDF is monotone increasing and satisfies

$$\lim_{\varepsilon \downarrow 0} \text{cdf}(x + \varepsilon) = \text{cdf}(x), \quad \lim_{x \rightarrow -\infty} \text{cdf}(x) = 0, \quad \lim_{x \rightarrow +\infty} \text{cdf}(x) = 1, \quad (18)$$

Thus every CDF is continuous from the right.

Proof. By (E6), $x \leq x'$ implies

$$[A \leq x'] - [A \leq x] \geq 0$$

so by (E6),

$$\Pr(A \leq x') - \Pr(A \leq x) = \langle [A \leq x'] - [A \leq x] \rangle = \langle [x < A \leq x'] \rangle = \Pr([x < A \leq x']) \geq 0.$$

and thus

$$\text{cdf}(x) \leq \text{cdf}(x').$$

To prove continuity from the right we note that the random variable

$$B(\varepsilon) := [x < A \leq x + \varepsilon]$$

vanishes at ω with $A(\omega) \leq x$ and for $\varepsilon < A(\omega) - x$ if $A(\omega) > x$, and is 1 otherwise. Therefore $B(\varepsilon) \downarrow 0$ for $\varepsilon \downarrow 0$. Thus

$$\langle B(\varepsilon) \rangle \downarrow 0 \quad \text{for } \varepsilon \downarrow 0$$

by (E5). But

$$\langle B(\varepsilon) \rangle = \Pr(x < A \leq x + \varepsilon) = \text{cdf}(x + \varepsilon) - \text{cdf}(x),$$

hence the first limit of (18) follows. The other limits are proved in the same way. \square

Let A be a random variable with CDF cdf . If f is a step function with finitely many jump points at $x_1 \leq \dots \leq x_n$ and f is continuous from the left then

$$\langle f(A) \rangle = f(x_1)\text{cdf}(x_1) + \sum_{k=1}^{n-1} f(x_{k+1})(\text{cdf}(x_{k+1}) - \text{cdf}(x_k)). \quad (19)$$

Indeed, the right hand side equals the expectation of

$$f(x_1)[A \leq x_1] + \dots + \sum f(x_k)[x_{k-1} < A \leq x_k] = f(A).$$

By taking a continuum limit in (19), we obtain the **Stieltjes integral representation**

$$\langle f(A) \rangle = \int_{-\infty}^{\infty} f(\xi) d\text{cdf}(\xi).$$

for the expectation of $f(A)$, for any function f that is continuous from the left. Thus the CDF contains all information about expectations of functions of a real random variable.

For a random variable A with continuously differentiable cumulative distribution function, the **distribution** or **density** ρ of A is defined as

$$\rho(\xi) = \frac{d}{d\xi} \text{cdf}(\xi).$$

It is always nonnegative since the CDF is monotone increasing. The CDF can be expressed in terms of the density as

$$\text{cdf}(\xi) = \int_{-\infty}^{\xi} \rho(\zeta) d\zeta$$

If the density $\rho(x)$ exists, it also carries all information about expectations of functions of A . Indeed, we have

$$\langle f(A) \rangle = \int_{-\infty}^{\infty} f(\xi) \rho(\xi) d\xi$$

since $d \text{cdf}(\xi) = \rho(\xi) d\xi$.

3.2 Description dependence of probabilities

Classical probabilities are dependent on the description used. The latter encode the assumed knowledge about the system under study. Note that the implied concept of **knowledge** is not the knowledge of a particular observer or person but an informal shorthand for what is modeled in a particular description. Thus the 'knowledge available', i.e., the knowledge encoded into a particular description, is an objective property of the description used to model the system, independent of who 'knows' or 'uses' it. Nothing mental is implied. A change of knowledge is therefore just a change of the model used to describe a particular system.

Sample expectations and hence sample probabilities have a clear operational meaning. But they are properties of the specific sample taken – changing the sample, e.g., when the values for new realizations become known, almost always changes the values of sample expectations and sample probabilities.

Cumulative distribution functions are easy to estimate on a sample S by the sample CDF

$$\text{cdf}_S(\xi) = \frac{\text{number of } \omega \in S \text{ with } x(\omega) \leq \xi}{|S|},$$

corresponding to the sample expectation (17). A sample CDF is always a step function with discontinuities at the $x(\omega)$ with $\omega \in S$. In many cases it is well approximated by a smoothed CDF, using one of many smoothing methods available. The derivative of the smoothed CDF then serves as an estimate for the density of a random variable A in a stochastic model in which A is treated as a random variable with density.

In the univariate case treated above in detail, going from a known sample to predictions for not yet known samples is based on the possibility to approximate sample distributions of different sample sizes by a single model distribution since their CDFs are very similar. This also holds in the multivariate case, though different techniques must be used to establish corresponding results. Finite sample properties can be proved using arguments basically similar to our proof of the weak law of large numbers (24); under reasonable assumptions, the uncertainty of the most relevant random variables scales again with $O(N^{-1/2})$. This is the contents of large N approximations that take the form of laws of large numbers and

central limit theorems. They guarantee enhanced predictability of the kind that the mean uncertainty is approximately the single case uncertainty divided by \sqrt{N} .

Thus the precise meaning of expectations and probabilities depends on which stochastic model is used for a given situation. To see what happens when we change the description we consider the concept of conditional expectation, which models the reweighing of evidence leading to a change of the description of a model. A **weight** is a nonzero random variable $P \geq 0$. Let

$$\Omega_P = \{\omega \in \Omega \mid P(\omega) > 0\}$$

We project the algebra \mathbb{E} of random variables $X : \Omega \rightarrow \mathbb{R}$ to an algebra \mathbb{E}_P of random variables $Y : \Omega_P \rightarrow \mathbb{R}$ by means of the homomorphism

$$\cdot|_{\Omega_P} : \mathbb{E} \rightarrow \mathbb{E}_P : X \mapsto X|_{\Omega_P}$$

that restricts all random variables $X \in \mathbb{E}$ to $X|_{\Omega_P} \in \mathbb{E}_P$. The **conditional expectation** of $X : \Omega \rightarrow \mathbb{R}$ with respect to the weight P is defined as

$$\langle X|_{\Omega_P} \rangle_P = \langle X \rangle_P := \frac{\langle X \cdot P \rangle}{\langle P \rangle};$$

note that $\langle P \rangle > 0$ by (E4). The resulting mapping $\langle \cdot \rangle_P : \mathbb{E}_P \rightarrow \mathbb{R}$ satisfies all expectation axioms: (E1)–(E3) and (E5) follow directly from the corresponding axioms for $\langle \cdot \rangle$. Only (E4) is nontrivial; $\langle X^2 \rangle_P = 0$ implies $\langle X^2 P \rangle = \langle P \rangle \langle X^2 \rangle_P = 0$. But $X^2 P \geq 0$, hence (E4) gives $X^2 P = 0$, i.e., $X(\omega)^2 P(\omega) = 0$ for all ω . Thus $X(\omega) = 0$ whenever $P(\omega) = 1$. This gives $X|_{\Omega_P} = 0$ and proves (E4). Therefore $\langle \cdot \rangle_P$ is a proper expectation for the experiments in Ω_P .

In particular, any statement A with positive probability $\Pr(A) > 0$ may be considered as a weight. Since Ω_A is the set of all those experiments where A is true, conditional expectation with respect to A is just expectation in the light of the assumed evidence that A is true. In particular, the probability for A being true changed to 1. More generally, the **conditional probability** of a statement B given the statement A is defined by

$$\Pr(B|A) := \langle B \rangle_A = \frac{\langle BA \rangle}{\langle A \rangle} = \frac{\Pr(A \wedge B)}{\Pr(A)};$$

the last equality holds since

$$BA(\omega) = B(\omega)A(\omega) = \begin{cases} 1 & \text{iff } A(\omega) \wedge B(\omega), \\ 0 & \text{otherwise.} \end{cases}$$

It is easily checked that for any statements A and B we have

$$\Pr(B \wedge A) = \Pr(B|A) \Pr(A), \tag{20}$$

$$\Pr(A|B) = \Pr(A) \frac{\Pr(B|A)}{\Pr(B)}. \tag{21}$$

(21) is often called the **Bayes theorem**. In this context, $\Pr(A)$ is called the **prior probability** of A , $\Pr(A|B)$ its **posterior probability**, and $\Pr(B|A)/\Pr(B)$ the **update ratio**. If

A and B are independent then $\langle AB \rangle = \langle A \rangle \langle B \rangle$, hence $\Pr(B|A)/\Pr(B) = \langle AB \rangle / \langle A \rangle \langle B \rangle = 1$. Thus the update ratio captures the degree to which the knowledge of B affects knowledge of A . Thus, Bayes theorem allows us to describe the change of c-probability of a class of statements when new information (namely the statement B) arrives and is accepted as valid. Bayes theorem is important to understand what it means to get new information when A is a statement of interest and B is information that becomes known. Bayes theorem tells how the probability of A changes under new insight. $\Pr(A)$ changes into $\Pr(A|B)$, so the probability of A must be multiplied by the update ratio $\Pr(B|A)/\Pr(B)$.

3.3 Deterministic and stochastic aspects

Some people are hoping to reintroduce determinism in some way, perhaps by means of hidden variables or something like that, but it just doesn't work according to the accepted ideas. I might add that personally I still have this prejudice against indeterminacy in basic physics. I have to accept it because we cannot do anything better at the present time. It may be that in some future development we shall be able to return to determinism, but only at the expense of giving up something else, some other prejudice which we hold to very strongly at the present time.

Paul Dirac, 1972 [12, p.7]

The Ehrenfest picture shows that the q-expectations behave like classical variables and are governed by a deterministic, Hamiltonian dynamics. Thus in the thermal interpretation, quantum physics is a deterministic theory.

Various theorems (beginning with BELL [3]) and associated experiments (beginning with ASPECT [2]) exclude the existence of certain kinds of deterministic theories explaining quantum mechanics through hidden variables that assume some form of locality, noncontextuality, or sharpness; cf. Subsection 4.5. Since the deterministic Ehrenfest dynamics for q-expectations follows from the Schrödinger equation, there can be no conflict with such no-go theorems for deterministic dynamics. Indeed, most q-expectations, for example those of momentum or kinetic energy, are nonlocal objects, and sharpness is in the thermal interpretation denied from the start. The germ of the new interpretation was actually my analysis in [32] of sharpness arguments in a number of no-go theorems.

A quantity A is considered to be **significant** if $\sigma_A \ll |\overline{A}|$, while it is considered as **noise** if $\sigma_A \gg |\overline{A}|$. If A is a quantity and \tilde{A} is a good numerical approximation of its value then $\Delta A := A - \tilde{A}$ is noise. Sufficiently significant quantities can be treated as deterministic; the analysis of noise is the subject of statistics.

Statistics is based on the idea of obtaining information about noisy quantities of a system by repeated **sampling** from a **population** of independent systems with identical preparation, but differing in noisy details not controllable by the preparation. In the present context, such systems are described by the same Hilbert space, the same set of quantities to be sampled, and the same state $\langle \cdot \rangle_0$. The quantities therefore belong to the algebra $\text{Lin } \mathbb{H}_S$ of linear operators on a Euclidean space \mathbb{H}_S dense in the Hilbert space of the system.

More precisely, stochastic features emerge when we consider a large sample of similar subsystems of a quantum system. For an ensemble of independent measurements on identi-

cally prepared systems, the consensus of all interpretations is that q-expectations represent (within the accuracy allowed by the law of large numbers) a statistical average of the measurement results.¹² In order that the thermal interpretation is viable we must therefore be able to show that the q-variance is indeed a statistical property of many independent measurements on identically prepared systems.

We regard the systems of the population considered as subsystems of a bigger system (e.g., the laboratory) whose set of quantities is given by the algebra $\text{Lin } \mathbb{H}$ of linear operators on a big Euclidean space \mathbb{H} dense in the Hilbert space of the big system. To model identically prepared subsystems we consider injective homomorphisms from $\text{Lin } \mathbb{H}_S$ into $\text{Lin } \mathbb{H}$ mapping each reference quantity $A \in \text{Lin } \mathbb{H}_S$ to the quantity $A_l \in \text{Lin } \mathbb{H}$ of the l th subsystem considered to be ‘identical’ with A . Of course, in terms of the big system, the A_l are not really identical; they refer to quantities distinguished by position and/or time. That the subsystems are **identically prepared** is instead modeled by the assumption

$$\langle A_k \rangle = \langle A_l \rangle \quad \text{for all } k \neq l, \quad (22)$$

and that they are **independent** by the assumption

$$\langle A_k A_l \rangle = \langle A_k \rangle \langle A_l \rangle \quad \text{for all } k \neq l. \quad (23)$$

The following result is fundamental for statistical considerations:

Theorem (Weak law of large numbers). For a sample of quantities A_l ($l = 1, \dots, N$) satisfying (22) and (23), the **mean** quantity

$$\widehat{A} := \frac{1}{N} \sum_{l=1}^N A_l$$

(which again is a quantity) satisfies for any l

$$\langle \widehat{A} \rangle = \langle A_l \rangle, \quad \sigma_{\widehat{A}} = \sigma_{A_l} / \sqrt{N}. \quad (24)$$

Proof. By (22) and (23), $\mu := \langle A_l \rangle$ and $\sigma := \sigma_{A_l}$ are independent of l , and we have

$$\begin{aligned} \langle \widehat{A} \rangle &= \frac{1}{N} (\langle A_1 \rangle + \dots + \langle A_N \rangle) = \frac{1}{N} (\mu + \dots + \mu) = \mu, \\ \langle \widehat{A}^* \widehat{A} \rangle &= \frac{1}{N^2} \left\langle \left(\sum_j A_j \right)^* \left(\sum_k A_k \right) \right\rangle = N^{-2} \sum_{j,k} \langle A_j^* A_k \rangle. \end{aligned} \quad (25)$$

¹² In order to take the q-variance alternatively as a time average of a single system, one would need to invoke an ergodic theorem stating that the time average equals the ensemble average. However such an ergodic theorem makes sense only semiclassically, and is valid only for very simple classical systems. Most deterministic systems are far from ergodic. (This is mentioned, e.g., in the statistical physics book by LANDAU & LIFSHITZ [29, Footnote 2, p.12].) From the eight examples of statistical models for deterministic systems given in Subsection 3.6 below, only two – cases (vi) and (vii) – have a generally valid ensemble interpretation in terms of ergodicity! Thus the interpretation of the q-variance as a time average is usually not warranted. This also means (Neumaier [36]) that – in contrast to what is usually done in the popular literature – the so-called vacuum fluctuations cannot be interpreted as fluctuations in time.

Now

$$\langle A_j^* A_j \rangle = \langle A_j \rangle^* \langle A_j \rangle + \sigma_{A_j}^2 = |\mu|^2 + \sigma^2,$$

and by (23) for $j \neq k$,

$$\langle A_j^* A_k + A_k^* A_j \rangle = 2 \operatorname{Re} \langle A_j^* A_k \rangle = 2 \operatorname{Re} \langle A_j \rangle^* \langle A_k \rangle = 2 \operatorname{Re} \mu^* \mu = 2|\mu|^2.$$

In the sum in (25), this leads to a contribution of $|\mu|^2 + \sigma^2$ for each of the N diagonal elements, and of $2|\mu|^2$ for each of the $\binom{N}{2}$ pairs of off-diagonal elements. Therefore

$$\langle \widehat{A}^* \widehat{A} \rangle = N^{-2} \left(N(|\mu|^2 + \sigma^2) + \binom{N}{2} 2|\mu|^2 \right) = N^{-1} \sigma^2 + |\mu|^2,$$

so that

$$\sigma_{\widehat{A}}^2 = \langle \widehat{A}^* \widehat{A} \rangle - \langle \widehat{A} \rangle^* \langle \widehat{A} \rangle = N^{-1} \sigma^2,$$

and the assertions follow. □

As a significant body of work in probability theory shows, the conditions under which $\sigma_{\widehat{A}} \rightarrow 0$ as $N \rightarrow \infty$ can be significantly relaxed; thus in practice, it is sufficient if (22) and (23) are approximately valid.

The significance of the weak law of large numbers lies in the fact that (24) becomes arbitrarily small as N becomes sufficiently large. Thus the uncertainty of quantities when averaged over a large population of identically prepared systems becomes arbitrarily small while the mean value reproduces the value of each quantity.

The weak law of large numbers implies that, in a context where many repeated experiments are feasible, states can be given a **frequentist** interpretation, in which $\overline{A} = \langle A \rangle$ is the **expectation** of A , empirically defined as an average over many realizations. In this case (and only in this case), the uncertainty σ_A becomes the standard deviation of A ; then it captures the absolute accuracy of the individual realizations.

Note that there are also many single systems with a sound statistical interpretation. Once one has a time series of a single system that empirically looks fluctuating, one can do valid statistics with it. Sun spot activity data or El Nino data (both time series for single physical systems) are traditional test data for statistical procedures. Single deterministic systems data with highly oscillating contributions – like the weather on Earth – are routinely explained and forecast in terms of a stochastic process. Thus, whenever the statistics of interest is that of a system's mean behavior in time (or in space), it is meaningful to do statistics on the single system. The time (or space) coordinates then label the sample under consideration. (See Footnote ¹² for the related notion of ergodicity.)

On the other hand, in equilibrium thermodynamics, where a tiny number of macroscopic observations on a single system completely determine its state to engineering accuracy, such a frequentist interpretation is inappropriate.

Thus, the thermal interpretation captures correctly the experimental practice, and determines the conditions under which deterministic and statistical reasoning are justified:

(SP) Statistical principle: *Deterministic reasoning is appropriate for all sufficiently significant quantities. Statistical reasoning is necessary for noisy quantities, and requires that these quantities are sufficiently similar and sufficiently independent to ensure that their mean is significant.*

In this way, statistical considerations naturally arise from the nonstatistical foundations, giving **statistics** the same role as in classical physics, namely as the *art and science of interpreting measurements*. Real experiments are (and must be) designed such that they allow one to determine approximately the relevant properties of the state under study, hence the values of all quantities of interest. The uncertainties in the experiments imply approximations, which, if treated probabilistically, need an *additional* probabilistic layer. Expectations from this secondary layer, which involve probabilistic statements about situations that are uncertain due to neglected but in principle observable details (cf. PERES [46]), happen to have the same formal properties as the values on the primary layer, though their physical origin and meaning is completely different.

3.4 What is probability?

In Subsection 3.1, we derived from (E1)–(E5) the traditional probabilistic machinery for single real random variables. More generally, it can be proved (see WHITTLE [59]) that even in the multivariate case, the above approach to classical probability is equivalent to the measure theoretical approach in the traditional axiomatic setting of Kolmogorov. In this equivalence, Ω is an abstract measure space, a stochastic model is a probability measure on Ω , and \mathbb{E} is a vector space of random variables with finite expectation.

The exposition in WHITTLE [59] (or, in more abstract terms, already in GELFAND & NAIMARK [15]) shows that, if the X_j are pairwise commuting, it is possible to define for any Gibbs state in the present sense, random variables X_j in Kolmogorov’s sense such that the expectation of all sufficiently regular functions $f(X)$ defined on the joint spectrum of X agrees with the value of f . It follows that in the pairwise commuting case, it is always possible to construct a probability interpretation for the quantities, completely independent of any assumed microscopic reality. (If the components of X do not commute, a probabilistic interpretation in the Kolmogorov sense is no longer possible because of the nonclassical uncertainty relations (16).)

The details (which the reader unfamiliar with measure theory may simply skip) are as follows. We may associate with every vector X of quantities with commuting components a time-dependent, monotone linear functional $\langle \cdot \rangle_t$ defining the **expectation**

$$\langle f(X) \rangle_t := \text{Tr } \rho(t) f(X)$$

at time t of arbitrary bounded continuous functions f of X . These functions define a commutative C^* -algebra $\mathbb{E}(X)$. The **spectrum** $\text{Spec } X$ of X is the set of all $*$ -homomorphisms (called **characters**) from $\mathbb{E}(X)$ to \mathbb{C} , and has the structure of a Hausdorff space, with the **weak- $*$ topology** obtained by calling a subset S of $\text{Spec } X$ closed if, for any pointwise convergent sequence (or net) contained in S , its limit is also in S . Now an expectation

functional satisfying (E1)–(E5) turns out to be equivalent to a multivariate probability measure $d\mu_t(X)$ (on the sigma algebra of Borel subsets of the spectrum Ω of X) defined by

$$\int d\mu_t(X)f(X) := \int \rho(t)f(X) = \langle f(X) \rangle_t.$$

Both Whittle's and Kolmogorov's foundation of classical probability theory are axiomatic and hence independent of the interpretation of the axioms. We may refer to probability as defined by Kolmogorov or Whittle as **c-probability**.

In generalization of this, we refer, for any Hermitian operators P satisfying $0 \leq P \leq 1$, to its expectation as the **q-probability**

$$\Pr(P) := \langle P \rangle$$

of P . As a special case, we call a quantity P satisfying $P^2 = P = P^*$ a **statement**; then $0 \leq P \leq 1$ follows by the spectral theorem. For a statement P , the uncertainty of its probability $p = \bar{P}$ is $\sigma_P = \sqrt{p(1-p)}$ since by (15), $\sigma_P^2 = \langle P^2 \rangle - \bar{P}^2 = p - p^2$. Another special case is the q-probability of a self-adjoint Hermitian q-observable A taking values in some open interval $]a, b[$ of real numbers, defined as

$$\Pr(A \in]a, b[) := \langle P_{]a, b[}(A) \rangle,$$

where $P = P_{]a, b[}(A)$ is the spectral projector of A to the interval $]a, b[$. Note that here $P^2 = P = P^*$, so that P is a statement – the formal equivalent of the informal statement 'A is in $]a, b[$ '.

Whittle's approach is essentially equivalent to the commutative case of the formal core of quantum mechanics, interpreted in statistical terms. Then q-probabilities and c-probabilities agree.

We discuss in more detail the important special case of binary tests, where Born's rule frequently applies essentially exactly. An **ideal binary measurement**, e.g., the click of a detector, is described by a statement P coding the presence (1) or absence (0) of a click. In particular, a **test for a state**¹³ ϕ with $\phi^*\phi = 1$ is an ideal binary measurement of $P = \phi\phi^*$; it is easily checked that this is a statement. By the above, such a test turns out positive with probability $p = \langle P \rangle$. In particular, if the system is in a pure state ψ then $p = \langle P \rangle = \psi^*P\psi = \psi^*\phi\phi^*\psi = |\phi^*\psi|^2$, hence

$$p = |\phi^*\psi|^2.$$

This is the well-known **squared probability amplitude** formula appearing in the scattering form of Born's rule as stated in Part I [38, Subsection 3.1]. Thus the scattering form of Born's rule appears as natural consequence rather than as a basic axiom.

¹³Note that a test for ϕ turns out positively with probability 1 if the measured system is in the pure state ϕ . However, it also turns out positively with a positive probability if the measured system is in a pure state different from ϕ , as long as it is not orthogonal to it. Thus calling it a 'test for ϕ ', though conventional, is something of a misnomer.

3.5 Probability measurements

When it uses probabilities, [...] science regards them [...] as measurable (and calculable) physical quantities like lengths, energies, and wavelengths. [...] The probability of a truly single event is intrinsically unmeasurable and [...] science has nothing to say about" it. "To obtain the value of a physical quantity, one must measure it a number of times, Each measurement contains an error, and the 'true' value is (usually) computed as the arithmetic mean of all measured values. [...] in a similar way", we "measure the relative frequency [...] of an event in a series of trials. Each relative frequency contains an error, and the 'true' probability is computed as the mean of the relative frequencies over a number of series. [...] nothing strange or inconsistent is left in the idea of probability as a measurable physical quantity.

Henry Margenau, 1950 [30, pp.250–252]

By its definition, the notions of q-expectations and q-probabilities belong to the formal core of quantum mechanics and are independent of any interpretation. But in the thermal interpretation all q-expectations, and in particular all q-probabilities, are among the be-ables. We discuss here basic aspects of their measurement; a more thorough discussion of measurement from the point of view of the thermal interpretation is given in Part III [39].

By the law of large numbers, q-expectations can be measured with in principle arbitrarily high accuracy by taking sample means of low accuracy measurements, whenever there is a device (the **preparation**) – e.g., a particle accelerator – that produces a large number of independent copies (realizations) of the same quantum system. The accuracy improves by a factor of \sqrt{N} , where N is the sample size.

In the same way, the q-probability p are approximately measurable as relative frequencies. As a consequence of the weak law of large numbers (24), the uncertainty of the **relative frequency** p_N , defined as the sample mean of ideal binary measurements in a sample of N independent realizations of the statement P , is $\sigma = \sigma_P \sqrt{N} = \sqrt{p(1-p)/N}$. This uncertainty approaches zero when the sample size N gets arbitrarily large. Thus measuring a probability by a relative frequency gives in principle arbitrarily accurate results.

This gives a fully adequate operational definition of probabilities without any logical problems, of the same quality as operational definitions of highly accurate length or time measurements: The q-probabilities are theoretical observables; they are measured as relative frequencies, to some reasonable accuracy that can be quantified by the associated uncertainty. We draw conclusions about sufficiently uncertain situations based on observed sample means and relative frequencies on a sample of significant size, and we quantify our remaining uncertainty by statistical safeguards (confidence intervals, etc.), well knowing that these sometimes fail. For example, the **5 σ -rule** for the discovery of elementary particles tries to guard against such failures. This view of probabilities as measurable entities is just the one described in the above quote by Margenau (if 'final' is read for 'true').

Thus probability has an objective interpretation precisely to the extent that objective protocols for taking the sample measurements i.e., how to distinguish a positive from a negative test, are agreed upon. Any subjectivity remaining lies in the question of deciding which

protocol should be used for accepting a measurement as 'correct'. Different protocols may give different results. Both classically and quantum mechanically, the experimental context needed to define the protocol influences the outcome. In particular, there is a big difference between the description of an event before (*predicting*) or after (*analyzing*) it occurs. This is captured rigorously in classical probability theory by conditional probabilities (cf. Subsection 3.2), and less rigorously in quantum physics by the so-called collapse of the wave function, where the description of the state of a particle is different before and after it passes a filter (polarizer, magnet, double slit, etc.). Thus we may view the **collapse** as the quantum analogue of the change of conditional probability when the context changes due to new information. Recognizing this removes another piece of strangeness from quantum physics.

3.6 The stochastic description of a deterministic system

A stochastic description of a deterministic system is a reduced deterministic description by moments rather than details. Formally, it is obtained by restricting an algebra of commuting, classical quantities describing a given deterministic system to the subalgebra of quantities completely symmetric in some properties declared **indistinguishable** for the purposes of the reduced (or coarse-grained) description. In the simplest case, where expectation is defined as sample expectation, the individual realizations over which the sample mean is taken are declared indistinguishable, with the consequence that only symmetric functions of realizations – hence functions of expectations – are available in the reduced descriptions.

In the terminology of knowledge discussed in Subsection 3.2, such a reduction amounts to forgetting or ignoring information known from the more detailed model. We have additional modeling uncertainty due to the lack of detail in the description used. This description is independent of a probabilistic interpretation.¹⁴ It just means that one only considers a limited family of well-behaved relevant quantities in place of the multitude of quantities in a more detailed description.

The analysis presented here allows one to apply statistical models to complicated *deterministic* situations – not only in physics – and to *single* complicated spatial events or time series. In each case, a suitable concept of expectation is introduced that, as a figure of speech, allows one to make probabilistic and other statistical statements about deterministic situations.

Important examples of statistical models for deterministic situations with increasingly random appearance are:

- (i) The deterministic but irregular sequence of prime numbers (TENNENBAUM [55]). Here experiments are the natural numbers, and expectations are introduced through a mathematically rigorous limit.
- (ii) Rounding errors in deterministic floating-point computations (VIGNES [56]). Here

¹⁴though via the construction of Subsection 3.4 below, it can always be given one in terms of c-probabilities

experiments are sequences of floating-point operations, and expectations are introduced through an empirical model for single rounding errors and an assumption of independence.

(iii) Texture in a single (hence fully determined) picture (HEEGER & BERGEN [20]). Here experiments are hypothetical images of the same size, and expectations are introduced through a mean over a (not well-defined) neighborhood of pixels.

(iv) Economic time series, e.g., prices of oil and electricity (GRANGER & NEWBOLD [18]). Here experiments are hypothetical scenarios for the time series, and expectations are introduced through a no longer fully well-defined time average.

(v) Tomorrow's weather prediction, when based on classical fluid mechanics (GNEITING & RAFTERY [17]). Here experiments are again hypothetical weather scenarios, and expectations are introduced through an even less well-defined space-time average.

(vi) Deciding for red or black by spinning a roulette wheel (HOPF [25]). Here expectations may be introduced through symmetry arguments (for an ideal wheel), or through ergodic theory.

(vii) The classical statistical mechanics of an ideal gas. Here BOLTZMANN [6] introduced expectations through an average over all particles.

(viii) The classical statistical mechanics of solids and fluids. Here GIBBS [16] introduced expectations through a fictitious average over many systems with identical macroscopic properties.

4 The thermal interpretation of quantum field theory

In the thermal interpretation, the fundamental description of reality is taken to be standard relativistic quantum field theory.¹⁵ On the formal (uninterpreted) level, the formal core of quantum physics is valid for both quantum mechanics and quantum field theory. But since the algebra of quantities considered is different, there are two essential differences between quantum mechanics and quantum field theory:

First, in place of position and momentum of finitely many particles in quantum mechanics one has in quantum field theory operators for fields. Each field ϕ has a space-time dependence that satisfy Galilei or Poincare invariance and causal commutation relations. Thus each field provides an infinitude of uncertain quantities. More precisely, since from a rigorous point of view, field operators $\phi(x, t)$ at spatial position x and time t are distribution-valued operators, the quantities in quantum field theory are **smearred fields**, local space-time integrals

$$\phi(f) = \int_{\Omega} f(x, t)^T \phi(x, t) dx dt$$

over local patches Ω in space-time, where f is a smooth **test function** (e.g., a Gaussian), and multipoint generalizations of these.

¹⁵It does not matter whether or not there is a deeper underlying structure such as that of string theory, in terms of which quantum field theory would be an effective theory only.

Second, unlike in quantum mechanics, position in quantum field theory is not an operator but a parameter, hence has no associated uncertainty. The uncertainty is instead in the quantities described by the details of the test functions f associated with real field measurements.

In this section we show that these differences strongly affect the relation between quantum field theory and reality. Among the beables of quantum field theory (discussed in Subsection 4.1) are smeared field expectations and pair correlation functions, which encode most of what is of experimental relevance in quantum field theory. Subsection 4.2 comments on relativistic quantum field theory at finite times, a usually much neglected topic essential for a realistic interpretation of the universe in terms of quantum field theory, given in Subsection 4.3. Finally we discuss notions of causality (Subsection 4.4) and nonlocality (Subsection 4.5) and their relation to the thermal interpretation.

4.1 Beables and observability in quantum field theory

The most directly observable (and hence obviously real) features of a macroscopic system modeled by quantum field theory are the q-expectations of smeared versions of the most important quantum fields, integrated over cells of macroscopic or mesoscopic size.

This identification can be made because statistical mechanics allows one to derive for the q-expectations of the fields the equations of state of equilibrium thermodynamics for cells of macroscopic size in thermal equilibrium and the hydromechanical equations for cells of mesoscopic size in local equilibrium. Both are known to yield excellent macroscopic descriptions of matter.

For macroscopic systems, one must necessarily use a coarse-grained description in terms of a limited number of parameters. In the quantum field theory of macroscopic objects, any averaging necessary for applying the law of large numbers is already done inside the definition of the macroscopic (i.e., smeared) operator to be measured. As shown in statistical mechanics, this is sufficient to guarantee very small uncertainties of macroscopic q-observables. Thus one does not need an additional averaging in terms of multiple experiments on similarly prepared copies of the system. This is the deeper reason why quantum field theory can make accurate predictions for single measurements on macroscopic systems.

In addition to the macroscopic ontology just described, the thermal interpretation also has a microscopic ontology concerning the reality of inferred entities. As in our earlier discussion of quantum mechanics, the thermal interpretation declares as real but not directly observable any q-expectation $\langle A(x, t) \rangle$ of operators and any **q-correlation**, the q-expectation of a product of operators at pairwise distinct points. More precisely, the q-expectations $\langle \phi(x, t) \rangle$ of fields are distributions that produce the – in principle approximately measurable – numbers $\langle \phi(f) \rangle$ when integrated over sufficiently smooth localized test functions f . Certain q-correlations are also measurable in principle by probing the state with external fields in linear or nonlinear response theory. See, e.g., HÄNGGI & THOMAS [19] for classical correlations and DAVIES & SPOHN [9] for quantum correlations.

Scattering experiments provide further observable information, about time-ordered multi-

point correlations of these fields. The related S-matrices also appear in microlocal kinetic descriptions of dilute macroscopic matter at the level of the Boltzmann equation or the Kadanoff–Baym equations. These are derived from the q-expectations of products of fields at two different space-time points. (The kinetics of the Boltzmann equation derived from the particle picture has long been replaced by more accurate Kadanoff–Baym equations derived from field theory.)

2-point correlations in quantum field theory are effectively classical observables; indeed, in kinetic theory they appear as the classical variables of the Kadanoff-Baym equations, approximate dynamical equations for the 2-point functions. After a Wigner transform and some further approximation (averaging over small cells in phase space), these turn into the classical variables of the Boltzmann equation. After integration over momenta and some further approximation (averaging over small cells in position space), these turn into the classical variables of the Navier-Stokes equation, hydromechanic equations that describe the behavior of macroscopic fluids. For macroscopic solids, one can use similar approximations to arrive at the equations of elasticity theory. The most detailed classical level, the Kadanoff-Baym equations, still contain the unsmearred ensemble means of field products.

According to the thermal interpretation, there is nothing in quantum field theory apart from q-expectations of the fields and q-correlations. The quantities accessible to an observer are those q-expectations and q-correlations whose arguments are restricted to the observer’s world tube. More precisely, what we can observe is contained in the least oscillating contributions to these q-expectations and q-correlations. The spatial and temporal high frequency part is unobservable due to the limited resolution of our instruments.

All macroscopic objects are objects describable by hydromechanics and elasticity theory; so their classical variables have the same interpretation. Thus the quantum-mechanical ensemble averages are classical variables. Moreover, because of the law of large numbers, $\langle f(x) \rangle \approx f(\langle x \rangle)$ for any sufficiently smooth function f of not too many variables. (These caveats are needed because high dimensions and highly nonlinear functions do not behave so well under the law of large numbers.) In particular, we get from Ehrenfest’s theorem (8) the standard classical Hamiltonian equations of motion for macroscopic objects.

Statistical mechanics shows that the uncertainties in the macroscopically relevant smeared fields scale with the inverse square root of the cell volume. This means that integrals over $\langle \Phi(x, t) \rangle$ are macroscopically meaningful only to an accuracy of order $V^{-1/2}$ where V is the volume occupied by the mesoscopic cell containing x , assumed to be homogeneous and in local equilibrium. This is the standard assumption for deriving from first principles hydrodynamical equations and the like. It means that the interpretation of a field gets more fuzzy as one reduces the amount of coarse graining – until at some point the local equilibrium hypothesis is no longer valid.

Everything deduced in quantum field theory about macroscopic properties of matter in local equilibrium or dilute matter in the kinetic regime follows, and one has a completely self-consistent setting. The transition to classicality is automatic and needs no deep investigations: The classical situation is simply the limit of a huge number of particles, where the law of large numbers discussed in Subsection 3.3 reduces the uncertainty to a level below

measurement accuracy.

Since quantum fields are quantities depending on a space-time argument, one can prepare or measure events at any particular space-time position at most once. Thus it is impossible to repeat measurements, and the standard statistical interpretation in terms of sufficiently many identically prepared systems is impossible. Therefore, the notion of ensemble cannot be understood as an actual repetition by repeated preparation.

In particular, the measurement of quantum fields is not covered by Born's rule in its standard measurement-based form. This is the reason why (unlike scattering applications) macroscopic applications of quantum field theory never invoke Born's rule. Indeed, we had concluded in Part I [38] that at finite times (i.e., outside its use in interpreting asymptotic S-matrix elements), Born's rule cannot be strictly true in relativistic quantum field theory, and hence not in nature.

4.2 Dynamics in quantum field theory

The description of dynamics in current relativistic quantum field theory textbooks is a delicate subject. In many such books, physical meaning is given only to scattering processes, i.e., the behavior at asymptotic times $t \rightarrow \pm\infty$, whose statistical properties are expressed in terms of time-ordered correlation functions. Textbooks commonly restrict their attention to the calculation of the low order contributions to the scattering amplitudes and how these are renormalized to give finite results. Questions about the quantum field dynamics at finite times are not discussed since the dynamics is deeply buried under the formal difficulties of the renormalization process needed to make relativistic quantum field theory work. Sometimes they are even claimed to be impossible to address!¹⁶ However, on the level of rigor customary in theoretical physics, quantum field dynamics at finite time is actually well-defined in terms of the so-called closed time path (CTP) approach; see, e.g., CALZETTA & HU [8].

Since the traditional Schrödinger picture breaks manifest Poincaré invariance, relativistic QFT is almost always treated in the Heisenberg picture. The Heisenberg dynamics on the fields is given by

$$\frac{\partial}{\partial x_\nu} \phi(x) = p_\nu \angle \phi(x),$$

where p_ν is the ν th operator component of the **4-momentum** vector p , defined as the generator of the translations of the Poincaré group. In particular, the physical Hamiltonian $H = cp_0$, where c is the speed of light, is obtained after the construction of the N -point functions (q-expectations of fields and q-correlations) as the operator generating the time shift of the fields.

¹⁶For example, SCHARF [51, introduction to Chapter 2] writes: "The more one thinks about this situation, the more one is led to the conclusion that one should not insist on a detailed description of the system in time. From the physical point of view, this is not so surprising, because in contrast to non-relativistic quantum mechanics, the time behavior of a relativistic system with creation and annihilation of particles is unobservable. Essentially only scattering experiments are possible, therefore we retreat to scattering theory. One learns modesty in field theory."

In the context of the thermal interpretation, a natural covariant generalization of the Ehrenfest equation (8) is obtained by considering, in place of the time-dependent q-expectations in the nonrelativistic setting discussed before, q-expectations dependent on a space-time location x in Minkowski space, and to require that the resulting space-time dependent q-expectations $\langle A \rangle_x$ satisfy the **covariant Ehrenfest equation**

$$\frac{\partial}{\partial x_\nu} \langle A \rangle_x = \langle p_\nu \angle A \rangle_x. \quad (26)$$

One easily concludes that for arbitrary space-time points x, y, z, w, \dots ,

$$\langle \phi(z) \rangle_x = \langle \phi(z + x - y) \rangle_y,$$

generalizing the nonrelativistic (10), and

$$\langle \phi(z)\phi(w)\dots \rangle_x = \langle \phi(z + x - y)\phi(w + x - y)\dots \rangle_y.$$

Thus the complete spacetime-dependence of q-expectations, and in particular their dynamics, is determined by the q-expectations at any particular fixed space-time position.

From the covariant Ehrenfest picture we may now deduce a **covariant Schrödinger picture**, by writing

$$\langle A \rangle_x := \text{Tr } \rho(x) A$$

with a space-time dependent density operator ρ . Then (26) becomes the **covariant von Neumann equation**

$$i\hbar \frac{\partial}{\partial x_\nu} \rho(x) = [p_\nu, \rho(x)].$$

The rank is preserved. Hence if ρ has rank 1, this is equivalent with writing $\rho(x) = \psi(x)\psi(x)^*$, where $\psi(x)$ satisfies the **covariant Schrödinger equation**

$$i\hbar \frac{\partial}{\partial x_\nu} \psi(x) = p_\nu \psi(x).$$

After rescaling, we see that in quantum field theory, time $t_\nu = x_\nu/c$ and energy $H_\nu = cp_\nu$ – not space and momentum – have become 4-dimensional:

$$i\hbar \frac{\partial}{\partial t_\nu} \psi(x) = H_\nu \psi(x).$$

4.3 The universe as a quantum system

Our solar system can be approximately treated classically, but from the fundamental point of view of quantum field theory it must be considered as a quantum system. The state of the solar system, when modeled by quantum fields, completely specifies what happens in any small space-time region within the solar system.

Traditional interpretations of Copenhagen flavor require that a quantum system is measured by an external classical apparatus. They cannot apply to the quantum field theory of

our solar system, say, since we do not have access to an external classical apparatus for measuring this system. The astronomers doing measurements on the solar system are part of the system measured – a situation outside the Copenhagen setting. The thermal interpretation has no such problems.

The thermal interpretation is even consistent with assigning a well-defined (though only superficially known) state to the whole universe, whose properties account for everything observable within this universe.

Unlike in conventional single-world interpretations of quantum physics, nothing in the thermal interpretation depends on the existence of measurement devices (which were not available in the very far past of the universe). Thus the thermal interpretation allows one to consider the single **universe** we live in as a quantum system, the smallest closed physical system containing us, hence strictly speaking the only system to which unitary quantum physics applies rigorously.

There is no longer a physical reason to question the existence of the state of the whole universe, even though all its details may be unknown for ever. Measuring all q-observables or finding its exact state is already out of the question for a small macroscopic quantum system such as a piece of metal. Thus, as for a metal, one must be content with describing the state of the universe approximately.

What matters for a successful physics of the universe is only that we can model (and then predict) those q-observables of the universe that are accessible to measurement at the temporal and spatial scales of human beings. Since all quantities of interest in a study of the universe as a whole are macroscopic, they have a tiny uncertainty and are well-determined even by an approximate state. For example, one could compute from a proposed model of the universe the uncertain values of the electromagnetic field at points where we can measure it, and (if the computations could be done) should get excellent agreement with the measurements.

Since every q-observable of a subsystem is also a q-observable of the whole system, the state of the universe must be compatible with everything we have ever empirically observed in the universe. This implies that the state of the universe is highly constrained since knowing this state amounts to having represented all physics accessible to us by the study of its subsystems. This constitutes a very stringent test of adequacy of a putative state of the universe.

Cosmology studies the state of the universe in a very coarse (and partly conjectured) approximation where even details at the level of galaxies are averaged over. Only for q-observables localized in the solar system we have a much more detailed knowledge.

Of course, a more detailed discussion of the state of the universe should include gravitation and hence would touch on the difficult, unsolved problem of quantum gravity. However, the thermal interpretation gives at least a consistent interpretational framework in which to discuss these questions - without having to worry about whether the concepts used to formulate these questions mean anything in the context of a quantum system without external observers.

In relativistic quantum field theory, the basic fields are local objects in the sense that smeared fields $\phi(f)$ and $\phi(g)$ commute whenever f and g have spacelike separated support. Nevertheless, a quantum state specifies the q-expectations $\langle\phi(f)\rangle$ for arbitrary smooth test functions f and the higher order moments (n -point correlation functions) $\langle\phi(f_1)\dots\langle\phi(f_n)\rangle$ for arbitrary smooth test functions f_1, \dots, f_n , hence makes statements about uncertain values at all space-time locations. Hence relativistic quantum field theory necessarily describes a complete universe. When gravitation is not modeled, everything happens in a Minkowski spacetime.

4.4 Relativistic causality

Phrases often found in the physical literature as 'disturbance of phenomena by observation' or 'creation of physical attributes of objects by measurements' represent a use of words like 'phenomena' and 'observation' as well as 'attribute' and 'measurement' which is hardly compatible with common usage and practical definition and, therefore, is apt to cause confusion. As a more appropriate way of expression, one may strongly advocate limitation of the use of the word phenomenon to refer exclusively to observations obtained under specified circumstances, including an account of the whole experiment.

Niels Bohr, 1948 [5]

In each experiment, irrespective of its history, there is only one quantum system, which may consist of several particles or other subsystems, created or annihilated at the various interventions.

Asher Peres and Daniel Terno, 2002 [47, p.98]:

We now consider relativistic causality in a Minkowski spacetime. By working with charts, everything said generalizes to the case of curved spacetime. Quantized spacetime is not discussed since there is no accepted framework for it.

A **point object** has, at any given time in any observer's frame, properties only at a single point, namely the point in the intersection of its world line and the spacelike hyperplane orthogonal to the observer's 4-momentum at the time (in the observer frame) under discussion. An **extended object** has properties that depend on more than one spacelike-separated space-time position. A **joint property** is a property that explicitly depends on more than one space-time location within the space-time region swept out by the extended object in the course of time.

Note that from a fundamental point of view there is no clear demarcation line that would tell when a system of particles (e.g., a molecule, or a solar system) should or should not be regarded as a single object. The thermal interpretation therefore treats arbitrary subsystems of a large system as a single object if they behave in some respect like a unity.

We may distinguish three Poincaré invariant definitions of causality.

- **Point causality:** Properties of a point object depend only on its closed past cones, and can influence only its closed future cones. This is used in special relativity, which discusses the motion of a single classical particle in a classical external field.

- **Separable causality:** Joint properties of an extended object consist of the combination of properties of their constituent points. This is intuitively assumed in all discussions of Bell-type nonlocality, and is in conflict with experiments involving highly entangled photons.
- **Extended causality:** Joint properties of an extended object depend only on the union of the closed past cones of their constituent parts, and can influence only the union of the closed future cones of their constituent parts. This is the version that can probably be derived from relativistic quantum field theory, where particles are localized excitations of the quantum field, and hence extended objects.

All three notions of causality agree on the causality properties of point objects ('point causality') but differ on the causality properties of extended objects. If one regards an entangled quantum system as a system of point particles one runs into lots of counterintuitive conceptual problems. If one regards an entangled quantum system as a single extended system in the above sense, all such difficulties disappear.

Extended causality is the form of causality appropriate for the thermal interpretation. It takes into account what was known almost from the outset of modern quantum physics - that quantum objects are intrinsically extended and must be treated as whole.

The extended system view gives the appropriate intuition. The violation of Bell inequalities in experiments such as those by ASPECT [2] (cf. Subsection 4.5) show that neither point causality or separable causality can be realized in nature. But extended causality is not ruled out by current experiments. EBERHARD & ROSS [13] gives a proof of causality from relativistic quantum field theory, in the sense that no faster than light communication is possible.

4.5 Nonlocal correlations and conditional information

Bell's theorem, together with experiments that prove that Bell inequalities are violated imply that reality modeled by deterministic process variables is intrinsically nonlocal. The thermal interpretation explicitly acknowledges that all quantum objects (systems and subsystems) have an uncertain, not sharply definable (and sometimes extremely extended) position, hence are intrinsically nonlocal. Thus it violates the assumptions of Bell's theorem and its variations.

Here we briefly discuss some related aspects of nonlocality from the point of view of the thermal interpretation.

We use the example of an entangled 2-photon state. In quantum physics, there is a definite concept of a system in a 2-photon state, but only a fuzzy one of 'two photons'. Attempting to literally interpret the two photons in a system with an entangled 2-photon state leads to paradoxes related to seemingly acausal nonlocal correlations.

Given the quantum mechanical 2-photon system together with the Schrödinger dynamics determined by the associated dispersion relation, Born's rule makes assertions about measurements anywhere in the universe at any future time! Something more nonlocal cannot be

conceived. It is therefore no surprise that intuition is violated. However, the thermal interpretation renounces the universal validity of Born's rule. Instead, the nonlocal correlations get their natural explanation in terms of extended causality and conditional information.

Consider the conceptual setting of a typical Bell-type experiment, where entangled 2-photon systems are subjected to measurements by two far away observers conventionally called Alice and Bob, at times and positions corresponding to spacelike distances. Their measurement results are later compared by another observer called Charles, say.

In an experiment checking Bell inequalities, an object in the form of a 2-photon system may be prepared at the source by parametric down-conversion and propagating freely in two opposite directions. Whatever Alice and Bob measure far away depends on the whole 2-photon system. According to the thermal interpretation, the object described by this 2-photon system is an extended object. Over long distances, the uncertainty intrinsic to the 2-photon system becomes huge. The object becomes vastly extended – so nonlocal that the assumptions in Bell's argument are obviously violated. It is not surprising that the conclusions can be violated, too.

Any meaningful use of the notion of causality depends on a notion of before and after, which does not exist in the case of correlations at spacelike distances. Because of the Lorentz invariance of all relativistic arguments, one cannot say that Alice's actions and observations cause (or affect) Bob's observations to be correlated once – as usually assumed – Alice's and Bob's position are causally unrelated. For in this case, there are always Lorentz frames in which Alice acts later than Bob observes, and others in which Bob acts later than Alice. So neither can be said to cause (or affect) the other observer (cf. PERES & TERNO [47, 48]). Whatever statistics can be made (by Bob or Charles) from data collected by Bob before Alice's choices or results become available to Charles – it will be completely unaffected by the behavior of Alice and her detector.

But something else from Alice becomes known to Bob faster than light – conditional information. **Conditional information** is information deduced from what – given past and present observations available to a local observer – is known from theory but is not observed itself by this observer. Additional observations may make conditional information more precise. But as long as part of the data in the condition is not yet known, nothing conclusive is known. Having about tomorrow's weather the conditional information that 'Should there be no clouds it will not rain' tells in fact nothing useful about the weather tomorrow, unless we have information about tomorrow's clouds.

Similarly, Bob gets conditional information of the kind: 'Should Alice have measured X then her result was Y.' Because Bob does not know whether the hypothesis holds, he knows nothing useful. Bob's claimed knowledge about the results of Alice's measurement is sound only if Alice actually measured something. If she instead took a nap, or if her detector failed because of a power outage, Bob concluded something wrongly.

Causality only demands that information flow is limited by the speed of light. Nothing in relativity forbids conditional information to be passed faster than light. For example, we know lots of conditional information about what can or cannot happen inside black holes although no information can flow out from there. Such conditional information is obtained from theory independent of observation. But theoretical conclusions apply instantaneously

and have no speed limit.

In Bell-type experiments, the conditional information and the correlations become actual only when someone (like Charles) has access to the actual data resolving the condition. It is easily seen that extended causality is observed.

One may ask, however, how it is possible that, in actual long-distance entanglement experiments performed in the past, when Charles checked the findings he found Alice's conditional information unconditionally satisfied. The reason for this is that, when preparing actual long-distance entanglement experiments, the experimenters make sure that nothing will interrupt the expected flow of events needed for a correct experimental performance. This means that they in fact prepare not only the source of the entangled photon pairs but also Alice's and Bob's environment and the whole environment the photons traverse during the experiment. This preparation must be careful enough to exclude all events that would cause unexpected changes to the intended protocol or remove the delicate entanglement. But this means that the preparation deposits in the past light cones of both Alice and Bob enough correlated classical information that influences the present of Alice and Bob when they perform their choices. This doesn't explain everything about the observed correlations but casts some doubt on the validity of the stringent assumptions made in derivations of Bell-type inequalities.

5 Conclusion

The thermal interpretation

- acknowledges that there is only one world;
- has no split between classical and quantum mechanics – the former emerges naturally as the macroscopic limit of the latter;
- is by design compatible with the classical ontology of ordinary thermodynamics;
- is description-dependent but observer-independent, hence free from subjective elements;
- is about both real systems and idealized systems, at every level of idealization;
- applies both to single quantum objects (like a quantum dot, a neutron star, or the universe) and to statistical populations;
- satisfies the principles of locality and Poincare invariance, as defined in relativistic quantum field theory;
- is compatible with relativistic extended causality;
- uses no concepts beyond what is taught in every quantum physics course;
- involves no philosophically problematic steps.

No other interpretation combines these merits. The thermal interpretation leads to a gain in clarity of thought. This results in saving a lot of time otherwise spent in the contemplation of aspects arising in traditional interpretations, which from the point of view of the thermal interpretation appear as meaningless or irrelevant side problems.

The thermal interpretation of quantum physics (including quantum mechanics, statistical mechanics and quantum field theory) is an interpretation of everything. It allows a consistent quantum description of the universe from the smallest to the largest levels of modeling, including its classical aspects.

These foundations are easily stated and motivated since they are essentially the foundations used everywhere for uncertainty quantification, just slightly extended to accommodate quantum effects by not requiring that q -observables commute.

In all traditional interpretations of quantum physics, the theoretical 'observables' are unobservable operators. It is no surprise that calling unobservable things observables causes apparent indeterminism. It was a historical misnomer that led to the strange, unfortunate situation in the foundations of quantum physics that persists now for nearly hundred years.

On the other hand, in the thermal interpretation of quantum physics, the theoretical observables (the beables in the sense of Bell) are the expectations and functions of them. They satisfy a deterministic dynamics. Some of these beables are practically (approximately) observable. In particular, q -expectations of Hermitian quantities and q -probabilities, the probabilities associated with appropriate self-adjoint Hermitian quantities, are among the theoretical observables. The q -expectations are approximately measured by reproducible single measurements of macroscopic quantities, or by sample means in a large number of observations on independent, similarly prepared microscopic systems. The q -probabilities are approximately measured by determining the relative frequencies of corresponding events associated with a large number of independent, similarly prepared systems.

This eliminates all foundational problems that were introduced into quantum physics by basing the foundations on an unrealistic concept of observables and measurement. With the thermal interpretation, the measurement problem turns from a philosophical riddle into a scientific problem in the domain of quantum statistical mechanics, namely how the quantum dynamics correlates macroscopic readings from an instrument with properties of the state of a measured microscopic system. This problem will be discussed in Part III [39].

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